Chapter 16
Moderately Thick Plates

In §3.2, we showed that the plane stress assumption

$$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$$

leads only to an approximate solution, except in the special case where $\nu = 0$. In particular, the resulting strains do not satisfy the full set of six compatibility equations. In this chapter, we shall show that an exact plane stress solution can be developed if we relax the assumption that the stresses do not vary through the thickness[1]. In other words, we allow the stress state to be three-dimensional.

As in Chapter 4, we can satisfy the two non-trivial equilibrium equations (4.1) by defining a stress function ϕ such that the non-zero stress components

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; , \tag{16.1}$$

where we note that ϕ will now generally be a function of all three coordinates x, y, z.

The non-zero strain components are then given by

$$e_{xx} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right) \; ; \quad e_{yy} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right) \; ;$$

$$e_{zz} = -\frac{\nu}{E}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \; ; \quad e_{xy} = -\frac{(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}$$

and substitution into the six compatibility equations (2.7) yields

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

[1] See A.E.H.Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th edn., Dover, 1944, §299 *et seq.*.

$$\begin{aligned}
\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right) &= \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\\
\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right) &= \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\\
(1+\nu)\frac{\partial^4\phi}{\partial x\partial y\partial z^2} &= -\nu\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\\
\frac{\partial^2}{\partial z\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) &= \frac{\partial^2}{\partial z\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0\;.
\end{aligned} \tag{16.2}$$

The problem is symmetrical with respect to the mid-plane of the plate so we expect ϕ to be an even function of z. Equations (16.2) therefore suggest a solution of the form

$$\phi(x,y,z)=\phi_1(x,y)+z^2\phi_2(x,y)\;. \tag{16.3}$$

Substituting this expression into (16.2) and equating coefficients of powers of z, we find that all the compatibility equations are satisfied by the choice

$$\phi_2=-\frac{\nu}{2(1+\nu)}\nabla^2\phi_1 \quad \text{and} \quad \nabla^2\phi_2=0\;,$$

implying also $\nabla^4\phi_1=0$. The non-zero stress components are then obtained as

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2}{\partial y^2}\left(\phi_1-\frac{\nu z^2}{2(1+\nu)}\nabla^2\phi_1\right)\\
\sigma_{yy} &= \frac{\partial^2}{\partial x^2}\left(\phi_1-\frac{\nu z^2}{2(1+\nu)}\nabla^2\phi_1\right)\\
\sigma_{xy} &= -\frac{\partial^2}{\partial x\partial y}\left(\phi_1-\frac{\nu z^2}{2(1+\nu)}\nabla^2\phi_1\right)\;,
\end{aligned} \tag{16.4}$$

where $\nabla^4\phi_1=0$. This representation reduces to the classical Airy function formulation when $\nu=0$.

16.1 Boundary conditions

The two-dimensionally biharmonic function ϕ_1 is sufficient to define two independent quantities at each point s on the boundary, typically the normal and shear components of the local traction $\boldsymbol{t}(s)$, but the stress components (16.4) imply that $\boldsymbol{t}$ will have the form

$$\boldsymbol{t}(s)=\boldsymbol{t}_1(s)+z^2\boldsymbol{t}_2(s)\;.$$

We can therefore satisfy the boundary conditions only in the weak sense

$$\int_{-c}^{c} \boldsymbol{t}(s,z)dz = \int_{-c}^{c} \boldsymbol{t}_0(s,z)dz \; , \tag{16.5}$$

where z is measured from the mid-plane of a plate of thickness $2c$, $\boldsymbol{t}_0(s,z)$ is the traction imposed in the physical problem, and $\boldsymbol{t}(s,z)$ is the corresponding value in the weak solution.

The resulting solution will therefore generally differ from the exact stress state in a region near the boundaries and the error will decay from these edges with a length scale related to the semi-thickness c. The astute reader might reasonably ask at this point whether the approximation involved is any better than that from the simple (z-independent) plane stress solution. We shall discuss this issue in §16.2 below, but first we illustrate the solution process for a simple example.

Example

We consider the problem of §5.2.1 and Figure 5.2 in which the rectangular cantilever $0<x<a$, $-b<y<b$, $-c<z<c$ is loaded by a traverse force F on the end $x=0$. The boundary conditions, modified from (5.11-5.14) are

$$\sigma_{xy} = 0 \;\; ; \;\; y = \pm b \tag{16.6}$$

$$\sigma_{yy} = 0 \;\; ; \;\; y = \pm b \tag{16.7}$$

$$\sigma_{xx} = 0 \;\; ; \;\; x = 0 \tag{16.8}$$

$$\int_{-c}^{c}\int_{-b}^{b} \sigma_{xy} dy dz = F \;\; ; \;\; x = 0 \; . \tag{16.9}$$

Since this is a 'correction' on the two-dimensional solution, we start by considering the stress function

$$\phi_1 = C_1 x y^3 + C_2 x y$$

from equation (5.19). However, note that we do not generally expect the constants C_1, C_2 to take the same values as in the two-dimensional solution, because of the boundary condition (16.5). It follows that $\nabla^2\phi_1 = 6C_1 xy$ and hence the stress components are

$$\sigma_{xx} = 6C_1 xy \; ; \quad \sigma_{yy} = 0 \; ; \quad \sigma_{xy} = -3C_1 y^2 - C_2 + \frac{3C_1 \nu z^2}{(1+\nu)} \; ,$$

from (16.4). These expressions satisfy the boundary conditions (16.7, 16.8), but (16.6) can be satisfied only in the weak sense

$$\int_{-c}^{c} \sigma_{xy}(x, \pm b, z) dz = 0 \ ,$$

from which

$$C_2 = -3C_1 b^2 + \frac{C_1 \nu c^2}{(1+\nu)} \ .$$

The end condition (16.9) then gives

$$C_1 = \frac{F}{8cb^3} \ ,$$

and the non-zero stress components are obtained as

$$\sigma_{xx} = \frac{3Fxy}{4cb^3} \ ; \quad \sigma_{xy} = \frac{F}{8cb^3}\left[3(b^2 - y^2) - \frac{\nu(c^2 - 3z^2)}{(1+\nu)}\right] \ .$$

We notice that the shear stress distribution is now predicted to vary with z over the cross-section, and in particular, the maximum shear stress occurs at the points $(x, 0, \pm c)$ and is

$$\tau_{\max} = \frac{3F}{8cb} + \frac{F\nu c}{4b^3(1+\nu)} \ . \tag{16.10}$$

In Chapter 18 we shall obtain a solution to this problem using a Fourier series representation to satisfy the strong condition (16.6). In particular, we shall show that the first term in this series is identical to the second 'corrective' term in equation (16.10). For the extreme case of a bar of square cross-section ($c = b$) and $\nu = 0.5$, it reduces the percentage error in the elementary two-dimensional solution from 18% to 1%.

16.2 Edge effects

The error associated with the weak form of the traction boundary condition (16.5) corresponds to the stress field due to locally self-equilibrated tractions of the form

$$\boldsymbol{t}(s, z) = \boldsymbol{C}(s)(c^2 - 3z^2) \ ; \quad -c < z < c \ ,$$

and based on Saint Venant's principle, we anticipate that the error will decay exponentially with distance from the boundary. For the normal stress component, the exponential decay rate λ can be estimated using the solution from §6.2.2. In particular, since the problem is symmetric about the mid-plane, $\lambda \approx 2.1/c$. For the shear traction, the corrective problem is one of antiplane strain (see Chapter 15 and particularly Problem 15.14) and the corresponding eigenvalue is $\lambda = \pi/c$.

For the example problem above, the maximum shear stress occurs far from the surface on which the weak boundary condition was imposed, so the solution given here represents a significant improvement on the two-dimensional solution. By contrast, for problems where the maximum stress is expected adjacent to a traction-free surface (notably stress-concentration problems such as §8.4.1), the extra complication of the three-dimensional solution is not justified.

16.3 Body force problems

Suppose a thin plate is subjected to a conservative body-force field $\boldsymbol{p} = -\boldsymbol{\nabla} V$, where the potential V is a function of the in-plane coordinates (x, y) only. Thus the body force is uniform through the thickness. As in Chapter 7, the equilibrium equations can then be satisfied by defining the stress components

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; .$$

If these equations are used in place of (16.1) in the derivation of equations (16.2), we obtain

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi &= -(1-\nu)\nabla^2 V \\
\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu\frac{\partial^2 \phi}{\partial y^2}\right) &= \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2V\right) \\
\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu\frac{\partial^2 \phi}{\partial x^2}\right) &= \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2V\right) \qquad (16.11) \\
(1+\nu)\frac{\partial^4 \phi}{\partial x \partial y \partial z^2} &= -\nu\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2V\right) \\
\frac{\partial^2}{\partial z \partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) &= \frac{\partial^2}{\partial z \partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \; ,
\end{aligned}$$

and it can be shown that these six equations have no solution for ϕ, except for the case where $\nabla^2 V$ is independent of x, y. However, this restricted class of body force problems includes the important cases of gravitational loading and uniform rotation within the xy-plane.

Gravitational loading

As in §7.2.1, gravitational loading is described by the potential $V = \rho g y$, where the y-axis defines the vertically upwards direction. Since V is a linear function of the coordinates, the terms involving V in (16.11) vanish, so the stress components are still given by (16.4) except for the addition of a term $+\rho g y$ in the expressions for σ_{xx} and σ_{yy}.

Uniform rotation

If the plate rotates about the z-axis at constant speed Ω, the body-force field is described by the potential

$$V = -\frac{1}{2}\rho\Omega^2(x^2+y^2) = -\frac{\rho\Omega^2 r^2}{2} \;,$$

from equation (7.17). In this case

$$\nabla^2 V = -2\rho\Omega^2 \;,$$

and it is easily verified that equations (16.11) are all satisfied by the particular solution

$$\begin{aligned}\phi_P &= \frac{\rho\Omega^2(1-\nu)}{32}(x^2+y^2)^2 - \frac{\rho\Omega^2\nu(1+\nu)}{4(1-\nu)}(x^2+y^2)z^2 \\ &= \frac{\rho\Omega^2(1-\nu)}{32}r^4 - \frac{\rho\Omega^2\nu(1+\nu)}{4(1-\nu)}r^2z^2 \;. \end{aligned} \tag{16.12}$$

The general solution is then obtained by superposing a homogeneous solution (i.e. a solution without body force) using equations (16.4).

Example

We illustrate the process for the problem of the circular disk $0 \le r < a$, $-c<z<c$, rotating at constant speed Ω about the z-axis, with traction-free boundary conditions. The particular solution is given by equation (16.12) and for the homogeneous solution we use the axisymmetric biharmonic function that is bounded at $r=0$ — i.e. $\phi_1 = Ar^2$. We then have

$$\phi_2 = -\frac{\nu}{2(1+\nu)}\nabla^2\phi_1 = -\frac{2\nu A}{(1+\nu)}$$

and hence

$$\phi = \phi_P + \phi_1 + z^2\phi_2 = \frac{\rho\Omega^2(1-\nu)}{32}r^4 - \frac{\rho\Omega^2\nu(1+\nu)}{4(1-\nu)}r^2z^2 + Ar^2 - \frac{2\nu A z^2}{(1+\nu)} \;.$$

The boundary condition in this problem is $\sigma_{rr}=0$ at $r=a$, so we calculate

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V = \frac{\rho \Omega^2 (1-\nu)}{8} r^2 - \frac{\rho \Omega^2 \nu (1+\nu)}{2(1-\nu)} z^2 + 2A - \frac{\rho \Omega^2 r^2}{2} .$$

The traction-free boundary condition can be satisfied only in the weak sense

$$\int_{-c}^{c} \sigma_{rr}(a,z) dz = 0 ,$$

from which we obtain

$$A = \frac{\rho \Omega^2}{4} \left[\frac{(3+\nu)a^2}{4} + \frac{\nu(1+\nu)c^2}{3(1-\nu)} \right] .$$

The final stress field is then given by

$$\sigma_{rr} = \rho \Omega^2 \left[\frac{(3+\nu)(a^2-r^2)}{8} + \frac{\nu(1+\nu)(c^2-3z^2)}{6(1-\nu)} \right]$$
$$\sigma_{\theta\theta} = \rho \Omega^2 \left[\frac{\{(3+\nu)a^2-(1+3\nu)r^2\}}{8} + \frac{\nu(1+\nu)(c^2-3z^2)}{6(1-\nu)} \right] .$$

The maximum tensile stress (and also the maximum von Mises stress) occurs at the origin and is

$$\sigma_{\max} = \rho \Omega^2 \left[\frac{(3+\nu)a^2}{8} + \frac{\nu(1+\nu)c^2}{6(1-\nu)} \right] .$$

This location is not close to the boundary where the weak boundary condition was applied, so the three-dimensional solution will give a good approximation to the exact result.

This is a particularly simple example, but it is clear that the same method can be easily applied to other geometries, including the rotating rectangular beam of §7.3.1.

16.4 Normal loading of the faces

Suppose that we solve the same two-dimensional boundary-value problem (i) under the plane strain assumptions, and (ii) using the plane stress assumptions, but including the three-dimensional correction described in this chapter. If we now construct the *difference* between these two solutions, the boundary conditions will generally involve normal loading σ_{zz} on the faces of the plate, but the other boundaries will be traction free within the limitations of the weak form of equation (16.5). This approach can be used to generate solutions of the problem where both sides of a thin plate are loaded by equal and opposite normal tractions σ_{zz}, subject to the restriction that these be two-dimensionally harmonic[2].

Suppose that a given two-dimensional plane strain solution is defined by the Airy function $\phi_1(x,y)$, so

$$\sigma_{zz}(x,y) = \nu(\sigma_{xx} + \sigma_{yy}) = \nu\nabla^2\phi_1 \ , \tag{16.13}$$

from (4.6). Since ϕ_1 is biharmonic, σ_{zz} must be harmonic.

If we now use the same function $\phi_1(x,y)$ to define a three-dimensional plane stress solution using equations (16.3, 16.1), the difference between the two solutions will then include the out-of-plane stresses (16.13), and in-plane stresses defined by the function

$$\phi = -z^2\phi_2(x,y) = \frac{\nu z^2}{2(1+\nu)}\nabla^2\phi_1 = \frac{z^2\sigma_{zz}}{2(1+\nu)} \ .$$

The in-plane stresses will not generally satisfy weak traction-free boundary conditions, but we can correct this by adding the extra term[3] $A\sigma_{zz}(x,y)$, where A is a constant chosen so as to satisfy the condition

$$\int_{-c}^{c} \phi \, dz = 0 \ .$$

We obtain

$$\phi = -\frac{(c^2 - 3z^2)\sigma_{zz}}{6(1+\nu)} \ , \tag{16.14}$$

after which the stresses are recovered from equations (16.1). The resulting stress field will clearly satisfy the condition that all surfaces with normals in the (x,y)-plane will have zero tractions in the weak sense.

16.4.1 Steady-state thermoelasticity

We showed in Chapter 14 that for a two dimensional body with steady-state heat conduction ($\nabla^2 T(x,y) = 0$), the plane strain solution involves no in-plane stresses, and the out-of-plane stress component is

$$\sigma_{zz} = -E\alpha T(x,y) \ ,$$

from equation (14.6). It follows immediately from equation (16.14) that the non-zero stress components for a traction-free plate of finite thickness $2c$ can

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[2] X-F, Li and Z-L. Hu (2020), Generalization of plane stress and plane strain states to elastic plates of finite thickness, *Journal of Elasticity*. These authors also considered problems where the normal surface displacements u_z are prescribed on the faces.

[3] Recall that σ_{zz} is harmonic, so $A\sigma_{zz}(x,y)$ is a legitimate stress function.

be defined through the stress function

$$\phi = \frac{(c^2 - 3z^2)E\alpha T}{6(1+\nu)}$$

and are

$$\sigma_{xx} = \frac{(c^2 - 3z^2)E\alpha}{6(1+\nu)} \frac{\partial^2 T}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{(c^2 - 3z^2)E\alpha}{6(1+\nu)} \frac{\partial^2 T}{\partial x^2} \; ;$$

$$\sigma_{xy} = -\frac{(c^2 - 3z^2)E\alpha}{6(1+\nu)} \frac{\partial^2 T}{\partial x \partial y} \; . \tag{16.15}$$

Problems

16.1. A curved beam $a < r < b$, $0 < \theta < \pi$, $-c < z < c$ is built in at $\theta = \pi$ and loaded by a tensile force F normal to the surface at $\theta = 0$, whose line of action passes through the origin. Find the distribution of shear stress $\sigma_{\theta r}$ on the cross section at $\theta = \pi/2$. What is the percentage difference in the maximum shear stress at this location, relative to the two-dimensional solution, if $b = 2a$ and $c = a/4$?

16.2. The beam $-b < y < b$, $0 < x < L$, $-c < z < c$ is built-in at the end $x = 0$ and loaded by a uniform shear traction $\sigma_{xy} = S$ on the upper edge, $y = b$, the remaining edges, $x = L$, $y = -b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x = L$. (This is the three-dimensional counterpart of Problem 5.1).

16.3. Show that equations (16.11) have solutions if and only if $\nabla^2 V$ is independent of x and y.

16.4. The beam $-b < y < b$, $0 < x < L$, $-c < z < c$ is built-in at the end $x = L$ and is subject only to gravitational loading $p_y = -\rho g$, all the remaining surfaces being traction-free. Find the complete stress field in the beam.

16.5. The rectangular block $-a < x < a$, $-b < y < b$, $-c < z < c$ rotates at uniform angular velocity Ω about the z-axis. Estimate the maximum tensile stress in the block if $a \gg b \gg c$.

16.6. An infinite elastic plate of thickness $2c$ contains a non-conducting circular inclusion of radius a whose elastic properties are the same as those in the rest of the plate. Find the magnitude and location of the maximum tensile stress in the plate if the inclusion perturbs a uniform heat flux $q_x = q_0$ and the extremities of the plate are traction-free.

Comment on the relation between your solution and that for a plate with a circular hole.

16.7. The solution of §16.4.1 is applied to a plate with a thermally-insulated traction-free boundary at $x = 0$, and the maximum value of the lateral stress σ_{yy} is predicted to be adjacent to this boundary. Show that the correction required to make this boundary exactly traction-free is not negligible, and obtain a more accurate expression for the local maximum stress in terms of in-plane derivatives of the temperature field.

16.8. The rectangular block $-a < x < a$, $-b < y < b$, $-c < z < c$ is initially at rest when equal and opposite forces F are applied on the ends $x = \pm a$, as in Figure 7.3. Find the stresses in the block just after the forces are applied, assuming that $a \gg b \gg c$.