PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2006) **462**, 1877–1896
doi:10.1098/rspa.2005.1646
Published online 21 March 2006

Three-dimensional elasticity problems for the prismatic bar

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A general solution is given to the three-dimensional linear elastic problem of a prismatic bar subjected to arbitrary tractions on its lateral surfaces, subject only to the restriction that they can be expanded as finite power series in the axial coordinate z. The solution is obtained by repeated differentiation of the tractions with respect to z, establishing a set of sub-problems $\mathcal{P}_j$. A recursive procedure is then developed for generating the solution to $\mathcal{P}_{j+1}$ from that for $\mathcal{P}_j$. This procedure involves three steps: integration of the stress and displacement fields $\mathcal{P}_j$ with respect to z, using an appropriate Papkovich–Neuber (P–N) representation; solution of two-dimensional in-plane and antiplane corrective problems for the tractions in $\mathcal{P}_{j+1}$ that are independent of z; and expression of these corrective solutions in P–N form. The method is illustrated by an example.

Keywords: three-dimensional elasticity; Papkovich–Neuber solution; prismatic bar

1. Introduction

One of the major achievements in the theory of linear elasticity is the establishment of a general solution to the two-dimensional problem for the prismatic bar within the formalism of complex variable theory (Stevenson 1943, 1945; Green & Zerna 1954; Muskhelishvili 1963). The problem naturally decomposes into two sub-problems: the *plane strain* problem (Milne-Thomson 1968), in which the stress and displacement fields are independent of distance z along the axis of the bar and the only permissible tractions and body forces lie in the xy-plane, and the *antiplane* problem (Milne-Thomson 1962), in which the ends of the bar are subjected to prescribed force and/or moment resultants, but the lateral surfaces of the bar are traction-free. Certain stress and displacement components in the antiplane problem have low-order polynomial dependence on z, but the solution reduces to that of a two-dimensional boundary-value problem on the bar cross-section.

Methods for the problem of the prismatic bar with more general loading are less well developed. Michell (1901) gave a solution for the case where tractions are applied that are independent of z and this case is also discussed by Love (1927) and Sokolnikoff (1956). Some problems with higher-order polynomial

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Received 22 September 2005
Accepted 15 December 2005

loading were considered by Almansi (1901*a*,*b*). However, no systematic procedure is available for solving problems of this class. In the present paper, we shall develop such a general procedure. The problem is reduced to the solution of a succession of purely two-dimensional problems (plane strain and antiplane), interspersed with partial integrations with respect to z, which are performed within the formalism of an appropriate form of the Papkovich–Neuber (P–N) solution. Closed-form solutions can be obtained to any problem in which the loading is expressible in terms of polynomials in z and for which the corresponding plane and antiplane two-dimensional problems can be solved.

The method provides an exact solution for a wide range of beam problems and has the advantage of generality in comparison with numerical methods. It can also be used to investigate the effect of three-dimensional loading on stress-concentrating features, such as holes and cracks.

2. General considerations

Consider the problem of a long bar of uniform cross-section under arbitrary loading, the only restriction being that the boundary conditions on the ends of the bar are satisfied only in the weak, force-resultant sense. We shall assume that the bar is aligned with the z-direction and that its cross-section Ω defines either the interior of a closed curve Γ in the xy-plane, or that part of the region interior to a closed curve Γ_0 that is also exterior to one or more closed curves Γ_1, Γ_2, etc. The following derivations and examples will be restricted to the former, simply-connected case, but it will be clear from the methods used that the additional complications associated with multiply-connected cross-sections occur only in the classical solution of the two-dimensional problem.

The case where the lateral surfaces of the bar are traction-free and the *only* loading is on the ends can be treated by classical methods. In particular, a combination of axial loading and bending moments on the ends is solved exactly by the elementary bending theory, whilst torsional and shear loading on the ends can be reduced to two-dimensional potential problems in the bar cross-section.

We denote the local outward normal to Γ by $\boldsymbol{n}$ and the corresponding tangent by t, where the rotation from $\boldsymbol{n}$ to $\boldsymbol{t}$ is counterclockwise when looking in the positive z-direction. The most general loading of these surfaces therefore comprise the three traction components T_n, T_t, T_z.

(*a*) *Power series solutions*

Consider the problem in which the three tractions T_n, T_t, T_z can be written as power series in z, i.e.

$$T_n = \sum_{i=1}^{m-1} f_i(s) z^{i-1}; \quad T_t = \sum_{i=1}^{m-1} g_i(s) z^{i-1}; \quad T_z = \sum_{i=1}^{m} h_i(s) z^{i-1}, \tag{2.1}$$

where f, g, h are arbitrary functions of the curvilinear coordinate s defining position on Γ. We shall denote this system of tractions by the symbol $\boldsymbol{T}^{(m)}$. Note that the in-plane tractions T_n, T_t are carried only up to the order z^{m-2}, whereas the out-of-plane traction T_z includes a term proportional to z^{m-1}. Practical cases where the highest-order term in all three tractions is the same (z^n say) can of course be treated by setting $m=n+2$ and $h_m(s)=0$.

Complete definition of the problem requires that we also specify the force resultants $\boldsymbol{F}^{(m)}(0), \boldsymbol{M}(0)^{(m)}$ at the plane $z=0$, where

$$F_x^{(m)}=\iint_\Omega \sigma_{zx}\,\mathrm{d}\Omega;\quad F_y^{(m)}=\iint_\Omega \sigma_{zy}\,\mathrm{d}\Omega;\quad F_z^{(m)}=\iint_\Omega \sigma_{zz}\,\mathrm{d}\Omega, \tag{2.2}$$

$$M_x^{(m)}=\iint_\Omega \sigma_{zz}y\,\mathrm{d}\Omega;\quad M_y^{(m)}=-\iint_\Omega \sigma_{zz}x\,\mathrm{d}\Omega;\quad M_z^{(m)}=\iint_\Omega (x\sigma_{zy}-y\sigma_{zx})\mathrm{d}\Omega. \tag{2.3}$$

We describe this problem as $\mathcal{P}_m$ and the resulting stress and displacement fields in the bar by $\boldsymbol{\sigma}^{(m)}$, $\boldsymbol{u}^{(m)}$, respectively. Note that $m=1$ corresponds to the non-trivial case, where T_z is proportional to z^0, i.e. out-of-plane tractions that are uniform along the bar.

(b) Superposition by differentiation

Suppose that the solution to a given problem $\mathcal{P}_m$ is known, i.e. that we have found stress and displacement components $\boldsymbol{\sigma}^{(m)}$, $\boldsymbol{u}^{(m)}$ that reduce to a particular set of polynomial tractions (2.1) on the lateral surfaces of the bar and that satisfy the quasi-static equations of elasticity in the absence of body forces, which we here represent in the symbolic form

$$\mathcal{L}(\boldsymbol{\sigma}^{(m)},\boldsymbol{u}^{(m)})=0, \tag{2.4}$$

where $\mathcal{L}$ is a set of linear differential operators. Differentiating (2.4) with respect to z, we have

$$\frac{\partial}{\partial z}\mathcal{L}(\boldsymbol{\sigma}^{(m)},\boldsymbol{u}^{(m)})=\mathcal{L}\left(\frac{\partial\boldsymbol{\sigma}^{(m)}}{\partial z},\frac{\partial\boldsymbol{u}^{(m)}}{\partial z}\right)=0. \tag{2.5}$$

It follows that the new set of stresses and displacements defined by differentiation as

$$\boldsymbol{\sigma}^{(m-1)}=l\frac{\partial\boldsymbol{\sigma}^{(m)}}{\partial z};\quad \boldsymbol{u}^{(m-1)}=l\frac{\partial\boldsymbol{u}^{(m)}}{\partial z}, \tag{2.6}$$

will also satisfy the equations of elasticity and will correspond to the tractions

$$T_n=l\sum_{i=1}^{m-2} if_{i+1}(s)z^{i-1};\quad T_t=l\sum_{i=1}^{m-2} ig_{i+1}(s)z^{i-1};\quad T_z=l\sum_{i=1}^{m-1} ih_{i+1}(s)z^{i-1}. \tag{2.7}$$

The constant l has dimensions of length and is introduced to ensure that $\boldsymbol{\sigma}^{(m-1)}$, $\boldsymbol{u}^{(m-1)}$ have the dimensions of stress and displacement, respectively. It clearly cancels in equations (2.5) and its magnitude can be taken as unity without loss of generality. We therefore omit it in the subsequent derivations.

This process of generating a new particular solution by differentiation with respect to a spatial coordinate can be seen as a form of linear superposition of the original solution on itself after an infinitesimal displacement in the z-direction. Since $\boldsymbol{u}^{(m)}(x,y,z)$ satisfies (2.4), so does $\boldsymbol{u}^{(m)}(x,y,z+a)$, since this represents the same field displaced a distance a in the negative z-direction. Superposing the two fields and multiplying by the dimensionless constant l/a, we can construct the new solution of (2.4),

$$\frac{l(\boldsymbol{u}^{(m)}(x,y,z+a)-\boldsymbol{u}^{(m)}(x,y,z))}{a},$$

which reduces to $(2.6)_{\text{(ii)}}$ in the limit $a\to 0$.

The tractions (2.7) resulting from this operation are clearly of the form $\boldsymbol{T}^{(m-1)}$. There will also generally be force and moment resultants on the end given by

$$\boldsymbol{F}^{(m-1)}(0)=\frac{\mathrm{d}\boldsymbol{F}^{(m)}}{\mathrm{d}z}(0);\quad \boldsymbol{M}^{(m-1)}(0)=\frac{\mathrm{d}\boldsymbol{M}^{(m)}}{\mathrm{d}z}(0). \tag{2.8}$$

Thus, the stress field $\partial\boldsymbol{\sigma}^{(m)}/\partial z$ is the solution of a problem of the class $\mathcal{P}_{m-1}$.

Repeating this operation m times, we find that the stress and displacement fields,

$$\boldsymbol{\sigma}^{(0)}=\frac{\partial^m\boldsymbol{\sigma}^{(m)}}{\partial z^m};\quad \boldsymbol{u}^{(0)}=\frac{\partial^m\boldsymbol{u}^{(m)}}{\partial z^m}, \tag{2.9}$$

correspond to the physical problem $\mathcal{P}_0$, in which the tractions $\boldsymbol{T}=0$ and the force resultants on the end are

$$\boldsymbol{F}^{(0)}(0)=\frac{\mathrm{d}^m\boldsymbol{F}^{(m)}}{\mathrm{d}z^m}(0);\quad \boldsymbol{M}^{(0)}(0)=\frac{\mathrm{d}^m\boldsymbol{M}^{(m)}}{\mathrm{d}z^m}(0). \tag{2.10}$$

This is a classical antiplane problem and the solution $\boldsymbol{\sigma}^{(0)}$ is such that $\sigma_{xx}^{(0)}=\sigma_{xy}^{(0)}=\sigma_{yy}^{(0)}=0$, $\sigma_{zx}^{(0)},\sigma_{zy}^{(0)}$ are independent of z and $\sigma_{zz}^{(0)}$ corresponds to a linearly varying bending moment and takes the form

$$\sigma_{zz}^{(0)}=(C_1x+C_2y+C_3)z+D_1x+D_2y+D_3. \tag{2.11}$$

From these results, it is clear that the solution of problem $\mathcal{P}_m$ possesses the following features:

(i) The stress and displacement components, $\boldsymbol{\sigma}^{(m)}$, $\boldsymbol{u}^{(m)}$ can be expressed as power series in z.

(ii) The highest-order terms in the stress components $\sigma_{xx}^{(m)},\sigma_{xy}^{(m)},\sigma_{yy}^{(m)}$ are of the order z^{m-2}, since after m differentiations with respect to z they are zero.

(iii) The highest-order terms in the stress components σ_{zx},σ_{zy} are of the order z^{m-1}, since after m differentiations with respect to z they are independent of z.

(iv) The highest-order terms in the stress component σ_{zz} are of the form

$$\sigma_{zz}=\frac{(C_1x+C_2y+C_3)z^{m+1}}{(m+1)!}+\frac{(D_1x+D_2y+D_3)z^m}{m!}, \tag{2.12}$$

from (2.9) and (2.11).

(v) The exact form of the highest-order term in the stress components $\sigma_{zx}^{(m)}$, $\sigma_{zy}^{(m)}$, $\sigma_{zz}^{(m)}$ depends only on the force and moment resultants $\boldsymbol{F}$, $\boldsymbol{M}$ associated with the highest-order term in the loading.

Conclusions (ii) and (iii) explain why we chose to terminate the traction series (2.1) at z^{m-2} in T_n, T_t and at z^{m-1} in T_z.

3. Solution of $\mathcal{P}_m$ by successive partial integration

The process of differentiation elaborated in §2*b* shows that for every problem $\mathcal{P}_m$, there exists a set of lower-order problems $\mathcal{P}_j$, $j=(0, m-1)$, for which

$$\boldsymbol{T}^{(j)} = \frac{\partial \boldsymbol{T}^{(j+1)}}{\partial z}; \quad \boldsymbol{\sigma}^{(j)} = \frac{\partial \boldsymbol{\sigma}^{(j+1)}}{\partial z}; \quad \boldsymbol{u}^{(j)} = \frac{\partial \boldsymbol{u}^{(j+1)}}{\partial z}. \tag{3.1}$$

Complete definition of the sub-problems $\mathcal{P}_j$ also requires that we specify the force resultants $\boldsymbol{F}^{(j)}(0), \boldsymbol{M}^{(j)}(0)$, but we shall find that it is not necessary to identify these resultants explicitly except in the final problem $\mathcal{P}_m$.

The lowest-order problem in this set, $\mathcal{P}_0$, can always be solved by classical methods, so the more general problem $\mathcal{P}_m$ can be solved recursively if we can devise a method to generate the solution of $\mathcal{P}_{j+1}$ from that of $\mathcal{P}_j$. This process clearly involves partial integration of $\boldsymbol{\sigma}^{(j)}$, $\boldsymbol{u}^{(j)}$ with respect to z. We obtain

$$\boldsymbol{\sigma}^{(j+1)} = \int \boldsymbol{\sigma}^{(j)} \mathrm{d}z + \boldsymbol{f}(x,y), \tag{3.2}$$

where $\boldsymbol{f}(x,y)$ is an arbitrary function of x, y only. This function is required to satisfy two conditions:

(i) the complete stress field $\boldsymbol{\sigma}^{(j+1)}$, including $\boldsymbol{f}(x,y)$, must satisfy the equations of elasticity in the strong sense, and
(ii) the stresses must reduce to the known tractions $\boldsymbol{T}^{(j+1)}$ on the lateral surfaces of the bar.

Condition (ii) involves only the coefficient of z^0 in $\boldsymbol{T}^{(j+1)}$, since the coefficients of higher-order terms will have been taken care of at an earlier stage in the recursive process. Similarly, the conditions imposed by the equations of elasticity can only arise in the lowest-order terms in the stress field.

Since methods for solving the two-dimensional problem $\mathcal{P}_0$ are well established, there are clear advantages in choosing a strategy for determining $\boldsymbol{f}(x,y)$ that will make use of these methods. We shall achieve this purpose by expressing the solution of each sub-problem $\mathcal{P}_j$ in the formalism of the P–N solution. Thus, the integration (3.2) will actually be performed on the P–N potentials, rather than directly on the stress components, and the requirement (i) that the solution satisfy the equations of elasticity will therefore be met by ensuring that the integrated potentials are three-dimensional harmonic functions. Correction of the zeroth-order term in the boundary tractions (ii) will then correspond to the combination of a plane and antiplane problem on the cross-section, which can be treated by classical methods. Since the P–N solution acts as the vehicle through which the solution is transmitted from stage j to $j+1$, we shall also need to express the solution of these two-dimensional problems in P–N form. The formalism needed for these steps will be developed in §4.

4. The Papkovich–Neuber solution

A general solution of the equations of elasticity without body forces can be written in the form

$$2\mu\boldsymbol{u} = -4(1-\nu)\psi + \boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{\psi} + \phi), \tag{4.1}$$

where μ, ν are the modulus of rigidity and Poisson's ratio, $\boldsymbol{r}$ is the position vector and $\boldsymbol{\psi}, \phi$ are three-dimensional harmonic vector and scalar potentials, respectively (Barber 2002, §18.3). We wish to perform the integrations (3.2) on the potentials $\boldsymbol{\psi}, \phi$ instead of on the stress and displacement components, but these two processes are not generally equivalent because of the dependence on z implied by $\boldsymbol{r}$ in (4.1).

This z-dependence can be avoided by eliminating the component ψ_z of the vector potential $\boldsymbol{\psi}$, which is therefore restricted to the xy-plane. A 'proof' that this can be done without loss of generality has been given by various authors, but Sokolnikoff (1956, p. 331 *et seq.*) showed that the proof fails unless the geometry of the body meets certain conditions. Fortunately, a sufficient condition for the elimination of ψ_z is that any straight line parallel to the z-axis cuts the boundary of the body in no more than two points (Eubanks & Sternberg 1956; see also Barber 2002, §18.4). This condition is clearly satisfied by the prismatic bar under consideration.

The displacements for the P–N solution can then be expanded as

$$\left.\begin{aligned} 2\mu u_x &= \frac{\partial\phi}{\partial x} + x\frac{\partial\psi_x}{\partial x} - (3-4\nu)\psi_x + y\frac{\partial\psi_y}{\partial x}, \\ 2\mu u_y &= \frac{\partial\phi}{\partial y} + x\frac{\partial\psi_x}{\partial y} + y\frac{\partial\psi_y}{\partial y} - (3-4\nu)\psi_y, \\ 2\mu u_z &= \frac{\partial\phi}{\partial z} + x\frac{\partial\psi_x}{\partial z} + y\frac{\partial\psi_y}{\partial z}, \end{aligned}\right\} \tag{4.2}$$

where ψ_x, ψ_y are the remaining components of $\boldsymbol{\psi}$ and the corresponding stress components can then be obtained from the strain–displacement and stress–strain relations as

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial x^2} + x\frac{\partial^2\psi_x}{\partial x^2} - 2(1-\nu)\frac{\partial\psi_x}{\partial x} + y\frac{\partial^2\psi_y}{\partial x^2} - 2\nu\frac{\partial\psi_y}{\partial y}, \tag{4.3}$$

$$\sigma_{xy} = \frac{\partial^2\phi}{\partial x\,\partial y} + x\frac{\partial^2\psi_x}{\partial x\,\partial y} - (1-2\nu)\frac{\partial\psi_x}{\partial y} + y\frac{\partial^2\psi_y}{\partial x\,\partial y} - (1-2\nu)\frac{\partial\psi_y}{\partial x}, \tag{4.4}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial y^2} + x\frac{\partial^2\psi_x}{\partial y^2} - 2\nu\frac{\partial\psi_x}{\partial x} + y\frac{\partial^2\psi_y}{\partial x^2} - 2(1-\nu)\frac{\partial\psi_y}{\partial y}, \tag{4.5}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} + x\frac{\partial^2\psi_x}{\partial z^2} - 2\nu\frac{\partial\psi_x}{\partial x} + y\frac{\partial^2\psi_y}{\partial z^2} - 2\nu\frac{\partial\psi_y}{\partial y}, \tag{4.6}$$

$$\sigma_{yz} = \frac{\partial^2\phi}{\partial y\,\partial z} + x\frac{\partial^2\psi_x}{\partial y\,\partial z} + y\frac{\partial^2\psi_y}{\partial y\,\partial z} - (1-2\nu)\frac{\partial\psi_y}{\partial z}, \tag{4.7}$$

$$\sigma_{zx} = \frac{\partial^2\phi}{\partial z\,\partial x} + x\frac{\partial^2\psi_x}{\partial z\,\partial x} - (1-2\nu)\frac{\partial\psi_x}{\partial z} + y\frac{\partial^2\psi_y}{\partial z\,\partial x}, \tag{4.8}$$

(see Barber 2002, ch. 19).

(*a*) *Complex variable form*

For our purposes, there is considerable advantage in restating the P–N solution in complex variable terms, using the notation $\zeta = x + \iota y, \bar{\zeta} = x - \iota y$ and

$$u = u_x + \iota u_y; \quad \Theta = \sigma_{xx} + \sigma_{yy}; \quad \Phi = \sigma_{xx} + 2\iota\sigma_{xy} - \sigma_{yy}; \quad \Psi = \sigma_{zx} + \iota\sigma_{zy},$$

as in Green (1949). With this notation, Θ is invariant with respect to in-plane coordinate transformation and Φ, Ψ transform according to the rules

$$\Phi_\alpha = \mathrm{e}^{-2\iota\alpha}\,\Phi; \quad \Psi_\alpha = \mathrm{e}^{-\iota\alpha}\,\Psi, \tag{4.9}$$

where Φ_α, Ψ_α are defined in a coordinate system rotated anticlockwise through an angle α with respect to x, y.

In the resulting expressions, ϕ will be left as a scalar potential, but ψ_x, ψ_y will be combined as

$$\psi = \psi_x + \iota\psi_y. \tag{4.10}$$

After routine manipulations, we obtain

$$2\mu u = 2\frac{\partial\phi}{\partial\bar{\zeta}} - (3-4\nu)\psi + \zeta\frac{\partial\bar{\psi}}{\partial\bar{\zeta}} + \bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}}, \tag{4.11}$$

$$2\mu u_z = \frac{\partial\phi}{\partial z} + \frac{1}{2}\left(\bar{\zeta}\frac{\partial\psi}{\partial z} + \zeta\frac{\partial\bar{\psi}}{\partial z}\right), \tag{4.12}$$

$$\Theta = -\frac{\partial^2\phi}{\partial z^2} - 2\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) - \frac{1}{2}\left(\zeta\frac{\partial^2\bar{\psi}}{\partial z^2} + \bar{\zeta}\frac{\partial^2\psi}{\partial z^2}\right), \tag{4.13}$$

$$\Phi = 4\frac{\partial^2\phi}{\partial\bar{\zeta}^2} - 4(1-2\nu)\frac{\partial\psi}{\partial\bar{\zeta}} + 2\zeta\frac{\partial^2\bar{\psi}}{\partial\bar{\zeta}^2} + 2\bar{\zeta}\frac{\partial^2\psi}{\partial\bar{\zeta}^2}, \tag{4.14}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} - 2\nu\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) + \frac{1}{2}\left(\bar{\zeta}\frac{\partial^2\psi}{\partial z^2} + \zeta\frac{\partial^2\bar{\psi}}{\partial z^2}\right), \tag{4.15}$$

$$\Psi = 2\frac{\partial^2\phi}{\partial\bar{\zeta}\,\partial z} - (1-2\nu)\frac{\partial\psi}{\partial z} + \zeta\frac{\partial^2\bar{\psi}}{\partial\bar{\zeta}\,\partial z} + \bar{\zeta}\frac{\partial^2\psi}{\partial\bar{\zeta}\,\partial z}, \tag{4.16}$$

where $\phi(\zeta, \bar{\zeta}, z)$ is a real three-dimensional harmonic function and $\psi(\zeta, \bar{\zeta}, z)$ is a complex three-dimensional harmonic function. In other words,

$$\frac{\partial^2\phi}{\partial z^2} + 4\frac{\partial^2\phi}{\partial\zeta\,\partial\bar{\zeta}} = 0; \quad \frac{\partial^2\psi}{\partial z^2} + 4\frac{\partial^2\psi}{\partial\zeta\,\partial\bar{\zeta}} = 0. \tag{4.17}$$

(*b*) *Development of harmonic potentials by integration*

The complex variable form is particularly useful in the integration process (3.2). For example, if we have a function ψ_j satisfying (4.17), we first integrate with respect to z, obtaining

$$\psi_{j+1} = \psi_{j+1}^P + f(\zeta, \bar{\zeta}), \tag{4.18}$$

where

$$\psi_{j+1}^{P} = \int \psi_j(\zeta, \bar{\zeta}, z)\mathrm{d}z, \tag{4.19}$$

is any partial integral (not necessarily harmonic) and f is an arbitrary function of $\zeta, \bar{\zeta}$. Substitution into (4.17) then yields

$$4\frac{\partial^2 f}{\partial\zeta\,\partial\bar{\zeta}} = -4\frac{\partial^2\psi_{j+1}^{P}}{\partial\zeta\,\partial\bar{\zeta}} - \frac{\partial^2\psi_{j+1}^{P}}{\partial z^2}. \tag{4.20}$$

The right-hand side of this equation is necessarily independent of z, since the z-derivative of ψ_{z+1}^{P} is ψ_j from (4.19) and this satisfies (4.17) *ex hyp.* A suitable function f can therefore always be obtained by integration with respect to ζ and $\bar{\zeta}$.

(*c*) *Relation to spherical harmonics*

A special category of harmonic function which will feature extensively in the subsequent solution process is that in which the lowest-order (two-dimensional) potential corresponds to a power of ζ. If we define the function

$$\chi_0^m(\zeta, \bar{\zeta}, z) = \zeta^m, \tag{4.21}$$

and apply the integration (4.18)–(4.20) procedure several times, we shall generate the sequence of functions,

$$\chi_1^m(\zeta, \bar{\zeta}, z) = z\zeta^m, \tag{4.22}$$

$$\chi_2^m(\zeta, \bar{\zeta}, z) = \frac{z^2\zeta^m}{2!} + \frac{\zeta^{m+1}\bar{\zeta}}{(-4)(m+1)}, \tag{4.23}$$

$$\chi_3^m(\zeta, \bar{\zeta}, z) = \frac{z^3\zeta^m}{3!} + \frac{z\zeta^{m+1}\bar{\zeta}}{(-4)(m+1)}, \tag{4.24}$$

$$\chi_4^m(\zeta, \bar{\zeta}, z) = \frac{z^4\zeta^m}{4!} + \frac{z^2\zeta^{m+1}\bar{\zeta}}{(-4)(m+1)(2!)} + \frac{\zeta^{m+2}\bar{\zeta}^2}{(-4)^2(m+1)(m+2)(2)}, \tag{4.25}$$

which are three-dimensionally harmonic and which satisfy the recurrence relations

$$\chi_{n+1}^m(\zeta, \bar{\zeta}, z) = \int \chi_n^m(\zeta, \bar{\zeta}, z)\mathrm{d}z; \quad \chi_n^{m+1}(\zeta, \bar{\zeta}, z) = (m+1)\int \chi_n^m(\zeta, \bar{\zeta}, z)\mathrm{d}\zeta. \tag{4.26}$$

These functions can be expressed in terms of spherical harmonics in the form

$$\chi_n^m(\zeta, \bar{\zeta}, z) = \frac{(-2)^m m! R^{m+n} P_{m+n}^m(z/R)}{(n+2m)!}(\zeta\bar{\zeta})^{m/2}, \tag{4.27}$$

where

$$R = \sqrt{z^2 + \zeta\bar{\zeta}} = \sqrt{x^2 + y^2 + z^2}, \tag{4.28}$$

is the distance from the origin and $P_n^m(\xi)$ is the Legendre function (see Barber 2002, §22.4). A few low-order potentials that we shall need later are

$$\chi_0^0 = 1; \quad \chi_1^0 = z; \quad \chi_2^0 = \frac{z^2}{2} - \frac{\zeta\bar{\zeta}}{4}; \quad \chi_3^0 = \frac{z^3}{6} - \frac{z\zeta\bar{\zeta}}{4}, \tag{4.29}$$

$$\chi_0^1 = \zeta; \quad \chi_1^1 = z\zeta; \quad \chi_2^1 = \frac{z^2\zeta}{2} - \frac{\zeta^2\bar{\zeta}}{8}; \quad \chi_3^1 = \frac{z^3\zeta}{6} - \frac{z\zeta^2\bar{\zeta}}{8}, \tag{4.30}$$

$$\chi_0^2 = \zeta^2; \quad \chi_1^2 = z\zeta^2; \quad \chi_2^2 = \frac{z^2\zeta^2}{2} - \frac{\zeta^3\bar{\zeta}}{12}; \quad \chi_3^2 = \frac{z^3\zeta^2}{6} - \frac{z\zeta^3\bar{\zeta}}{12}. \tag{4.31}$$

5. The two-dimensional problem

To start the recursive process, we need to obtain the general solution to the problem $\mathcal{P}_0$, in which the lateral tractions are zero, for the specific cross-section Ω. Physically, this corresponds to the problem in which the bar is loaded only by general (and as yet unspecified) force and moment resultants $\boldsymbol{F}(0), \boldsymbol{M}(0)$ at the end $z=0$. This is of course a classical two-dimensional elasticity problem, but we need here to cast the solution in a form consistent with the P–N solution of §4 in order to facilitate the three-dimensional solution procedure.

(a) Pure bending and axial force

Of the six force and moment resultants, three (F_z, M_x, M_y) correspond to the problems of axial loading and pure bending and have the exact elementary solution,

$$\sigma_{zz} = A_0\zeta + \bar{A}_0\bar{\zeta} + B_0, \tag{5.1}$$

where A_0 is a complex constant related to $M_x + \imath M_y$ and B_0 is a real constant related to F_z. All the remaining stress components are zero ($\Theta = \Phi = \Psi = 0$). It follows that no tractions are implied on the surface Γ for any cross-section Ω, and hence that this solution is independent of Ω. It is readily verified by substitution in (4.11)–(4.16) that this elementary stress field is generated by the choice of potentials

$$\phi = \frac{(1-2\nu)}{2(1-\nu^2)}(A_0\chi_2^1 + \bar{A}_0\bar{\chi}_2^1) + \frac{B_0\chi_2^0}{(1+\nu)}, \tag{5.2}$$

$$\psi = \frac{\bar{A}_0\chi_2^0}{(1-\nu^2)} - \frac{(1-2\nu)A_0\chi_0^2}{8(1-\nu^2)} - \frac{B_0\chi_0^1}{4(1+\nu)}. \tag{5.3}$$

(b) Shear, torsion and push

The remaining three resultants (F_x, F_y, M_z) correspond to the transmission of a torque and a shear force along the bar and involve non-zero values of Ψ. They therefore require the solution of an antiplane problem in order to render the lateral surfaces traction-free and the solution depends on the cross-section Ω. Various methods exist for the solution of this problem, the most popular involving the use of the Prandtl stress function (Barber 2002, ch. 16 and 17).

For our purposes, the most convenient method is first to generate a particular solution that is independent of Ω and then superpose a corrective potential to render the surface Γ free of tractions. For the shear problem (F_x, F_y), the bending moment must increase linearly with z and an appropriate particular solution is easily obtained by using the recurrence relation $(4.26)_{(\mathrm{i})}$ to generate harmonic partial integrals of (5.2) and (5.3) as

$$\phi^{(s)} = \frac{(1-2\nu)}{2(1-\nu^2)}(A_0\chi_3^1 + \bar{A}_0\bar{\chi}_3^1) + \frac{B_0\chi_3^0}{(1+\nu)}, \tag{5.4}$$

$$\psi^{(s)} = \frac{\bar{A}_0\chi_3^0}{(1-\nu^2)} - \frac{(1-2\nu)A_0\chi_1^2}{8(1-\nu^2)} - \frac{B_0\chi_1^1}{4(1+\nu)}. \tag{5.5}$$

The corresponding non-zero stress components are

$$\sigma_{zz} = A_0 z\zeta + \bar{A}_0 z\bar{\zeta} + B_0 z, \tag{5.6}$$

$$\Psi = -\frac{A_0(1+2\nu)\zeta^2}{4(1+\nu)} - \frac{\bar{A}_0\zeta\bar{\zeta}}{2(1+\nu)} - \frac{B_0\zeta}{2}, \tag{5.7}$$

from (5.4), (5.5) and (4.11)–(4.16). This solution is actually rather more general than is required for the shear problem, since the real constant B_0 corresponds to an axial force F_z that varies linearly with z and these terms are necessarily cancelled by the corrective solution when the surface Γ is traction-free. However, there is some advantage in retaining them here, since we shall be concerned with more general loading of Γ. In fact, the terms involving B_0 in (5.5) correspond to what Milne-Thomson (1962) calls the 'push' problem.

The torsion problem also involves z-independent non-zero values of the stress component Ψ, but in this case $\sigma_{zz}=0$. The appropriate particular solution is

$$\phi^{(t)} = 0; \quad \psi^{(t)} = -\frac{\imath C_0\chi_1^1}{2(1-\nu)}, \tag{5.8}$$

and the only non-zero stress component is

$$\Psi = \imath C_0\zeta, \tag{5.9}$$

where C_0 is a real constant.

(c) *The corrective antiplane solution*

Equations (5.7) and (5.9) define non-zero values of Ψ, and hence imply non-zero tractions T_z on Γ, which, however, are independent of z. In the three-dimensional problem, we may indeed have non-zero tractions T_z, but these will not generally correspond to those of the particular solution, and it is therefore necessary to superpose a corrective solution in which the tractions are changed without affecting the force resultants implied by the constants A_0, B_0, C_0. Thus, the corrective solution satisfies the conditions

$$\Theta = \Phi = \sigma_{zz} = 0; \quad \frac{\partial\Psi}{\partial z} = 0, \tag{5.10}$$

and it is an antiplane solution in the sense of Barber (2002, ch. 15).

A convenient representation in P–N form is obtained by writing

$$\phi = z(h+\bar{h}) - \frac{z(\zeta h' + \overline{\zeta h}')}{4(1-\nu)}; \quad \psi = \frac{z\bar{h}'}{2(1-\nu)}, \tag{5.11}$$

in (4.11)–(4.16), where $h(\zeta)$ is a function only of the complex variable ζ, $\bar{h} \equiv h(\bar{\zeta})$ and $\bar{h}' \equiv \partial\bar{h}/\partial\bar{\zeta}$. We then obtain

$$u = 0; \quad \Theta = \Phi = \sigma_{zz} = 0, \tag{5.12}$$

$$2\mu u_z = h + \bar{h}; \quad \Psi = \bar{h}'. \tag{5.13}$$

It follows that

$$\sigma_{zx} = \frac{\partial h_y}{\partial y}; \quad \sigma_{zy} = -\frac{\partial h_y}{\partial x}, \tag{5.14}$$

where $h_y = \mathrm{Im}(h)$, and hence h_y is equivalent to the real Prandtl stress function. In particular, the traction

$$T_z = \sigma_{zn} = \frac{\partial h_y}{\partial t}, \tag{5.15}$$

where t is a coordinate locally tangential to the boundary Γ (see Barber 2002, §16.1). Since these tractions are known everywhere except for the as yet unknown function h_y, we can integrate around Γ to determine h_y at all points on the boundary. It is then a standard boundary-value problem to determine the real harmonic function h_y in Ω, and hence the analytic function of which is the imaginary part. It is important that the integration of (5.15) should yield a single-valued function of t. This is equivalent to the requirement that the tractions in the pure antiplane problem should sum to a zero axial force F_z. If the applied tractions do not meet this condition, F_z will be a linear function of z, and hence B_0 will be non-zero. Thus, the single-valued condition on h_y serves to determine the constant B_0.

(*d*) *The in-plane solution*

In higher-order problems, we shall also need to make corrections for zeroth-order in-plane tractions T_n, T_t, using an appropriate form of the two-dimensional plane strain equations. If we use the complex form of the P–N solution (4.11)–(4.16) and define plane harmonic functions through

$$\phi = f(\zeta) + f(\bar{\zeta}); \quad \psi = g(\zeta), \tag{5.16}$$

where f, g are functions of the complex variable ζ, we obtain

$$2\mu u = -(3-4\nu)g + \zeta\bar{g}' + 2\bar{f}'; \quad 2\mu u_z = 0. \tag{5.17}$$

This is identical to the classical form of the plane strain complex variable solution, except for the term $2\bar{f}'$, which can be replaced if desired by a new function of $\bar{\zeta}$. The corresponding stress components are given by

$$\Theta = -2(g' + \bar{g}'), \tag{5.18}$$

$$\Phi = 2(\zeta\bar{g}'' + 2\bar{f}''), \tag{5.19}$$

$$\sigma_{zz} = -2\nu(g' + \bar{g}'), \tag{5.20}$$

$$\Psi = 0. \tag{5.21}$$

In some cases, the plane strain problem is easier to solve in the context of the real Airy stress function φ, which satisfies the biharmonic equation

$$\nabla^4\varphi = 0. \tag{5.22}$$

If this is done, a complex variable form of $\varphi(x,y)$ can be obtained by substituting $x=(\zeta+\bar{\zeta})/2$, $y=-\imath(\zeta-\bar{\zeta})/2$, and separating terms to express the real function φ in the form

$$\varphi(\zeta,\bar{\zeta}) = f_1(\zeta) + f_1(\bar{\zeta}) + \bar{\zeta}f_2(\zeta) + \zeta f_2(\bar{\zeta}), \tag{5.23}$$

which must be possible in view of (4.17) and (5.22). If this separation is not obvious, we can construct the function

$$\varphi_1 = \int \frac{\partial\varphi}{\partial\bar{\zeta}}\,\mathrm{d}\bar{\zeta}, \tag{5.24}$$

where the integration is performed treating ζ as a constant, so that at most an arbitrary constant of integration is introduced. With this construction, φ_1 will contain only those terms in φ that depend on $\bar{\zeta}$, and we conclude that

$$f_1(\zeta) = \varphi - \varphi_1. \tag{5.25}$$

Once f_1 (and hence $\bar{f}_1$) is determined, it is straightforward to find f_2 from (5.23). Comparison of the well-known expressions for the stress components,

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2}; \quad \sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\,\partial y}; \quad \sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2}, \tag{5.26}$$

with (5.18) and (5.19) then shows that $f=-f_1$, $g=-2f_2$, and hence

$$\phi = -f_1(\zeta) - f_1(\bar{\zeta}); \quad \psi = -2f_2(\zeta), \tag{5.27}$$

from (5.16).

6. Solution procedure

We are now in a position to summarize the solution procedure for the three-dimensional problem $\mathcal{P}_m$. We first differentiate the tractions $\boldsymbol{T}$ m times with respect to z in order to define the sub-problems $\mathcal{P}_j$, $j=(1,m)$. We shall be particularly interested in the zeroth-order tractions in each sub-problem, i.e. the terms in $\boldsymbol{T}^{(j)}$ that are independent of z, since these are the only terms that are active in the incremental solution.

Suppose the potentials ϕ_j, ψ_j corresponding to problem $\mathcal{P}_j$ are known. In other words, if we substitute these potentials into equations (4.11)–(4.16), the resulting stresses satisfy the boundary conditions on the tractions $\boldsymbol{T}^{(j)}$ on the lateral surfaces in problem $\mathcal{P}_j$. This solution will also contain two undetermined constants A_{j-1}, C_{j-1}, representing force and moment resultants on the end $z=0$.

To move up to the solution of problem $\mathcal{P}_{j+1}$, we proceed as follows:

(i) We define new potentials by adding in the zeroth-order solution (5.2) and (5.3) with new constants A_j, B_j, i.e.

$$\phi = \phi_j + \frac{(1-2\nu)}{2(1-\nu^2)}(A_j\chi_2^1 + \bar{A}_j\bar{\chi}_2^1) + \frac{B_j\chi_2^0}{(1+\nu)}, \tag{6.1}$$

$$\psi = \psi_j + \frac{\bar{A}_j \chi_2^0}{(1-\nu^2)} - \frac{(1-2\nu) A_j \chi_0^2}{8(1-\nu^2)} - \frac{B_j \chi_0^1}{4(1+\nu)}. \tag{6.2}$$

(ii) We next integrate ϕ, ψ with respect to z as in §4*b* and add in the torsion solution (5.8), again with a new constant C_j. Note that any terms in these potentials of the form χ_n^m integrate simply to χ_{n+1}^m in view of (4.26).

(iii) At this stage, $\boldsymbol{\Theta}, \boldsymbol{\Phi}$ will generally contain terms up to order $j-1$ in z and $\boldsymbol{\Psi}$ will contain terms up to order j. However, all except the zeroth-order terms will satisfy the boundary conditions on the lateral surfaces, since both the tractions and the stresses were obtained by one integration with respect to z from the given solution $\mathcal{P}_j$. To satisfy the boundary conditions on the zeroth-order terms in $T_n^{(j+1)}$, $T_t^{(j+1)}$, we add in the in-plane solution from §5*d* and solve an in-plane boundary-value problem exactly as in the two-dimensional case. Solvability of this in-plane problem requires that the corrective tractions be self-equilibrated in the plane and this provides a condition for determining the constants A_{j-1}, C_{j-1}.

(iv) To satisfy the conditions on the zeroth-order term in the out-of-plane tractions $T_z^{(j+1)}$, we add in the antiplane corrective solution from (5.11) and determine the function $h(\zeta)$, using the procedure outlined in §5*c*. This solution is only possible if the tractions associated with h alone are self-equilibrating, and hence the solution at this stage will also determine the constant B_j. This completes the solution of problem $\mathcal{P}_{j+1}$, except for the constants A_j, C_j.

This recursive procedure can be started at $j=0$ by assuming $\phi_0 = \psi_0 = 0$ in equations (6.1) and (6.2). After repeating the procedure m times, a solution will be obtained that satisfies the traction conditions $\boldsymbol{T}$ completely and which contains two free constants A_m, C_m. To complete the solution, we once again add in the zeroth-order solution, as in equations (6.1) and (6.2), using constants A_{m+1}, B_{m+1}. Finally, we determine the constants $A_m, C_m, A_{m+1}, B_{m+1}$ from the end conditions (2.2) and (2.3). More specifically, the conditions on $F_z(0), M(0)$ determine B_{m+1}, C_m, respectively, the conditions on $F_x(0), F_y(0)$ determine A_m (which is a complex constant, and hence has two degrees of freedom) and the conditions on $M_x(0), M_y(0)$ determine A_{m+1}.

7. Body forces

The method is readily extended to problems involving body forces $\boldsymbol{p}$ that exhibit polynomial dependence on z and arbitrary dependence on x, y. The simplest way to do this is first to seek a particular solution of the body force problem, i.e. a solution that satisfies the governing equations without regard to the boundary conditions. The stress components will also have polynomial dependence on z and the boundary conditions can therefore be corrected by superposing the solution of an appropriate problem of form $\mathcal{P}_m$ without body forces. Note that in this case, the corrective problem involves no body forces, and hence no modification is required in the above solution procedure. Since body forces most often arise from gravitational or inertia loading, the particular solution will generally exhibit low-order polynomial behaviour in all three coordinates.

Alternatively, if the body-force field is conservative, we can express it in the form

$$\boldsymbol{p} = -\boldsymbol{\nabla} V, \tag{7.1}$$

where V is a scalar potential. We can then use the P–N representation (4.1) with

$$\nabla^2\phi = \frac{(1-2\nu)\,V}{(1-\nu)}; \quad \nabla^2\psi = 0, \tag{7.2}$$

(Barber 2002, §18.5.1). The expressions for the stresses are then modified by the addition of the term $\nu V/(1-\nu)$ into the normal stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ only. In particular, Φ, Ψ are unchanged from (4.14) and (4.16), while Θ, σ_{zz} become

$$\Theta = \frac{2\nu V}{(1-\nu)} - \frac{\partial^2\phi}{\partial z^2} - 2\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) - \frac{1}{2}\left(\zeta\frac{\partial^2\bar{\psi}}{\partial z^2} + \bar{\zeta}\frac{\partial^2\psi}{\partial z^2}\right), \tag{7.3}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial z^2} - 2\nu\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) + \frac{1}{2}\left(\bar{\zeta}\frac{\partial^2\psi}{\partial z^2} + \zeta\frac{\partial^2\bar{\psi}}{\partial z^2}\right). \tag{7.4}$$

With these modifications, the procedure of §6 can be applied directly to conservative body-force problems.

8. Example: the cylinder loaded by a linearly increasing concentrated tangential force

To illustrate the procedure, we consider the example of a solid cylinder of unit radius, loaded by a concentrated tangential force that is a linear function of z, as shown in figure 1. The end $z=0$ is unloaded, so the six force resultants of equations (2.2) and (2.3) are all zero. The tractions for this problem are

$$T_r = 0; \quad T_\theta = F_0 z\delta(\theta); \quad T_z = 0, \tag{8.1}$$

where the components are now conveniently defined in cylindrical polar coordinates (r, θ, z). Comparison with (2.1) shows that this is a problem of class $\mathcal{P}_3$. The sub-problem $\mathcal{P}_2$ corresponds to the tractions

$$T_r = 0; \quad T_\theta = F_0\delta(\theta); \quad T_z = 0, \tag{8.2}$$

and the tractions in $\mathcal{P}_1$ are zero.

We start at step (i) with $\phi_0 = \psi_0 = 0$, and after integration and adding the torsion solution the potentials are

$$\phi = \frac{(1-2\nu)}{2(1-\nu^2)}(A_0\chi_3^1 + \bar{A}_0\bar{\chi}_3^1) + \frac{B_0\chi_3^0}{(1+\nu)}, \tag{8.3}$$

$$\psi = \frac{\bar{A}_0\chi_3^0}{(1-\nu^2)} - \frac{(1-2\nu)A_0\chi_1^2}{8(1-\nu^2)} - \frac{B_0\chi_1^1}{4(1+\nu)} - \frac{\imath C_0\chi_1^1}{2(1-\nu)}, \tag{8.4}$$

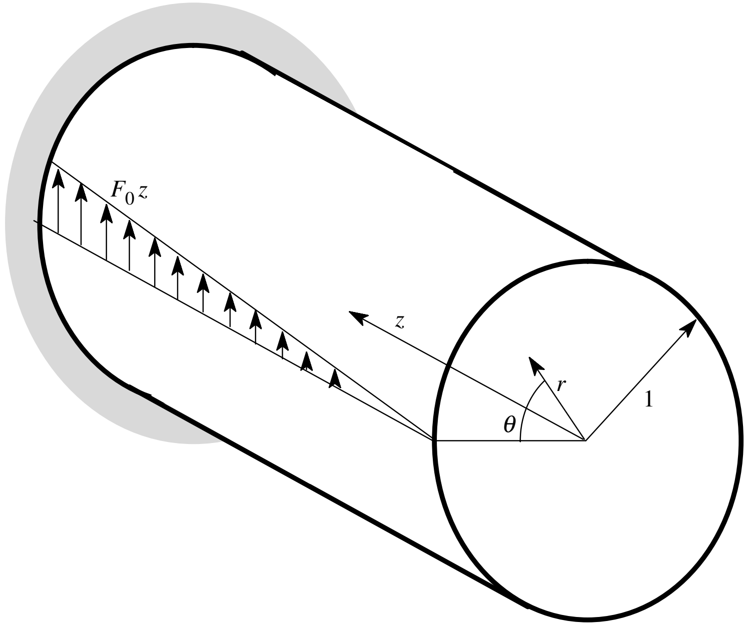

Figure 1. Cylinder with a linearly varying concentrated tangential force.

from equations (5.5) and (5.8). The only non-zero stresses at this stage are σ_{zz} and

$$\Psi = -\frac{A_0(1+2\nu)\zeta^2}{4(1+\nu)} - \frac{\bar{A}_0\zeta\bar{\zeta}}{2(1+\nu)} - \frac{B_0\zeta}{2} + \iota C_0\zeta, \tag{8.5}$$

from (4.11)–(4.16) or (5.7) and (5.9). On the boundary $r=1$, we have

$$\sigma_{zr} + \iota\sigma_{z\theta} = \mathrm{e}^{-\iota\theta}\Psi(1,\theta) = -\frac{A_0(1+2\nu)\mathrm{e}^{\iota\theta}}{4(1+\nu)} - \frac{\bar{A}_0\,\mathrm{e}^{-\iota\theta}}{2(1+\nu)} - \frac{B_0}{2} + \iota C_0, \tag{8.6}$$

and hence the antiplane traction is

$$T_z = \sigma_{zr}(1,\theta) = -\frac{(3+2\nu)(\mathrm{Re}\{A_0\}\cos\theta - \mathrm{Im}\{A_0\}\sin\theta)}{4(1+\nu)} - \frac{B_0}{2} + \frac{\partial h_y}{\partial\theta}, \tag{8.7}$$

where we have added in the antiplane corrective term from (5.15). This traction is zero in $\mathcal{P}_1$ and the resulting expression leads to an integrable expression for h_y if and only if $B_0=0$. Elementary calculations then show that the traction-free boundary condition can be satisfied by writing

$$h = \frac{(3+2\nu)A_0\zeta}{4(1+\nu)}, \tag{8.8}$$

and hence

$$\phi_1 = \frac{(1-2\nu)}{2(1-\nu^2)}(A_0\chi_3^1 + \bar{A}_0\bar{\chi}_3^1) + \frac{(3-4\nu)(3+2\nu)(A_0\chi_1^1 + \bar{A}_0\bar{\chi}_1^1)}{16(1-\nu^2)}, \tag{8.9}$$

$$\psi_1 = \frac{\bar{A}_0\chi_3^0}{(1-\nu^2)} - \frac{(1-2\nu)A_0\chi_1^2}{8(1-\nu^2)} - \frac{\iota C_0\chi_1^1}{2(1-\nu)} + \frac{(3+2\nu)\bar{A}_0\chi_1^0}{8(1-\nu^2)}, \tag{8.10}$$

defines the complete solution to $\mathcal{P}_1$ in P–N form. This of course represents the well-known solution for the cylinder loaded at the end by a shear force and a torque, but we include it here to illustrate the recursive procedure and particularly the way in which the antiplane solution is incorporated into the P–N formalism.

The next stage is once more to add in the zeroth-order solution (5.3) with new constants A_1, B_1, perform a further integration with respect to z, use equations (4.18)–(4.20) and (4.26) to determine the function $f(\zeta, \bar{\zeta})$ required for each of the potentials ϕ, ψ to be three-dimensionally harmonic and add in the torsion solution (5.8) with a new constant C_1. We obtain

$$\begin{aligned}\phi = &\frac{(1-2\nu)}{2(1-\nu^2)}(A_0\chi_4^1 + \bar{A}_0\bar{\chi}_4^1) + \frac{(3-4\nu)(3+2\nu)(A_0\chi_2^1 + \bar{A}_0\bar{\chi}_2^1)}{16(1-\nu^2)} \\ &+ \frac{(1-2\nu)}{2(1-\nu^2)}(A_1\chi_3^1 + \bar{A}_1\bar{\chi}_3^1) + \frac{B_1\chi_3^0}{(1+\nu)},\end{aligned} \tag{8.11}$$

$$\psi = \frac{\bar{A}_0\chi_4^0}{(1-\nu^2)} - \frac{(1-2\nu)A_0\chi_2^2}{8(1-\nu^2)} - \frac{\iota C_0\chi_2^1}{2(1-\nu)} + \frac{(3+2\nu)\bar{A}_0\chi_2^0}{8(1-\nu^2)} + \frac{\bar{A}_1\chi_3^0}{(1-\nu^2)}$$

$$- \frac{(1-2\nu)A_1\chi_1^2}{8(1-\nu^2)} - \frac{B_1\chi_1^1}{4(1+\nu)} - \frac{\iota C_1\chi_1^1}{2(1-\nu)}. \tag{8.12}$$

Substituting this partial solution into equations (4.13) and (4.14), we obtain the non-zero (but z-independent) in-plane stress components

$$\Theta = \frac{[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)](A_0\zeta + \bar{A}_0\bar{\zeta})}{16(1-\nu^2)}, \tag{8.13}$$

$$\Phi = \frac{(1+4\nu)A_0\zeta^3}{24(1+\nu)} + \frac{[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)]\bar{A}_0\zeta}{16(1-\nu^2)} - \frac{\iota C_0\zeta^2}{2}, \tag{8.14}$$

and the corresponding tractions on the surface $r=1$ are

$$T_r + \iota T_\theta = \frac{(\Theta + \Phi\,\mathrm{e}^{-2\iota\theta})}{2} = -\frac{(19-18\nu-16\nu^2)A_0\,\mathrm{e}^{\iota\theta}}{96(1-\nu^2)} - \frac{\bar{A}_0\,\mathrm{e}^{-\iota\theta}}{4} - \frac{\iota C_0}{4}, \tag{8.15}$$

using (4.9).

In order to satisfy the traction conditions $(8.2)_{(\mathrm{i,ii})}$, we need to superpose the solution of an in-plane problem corresponding to the loading of a circular disc by a concentrated tangential force F_0 per unit length, equilibrated by tractions of the form (8.15). The Airy stress function,

$$\varphi = \frac{F_0 r\theta\cos\theta}{\pi} = -\frac{\iota F_0(\zeta+\bar{\zeta})(\ln(\zeta) - \ln(\bar{\zeta}))}{4\pi}, \tag{8.16}$$

represents a force F_0 in the y-direction at the origin, so the concentrated force at the point $(1,0,z)$ in problem $\mathcal{P}_2$ can be described by the function

$$\varphi = -\frac{\iota F_0(\zeta+\bar{\zeta}-2)(\ln(1-\zeta)-\ln(1-\bar{\zeta}))}{4\pi}. \tag{8.17}$$

Comparing this with (5.23), we see that

$$f_1(\zeta) = -\frac{\iota F_0(\zeta-2)\ln(1-\zeta)}{4\pi}; \quad f_2(\zeta) = -\frac{\iota F_0 \ln(1-\zeta)}{4\pi}. \tag{8.18}$$

In view of equations (5.27), we therefore add the corrective terms,

$$\phi_c = \frac{\iota F_0[(\zeta-2)\ln(1-\zeta)-(\bar{\zeta}-2)\ln(1-\bar{\zeta})]}{4\pi}, \tag{8.19}$$

$$\psi_c = \frac{\iota F_0 \ln(1-\zeta)}{2\pi} + \iota D_0\chi_0^2, \tag{8.20}$$

to the stress functions of equations (8.11) and (8.12), where D_0 is an arbitrary constant and the additional term $\iota D_0\chi_0^2$ corresponds to the Airy function $D_0 r^3 \sin\theta$, which is the only non-trivial degree of freedom in the Michell solution with the appropriate Fourier dependence. We then repeat the procedure of equations (8.13)–(8.15) using the modified stress function and enforce the traction-free condition for $\theta\neq 0$ by setting the coefficients of each power of exp $(\iota\theta)$ in the numerator of $T_r+\iota T_\theta$ to zero, with the result

$$A_0 = \frac{2\iota F_0}{\pi}; \quad C_0 = -\frac{2F_0}{\pi}; \quad D_0 = -\frac{F_0(19-18\nu-16\nu^2)}{96\pi(1-\nu^2)}. \tag{8.21}$$

The constants A_0, C_0 relate to the moment resultants transmitted through the cross-section, and hence could have been determined without the solution of the in-plane boundary-value problem. However, in the present procedure, it is not necessary to use equilibrium arguments. Constants related to transmitted force resultants are determined as conditions of solvability at a higher stage in the recursive procedure.

Using (4.16), (8.11), (8.12), (8.19)–(8.21), the antiplane stress components are given by

$$\begin{aligned}\Psi = &-\frac{\iota F_0 z[(1+2\nu)\zeta^2+4(1+\nu)\zeta-2\zeta\bar{\zeta}+(3+2\nu)]}{2\pi(1+\nu)}\\ &-\frac{A_1(1+2\nu)\zeta^2}{4(1+\nu)}-\frac{\bar{A}_1\zeta\bar{\zeta}}{2(1+\nu)}-\frac{B_1\zeta}{2}+\iota C_1\zeta.\end{aligned} \tag{8.22}$$

The first (z-varying) term necessarily corresponds to zero tractions on the surface $r=1$, since it was obtained by integrating stresses which satisfy this condition using equations (8.7) and (8.8). The remaining (z-independent) terms in (8.22) are identical in form to those in (8.5), and the traction-free condition can therefore be satisfied by setting $B_1=0$ and adding the function $zh'/2(1-\nu)$ into ψ, where

$$h = \frac{(3+2\nu)A_1\zeta}{4(1+\nu)}. \tag{8.23}$$

Thus, the solution to problem $\mathcal{P}_2$ is defined by the potentials

$$\phi = \frac{\iota F_0(1-2\nu)(\chi_4^1-\bar{\chi}_4^1)}{\pi(1-\nu^2)} + \frac{\iota F_0(3-4\nu)(3+2\nu)(\chi_2^1-\bar{\chi}_2^1)}{8\pi(1-\nu^2)}$$

$$+\frac{\iota F_0[(\zeta-2)\ln(1-\zeta)-(\bar{\zeta}-2)\ln(1-\bar{\zeta})]}{4\pi} + \frac{(1-2\nu)}{2(1-\nu^2)}(A_1\chi_3^1+\bar{A}_1\bar{\chi}_3^1), \quad (8.24)$$

$$\psi = \frac{\iota F_0}{4\pi(1-\nu^2)}[-8\chi_4^0-(1-2\nu)\chi_2^2+4(1+\nu)\chi_2^1-(3+2\nu)\chi_2^0]$$

$$+\frac{\iota F_0\ln(1-\zeta)}{2\pi} - \frac{\iota F_0(19-18\nu-16\nu^2)\chi_0^2}{96\pi(1-\nu^2)} + \frac{\bar{A}_1\chi_3^0}{(1-\nu^2)} - \frac{(1-2\nu)A_1\chi_1^2}{8(1-\nu^2)}$$

$$-\frac{\iota C_1\chi_1^1}{2(1-\nu)} + \frac{(3+2\nu)A_1\chi_1^0}{8(1-\nu^2)}. \quad (8.25)$$

The final stage of the solution starts with the superposition of the zeroth-order potentials (5.3) with new constants A_2, B_2, a further integration with respect to z, using equations (4.18)–(4.20) and (4.26) to determine the function $f(\zeta,\bar{\zeta})$ required for each of the potentials ϕ,ψ to be three-dimensionally harmonic and the addition of the torsion solution (5.8) with a new constant C_2. Substitution into (4.11)–(4.16) and computation of the in-plane tractions, as in the analysis leading to equation (8.15), yield

$$T_r+\iota T_\theta = \iota F_0 z\delta(\theta) - \frac{(19-18\nu-16\nu^2)A_1\,\mathrm{e}^{\iota\theta}}{96(1-\nu^2)} - \frac{\bar{A}_1\,\mathrm{e}^{-\iota\theta}}{4} - \frac{\iota C_1}{4}, \quad (8.26)$$

which satisfies the boundary conditions $(8.1)_{(\mathrm{i,ii})}$ with the trivial choice $A_1=C_1=0$. We could, of course, have inferred this from the fact that the in-plane tractions in $\mathcal{P}_3$ have no z-independent terms, and hence the moment resultants cannot contain quadratic terms in z.

Using this result, the out-of-plane tractions are then found to be

$$T_z = -\frac{F_0(1-\nu)\ln(2(1-\cos(\theta))\sin\theta}{2\pi} + \frac{F_0(25+38\nu+16\nu^2)\sin\theta}{48\pi(1+\nu)}$$

$$+\frac{F_0\nu(\pi-\theta)\cos\theta}{2\pi} - \frac{(3+2\nu)(\mathrm{Re}\{A_2\}\cos\theta-\mathrm{Im}\{A_2\}\sin\theta)}{4(1+\nu)} - \frac{B_2}{2}. \quad (8.27)$$

Integrating with respect to θ to determine the boundary values of the corrective potential h_y shows that a single-valued function is obtained if and only if $B_2=0$, and with this choice the required traction-free condition can be restored using the function

$$h = D_1\zeta\ln(1-\zeta) + D_2\ln(1-\zeta) + \frac{D_3\ln(1-\zeta)}{\zeta} + D_4\zeta, \quad (8.28)$$

where

$$D_1 = -\frac{\iota F_0}{2\pi}; \quad D_2 = \frac{\iota F_0(1-\nu)}{\pi}; \quad D_3 = -\frac{\iota F_0(1-2\nu)}{2\pi};$$
$$D_4 = \frac{\iota F_0(49+62\nu+16\nu^2)}{48\pi(1+\nu)} + \frac{(3+2\nu)A_2}{4(1+\nu)}. \tag{8.29}$$

The appropriate additional terms in ϕ, ψ are obtained by substituting (8.28) into (5.11).

It remains to satisfy the weak traction-free conditions on the end $z=0$. The six force and moment resultants are obtained from (2.2) and (2.3) as

$$F_x(0) + \iota F_y(0) = -\frac{\iota F_0(15+22\nu) + 12\pi \bar{A}_2}{48(1+\nu)}; \quad F_z(0) = 0, \tag{8.30}$$

$$M_x(0) + \iota M_y(0) = 0; \quad M_z(0) = \frac{F_0(7-6\nu)}{12} + \frac{\pi C_2}{2}, \tag{8.31}$$

and hence the weak conditions can be satisfied by the choice

$$A_2 = \frac{\iota F_0(15+22\nu)}{12\pi}; \quad C_2 = -\frac{F_0(7-6\nu)}{6\pi}. \tag{8.32}$$

Note that if the resultants F_z, M_x, M_y had not been found to be zero, it would have been necessary to add in the zeroth-order potentials (5.2) and (5.3) with new multipliers A_3, B_3. The final solution of the example problem is defined by the potentials

$$\begin{aligned}\phi = {} & \iota F_0 z\left(\zeta - \frac{(3-4\nu)\zeta}{8\pi(1-\nu)} + \frac{(1-\nu)}{\pi} - 2 - \frac{(1-2\nu)(5-4\nu)}{8\pi(1-\nu)\zeta}\right)\ln(1-\zeta) \\ & + \frac{\iota F_0}{96\pi(1-\nu^2)}(96(1-2\nu)\chi_5^1 + 8(21-34\nu)(1+\nu)\chi_3^1 \\ & + (153+61\nu-226\nu^2-120\nu^3)\chi_1^1 - 12(1-2\nu)(1+\nu)\chi_1^0),\end{aligned} \tag{8.33}$$

$$\begin{aligned}\psi = {} & \frac{\iota F_0 z}{2\pi}\left(\ln(1-\zeta) + \frac{\ln(1-\bar{\zeta})}{2(1-\nu)} - \frac{(1-2\nu)\ln(1-\bar{\zeta})}{2(1-\nu)\bar{\zeta}^2} - \frac{(1-2\nu)}{2(1-\nu)\bar{\zeta}}\right) \\ & + \frac{\iota F_0}{48\pi(1-\nu^2)}(-96\chi_5^0 - 12(1-2\nu)\chi_3^2 + 48(1+\nu)\chi_3^1 - 16(6+7\nu)\chi_3^0 \\ & - (17-30\nu)(1+\nu)\chi_1^2 + 4(7-6\nu)(1+\nu)\chi_1^1 - (35+67\nu+30\nu^2)\chi_1^0),\end{aligned} \tag{8.34}$$

and the stress and displacement components can be recovered by substitution into equations (4.11)–(4.16).

9. Discussion and conclusions

We have deliberately chosen a fairly straightforward example to illustrate the method, because, as the reader can see, the algebraic expressions involved rapidly become rather lengthy. However, it should be emphasized that the operations involved are all essentially routine and present no serious challenge to symbolic processors, such as MATHEMATICA or MAPLE, in cases where closed-form solutions can be obtained.

The solution procedure intersperses solutions of classical two-dimensional (plane strain and antiplane) boundary-value problems, with integrations performed in the P–N formalism. These integrations are all of an extremely simple nature, and hence solutions can be obtained for any bar for which a general solution exists to the two-dimensional plane strain and antiplane problems. In particular, this category includes all bar cross-sections that can be conformally mapped to the unit circle. The two-dimensional solutions can also be solved using real stress functions if desired, since these are easily converted to complex form to permit the necessary P–N integrations. Thus, the proposed procedure provides a quite general solution to the problem of the prismatic bar loaded by tractions that permit a polynomial expansion in the axial coordinate z.

The method can easily be extended to include body forces, provided these can be described by a body-force potential.

References

Almansi, E. 1901*a* Sopra la deformazione dei cilinri solecitati lateralmente. *Atti Real Accad. Naz. Lincei Rend, Cl. Sci Fis., Mat Nat.* Ser. 5 **10**, 333–338.

Almansi, E. 1901*b* Sopra la deformazione dei cilinri solecitati lateralmente. *Atti Real Accad. Naz. Lincei Rend, Cl. Sci Fis., Mat Nat.* Ser. 5 **10**, 400–408.

Barber, J. R. 2002 *Elasticity*, 2nd edn. Dordrecht: Kluwer.

Eubanks, R. A. & Sternberg, E. 1956 On the completeness of the Boussinesq–Papkovich stress functions. *J. Rat. Mech. Anal.* **5**, 735–746.

Green, A. E. 1949 The elastic equilibrium of isotropic plates and cylinders. *Proc. R. Soc. A* **195**, 533–552.

Green, A. E. & Zerna, W. 1954 *Theoretical elasticity.* Oxford: Clarendon Press.

Love, A. E. H. 1927 *A treatise on the mathematical theory of elasticity*, 4th edn. Cambridge: Cambridge University Press.

Michell, J. H. 1901 The theory of uniformly loaded beams. *Q. J. Math.* **32**, 28–42.

Milne-Thomson, L. M. 1962 *Antiplane elastic systems.* Berlin: Springer.

Milne-Thomson, L. M. 1968 *Plane elastic systems*, 2nd edn. Berlin: Springer.

Muskhelishvili, N. I. 1963 *Some basic problems of the mathematical theory of elasticity.* Groningen: Noordhoff.

Sokolnikoff, I. S. 1956 *Mathematical theory of elasticity*, 2nd edn. New York: McGraw-Hill.

Stevenson, A. C. 1943 Some boundary problems of two-dimensional elasticity. *Phil. Mag.* **34**, 766–793.

Stevenson, A. C. 1945 Complex potentials in two-dimensional elasticity. *Proc. R. Soc. A* **184**, 129–179.