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Sketch of solutions - Part 2

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Solution:

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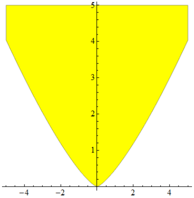

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$$\begin{bmatrix} 2 & 1 & 0 & -b & -d & -c \\ 1 & 2b & d & 0 & e & f \\ 0 & d & 2c & -e & -f & 0 \\ -b & 0 & -e & 2 & 1 & a \\ -d & e & -f & 1 & -1-2a & 0 \\ -c & f & 0 & a & 0 & 5 \end{bmatrix} \succeq 0$$

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For instance $a = -1, b = 1, c = d = e = f = 0$.

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Some (not so) helpful Yalmip code:

```
sdpvar x y z l
p=x^2*y^2+x^2*z^2+y^2*z^2+1-4*x*y*z;
q=x^4*y^2+y^4+x^2-3*x^2*y^2;

F=sos((1+x^2+y^2+z^2)*(p-l));
solvesos(F,-l,[],l)
double(l)
sdisplay(sosd(F))
```

Sketch of solutions - Part 5

9 A trignometric polynomial of degree d is a function of the form

$$p(\theta) = a_0 + \sum_{k=1}^{d} (a_k \cos(k\theta) + b_k \sin(k\theta)).$$

a Show that if d is even $p(\theta) \geq 0$ for all θ if and only if $p(\theta) = p_1(\theta)^2 + p_2(\theta)^2$ for some trignometric polynomials p_1 and p_2.

b Write a semidefinite program to certify the nonnegativity of

i $p(\theta) = 4 - \sin(\theta) + \sin(2\theta) - 3\cos(2\theta)$

ii $p(\theta) = 5 - \sin(\theta) + \sin(2\theta) - 3\cos(3\theta)$

Compute their sos certificates.

Solution idea:

Use the parametrization $\theta = 2\arctan(x)$ or, equivalently, $\cos(\theta) = \frac{1-x^2}{1+x^2}$ and $\sin(\theta) = \frac{2x}{1+x^2}$, and reduce to one variable case.