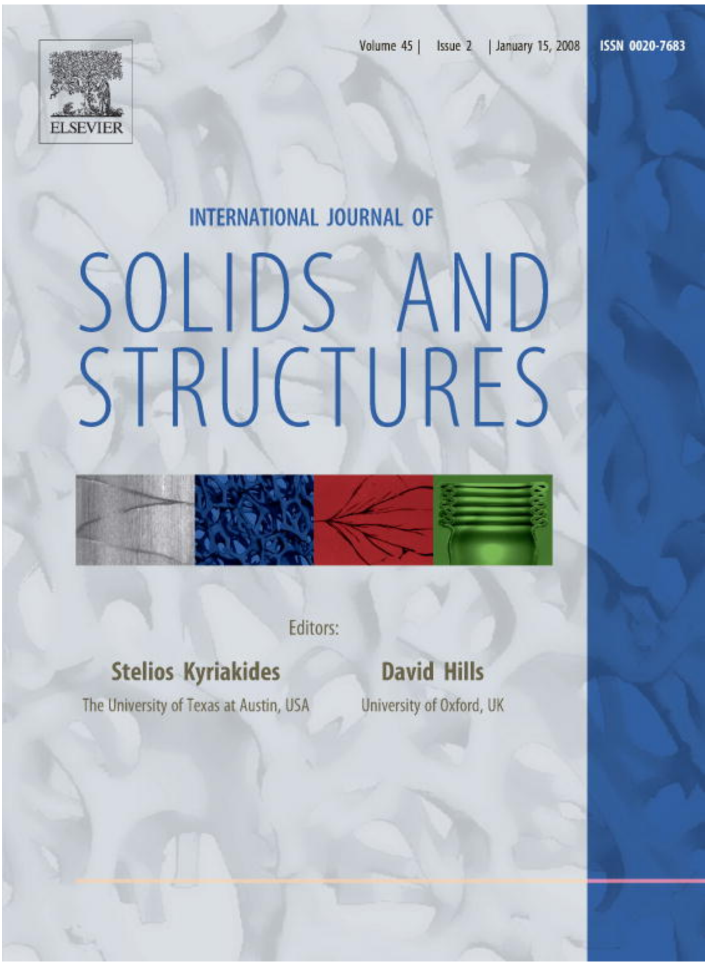
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Stresses in accreted planetary bodies

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Abstract

An explicit analytical solution is obtained for the stress field in an accreted triaxial ellipsoid under the influence of self-gravitation and rotation. Material is assumed to attach to the surface of the accreting body in a stress-free state, after which it behaves elastically. The results differ significantly from the classical elasticity solutions that are based on the assumption that the body is fully formed before the loading is applied. These results are relevant to the strengths of accreted planetary bodies such as comets and asteroids.

The solution allows both the magnitude and direction of the angular velocity to be a general function of the time-like parameter defining the progress of accretion. Simple closed-form expressions are given for two special cases—the ellipsoid accreting at constant angular velocity and the sphere accreting with an angular velocity vector that precesses through 90° during the accretion process. A Mathematica notebook permitting the solution of other problems can be downloaded from the website http://www-personal.umich.edu/jbarber/ellipsoid.nb.

Keywords: Accretion; Elasticity; Spherical harmonics; Residual stresses; Asteroids; Comets; Planets; Rotational stresses

1. Introduction

The stress field in an object depends on the manner in which it was constructed. For example, the stresses in a body that has solidified from a melt are influenced by its original liquid state in which the stresses are hydrostatic (Pedroso and Domoto, 1973); calculation of the stresses in dam embankments must account for the fact that these structures are constructed by placing pre-formed blocks one on top of the other (Clough and Woodward, 1967); stresses in biological tissues are influenced by growth emanating from the material's bulk (e.g., Rodriguez et al., 1994); and stresses in spherulites are influenced by transformational strains during the growth process (Dryden, 1987; Burns, 1996). In this paper, we shall investigate how the growth process affects the stress fields of small planetary bodies.

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4.6 Billion years ago, all the mass currently contained within the planets, moons, asteroids, and comets of the solar system was in the form of dust and gas that orbited the Sun in a large cloud called the solar nebula. The growth of kilometer sized objects from sub-micron sized dust grains occurred by the collisional and gravitational evolution of a swarm of particles. Growth in this manner, or growth by the continual deposition of material onto an object's surface, is known as a process of accretion (e.g., Weidenschilling, 2000). The only forays into calculating the stress fields of accreted planetary bodies were made by Brown and Goodman (1963) and Kadish et al. (2005). Brown and Goodman (1963) found the stress field of an accreted sphere under the influence of self-gravitation, and made brief mention of the relationship between their results and the stress field of the Earth. Kadish et al. (2005) extended Brown and Goodman's results to include the effect of loading due to rotation, and applied the results to investigate possible disruption mechanisms of small planetary bodies. They found that the stress field is significantly influenced by the history of the rotational speed during the accretion process.

Of course, not all objects of the solar system are spheres and a better approximation is to calculate an ellipsoidal fit to these objects. In this paper, we extend the solution of Brown and Goodman (1963) and Kadish et al. (2005) to that of an accreted ellipsoid.

2. Stress field in an accreted body

We consider the accretion of a triaxial ellipsoid whose shape remains constant while its size grows. The semi-axes of the final ellipsoid after growth is completed are denoted by a_i, $i = 1, 2, 3$ and we define a dimensionless effective radius r as

$$r \equiv \sqrt{\frac{x_i^2}{a_i^2}}, \tag{1}$$

where x_i are coordinates of a body-fixed Cartesian coordinate system aligned with the semi-axes of the ellipsoid and the summation convention is implied. The instantaneous boundary of the body at time t is then defined by $r = s(t)$ where s is called the growth parameter.

The ellipsoid grows from $s = 0$ to $s = 1$ due to the accretion of particles in an orbiting dust cloud. These particles add mass and angular momentum to the accreted body, and hence may change its angular velocity vector $\mathbf{\Omega}(s)$, which therefore becomes a function of the growth parameter s. Dones and Tremaine (1993) discuss the form that this function may take, based on the accumulation of mass and angular momentum of an accreting body in a cloud of particles. The accretion process is extremely slow—accretion of a typical asteroid may take 10^6 years—so it is reasonable to neglect the inertia forces associated with angular acceleration due to accretion.

If an ellipsoidal body rotates about a non-principal axis and experiences no external forces, the inertia forces associated with centripetal acceleration have a non-zero torque resultant about a perpendicular axis, causing the axis of rotation to precess or 'nutate'. In a frame of reference fixed in the body, the instantaneous axis of rotation traces out a cone centered on the major principal axis. The resulting body forces and the oscillating elastic stress field that they generate were given by Sharma et al. (2005), who used their results to estimate the rate at which the nutation would become damped out, causing the axis of rotation to approach the major principal axis.

In the present paper, we shall restrict attention to cases where the accelerations due to the time derivative $\dot{\mathbf{\Omega}}$ of the angular velocity vector can be neglected in comparison with the centripetal accelerations. This will be a reasonable approximation if the nutation period is long relative to that of diurnal rotation, which requires either that the ellipticity of the body be not too large, or that the axis of rotation be at (in which case there is no nutation) or near to a principal axis.[1]

[1] It is perhaps worth noting that if these conditions are not met and there is rapid nutation and associated periodically varying accelerations, the effect of the accretion process would be to develop a spatially periodic residual stress field. However, because the accretion process is so slow, the spatial wavelength of this field would be relatively short and may in fact be smaller than the size of the typical accreted particle.

With this simplification, the body force vector $\boldsymbol{f}$ can be derived from a body force potential

$$V(s) = V_0 + V_i(s)x_i^2 \tag{2}$$

by $\boldsymbol{f} = -\nabla V$, where

$$\begin{aligned} V_0 &= -\pi\rho^2 G a_1 a_2 a_3 K_0 \\ V_i(s) &= \pi\rho^2 G a_1 a_2 a_3 K_i - \frac{1}{2}\rho\left(\Omega_k(s)\Omega_k(s) - \Omega_i^2(s)\right) \end{aligned} \tag{3}$$

and

$$\begin{aligned} K_0 &= \int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} \\ K_i &= \int_0^\infty \frac{\mathrm{d}u}{(a_i^2+u)\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} \end{aligned}$$

(Chandrasekhar, 1969), where ρ is the density of the material and $G = 66.7 \times 10^{-12}\ \mathrm{Nm^2/kg^2}$ is the universal gravitational constant.

The general quadratic form (2) can also be used to describe 'tidal forces' due the gravitational attraction of the accreting body to another massive body, subject to the restrictions that (i) the massive body is a homogeneous sphere (or its inhomogeneity is spherically symmetric), (ii) a semi-axis of the accreting body is always pointed towards the attracting body's center of mass, and (iii) the distance between the bodies is constant (i.e., the orbit is circular) and sufficiently large for the attractive body force to be approximated as a second-order Taylor series expansion about the accreting body's center of mass (Chandrasekhar, 1969).

We assume that the accreting particles are much smaller than the growing ellipsoid so that the latter can at all times be treated as a continuum, that the growth parameter is a continuous, monotonically increasing function $s(t)$ of time, and that the impact of the accreting particles is "soft" so that collisional and other dynamical effects can be ignored. If it is also assumed that the material behaves elastically once it has accreted, then only the time derivative $\dot{\boldsymbol{\sigma}}$ of the total stress field is required to satisfy the compatibility equations (e.g., Kadish et al., 2005). Integrating $\dot{\boldsymbol{\sigma}}$ with respect to time, we deduce that the most general stress field of an accreting elastic body takes the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^c(x_1, x_2, x_3, t) + \boldsymbol{\sigma}^o(x_1, x_2, x_3), \tag{4}$$

where the time-varying term $\boldsymbol{\sigma}^c(x_1, x_2, x_3, t)$ is required to satisfy the equations of elasticity (equilibrium and compatibility), but the "initial stress" term $\boldsymbol{\sigma}^o(x_1, x_2, x_3)$ is only required to satisfy the equilibrium condition. Notice that the incompatible part of the stress field must be time independent, since if it were not, the time derivative $\dot{\boldsymbol{\sigma}}$ would also be incompatible. The decomposition of Eq. (4) is not unique, since a term that is both compatible and independent of time could be included in either $\boldsymbol{\sigma}^c$ or $\boldsymbol{\sigma}^o$.

To explain the physical basis of the residual stress term $\boldsymbol{\sigma}^o$, consider a body that is deformed elastically by some system of external loads and suppose that a thin unstretched layer of the same material is attached while the original body is still deformed. If the loads are now removed, the composite body will remain in a state of residual stress. In fact, Brown and Goodman (1963) developed their solution precisely in this manner, by considering the effect of attaching a small but finite unstressed layer and then proceeding to the limit as the thickness of this layer tends to zero. The present solution is mathematically equivalent to their procedure.

2.1. Time-varying stress field

The time-varying compatible stress field $\boldsymbol{\sigma}^c$ is conveniently expressed in terms of three scalar potential functions φ, ω, ψ through the equations

$$\sigma^c_{11} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\varphi}{\partial x_1^2} + x_3\frac{\partial^2\omega}{\partial x_1^2} - 2\nu\frac{\partial\omega}{\partial x_3} + 2\frac{\partial^2\psi}{\partial x_1\partial x_2} \tag{5}$$

$$\sigma^c_{22} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\varphi}{\partial x_2^2} + x_3\frac{\partial^2\omega}{\partial x_2^2} - 2\nu\frac{\partial\omega}{\partial x_3} - 2\frac{\partial^2\psi}{\partial x_1\partial x_2} \tag{6}$$

$$\sigma^c_{33} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\varphi}{\partial x_3^2} + x_3\frac{\partial^2\omega}{\partial x_3^2} - 2(1-\nu)\frac{\partial\omega}{\partial x_3} \tag{7}$$

$$\sigma^c_{12} = \frac{\partial^2\varphi}{\partial x_1\partial x_2} + x_3\frac{\partial^2\omega}{\partial x_1\partial x_2} + \frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2} \tag{8}$$

$$\sigma^c_{23} = \frac{\partial^2\varphi}{\partial x_2\partial x_3} + x_3\frac{\partial^2\omega}{\partial x_2\partial x_3} - (1-2\nu)\frac{\partial\omega}{\partial x_2} - \frac{\partial^2\psi}{\partial x_3\partial x_1} \tag{9}$$

$$\sigma^c_{31} = \frac{\partial^2\varphi}{\partial x_3\partial x_1} + x_3\frac{\partial^2\omega}{\partial x_3\partial x_1} - (1-2\nu)\frac{\partial\omega}{\partial x_1} + \frac{\partial^2\psi}{\partial x_2\partial x_3}. \tag{10}$$

This solution of the equations of elasticity was first introduced by Boussinesq (1885) and formalized by Green and Zerna (1968), who described the contributions of the three functions as Solutions A, B, and E, respectively (see also Barber, 2002, Chapter 19). The equilibrium equations require that φ, ω, ψ satisfy the equations

$$\nabla^2\varphi = \frac{(1-2\nu)V}{1-\nu}; \quad \nabla^2\omega = 0; \quad \nabla^2\psi = 0. \tag{11}$$

A particular solution to Eq. (11:i) is

$$\varphi^p = \frac{(1-2\nu)}{12(1-\nu)}\left(2V_0 x_i x_i + V_i(s)x_i^4\right). \tag{12}$$

Suitable potentials for ω, ψ and for the homogenous solution to Eq. (11:i) are chosen based on the fact that Eqs. (12, 2) result in stresses that are constant or quadratic in x_i when substituted into Eqs. (5)–(10). Hence, we choose the following forms because they will also yield stresses that are constant and quadratic in the spatial variables

$$\begin{aligned}&\varphi^h = \mathcal{F}_i(s)x_i^2 + \mathcal{G}_{ij}(s)x_i^2x_j^2, \quad \omega = \mathcal{H}_0(s)x_3 + \mathcal{H}_i(s)x_3x_i^2,\\ &\psi = \left(\mathcal{L}_0(s) + \mathcal{L}_i(s)x_i^2\right)x_1x_2,\end{aligned} \tag{13}$$

where $\mathcal{F}_i, \mathcal{G}_{ij}, \mathcal{H}_i, \mathcal{L}_i$ are functions of time and hence s. Also, we can set $\mathcal{G}_{ij} = \mathcal{G}_{ji}$ without loss of generality. The potentials (13) are required to be harmonic, leading to the conditions

$$\begin{aligned}&\sum_{i=1}^3 \mathcal{F}_i(s) = 0, \quad 2\mathcal{G}_{\alpha\alpha}(s) + \sum_{i=1}^3 \mathcal{G}_{i\alpha}(s) = 0, \quad \sum_{i=1}^3 \mathcal{H}_i(s) + 2\mathcal{H}_3(s) = 0,\\ &3\sum_{i=1}^3 \mathcal{L}_i(s) - 2\mathcal{L}_3(s) = 0,\end{aligned} \tag{14}$$

where no summation is implied in the term $\mathcal{G}_{\alpha\alpha}$. From this point on, the functional dependence on s of the calligraphic symbols $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{L}$ will be omitted in the interests of brevity.

The relations (14) can be used to eliminate the functions $\mathcal{F}_1, \mathcal{G}_{11}, \mathcal{G}_{22}, \mathcal{G}_{33}, \mathcal{H}_3, \mathcal{L}_1$ in (5)–(10), giving

$$\sigma_{11}^c = -2\left[\mathcal{F}_2 + \mathcal{F}_3 + v\mathcal{H}_0 - \mathcal{L}_0 - \frac{(1+v)V_0}{6(1-v)}\right] - 2\left(2\mathcal{G}_{12} + 2\mathcal{G}_{31} + v\mathcal{H}_1 + 3\mathcal{L}_2 + \mathcal{L}_3 - \frac{V_1(s)}{2}\right)x_1^2 + 2\left(2\mathcal{G}_{12} - v\mathcal{H}_2 + 3\mathcal{L}_2 + \frac{vV_2(s)}{2(1-v)}\right)x_2^2 + 2\left(2\mathcal{G}_{31} + (1+v)\mathcal{H}_1 + v\mathcal{H}_2 + \mathcal{L}_3 + \frac{vV_3(s)}{2(1-v)}\right)x_3^2 \quad (15)$$

$$\sigma_{22}^c = 2\left[\mathcal{F}_2 - v\mathcal{H}_0 - \mathcal{L}_0 + \frac{(1+v)V_0}{6(1-v)}\right] + 2\left(2\mathcal{G}_{12} - v\mathcal{H}_1 + 3\mathcal{L}_2 + \mathcal{L}_3 + \frac{vV_1(s)}{2(1-v)}\right)x_1^2 - 2\left(2\mathcal{G}_{12} + 2\mathcal{G}_{23} + v\mathcal{H}_2 + 3\mathcal{L}_2 - \frac{V_2(s)}{2}\right)x_2^2 + 2\left(2\mathcal{G}_{23} + v\mathcal{H}_1 + (1+v)\mathcal{H}_2 - \mathcal{L}_3 + \frac{vV_3(s)}{2(1-v)}\right)x_3^2 \quad (16)$$

$$\sigma_{33}^c = 2\mathcal{F}_3 - 2(1-v)\mathcal{H}_0 + \frac{(1+v)V_0}{3(1-v)} + 2\left[2\mathcal{G}_{31} - (1-v)\mathcal{H}_1 + \frac{vV_1(s)}{2(1-v)}\right]x_1^2 + 2\left[2\mathcal{G}_{23} - (1-v)\mathcal{H}_2 + \frac{vV_2(s)}{2(1-v)}\right]x_2^2 - 2\left[2\mathcal{G}_{31} + 2\mathcal{G}_{23} + v(\mathcal{H}_1 + \mathcal{H}_2) - \frac{V_3(s)}{2}\right]x_3^2 \quad (17)$$

$$\sigma_{12}^c = 2(4\mathcal{G}_{12} + 6\mathcal{L}_2 + \mathcal{L}_3)x_1x_2 \quad (18)$$

$$\sigma_{23}^c = 2(4\mathcal{G}_{23} + 2v\mathcal{H}_2 - \mathcal{L}_3)x_2x_3 \quad (19)$$

$$\sigma_{31}^c = 2(4\mathcal{G}_{31} + 2v\mathcal{H}_1 + \mathcal{L}_3)x_3x_1. \quad (20)$$

This solution satisfies the equations of equilibrium and compatibility for all (x_1, x_2, x_3, s). The remaining unknown functions will be eliminated using the boundary conditions in Section 2.3.

2.2. Initial stress field

The initial stress field $\boldsymbol{\sigma}^o$ is not required to satisfy the equations of compatibility, but must satisfy the equilibrium equations in the absence of body force (since the body forces were already taken into account in the term $\boldsymbol{\sigma}^c$). A suitable general form is

$$\sigma_{11}^o = f_1(r) + g_{1i}(r)x_i^2; \quad \sigma_{22}^o = f_2(r) + g_{2i}(r)x_i^2; \quad \sigma_{33}^o = f_3(r) + g_{3i}(r)x_i^2 \quad (21)$$

$$\sigma_{12}^o = h_{12}(r)x_1x_2; \quad \sigma_{23}^o = h_{23}(r)x_2x_3; \quad \sigma_{31}^o = h_{31}(r)x_3x_1, \quad (22)$$

where f_i, g_{ji}, h_{ji} are arbitrary functions of r. This form of the initial stress field is chosen because Eq. (4) (the sum of Eqs. (15)–(20) and Eqs. (21 and 22)) must satisfy homogeneous boundary conditions on the ellipsoid's surface $r = s$. We shall show in Section 2.4 that Eqs. (21 and 22) are sufficiently general to provide a solution satisfying the stated governing equations and boundary conditions for any history of body force described by a potential of the form (2).

Substituting Eqs. (21 and 22) into the equilibrium equations and noting that

$$\frac{\partial f(r)}{\partial x_i} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x_i},$$

we obtain

$$r\{2g_{11}(r) + h_{12}(r) + h_{31}(r)\} + \frac{f_1'(r)}{a_1^2} + \frac{g_{11}'(r)x_1^2}{a_1^2} + \left\{\frac{g_{12}'(r)}{a_1^2} + \frac{h_{12}'(r)}{a_2^2}\right\}x_2^2 + \left\{\frac{g_{13}'(r)}{a_1^2} + \frac{h_{31}'(r)}{a_3^2}\right\}x_3^2 = 0$$

$$r\{2g_{22}(r) + h_{12}(r) + h_{23}(r)\} + \frac{f_2'(r)}{a_2^2} + \left\{\frac{g_{21}'(r)}{a_2^2} + \frac{h_{12}'(r)}{a_1^2}\right\}x_1^2 + \frac{g_{22}'(r)x_2^2}{a_2^2} + \left\{\frac{g_{23}'(r)}{a_2^2} + \frac{h_{23}'(r)}{a_3^2}\right\}x_3^2 = 0$$

$$r\{2g_{33}(r) + h_{31}(r) + h_{23}(r)\} + \frac{f_3'(r)}{a_3^2} + \left\{\frac{g_{31}'(r)}{a_3^2} + \frac{h_{31}'(r)}{a_1^2}\right\}x_1^2 + \left\{\frac{g_{32}'(r)}{a_3^2} + \frac{h_{23}'(r)}{a_2^2}\right\}x_2^2 + \frac{g_{33}'(r)x_3^2}{a_3^2} = 0.$$

These equations can be satisfied by equating the coefficients of the spatial variables x_1, x_2, x_3 to zero in each equation. This gives 12 first-order ordinary differential equations that can be used to eliminate f_i, g_{ji} in terms of the remaining functions h_{ji} through the relations

$$g_{11}(r) = A_{11}; \quad g_{12}(r) = A_{12} - \frac{a_1^2 h_{12}(r)}{a_2^2}; \quad g_{13}(r) = A_{13} - \frac{a_1^2 h_{31}(r)}{a_3^2} \tag{23}$$

$$g_{21}(r) = A_{21} - \frac{a_2^2 h_{12}(r)}{a_1^2}; \quad g_{22}(r) = A_{22}; \quad g_{23}(r) = A_{23} - \frac{a_2^2 h_{23}(r)}{a_3^2} \tag{24}$$

$$g_{31}(r) = A_{31} - \frac{a_3^2 h_{31}(r)}{a_1^2}; \quad g_{32}(r) = A_{32} - \frac{a_3^2 h_{23}(r)}{a_2^2}; \quad g_{33}(r) = A_{33} \tag{25}$$

$$f_1(r) = -a_1^2 r^2 A_{11} + B_1 + a_1^2 \int_r^s \{h_{12}(p) + h_{31}(p)\} p\,\mathrm{d}p \tag{26}$$

$$f_2(r) = -a_2^2 r^2 A_{22} + B_2 + a_2^2 \int_r^s \{h_{12}(p) + h_{23}(p)\} p\,\mathrm{d}p \tag{27}$$

$$f_3(r) = -a_3^2 r^2 A_{33} + B_3 + a_3^2 \int_r^s \{h_{31}(p) + h_{23}(p)\} p\,\mathrm{d}p, \tag{28}$$

where A_{ij}, B_i are constants of integration and p is a dummy variable of integration.

2.3. Stress boundary conditions

At any given time t, the boundary $r = s(t)$ of the partially accreted body must be traction-free. However, we also assume that the accretion process occurs in such a manner that each increment of new material is laid down in a stress-free state. There can be no sudden change in stress state as a particle resting on the surface becomes a part of the accreting body and hence we must impose the stronger condition that all six stress components are instantaneously zero at $r = s$ (Naumov, 1994; Kadish et al., 2005).

To explain this stronger boundary condition, we envisage a situation in which recently accreted particles near the surface simply rest against each other as in a granular medium, so that particles in the surface layer transmit negligible forces. However, as more particles accrete, those deeper below the surface will come under increasing pressure from the weight of those above and sintering and diffusion processes will lead to the development of a more cohesive structure (Coble, 1970). The additional equations implied in the stress-free boundary condition will suffice to determine the initial stress field of Section 2.2, since every point in the fully accreted body was at some point in time located on the boundary of the partially accreted body. This procedure is similar to that used to determine the residual stress field in a body solidified from a melt (Pedroso and Domoto, 1973).

To impose the stress-free boundary conditions, the time-varying stress field (15)–(20) is added to the initial stress field (21 and 22) after using the substitutions (23)–(28). The instantaneous boundary is defined by the equation $r = s$ and hence

$$x_3^2 \rightarrow a_3^2\left(s^2 - x_1^2/a_1^2 - x_2^2/a_2^2\right)$$

from (1). This result is used to eliminate x_3 in the expressions for the stress components, which then read:

$$\begin{aligned}
\sigma_{11} = {} & -2(\mathcal{F}_2 + \mathcal{F}_3 + v\mathcal{H}_0 - \mathcal{L}_0) + \frac{(1+v)V_0}{3(1-v)} + B_1 \\
& - \left[(A_{11} - h_{31})a_1^2 + \left(4\mathcal{G}_{31} + 2(1+v)\mathcal{H}_1 + 2v\mathcal{H}_2 + 2\mathcal{L}_3 + A_{13} + \frac{vV_3(s)}{(1+v)}\right)a_3^2\right]s^2 \\
& - \left[2(2\mathcal{G}_{12} + 2\mathcal{G}_{31} + v\mathcal{H}_1 + 3\mathcal{L}_2 + \mathcal{L}_3) - A_{11} - V_1(s) - h_{31} + \frac{a_3^2}{a_1^2}\left(4\mathcal{G}_{31} + 2(1+v)\mathcal{H}_1 + 2v\mathcal{H}_2 + 2\mathcal{L}_3 + \frac{vV_3(s)}{(1-v)} + A_{13}\right)\right]x_1^2 \\
& + \left[2(2\mathcal{G}_{12} + 3\mathcal{L}_2 - v\mathcal{H}_2) + A_{12} + \frac{vV_2(s)}{(1-v)} - \frac{a_1^2}{a_2^2}(h_{12} + h_{31}) - \frac{a_3^2}{a_2^2}\left(4\mathcal{G}_{31} + 2(1+v)\mathcal{H}_1 + 2v\mathcal{H}_2 + 2\mathcal{L}_3 + A_{13} + \frac{vV_3(s)}{(1-v)}\right)\right]x_2^2
\end{aligned} \tag{29}$$

$$\sigma_{22}=2(\mathcal{F}_2-\nu\mathcal{H}_0-\mathcal{L}_0)+B_2+\frac{(1+\nu)V_0}{3(1-\nu)}$$
$$-\left[(A_{22}+h_{23})a_2^2-\left(4\mathcal{G}_{23}+\nu\mathcal{H}_1+2(1+\nu)\mathcal{H}_2-2\mathcal{L}_3+A_{23}+\frac{\nu V_3(s)}{(1-\nu)}\right)a_3^2\right]s^2$$
$$+\left[2(2\mathcal{G}_{12}+\nu\mathcal{H}_1+3\mathcal{L}_2+\mathcal{L}_3)+A_{21}+\frac{\nu V_1(s)}{(1-\nu)}-\frac{a_2^2}{a_1^2}(h_{12}-h_{23})\right.$$
$$\left.-\frac{a_3^2}{a_1^2}\left(4\mathcal{G}_{23}+2\nu\mathcal{H}_1+2(1+\nu)\mathcal{H}_2+2\mathcal{L}_3+A_{23}+\frac{\nu V_3(s)}{(1-\nu)}\right)\right]x_1^2$$
$$-\left[2(2\mathcal{G}_{12}+2\mathcal{G}_{23}+\nu\mathcal{H}_2+3\mathcal{L}_2)-A_{22}-V_2(s)-h_{23}\right.$$
$$\left.-\frac{a_3^2}{a_2^2}\left(4\mathcal{G}_{23}+2\mathcal{H}_2-2\mathcal{L}_3+2\nu\mathcal{H}_1+2\nu\mathcal{H}_2+A_{23}+\frac{\nu V_3(s)}{(1-\nu)}\right)\right]x_2^2 \quad (30)$$

$$\sigma_{33}=2(\mathcal{F}_3-(1-\nu)\mathcal{H}_0)+B_3+\frac{(1+\nu)V_0}{3(1-\nu)}-(4\mathcal{G}_{31}+4\mathcal{G}_{23}+2\nu\mathcal{H}_1+2\nu\mathcal{H}_2-V_3(s))a_3^2s^2$$
$$\times\left[2(2\mathcal{G}_{31}-(1-\nu)\mathcal{H}_1)+A_{31}+\frac{\nu V_1(s)}{1-\nu}+\frac{a_3^2}{a_1^2}(4\mathcal{G}_{31}+4\mathcal{G}_{23}+2\nu\mathcal{H}_1+2\nu\mathcal{H}_2-A_{33}-h_{31}-V_3(s))\right]x_1^2$$
$$\times\left[2(2\mathcal{G}_{23}-(1-\nu)\mathcal{H}_2)+A_{32}+\frac{\nu V_2(s)}{(1-\nu)}+\frac{a_3^2}{a_2^2}(4\mathcal{G}_{31}+4\mathcal{G}_{23}+2\nu\mathcal{H}_1+2\nu\mathcal{H}_2-A_{33}-h_{23}-V_3(s))\right]x_2^2 \quad (31)$$

$$\sigma_{12}=[2(4\mathcal{G}_{12}+6\mathcal{L}_2+\mathcal{L}_3)+h_{12}]x_1x_2 \quad (32)$$
$$\sigma_{31}=[2(4\mathcal{G}_{31}+\mathcal{L}_3+2\nu\mathcal{H}_1)+h_{31}]x_3x_1 \quad (33)$$
$$\sigma_{23}=[2(4\mathcal{G}_{23}-\mathcal{L}_3+2\nu\mathcal{H}_2)+h_{23}]x_2x_3 \quad (34)$$

We recall that the calligraphic symbols $\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{L}$ are functions of time and hence of s, whereas the h_{ij} are functions of the position variable r. However, at the instantaneous boundary $r=s$, all these expressions become functions of s and hence each stress component in Eqs. (29)–(34) has the form a polynomial in x_1, x_2 with coefficients that are functions of s. By setting each of these coefficients to zero, we obtain 12 linear algebraic equations for the remaining unknown functions, with s as a parameter. These equations are not linearly independent and it proves possible to satisfy them without determining all the unknown functions. This arises because of the previously noted ambiguity in the decomposition (4) and it can be verified that the remaining undetermined functions can be given arbitrary values without affecting the expressions for the total stress components.

2.4. Complete stress field

Once the functions $\mathcal{F}_i,\mathcal{G}_{ij},\mathcal{H}_i,\mathcal{L}_i,h_{ij}$ have been determined, the time-varying stress field $\boldsymbol{\sigma}^c$ can be determined from Eqs. (15)–(20) and the initial stress field $\boldsymbol{\sigma}^o$ from (21 and 22), using (23)–(28). The complete stress field is then recovered by combining these expressions as in (4). Notice that depending on the variation of the angular velocity $\boldsymbol{\Omega}$ with time, the initial stress and hence the total stress can have fairly general (not necessarily quadratic) dependence on the spatial coordinates x_1, x_2, x_3.

The stress field can be written as closed-form expressions involving integrals of the components of angular velocity $V_i(s)$, but because of the number of parameters involved, the resulting expressions are too lengthy to present here. They are however readily obtained using a symbolic manipulation program such as Mathematica or Maple and an appropriate code is available for download at the website http://www-personal.umich.edu/jbarber/ellipsoid.nb. In the following section, we shall present the results of some special cases.

3. Results

The classical solution for the stresses in a rotating ellipsoidal body is that due to Chree (1895), who implicitly assumed that the initial stress $\boldsymbol{\sigma}_o$ is zero—in other words, if the body were somehow brought to rest and gravitational loading 'turned off', it would be left in a stress-free state. This is certainly not the case for an accreted body, as demonstrated by the solution for the sphere (Kadish et al., 2005). Instead, the stress state is significantly influenced by the history of rotation during the accretion process.

Dones and Tremaine (1993) discussed the way in which the angular velocity might be influenced by the dynamics of an accreting system of particles and suggested that it may take the power-law form

$$\mathbf{\Omega}(s) = \mathbf{\Omega}_f s^n,$$

where $\mathbf{\Omega}_f = \mathbf{\Omega}(1)$ is the angular velocity when growth is complete. They concluded that the value of n would lie in the range $0 \leqslant n \leqslant 2$ and depend on the magnitude of the partially accreted body's gravitational field relative to that of the Sun and the velocity dispersion of the accreting particles. When the velocity dispersion is high and gravitation is strong, n is predicted to approach 2 and the angular velocity increases during growth, whereas for other combinations of these parameters n is predicted to be zero and the angular velocity is constant. Some results for the sphere with a quadratically varying angular velocity were given by Kadish et al. (2005).

3.1. Constant angular velocity

For constant angular velocity $\mathbf{\Omega}$ about an arbitrary axis, the solution for the stress field simplifies to the quadratic form

$$\sigma_{11} = \left(\frac{\rho(\Omega_2^2 + \Omega_3^2)}{2} - \pi\rho^2 G a_1 a_2 a_3 K_1\right) a_1^2 (1 - r^2) \tag{35}$$

$$\sigma_{22} = \left(\frac{\rho(\Omega_3^2 + \Omega_1^2)}{2} - \pi\rho^2 G a_1 a_2 a_3 K_2\right) a_2^2 (1 - r^2) \tag{36}$$

$$\sigma_{33} = \left(\frac{\rho(\Omega_1^2 + \Omega_2^2)}{2} - \pi\rho^2 G a_1 a_2 a_3 K_3\right) a_3^2 (1 - r^2) \tag{37}$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0. \tag{38}$$

The off-diagonal stresses are all zero, so the diagonal stresses are also the principal stresses. The stress field reduces to this simple form because the body force coefficients V_i are not functions of the growth parameter. Notice that even in this simple case, the stress field is significantly different from that given by Chree (1895).

We notice from Eqs. (35)–(38) that the stresses due to gravitation alone ($\Omega_i = 0$) are always compressive, but except in the limiting case of the sphere, they are not everywhere hydrostatic. Thus, gravitational effects alone could conceivably cause failure through a shear-derived yield criterion such as Mises or Mohr-Coulomb. The rotation Ω_i tends to produce superposed tensile stresses and hence to reduce the magnitude of the compressive stress components.

3.1.1. Example

To illustrate the qualitative effects of ellipticity on the resulting stress field, we compare the Mises stress

$$\sigma_m = \sqrt{\frac{(3\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})}{2}}$$

in an accreted sphere with that in an accreted ellipsoid with the same mass, density, and angular momentum. The comparison sphere has a radius of 25 km, density 2500 kg/m^3, and angular velocity 2×10^{-4} rad/s, which is assumed constant throughout the accretion process. These values are arbitrary in the sense that they do not correspond to any one particular object, but are representative in that they fall within the range of observed asteroid sizes, densities, and angular velocities (Hilton, 2002; Pravec et al., 2002). The ellipsoid has principal semi-axis ratios $a_2/a_1 = 1$, $a_3/a_1 = 0.6$ and rotates about the x_3-axis, which is the axis with the maximum inertia.

Figs. 1(a and b) shows contour plots of Mises stress in the sphere and the ellipsoid, respectively. The stress in the ellipsoid varies quadratically with r (see Eqs. (35)–(38)) from a maximum of 186.7 kPa at the center to

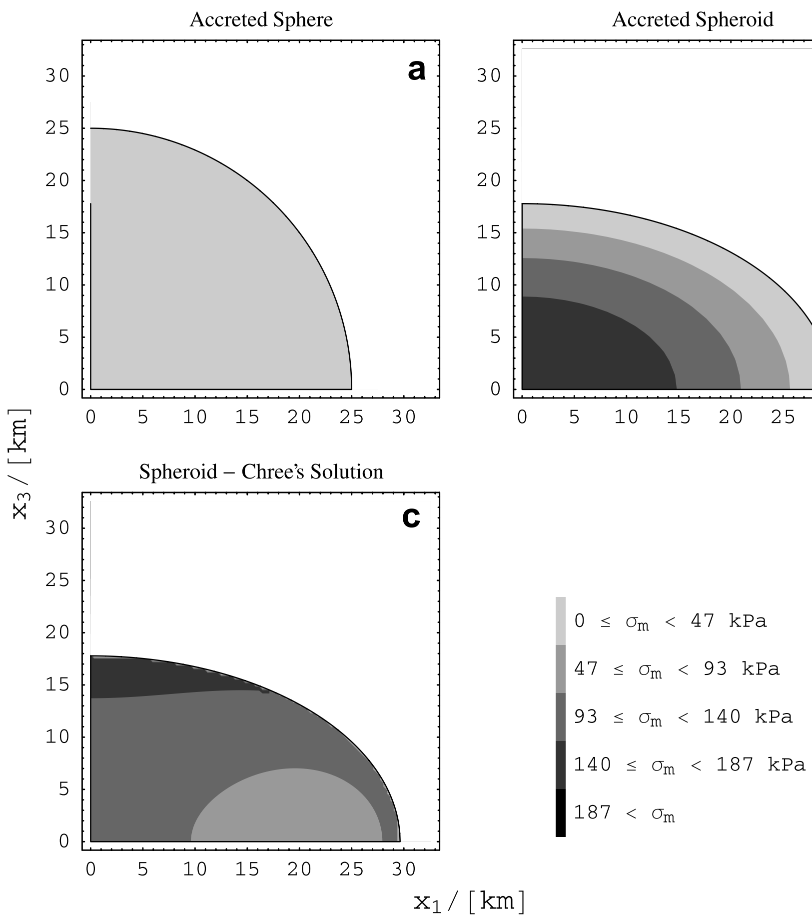

Fig. 1. Contour plot of the Mises stress in (a) an accreted sphere, (b) an accreted ellipsoid, and (c) an ellipsoid which is assembled before the body forces are applied (Chree's solution).

zero at the boundary. The stress in the sphere also varies quadratically with r, but the gradation in the contour plot is insufficiently fine to show this, since the maximum at the center is only 31.25 kPa.

Chree's calculation for a 'preformed' ellipsoid with the same properties and rotation rate are shown in Fig. 1(c). Comparison of Figs. 1(b and c) shows that Chree's solution predicts a totally different qualitative form for the stress field. The maximum occurs at the poles and is 164.9 kPa, while the minimum occurs at the interior point ($x_1 = 20.9$ km, $x_3 = 0$ km) on the equatorial plane and is 73.7 kPa. The present authors maintain that Chree's solution is unrealistic for any practical planetary body, since its predictions differ so much from the accreted solution and no scientific mechanism can be envisaged by which the body could be assembled before the body forces were applied.

3.2. Variable angular velocity

To illustrate the range of problems that can be solved with variable angular velocity, while keeping the expressions for stress components tolerably brief, we consider the special case of a sphere $a_1 = a_2 = a_3 = a$ which rotates at a constant speed Ω, but whose axis of rotation precesses uniformly during accretion from the x_3-axis at $s = 0$ to the x_1-axis at $s = 1$. It then follows that

$$\Omega_1 = \Omega \sin\left(\frac{\pi s}{2}\right); \quad \Omega_2 = 0; \quad \Omega_3 = \Omega \cos\left(\frac{\pi s}{2}\right).$$

The final stress field is obtained as

$$\sigma_{11}=\frac{\rho\Omega^2(1+v)}{4(7+5v)}\left[(a^2-R^2)-2\left(\frac{a^2}{\pi^2}+x_2\right)\left\{1+\cos\left(\frac{\pi R}{a}\right)\right\}-\frac{2aR}{\pi}\sin\left(\frac{\pi R}{a}\right)\right]-\frac{2\pi\rho^2G(a^2-R^2)}{3}$$

$$\sigma_{22}=\frac{\rho\Omega^2}{2}\left[(a^2-R^2)-\frac{(x_1^2-x_3^2)(1+v)}{(7+5v)}\left\{1+\cos\left(\frac{\pi R}{a}\right)\right\}\right]-\frac{2\pi\rho^2G(a^2-R^2)}{3}$$

$$\sigma_{33}=\frac{\rho\Omega^2}{4(7+5v)}\left[(13+9v)(a^2-R^2)+2\left(\frac{a^2}{\pi^2}+x_2^2\right)(1+v)\left\{1+\cos\left(\frac{\pi R}{a}\right)\right\}+\frac{2aR(1+v)}{\pi}\sin\left(\frac{\pi R}{a}\right)\right]-\frac{2\pi\rho^2G(a^2-R^2)}{3}$$

$$\sigma_{12}=\frac{\rho\Omega^2(1+v)x_1x_2}{(7+5v)}\cos^2\left(\frac{\pi R}{2a}\right)$$

$$\sigma_{23}=-\frac{\rho\Omega^2(1+v)x_2x_3}{(7+5v)}\cos^2\left(\frac{\pi R}{2a}\right)$$

$$\sigma_{31}=0$$

where

$$R=\sqrt{x_1^2+x_2^2+x_3^2}$$

is the distance from the origin.

4. Conclusions

The stress field has been derived for an accreted, self-gravitating triaxial ellipsoid whose angular velocity may vary in a fairly general way during growth. The only restriction imposed is that the nutation period (if any) be long relative to the period of diurnal rotation. It is assumed growth occurs in such a way that the ellipsoid's shape remains constant, that the body can at all time be treated as a continuum, and that all deformations are infinitesimal and elastic. The most general state of stress is available in closed form, but it is too long to present explicitly. For the special case where the ellipsoid's angular velocity is constant (not a function of time) during growth, the stress field takes the simple form of Eqs. (35)–(38).

The results show that ellipticity has a significant effect on the state of stress compared with an accreted sphere with similar mass, density, and angular momentum. Also, comparison of the stress field with the classical solution due to Chree (1895) shows that the accretion process has a major qualitative effect on the stress field. In particular, Chree's solution, which assumes that the body was somehow preformed before the body forces due to gravitation and rotation were applied, predicts different locations for the maximum Mises stress.

To illustrate the application of the method to more complex rotation histories, the solution is given for a sphere whose angular velocity precesses during the accretion process. The authors anticipate that the astronomical community would like to see the implications of the present theory for a wider range of astronomical bodies and for this purpose a user-friendly Mathematica™ notebook permitting the solution of other problems can be downloaded from the website http://www-personal.umich.edu/jbarber/ellipsoid.nb. It is not necessary to understand the mathematical structure of the solution in order to use this resource.

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