Introduction to Internal Gravity Waves

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LECTURE NOTES

Contents

Chapter 1

Governing Equations

The full governing equations (Navier-Stokes equations) for an incompressible fluid are given by

$$\rho_* \left(\frac{Du}{Dt} - fv \right) = -\frac{\partial p}{\partial x} \tag{1.1}$$

$$\rho_* \left(\frac{Dv}{Dt} + fu \right) = -\frac{\partial p}{\partial y} \tag{1.2}$$

$$\rho_* \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho \tag{1.3}$$

$$\frac{D\rho}{Dt} = -w \frac{\rho_0}{dz} \tag{1.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1.5}$$

where (u, v, w) are the velocity components in the x, y, z directions respectively, p is pressure, ρ_* is a constant density, $\rho(x, y, z, t)$ is the total density, $\rho_0(z)$ is the background density, g is acceleration due to gravity and f is the Coriolis frequency due to rotation of the Earth. The material derivative D/Dt is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

1.1 Simplifications:

Consider a two dimensional flow in the $xz-$plane (see Figure 1.1), without the Coriolis frequency and with constant density ($\rho_0 = \rho =$ constant), then the governing equations reduce to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.6}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{1.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.8}$$

The vertical coordinate is measured upward from the undisturbed free surface, and the free surface displacement is $\eta(x, t)$. The Coriolis frequency is

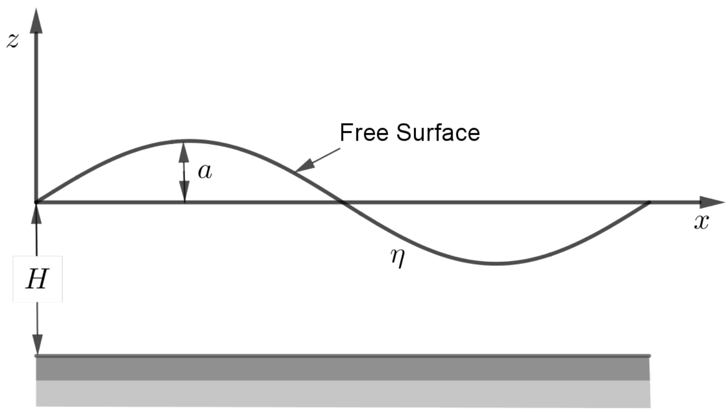

Figure 1.1: Surface wave

neglected by assuming that the frequency of the waves is large compared to the Coriolis frequency such that the waves are not affected by the Earth's rotation. Viscous effects have also been neglected because they do not have significant effect on the wave propagation. The motion is generated from rest by wind action or dropping a stone in the water body. The resulting motion is **irrotational**, by the Kelvin's circulation theorem.

To further simplify the equations motion, it is assumed that the amplitude a of oscillation of the free surface is small. That is, both a/λ and a/H are much smaller than one.

- $a/\lambda << 1$ implies that the slope of the sea/water surface is small.
- $a/H << 1$ implies that the instantaneous depth is not significantly different from the undisturbed depth.

These small amplitude assumptions allows for the problem to be linearized. Under the small amplitude assumption, the equations become

$$\begin{aligned} \frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{w\frac{\partial u}{\partial z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial w}{\partial t} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{w\frac{\partial w}{\partial z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \end{aligned}$$

Thus, the simplified equations to solve are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.9}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{1.10}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1.11}$$

Before proceeding further, we give a few definitions and concepts which allows for further simplification of the equations.

Definition 1 (Vorticity). The vorticity vector ω of a fluid element with velocity vector $\mathbf{u} = (u, v, w)$ is defined as

$$\omega = \nabla \times \mathbf{u} \tag{1.12}$$

with components:

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{1.13}$$

Remark. Vorticity is a measure of the local rotation of a fluid element. It is related to the concept of **circulation** which is the line integral of the tangental component of velocity around a closed contour.

Definition 2 (Irrotational Flow). A fluid motion is said to be irrotational if the vorticity is equal to zero

$$\omega = \nabla \times \mathbf{u} = 0, \tag{1.14}$$

which requires that

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \qquad i \neq j, \tag{1.15}$$

where u_i and u_j denote the velocity components in the x_i and x_j coordinates respectively.

Remark. If the flow is irrotational, the velocity vector can be written as the gradient of a scalar function $\phi(\mathbf{x}, t)$. This is because

$$u_i = \frac{\partial \phi}{\partial x_i} \tag{1.16}$$

satisfies the condition of irrotationality in equation (1.14).

For example, in our $2D$ flow in the $xz-$plane (i.e., x_1x_3-plane), irrotationality implies that

$$\frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1}$$

$$\Longrightarrow \boxed{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,} \tag{1.17}$$

and the velocity field satisfies

$$u = \frac{\partial \phi}{\partial x}, \qquad w = \frac{\partial \phi}{\partial z} \tag{1.18}$$

Further simplifications

Substituting (1.18) into the continuity equation (1.11) results in the Laplace equation

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.} \tag{1.19}$$

Also, from equation (1.10), we have

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

and inter-changing derivatives gives

$$\Longrightarrow \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) = -\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) - g,$$

$$\Longrightarrow \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho}\right) = -g.$$

Integrating both sides with respect to z results in the linearized **Bernoulli equation**

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = 0.} \tag{1.20}$$

Note that equation (1.9) also satisfies the Bernoulli equation, where the resulting integration constant, say $C(z)$, must be equated to $C(z) = -gz$. To solve the Laplace equation, we need to specify boundary conditions at the free surface and at the bottom.

Boundary conditions

Bottom boundary condition:

At the bottom, we specify zero normal velocity such that

$$w = \frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -H \tag{1.21}$$

Kinematic boundary condition:

The kinematic boundary condition at the free surface states that the fluid particle never leaves the surface, that is

$$\frac{D\eta}{Dt} = w_\eta, \quad \text{at} \quad z = \eta, \tag{1.22}$$

where the material derivative is $D/Dt = \partial/\partial t + u(\partial/\partial x)$, and w_η is the vertical component of fluid velocity at the free surface. In other words,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta}, \tag{1.23}$$

which is a non-linear equation. Recall that $\eta = \eta(x,t)$. For small amplitude waves both u and $\partial\eta/\partial x$ are small, so that the quadratic term $u\partial\eta/\partial x$ is one order smaller than the other terms in (1.23) so we neglect $u\partial\eta/\partial x$ and get

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta}. \tag{1.24}$$

We can simplify this condition further by evaluating the right side at $z = 0$ rather than at the free surface. This can be justified by performing a Taylor series expansion of $\partial\phi/\partial z$ about $z = 0$:

$$\frac{\partial \phi}{\partial z}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=0} + \eta\frac{\partial^2 \phi}{\partial^2 z} + \cdots \approx \frac{\partial \phi}{\partial z}\Big|_{z=0} \tag{1.25}$$

Thus, to first order of accuracy, $\partial\phi/\partial z$ can be evaluated at $z = 0$ and (1.24) becomes

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0. \tag{1.26}$$

Dynamic boundary condition:

There is a dynamic boundary condition that the pressure just below the free surface is always equal to the ambient (atmospheric) pressure. Taking the ambient pressure to be zero results in

$$p = 0 \qquad \text{at} \quad z = \eta. \tag{1.27}$$

Substituting into the Bernoulli equation (1.20) yields

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = \eta \tag{1.28}$$

As in the case of the kinematic condition, for small amplitude waves, the term $\partial\phi/\partial t$ can be evaluated at $z = 0$ instead of at $z = \eta$ so that

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = 0 \tag{1.29}$$

Solution of the Problem

Summarizing, we need to solve the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{1.30}$$

in the interior of the domain, subject to the conditions

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -H \tag{1.31}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \tag{1.32}$$

$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0 \tag{1.33}$$

To apply the boundary conditions, we need to assume a form for $\eta(x,t)$. We assume the simplest case of a sinusoidal component with wave number k and frequency ω, such that

$$\eta = a\cos(kx - \omega t) \tag{1.34}$$

Remark. A strong motivation for studying sinusoidal waves is that an arbitrary disturbance can be decomposed into various sinusoidal components by Fourier analysis, and the response of the system to an arbitrary small disturbance is the sum of the responses to the various sinusoidal components.

We solve the Laplace equation by assuming a separable solution of the form $\phi(x,z,t) = \psi(z)\Phi(x,t)$. Since we assumed a cosine dependence of η on $(kx-\omega t)$, the conditions in (1.32) and (1.33) imply that $\Phi(x,t)$ must be sine function of $kx-\omega t$. Thus, we assume a solution in the form

$$\phi = \psi(z)\sin(kx-\omega t), \tag{1.35}$$

where $\psi(z)$ and $\omega(k)$ are to be determined. Substituting (1.35) into the Laplace equation (1.30) gives

$$-k^2\psi(z)\sin(kx-\omega t) + \psi''(z)\sin(kx-\omega t) = 0,$$
$$\frac{d^2\psi}{dz^2} - k^2\psi = 0. \tag{1.36}$$

This is a second order ordinary differential equation with a general solution

$$\psi(z) = Ae^{kz} + Be^{-kz}.$$

Thus, the velocity potential ϕ in (1.35) is given by

$$\phi = \left(Ae^{kz} + Be^{-kz}\right)\sin(kx-\omega t). \tag{1.37}$$

The constants A and B are determined from the conditions in (1.31) and (1.32). Now,

$$\frac{\partial\phi}{\partial z} = \left(kAe^{kz} - kBe^{-kz}\right)\sin(kx-\omega t)$$

Applying (1.31):

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at} \quad z = 0$$

gives

$$\left(kAe^{-kH} - kBe^{kH}\right)\sin(kx-\omega t) = 0,$$
$$\Longrightarrow k\left(Ae^{-kH} - Be^{kH}\right)\sin(kx-\omega t) = 0.$$

For a nontrivial solution,

$$\sin(kx-\omega t) \neq 0 \Longrightarrow Ae^{-kH} = Be^{kH}$$

$$\Longrightarrow B = Ae^{-2kH} \tag{1.38}$$

We next apply condition (1.32). But before we do that, consider the following remark.

Remark. Suppose we applied condition (1.32) at $z = \eta$ instead of the linearized form at $z = 0$. Then from (1.37) we get

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = k\left(Ae^{k\eta} - Be^{-k\eta}\right)\sin(kx - \omega t)$$

For a small slope of the free surface, $k\eta << 1$, we can set $e^{k\eta} \approx e^{-k\eta} = 1$. This is effectively what we are doing by applying the surface boundary conditions at $z = 0$ instead of at $z = \eta$, which was justified using Taylor series expansions.

Substituting (1.34) and (1.37) into (1.32) gives

$$k\left(Ae^{kz} - Be^{-kz}\right)\sin(kx - \omega t) = a\omega\sin(kx - \omega t)$$

At $z = 0$ we have

$$\begin{aligned} k(A - B) &= a\omega \qquad (1.39)\\ \Longrightarrow A - B &= \frac{a\omega}{k}\\ \Longrightarrow B &= A - \frac{a\omega}{k} \end{aligned}$$

Employing (1.38) results in

$$\begin{aligned} A - \frac{a\omega}{k} &= Ae^{-2kH}\\ \Longrightarrow A\left(1 - e^{-2kH}\right) &= \frac{a\omega}{k}\\ \Longrightarrow A &= \frac{a\omega}{k\left(1 - e^{-2kH}\right)} \qquad (1.40) \end{aligned}$$

and

$$B = \frac{a\omega e^{-2kH}}{k\left(1 - e^{-2kH}\right)} \qquad (1.41)$$

From (1.37) we finally get the velocity potential

$$\begin{aligned} \phi &= \left[\frac{a\omega e^{kz}}{k\left(1 - e^{-2kH}\right)} + \frac{a\omega e^{-2kH}}{k\left(1 - e^{-2kH}\right)}e^{-kz}\right]\sin(kx - \omega t).\\ \phi &= \frac{a\omega}{k}\left[\frac{e^{kz}}{\left(1 - e^{-2kH}\right)} + \frac{e^{-k(z+2H)}}{\left(1 - e^{-2kH}\right)}\right]\sin(kx - \omega t). \qquad (1.42) \end{aligned}$$

Using the fact that

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$1 - e^{-2kH} = 1 - \frac{e^{-kH}}{e^{kH}} = \frac{e^{kH} - e^{-kH}}{e^{kH}}$$

Thus,

$$\phi = \frac{a\omega}{k}\left[\frac{e^{k(z+H)}}{(e^{kH} - e^{-kH})} + \frac{e^{(-kz-2kH)} \cdot e^{kH}}{(e^{kH} - e^{-kH})}\right]\sin(kx - \omega t)$$

$$\Longrightarrow \phi = \frac{a\omega}{k}\left[\frac{e^{k(z+H)} + e^{-k(z+H)}}{e^{kH} - e^{-kH}}\right]\sin(kx - \omega t)$$

$$\therefore \boxed{\phi = \frac{a\omega}{k}\frac{\cosh[k(z+H)]}{\sinh(kH)}\sin(kx - \omega t)} \tag{1.43}$$

The velocity components $u = \partial\phi/\partial x$ and $w = \partial\phi/\partial z$ are given by

$$\boxed{u = a\omega\frac{\cosh[k(z+H)]}{\sinh(kH)}\cos(kx - \omega t)} \tag{1.44}$$

$$\boxed{w = a\omega\frac{\sinh[k(z+H)]}{\sinh(kH)}\sin(kx - \omega t)} \tag{1.45}$$

Remark. Note that since $\eta = a\cos(kx - \omega t)$, we see that the u velocity is in phase with the displacement while the w velocity is 90° out of phase with η, as shown in Figure 1.2a. Snapshots of ϕ, u and w are depicted in Figures 1.2b-d.

Dispersion relation

Note that we solved the Laplace equation by using only the bottom and kinematic conditions (1.31) and (1.32); without employing the dynamic condition (1.33). Application of the dynamic condition (1.33) results in a relation between the wave number k and frequency ω. Substituting (1.34) and (1.43) into condition (1.33), we have

$$\frac{-a\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)}\cos(kx - \omega t) = -ga\cos(kx - \omega t)$$

$$\Longrightarrow \frac{\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)} = g$$

$$\Longrightarrow \omega^2 = gk\tanh(kH)$$

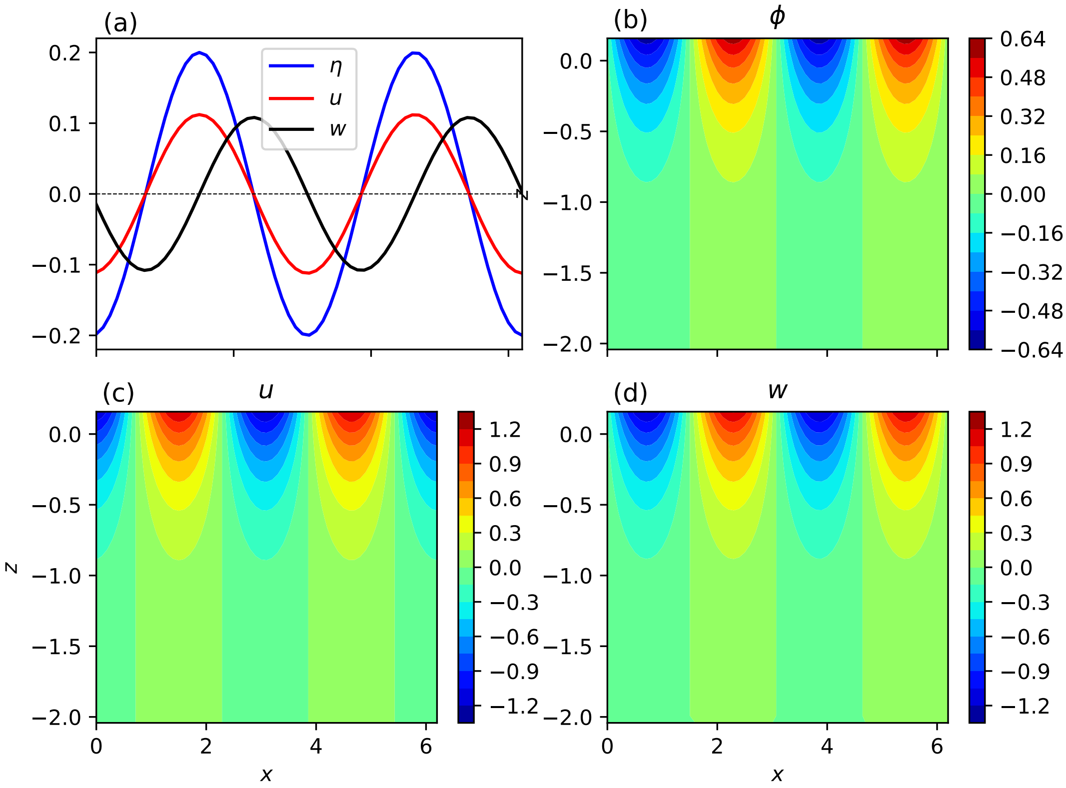

Figure 1.2: (a) Phase relationships between u, w and η, and snapshots of (b) velocity potential, ϕ, (c) u velocity, and (d) w velocity.

$$\boxed{\omega = \sqrt{gk \tanh(kH)}} \quad \text{or} \quad T = \sqrt{\frac{2\pi\lambda}{g} \coth\left(\frac{2\pi H}{\lambda}\right)}, \tag{1.46}$$

where T is the wave period. The wave speed $c = \omega/k$ is related to the wave size by

$$\boxed{c = \sqrt{\frac{g}{k} \tanh(kH)} = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi H}{\lambda}}} \tag{1.47}$$

These equations show that the speed of propagation of a wave component depends on its wavenumber. Waves for which c is a function of k are said to be *dispersive* since they separate into individual components. The relationship in (1.46) or (1.47) where ω is a function of k is called a **dispersion relation**. This is because it expresses the nature of the dispersive process.

1.2 Approximations for Deep and Shallow Water

The analysis in the previous section is applicable irrespective of the relationship between the magnitude of λ and the water depth H. Some interesting simplifications arise for shallow water ($H/\lambda << 1$) and deep water ($H/\lambda >> 1$). We derive approximations for the phase speed for which (1.47) takes simple forms. The behaviour of hyperbolic functions (Figure 1.3) as $x \to \infty$ and as $x \to 0$ be employed for the simplifications.

Deep Water Approximation

In deep water $H/\lambda >> 1$, we use the fact that $\tanh x \to 1$ as $x \to \infty$ to approximate (1.47) by

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{k}}. \tag{1.48}$$

We note that x does not need to be very large for the approximation $\tanh x \to 1$ to be valied since $\tanh x = 0.96403$ for $x = 2$. Thus, with 2% accuracy, the approximation (1.48) is valied for $H > 0.32\lambda$ (i.e., $kH > 2$). Therefore, surface waves are classified as *deep water waves* if the depth is more than one-third of the wavelength. We see here that deep water waves are dispersive since the phase speed depends on the wavelength.

Remark. 1. The 2% accuracy is computed from

$$\frac{\sqrt{\frac{g\lambda}{2\pi}} - \sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi H}{\lambda}}}{\sqrt{\frac{g\lambda}{2\pi}}} = \frac{1 - \sqrt{\tanh\frac{2\pi H}{\lambda}}}{1} = 1 - \sqrt{0.96403} \approx 2\%,$$

for $2\pi H/\lambda = 2$.

2. As an example, the dominant period of surface waves generated by winds in the ocean is approximately 10 s which corresponds to a wavelength of about 150m (using equation 1.46). The water depth on a typical continental shelf is about 100 m and about 4 km in the open ocean. This means that over the continental shelf, $H/\lambda \approx 100/150 = 0.67 > 0.32$. So the dominant wind waves in the ocean, even over the continental shelf, act as deep-water waves and do not feel the effects of the ocean bottom until they arrive near the coastline. In contrast, gravity waves with very long wavelengths or tsunamis generated by tidal

forces or earthquakes act as shallow water waves as discussed next. Such waves may have wavelengths of hundreds of kilometers.

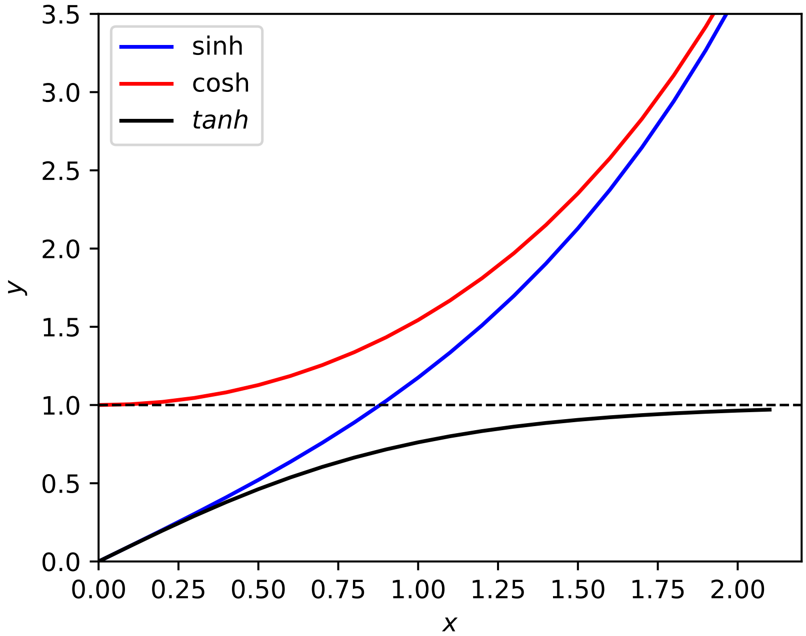

Figure 1.3: Hyperbolic functions.

Shallow Water Approximation

From Figure 1.3 we see that $\tanh x \approx x$ for $x \to 0$. Thus, for $H/\lambda << 1$, we have

$$\tanh \frac{2\pi H}{\lambda} \approx \frac{2\pi H}{\lambda}$$

and the phase speed simplifies to the relation

$$c = \sqrt{gH}. \tag{1.49}$$

The approximation gives better than 3% accuracy if $H < 0.07\lambda$. Thus, surface waves are regarded as *shallow-water waves* if the water depth is less than 7% of the wavelength. In other words, they have to be 14 times longer than the water depth. Equation (1.49) shows that the phase speed of shallow-water waves are independent of wavelength and increases with water depth. So shallow-water waves are *nondispersive*.

1.3 Interfacial Waves

We next consider the case of waves that exists in the interface between two fluids with different densities. Such density discontinuities can often exist in the ocean as a result of the sun heating the upper layers. This can also occur in an estuary, that is, a river mouth, or in a fjord whereby less dense fresh water flows over denser oceanic water. In this section, we consider an idealized situation of a finite layer of density ρ_1 over an infinitely deep layer of density ρ_2. This situation allows for both surface waves and waves at the interface. So we expect two modes of oscillation; the first mode is the case in which the free surface is in phase with the interface and a second mode in which they are out of phase or oppositely directed.

Figure 1.4 depicts the setup, where H is the thickness of the upper layer, and the origin is at the mean position of the free surface. As in the one-layer case, the Laplace equation applies in both layers such that

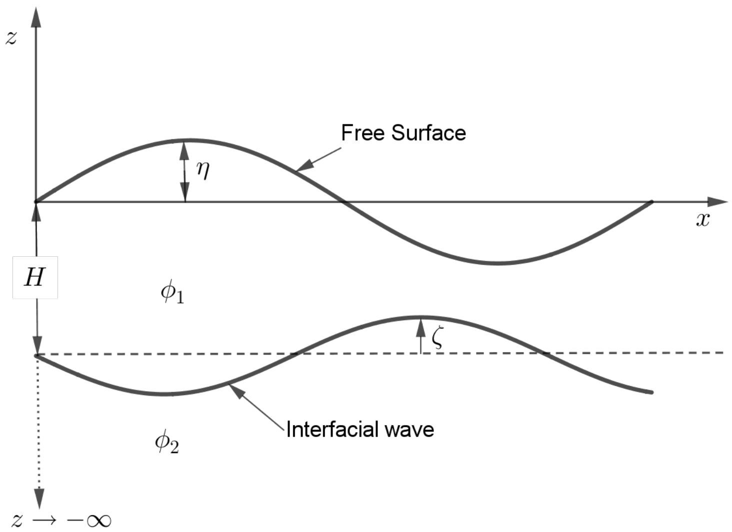

Figure 1.4: Surface and interfacial waves

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \tag{1.50}$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \tag{1.51}$$

with the conditions

$$\phi_2 \to 0 \qquad \text{at} \quad z \to -\infty \tag{1.52}$$
$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{at} \quad z = 0 \tag{1.53}$$
$$\frac{\partial \phi_1}{\partial t} = -g\eta \qquad \text{at} \quad z = 0 \tag{1.54}$$
$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad \text{at} \quad z = -H \tag{1.55}$$
$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g\zeta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g\zeta \qquad \text{at} \quad z = -H \tag{1.56}$$

We assume a free surface displacement η, and interface displacement ζ, of the form:

$$\eta = ae^{i(kx-\omega t)} \tag{1.57}$$
$$\zeta = be^{i(kx-\omega t)} \tag{1.58}$$

Only the real part of the right hand side of (1.57) and (1.58) are meant. The constant a can be regarded as real but the constant b should be left complex since η and ζ may not be in phase. From our previous analysis in equations (1.35)-(1.37), it is clear that ϕ_1 and ϕ_2 must be of the form

$$\phi_1 = \left(Ae^{kz} + Be^{-kz}\right) e^{i(kx-\omega t)} \tag{1.59}$$
$$\phi_2 = Ce^{kz}e^{i(kx-\omega t)} \tag{1.60}$$

The form (1.60) is chosen to satisfy (1.52). We are left with the problem of finding A, B and C, as well as the relationship between amplitudes a and b. Applying (1.53) we have

$$\frac{\partial \phi_1}{\partial z} = k\left(Ae^{kz} - Be^{-kz}\right) e^{i\theta} = -ia\omega e^{i\theta},$$

and at $z = 0$, we get

$$k(A - B) = -ia\omega$$
$$\implies A - B = -\frac{ia\omega}{k} \tag{1.61}$$

where $\theta = kx - \omega t$. Applying (1.54), we have

$$\frac{\partial \phi_1}{\partial t} = -i\omega\left(Ae^{kz} + Be^{-kz}\right) e^{i\theta} = -ga\omega e^{i\theta},$$

and at $z = 0$, we get

$$A + B = \frac{ga}{i\omega} = -i\frac{ag}{\omega} \tag{1.62}$$

Summing (1.61) and (1.62) results in

$$\begin{aligned} 2A &= i\left(-\frac{ag}{\omega} - \frac{a\omega}{k}\right) \\ \Longrightarrow A &= -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \qquad (1.63) \\ \Longrightarrow B = -\frac{iag}{\omega} - A &= -\frac{iag}{\omega} + \frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \\ \Longrightarrow B &= \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) \qquad (1.64) \end{aligned}$$

From (1.55):

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z}$$

$$\Longrightarrow k\left(Ae^{kz} - Be^{-kz}\right)e^{i\theta} = -kCe^{kz}\omega e^{i\theta}$$

At $z = -H$, we have

$$Ae^{-kH} - Be^{kH} = Ce^{-kH}$$

$$\Longrightarrow C = A - Be^{2kH}$$

$$C = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right)e^{2kH} \tag{1.65}$$

To determine the relationship between a and b, we may use either

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \zeta}{\partial t} \quad \text{or} \quad \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t}$$

from equation (1.55). Employing the latter equation for simplicity, we have

$$kCe^{kz}e^{i\theta} = -i\omega b e^{i\theta}$$

and at $z = -H$:

$$b = \frac{k}{-i\omega}Ce^{-kH} = \frac{ik}{\omega}Ce^{-kH}$$

$$\Longrightarrow b = \frac{ik}{\omega}\left\{-\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right)e^{2kH}\right\}e^{-kH}$$

$$\therefore b = \frac{a}{2}\left(1 + \frac{gk}{\omega^2}\right)e^{-kH} + \frac{a}{2}\left(1 - \frac{gk}{\omega^2}\right)e^{kH} \tag{1.66}$$

Recapitulating, the velocity potentials are given by (1.59)-(1.60), with the constants given by

$$A = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \tag{1.67}$$

$$B = \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) \tag{1.68}$$

$$C = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right)e^{2kH} \tag{1.69}$$

and

$$\therefore b = \frac{a}{2}\left(1 + \frac{gk}{\omega^2}\right)e^{-kH} + \frac{a}{2}\left(1 - \frac{gk}{\omega^2}\right)e^{kH}. \tag{1.70}$$

1.3.1 Dispersion Relation

For the dispersion relation, we employ equation (1.56). After a few pages of algebra, the result can be written as (see Appendix **??**)

$$\left(\frac{\omega^2}{gk} - 1\right)\left\{\frac{\omega^2}{gk}\left[\rho_1 \sinh(kH) + \rho_2 \cosh(kH)\right] - (\rho_2 - \rho_1)\sinh(kH)\right\} = 0. \tag{1.71}$$

Equation (1.71) shows that there are two possible roots, resulting in the barotropic or surface mode and the baroclinic or internal modes as discussed below.

Barotropic or Surface Mode

From (1.71), one of the roots is given by

$$\left(\frac{\omega^2}{gk} - 1\right) = 0$$

$$\therefore \omega^2 = gk. \tag{1.72}$$

This is the same dispersion relation we obtained for a deep-water water wave. Equation (1.70) shows that

$$b = ae^{-kH} \tag{1.73}$$

which implies that the amplitude at the interface is smaller than that at the surface by a factor e^{-kH}. Also, equation (1.73) together with (1.57)-(1.58) shows that the motions of the free surface and the interface are locked in phase. That is, they go up or down simultaneously. This mode is similar to a gravity wave propagating on the free surface of the upper liquid, in which the motion decays as e^{-kz} from the free surface. It is called the **barotropic mode**.

Baroclinic or Internal Mode

From (1.71), the second root is given by

$$\frac{\omega^2}{gk}\left[\rho_1 \sinh(kH) + \rho_2 \cosh(kH)\right] - (\rho_2 - \rho_1)\sinh(kH) = 0$$

$$\Longrightarrow \omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh(kH)}{\rho_1 \sinh(kH) + \rho_2 \cosh(kH)} \tag{1.74}$$

We can determine the relation between a and b by substituting (1.74) into (1.70). From (1.74) we have

$$\frac{gk}{\omega^2} = \frac{\rho_2 \cosh kH + \rho_1 \sinh kH}{(\rho_2 - \rho_1)\sinh kH},$$

and so

$$1 - \frac{gk}{\omega^2} = 1 - \frac{\sinh kH + (\rho_2/\rho_1)\cosh kH}{[(\rho_2 - \rho_1)/\rho_1]\sinh kH}$$

and

$$1 + \frac{gk}{\omega^2} = 1 + \frac{\sinh kH + (\rho_2/\rho_1)\cosh kH}{[(\rho_2 - \rho_1)/\rho_1]\sinh kH}.$$

Substituting the last two equations into (1.70) and after some algebra gives

$$b = -a\left(\frac{\rho_1}{\rho_2 - \rho_1}\right)(\cosh kH + \sinh kH) = -a\left(\frac{\rho_1}{\rho_2 - \rho_1}\right)e^{kH}$$

$$\Longrightarrow a = -b\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)e^{-kH}. \tag{1.75}$$

From (1.57) and (1.58), we have

$$\eta = \frac{a}{b}\zeta$$

$$\boxed{\therefore \eta = -\zeta\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)e^{-kH}.} \tag{1.76}$$

Remark.

- Equation (1.76) shows that η and ζ have opposite signs, and if the density difference is small, the interface displacement is much larger than the surface displacement. This is one reason why it is important to study internal waves because under nonlinear conditions, the waves can break and mix the interior of the ocean thereby changing the local stratification. Secondly, large amplitude internal waves carry more energy as they propagate away from where they are generated.
- This mode is called the **baroclinic** or **internal mode**. This is because the surfaces of constant pressure and density do not coincide.
- It can also be shown that the horizontal velocity changes sign across the interface as depicted in Figure 1.5.
- The analysis above shows that the presence of a density difference generates a motion that is quite different from the barotropic behaviour.

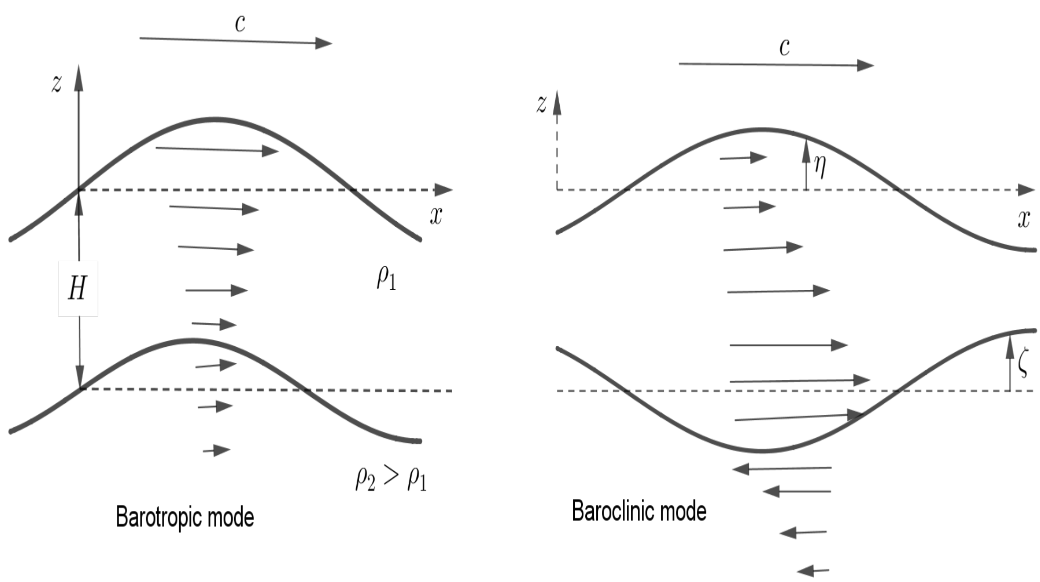

Figure 1.5: The two modes of motion of a layer of fluid overlying an infinitely deep layer.

Chapter 2

Internal Gravity Waves

In this chapter, we turn our attention to internal gravity waves in a continuously stratified environment.

2.1 Governing Equations

We consider the governing equations for an incompressible fluid under the Boussinesq approximation:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x} \tag{2.1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial y} \tag{2.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_*} \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.4}$$

$$\frac{D\rho}{Dt} = -w\frac{d\rho_0}{dz} \tag{2.5}$$

where (u, v, w) are the velocity components in the x, y, z directions respectively, p is pressure, $\rho(x, y, z, t)$ is the total density, $\rho_0(z)$ is the background density, g is acceleration due to gravity and f is the Coriolis frequency due to rotation of the Earth. The material derivative D/Dt is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Considering small amplitude motions, we neglect the nonlinear terms, and obtain

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x} \tag{2.6}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial y} \tag{2.7}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_*} \tag{2.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.9}$$

$$\frac{\partial \rho}{\partial t} = -w\frac{d\rho_0}{dz} \tag{2.10}$$

To obtain equations governing the propagation of internal gravity waves, we assume that the flow is superimposed on a "background" state (i.e. it is a perturbation of a known static background) with only vertical dependences such that

$$p = p_0(z) + p'(x, y, z, t) \tag{2.11}$$

$$\rho = \rho_0(z) + \rho'(x, y, z, t) \tag{2.12}$$

and the static background density and pressure are in **hydrostatic balance**:

$$\frac{dp_0}{dz} = -g\rho_0 \tag{2.13}$$

Substituting (2.11)-(2.12) into (2.6)-(2.10), equations (2.6)-(2.7) become

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p'}{\partial x} \tag{2.14}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p'}{\partial y} \tag{2.15}$$

From equation (2.8) we have

$$\begin{aligned}\frac{\partial w}{\partial t} &= -\frac{1}{\rho_*}\frac{dp_0}{dz} - \frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g(\rho_0 + \rho')}{\rho_*} \\ &= -\frac{1}{\rho_*}\frac{dp_0}{dz} - \frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g\rho_0}{\rho_*} - \frac{g\rho'}{\rho_*}\end{aligned}$$

applying (2.13) results in

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g\rho'}{\rho_*} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} + b \tag{2.16}$$

where

$$b = -\frac{g\rho'}{\rho_*} \tag{2.17}$$

From (2.10) we have

$$\frac{\partial \rho'}{\partial t} = -w\frac{d\rho_0}{dz}$$

$$\Longrightarrow \frac{\partial \rho'}{\partial t} = -\frac{\rho_* w}{g}\frac{g}{\rho_*}\frac{d\rho_0}{dz} = \frac{\rho_* w}{g}N^2$$

$$\boxed{\frac{\partial \rho'}{\partial t} - \frac{\rho_* w}{g}N^2 = 0} \tag{2.18}$$

$$\boxed{\therefore \frac{\partial b}{\partial t} + wN^2 = 0} \tag{2.19}$$

where we used (2.17) and the **buoyancy frequency** or **Brunt-Väisälä frequence**, N, is defined as

$$N^2 = -\frac{g}{\rho_*}\frac{d\rho_0}{dz}. \tag{2.20}$$

It describes the frequency that a vertically displaced fluid will oscillate at in an environment with background stratification given by $d\rho_0/dz$. This is briefly derived from first principles below.

Buoyancy frequence

Consider fluid in a background density, $\rho_0(z)$, that is continuously decreasing with height as depicted in Figure 2.1. Consider a fluid parcel of density $\rho_* = \rho_0(z_0)$ situated initially at a vertical level z_0. If the parcel is displaced vertically by a small distance δ_z, it will maintain its density which is different from that of the surrounding fluid (by neglecting thermodynamic effects). It therefore experiences a buoyancy force. Newton law predicts that

$$mass \times acceleration = Force_{buoyancy}$$

$$\Longrightarrow \rho_* \frac{d^2\delta_z}{dt^2} = -\delta_\rho g$$

where δ_ρ is the density difference between the fluid parcel and the surrounding fluid at its displaced position. Because the displacement $|\delta_z|$ is small, the density difference can be written in terms of δ_z to get

$$\delta_\rho \approx -\frac{d\rho_0}{dz}\delta_z$$

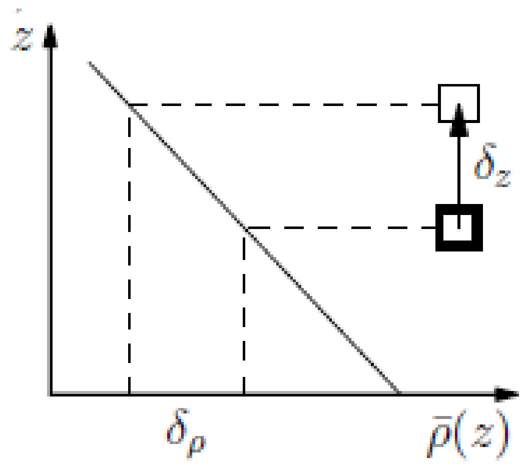

Figure 2.1: Illustration of fluid displaced vertically upward in a linearly stratified background density.

Therefore, we get

$$\rho_* \frac{d^2\delta_z}{dt^2} = g\frac{d\rho_0}{dz}\delta_z \quad \Longrightarrow \frac{d^2\delta_z}{dt^2} - \frac{g}{\rho_*}\frac{d\rho_0}{dz}\delta_z = 0.$$

This can be written as the "spring equation":

$$\frac{d^2\delta_z}{dt^2} + N^2\delta_z = 0$$

where N is the buoyancy frequency (2.20).

In summary, the equations governing linear internal wave motions in a continuously stratified ambient are given by

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p'}{\partial x} \tag{2.21}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p'}{\partial y} \tag{2.22}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} + b \tag{2.23}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.24}$$

$$\frac{\partial b}{\partial t} + wN^2 = 0 \tag{2.25}$$

We next derive a single equation for the vertical velocity w, from (2.21)-(2.25). Taking the derivative $\partial/\partial z$ of (2.22) and $\partial/\partial y$ of (2.23) gives

$$\frac{\partial}{\partial z}(2.22) \to \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}\right) + f\frac{\partial u}{\partial z} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right)$$
$$\frac{\partial}{\partial y}(2.23) \to \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}\right) = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial b}{\partial y}$$

Subtracting the first equation from the second gives

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - f\frac{\partial u}{\partial z} = \frac{\partial b}{\partial y}$$
$$\implies \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = f\frac{\partial u}{\partial z} + \frac{\partial b}{\partial y} \tag{2.26}$$

Similarly, taking $\partial/\partial z$ of (2.21) and $\partial/\partial x$ of (2.23) gives

$$\frac{\partial}{\partial z}(2.21) \to \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right) - f\frac{\partial v}{\partial z} = -\frac{1}{\rho_*}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right)$$
$$\frac{\partial}{\partial x}(2.23) \to \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) = -\frac{1}{\rho_*}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial b}{\partial x}$$

Subtracting the second equation from the first gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - f\frac{\partial v}{\partial z} = -\frac{\partial b}{\partial x}$$
$$\implies \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = f\frac{\partial v}{\partial z} - \frac{\partial b}{\partial x} \tag{2.27}$$

Finally, taking $\partial/\partial y$ of (2.21) and $\partial/\partial x$ of (2.22) gives

$$\frac{\partial}{\partial y}(2.21) \to \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - f\frac{\partial v}{\partial y} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$$
$$\frac{\partial}{\partial x}(2.22) \to \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) + f\frac{\partial u}{\partial x} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$$

The second equation minus the first gives

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$
$$\implies \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = f\frac{\partial w}{\partial z} \tag{2.28}$$

where we used the continuity equation (2.24).

Remark. We note that the left hand side of (2.26)-(2.28) shows the vorticity $\nabla \times \mathbf{u}$. The vorticity is a measure of the rotation of water parcels, that is, their change in orientation. This is different from a parcel traversing, e.g., a circle in a translational movement.

Now $\partial^2/\partial y\partial t$ of (2.26) and $\partial^2/\partial x\partial t$ of (2.27) yield:

$$\frac{\partial^2}{\partial y\partial t}(2.26) \to \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 v}{\partial y\partial z}\right] = f\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial y\partial z} + \frac{\partial}{\partial t}\frac{\partial^2 b}{\partial y^2} \tag{2.29}$$

$$\frac{\partial^2}{\partial x\partial t}(2.27) \to \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u}{\partial x\partial z} - \frac{\partial^2 w}{\partial^2 x^2}\right] = f\frac{\partial}{\partial t}\frac{\partial^2 v}{\partial x\partial z} - \frac{\partial}{\partial t}\frac{\partial^2 b}{\partial x^2} \tag{2.30}$$

Equation (2.29) minus (2.30) gives

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial^2 y^2} - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = f\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2}\right)$$

Thus,

$$\frac{\partial^2}{\partial t^2}\left[\nabla_h^2 w + \frac{\partial^2 w}{\partial z^2}\right] = f\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial t}\nabla_h^2 b$$

where we used (2.24) and

$$\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The Coriolis term can be re-written using (2.28) and the last (buoyancy) term using (2.25) to get

$$\frac{\partial^2}{\partial t^2}\left[\nabla_h^2 w + \frac{\partial^2 w}{\partial z^2}\right] + f\frac{\partial}{\partial z}\left(f\frac{\partial w}{\partial z}\right) + \nabla_h^2 w N^2 = 0$$

$$\therefore \boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_h^2 w = 0}, \tag{2.31}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2.2 Boundary Conditions

To solve equation (2.31), we need to specify boundary conditions at the surface and bottom of our domain. The surface and bottom boundaries then acts like a "waveguide" to internal waves that can travel unboundedly in the horizontal direction. The simplest surface condition employs the so-called **rigid-lid approximation** in which the vertical velocity is taken to be zero at the mean level $z = 0$:

$$w = 0 \quad \text{at} \quad z = 0. \tag{2.32}$$

The simplest bottom boundary condition is obtained by assuming a flat bottom, with water depth H, such that

$$w = 0 \quad \text{at} \quad z = -H. \tag{2.33}$$

The details of the rigid-lid approximation and the bottom boundary condition are given next.

2.2.1 Surface Condition: Rigid-lid Approximation

As seen in the case for interfacial waves, both surface waves and internal waves exist in our domain but we wish to focus on internal waves. However, the presence of internal waves results in small elevations and depressions of the surface due to pressure gradients effected by the propagation of internal waves. Therefore, we cannot simply assume that the surface is still. Nevertheless, the surface vertical elevations induced by an internal wave are very small compared to those in the interior of the ocean. This is because the force/energy that is used to produce large isopycnal displacements in the ocean's interior is only able to generate very small surface displacements as a result of the large density gradient between air and sea water, in contrast to smaller density gradients in the interior. This fact is used to simplify the boundary conditions to obtain the "rigid-lid approximation".

Let the free surface be described by

$$z = \eta(t, x, y), \tag{2.34}$$

so we get

$$w(t, x, y, \eta) = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}. \tag{2.35}$$

Assuming the atmospheric pressure, p_a, is constant, then at the surface:

$$p(t, x, y, \eta) = p_0(\eta) + p'(t, x, y, \eta) = p_a, \tag{2.36}$$

where we used (2.11). A Taylor expansion about $z = 0$ for both (2.35) and (2.36) yields

$$w(t,x,y,0) + \eta\frac{\partial w}{\partial z} + \cdots = \frac{\partial \eta}{\partial t} + u(t,x,y,0)\frac{\partial \eta}{\partial x} + v(t,x,y,0)\frac{\partial \eta}{\partial y} + \cdots$$

$$p_0(0) + p'(t,x,y,0) + \eta\frac{dp_0}{dz} + \eta\frac{\partial p'}{\partial z} + \cdots = p_a$$

For small amplitude motions, we neglect the product of perturbation terms (i.e. those involving u, v, w, η and p'), to get

$$w = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \tag{2.37}$$

$$p_0 + p' + \eta\frac{dp_0}{dz} = p_a \quad \text{at} \quad z = 0 \tag{2.38}$$

Combining (2.37) and (2.38) by using the hydrostatic balance (2.13) gives

$$\frac{\partial p'}{\partial t} = wg\rho_0 \quad \text{at} \quad z = 0. \tag{2.39}$$

Equation (2.39) can be nondimensionalized, by letting $\rho_0(0) = \rho_*$, $w = W\hat{w}$, $p' = P\hat{p}'$, $t = T\hat{t}$, to get an equation of the form

$$\frac{C_i^2}{C_{sf}^2}\frac{\partial \hat{p}'}{\partial \hat{t}} = \hat{w} \quad \text{at} \quad z = 0. \tag{2.40}$$

where $C_i = L/T$ is a measure of the phase speed of internal waves and C_{sf} is that of surface gravity waves ($C_{sf}^2 = gH$, H is water depth). Thus, C_{sf} is proportional to gravity g while C_i is proportional to the so-called "reduced gravity", g' (e.g., $g' = (\rho_2 - \rho_1)/\rho_2$ for a two-layer fluid). Therefore $C_i << C_{sf}$ such that

$$\epsilon = \frac{C_i^2}{C_{sf}^2} << 1$$

and (2.40) becomes

$$\epsilon\frac{\partial \hat{p}'}{\partial \hat{t}} = \hat{w} \quad \text{at} \quad z = 0. \tag{2.41}$$

We cannot just assume that the left hand side of (2.41) is smaller than the righthand side; instead, we need to perform a perturbation expansion on $\hat{p}'$ and $\hat{w}$ in a series by letting

$$\hat{p}' = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots$$

$$\hat{w} = w^{(0)} + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \cdots$$

where each of the $p^{(i)}$ and $w^{(i)}$ are of order one. Substituting the expansions into (2.41) gives

$$\epsilon \frac{\partial p^{(0)}}{\partial \hat{t}} + \epsilon^2 \frac{\partial p^{(1)}}{\partial \hat{t}} + \epsilon^3 \frac{\partial p^{(2)}}{\partial \hat{t}} + \cdots = w^{(0)} + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \cdots$$

To lowest order, we get the approximation

$$w^{(0)} = 0 \quad \text{at} \quad z = 0$$

which, for simplicity, is usually written as

$$w = 0 \quad \text{at} \quad z = 0. \tag{2.42}$$

Keep in mind that w in (2.42) is actually an approximate variable and not the w in the original equations.

2.2.2 Bottom Condition

The boundary condition at the bottom is much simpler to derive, it is that of no normal flow. The bottom is described by

$$z = -h(x, y).$$

Thus, from (2.35), and replacing η with h gives the boundary condition

$$w = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \quad \text{at} \quad z = -h. \tag{2.43}$$

Assuming the bottom is horizontal, with water depth H, then we get the simple condition (2.33):

$$w = 0 \quad \text{at} \quad z = -H. \tag{2.44}$$

Before delving into the various ways of solving (2.31), we will use it to derive the dispersion of relation of internal waves, from which we will discuss some of their important basic properties.

2.3 Basic Properties of Internal Waves

Dispersion relation

The approximations that culminated in the derivation of (2.31) makes the horizontal plane isotropic: northward propagating waves ($y-$direction), say, behave the same way as eastward propagating ones ($x-$direction). This can be seen by assuming the waves are sinusoidal and substituing $w \sim \exp(-i\omega t)$ into (2.31). So we consider waves propagating in the $x-$direction, taking $\partial/\partial y = 0$, and get the equation

$$\boxed{\frac{\partial^2}{\partial t^2}w_{xx} + f^2 w_{zz} + N^2 w_{xx} = 0}, \tag{2.45}$$

Consider a wave for which

$$w = w_0 e^{i(kx+mz-\omega t)} \tag{2.46}$$

where w_0 is the amplitude of fluctuations. Substituting (2.46) into (2.45) gives the dispersion relation

$$\omega^2 = \frac{f^2 m^2 + k^2 N^2}{k^2 + m^2}. \tag{2.47}$$

For simplicity of discussing the basic features of internal waves, we consider waves in a non-rotating frame of reference, $f = 0$, so the dispersion relation becomes

$$\boxed{\omega^2 = \frac{k^2}{k^2 + m^2} N^2}. \tag{2.48}$$

The wave frequency is taken to be positive, and the direction of propagation is determined by the wavenumber vector

$$\vec{K} = \mathbf{K} = (k,\, m) = |\mathbf{K}|(\cos\theta,\, \sin\theta), \tag{2.49}$$

where θ is the angle between the phase velocity vector $\mathbf{c}$ (and therefore $\mathbf{K}$) and the horizontal direction as illustrated in Figure 2.2. Thus, the dispersion relation can also be written as

$$\boxed{\omega = N\cos\theta}, \tag{2.50}$$

and

$$\theta = \tan^{-1}(m/k), \quad -\pi/2 \le \theta \le \pi/2 \tag{2.51}$$

and θ is restricted to lie between $-\pi/2$ and $\pi/2$ so that the frequency in (2.50) is non-negative.

Remark. Equation (2.50) shows that

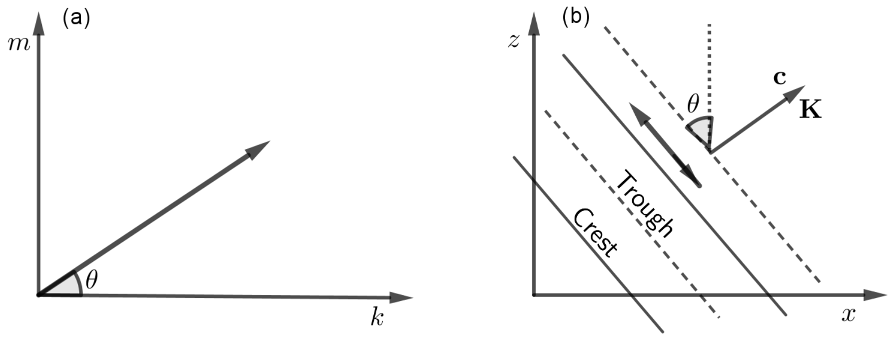

Figure 2.2: Schematic illustration of (a) internal wavevector, $\mathbf{K} = (k,\, m)$ and (b) motion along lines of constant phase and corresponding direction of wavenumber vector.a

- the frequency of an internal wave in a stratified fluid depends only on the *direction* of the wavenumber vector and not on the magnitude of the wavenumber. This is in contrast with surface and interfacial waves, for which frequency depends only on the magnitude.
- the frequency lies in the range $0 < \omega \leq N$. Therefore N is the maximum possible frequency of internal waves in a stratified fluid.

Particle or fluid motion

Suppose the fluid motion is given by (2.46), with similar expressions for u such that

$$\frac{\partial w}{\partial z} = imw_0 e^{i(kx+mz-\omega t)} = imw$$

Thus, from the continuity equation we have

$$iku + imw = 0 \quad \Longrightarrow ku + mw = 0$$

$$\boxed{\mathbf{K}\cdot\mathbf{u} = 0} \tag{2.52}$$

where $\mathbf{u} = (u,\, w)$, showing that the **particle motion is perpendicular to the wavenumber vector** as illustrated in Figure 2.2b (in physical space). In other words, fluid motion is parallel to lines of constant phase lines. So the angle θ in (2.50) can now be interpreted as the angle between the particle motion and the vertical direction (Figure 2.2b). In real space, it is measured counterclockwise from the vertical. In general, the sign of θ is determined by the sign of the ratio m/k (see equation 2.51); for a wave propagating to the right ($k > 0$), θ is positive if crests move upward ($m > 0$) and negative if crests move downward ($m < 0$).

The angle θ is also a measure of the wave frequency relative to the buoyancy frequency (2.50). So if the frequency is known, we can compute

$$\theta = \pm \cos^{-1}(\omega/N), \tag{2.53}$$

where the sign is determined by the sign of m/k.

Remark (Limiting Cases). •

- The maximum frequency $\omega = N$ occurs when $\theta = 0$, that is when the particles move up and down vertically. This happens for $m = 0$ (see equation 2.48), showing that the motion is independent of the $z-$coordinate. So the fastest frequency waves have infinitely large vertical wave lengths (since $\lambda_m = 2\pi/m$), with lines of constant phase lying parallel to the z-axis.

- The frequency $\omega = 0$ for $\theta = \pi/2$, that is when the particle motion is purely horizontal. So waves with nearly zero frequency have crests that lie almost parallel to the horizontal axis.

Phase and group velocity

For a one-dimensional wave having structure in the x-direction alone, the phase speed is defined by

$$c_p = \frac{\omega}{k} = \frac{\lambda}{T}$$

which means the crest moves one wavelength λ in the time of one wave period T. For waves having structure in two or three dimensions, the **phase velocity** can be described by imagining setting on a crest and moving in the direction of the wavenumber vector. Thus, the phase velocity is defined by

$$\mathbf{c} = \frac{\omega}{|\vec{k}|}\hat{\mathbf{k}} = \frac{\omega}{|\vec{k}|^2}\vec{k} = \frac{\omega}{|\vec{k}|^2}(k,\, m) \tag{2.54}$$

where $\hat{\mathbf{k}} = \vec{k}/|\vec{k}|$ is the unit vector in the direction of $\vec{k}$, as shown in Figure 2.3.

In two dimensions, the group velocity c_g is defined as

$$\mathbf{c_g} = \frac{\partial \omega}{\partial k}\mathbf{i}_x + \frac{\partial \omega}{\partial m}\mathbf{i}_z = \left(\frac{\partial \omega}{\partial k},\, \frac{\partial \omega}{\partial m}\right) \tag{2.55}$$

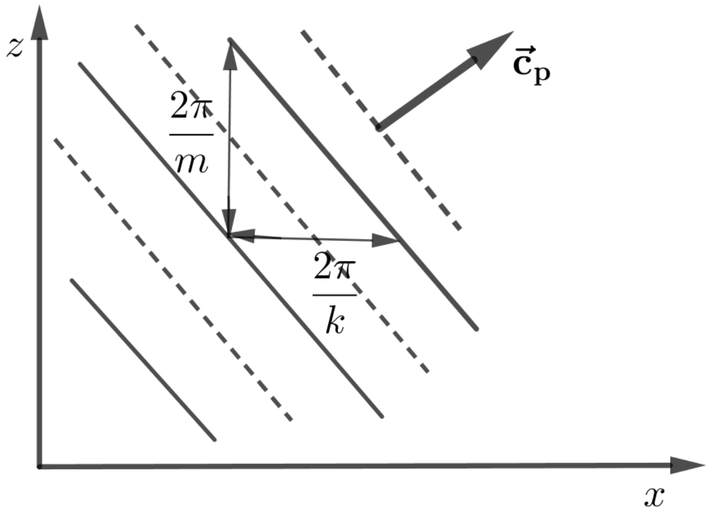

Figure 2.3: Schematic illustration of phase velocity and the wavelengths in the x and z directions.

where $\mathbf{i}_x$ and $\mathbf{i}_z$ are unit vectors in the x and z directions respectively. From the dispersion relation (2.48) we have

$$\frac{\partial \omega}{\partial k} = \frac{Nm^2}{(k^2+m^2)^{3/2}} = \frac{Nm^2}{|\vec{k}|^3}$$

$$\frac{\partial \omega}{\partial m} = \frac{-Nmk}{(k^2+m^2)^{3/2}} = \frac{-Nmk}{|\vec{k}|^3}$$

$$\therefore \mathbf{c}_g = \frac{Nm}{|\vec{k}|^3}(m, -k) \tag{2.56}$$

Thus, from (2.54) and (2.56), we find that

$$\boxed{\mathbf{c}_g \cdot \mathbf{c} = 0} \tag{2.57}$$

showing that the phase and group velocity vectors of internal waves are perpendicular. This means that while crests are moving in the direction of the wavenumber vector, the wave packet as a whole is moving in a direction that is parallel to the crests. From (2.49) and 2.50, the phase and group velocity may written in the form

$$\mathbf{c} = \frac{N}{|\vec{k}|}(\cos^2\theta,\, \sin\theta\cos\theta) = \frac{N\cos\theta}{|\vec{k}|}(\cos\theta,\, \sin\theta) \tag{2.58}$$

$$\mathbf{c_g} = \frac{N}{|\vec{k}|}(\sin^2\theta,\, -\sin\theta\cos\theta) = \frac{N\sin\theta}{|\vec{k}|}(\sin\theta,\, -\cos\theta) \tag{2.59}$$

This shows that the horizontal components of $\mathbf{c}$ and $\mathbf{c_g}$ are in the same direction, while their vertical components are opposite. Thus, if crests move upward the wavepacket as a whole moves downward and vice versa, as depicted in Figure 2.4.

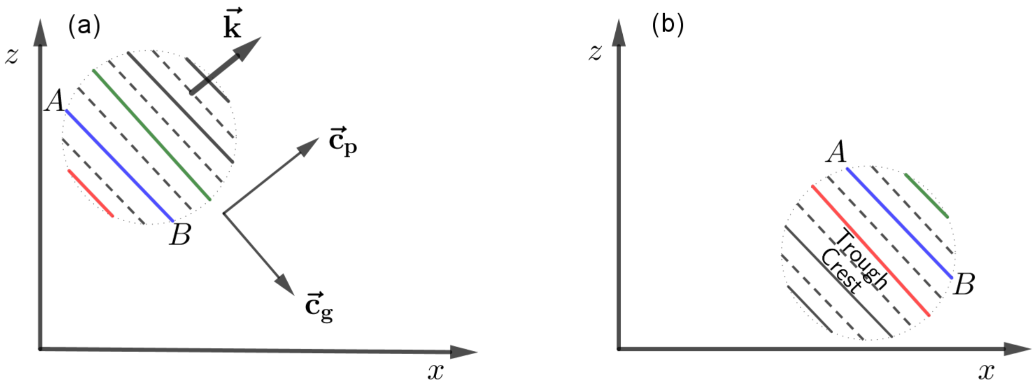

Figure 2.4: Schematic illustration of the phase and group velocity of a wavepacket at (a) an early time and (b) later time. The phase line AB in (a) has moved through the packet to the new position in (b).

The St. Andrews Cross

As mentioned previously, the group velocity shows the direction of propagation of energy of a sinusoidal component. This was first illustrated in a laboratory experiment by Mowbray & Rarity (1967) by oscillating a cylinder at frequeny ω in a tank filled with uniformly stratifed salt water. The energy of the wave was found to radiate outward along four beams in a cross-shaped pattern often referred to as a 'St. Andrews Cross' because of its resemblance to the Scottish flag (see Figure 2.5). The structure of the wave beams is a superposition of plane waves having different spatial structure but identical frequencies. This will be discussed later under vertical normal modes. Because the buoyancy frequency and oscillation frequency are fixed, the phase lines must be oriented at a fixed angle $\theta = \cos^{-1}(\omega/N)$ to the vertical, independent of the size of the cylinder or its cross-sectional shape. Conical wave beams generated by a plume impinging on a linearly stratified fluid are shown in Figure 2.6, showing a pattern similar to the bottom half of the St. Andrews Cross.

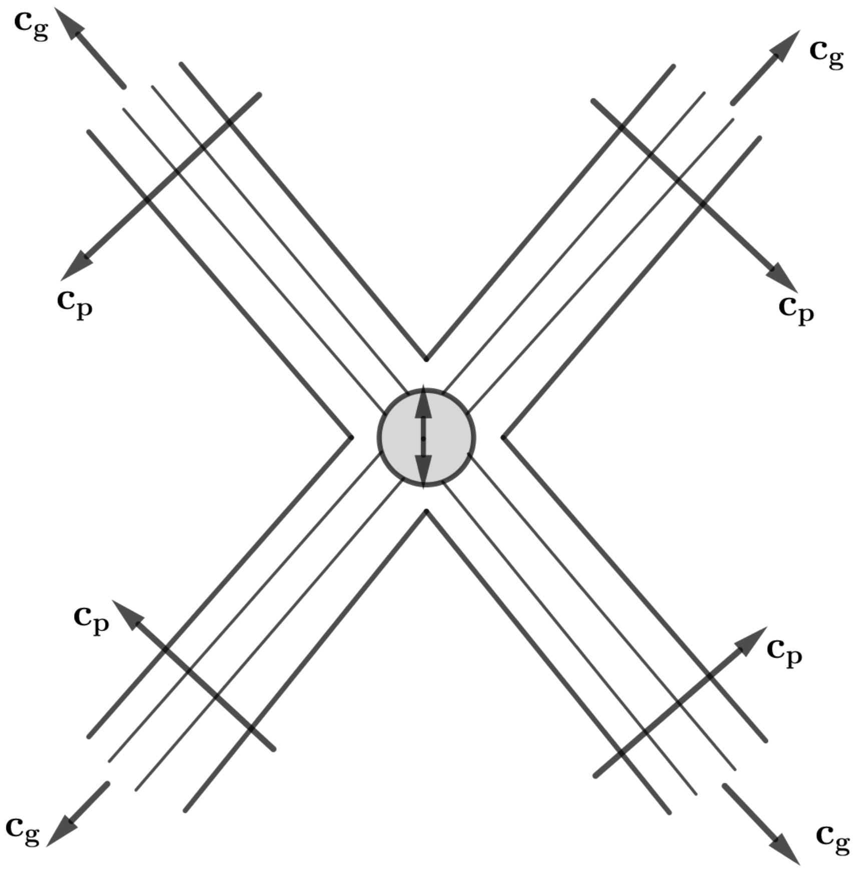

Figure 2.5: Schematic illustration of the St. Andrews Cross showing the directions of $\mathbf{c_p}$ and $\mathbf{c_g}$ for the four wave beams.

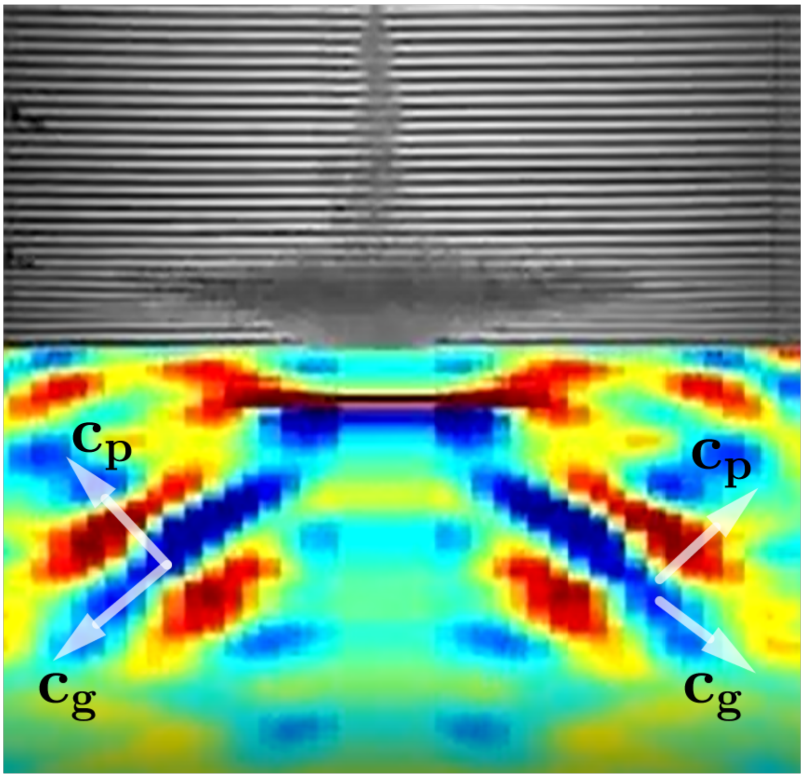

Figure 2.6: Direction of propagation of conical wave beams generated by a plume in a linearly stratified fluid.

Chapter 3

Propagation of Internal Gravity Waves: Vertical normal modes

3.1 Solution Approach

Here we return to finding solutions to equation (2.31)

$$\therefore \boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_h^2 w = 0}, \tag{3.1}$$

using the method of vertical normal modes subject to the boundary conditions:

$$w = 0 \quad \text{at} \quad z = 0. \tag{3.2}$$

$$w = 0 \quad \text{at} \quad z = -H. \tag{3.3}$$

We consider waves propagating in the $x-$direction ($\partial/\partial y = 0$) since the problem is horizontally isotropic, but we maintain f. We seek solutions of the form

$$w = W(z)e^{i(kx-\omega t)}, \tag{3.4}$$

where the frequency ω is taken to be positive. Substituting into (3.1) gives the ordinary differential equation for W:

$$W''(z) + k^2\frac{N^2(z) - \omega^2}{\omega^2 - f^2}W(z) = 0 \tag{3.5}$$

Employing the boundary conditions (3.2)-(3.3), we get the boundary conditions for W to be

$$W = 0 \quad \text{at} \quad z = 0, -H. \tag{3.6}$$

Thus, (3.5)-(3.6) forms a Sturm-Liouville problem, which, for a fixed frequency ω, has an infinite number of solutions W_n (called eigenfunctions or vertical modes) with corresponding eigenvalues k_n. From the governing equations (2.21)-(2.25), the other variables (u, v, p, b) with their $W(z)$ counterparts ($U(z)$, $V(z)$, $P(z)$, $B(z)$) can be written in terms of W:

$$U = \frac{i}{k}W'; \quad V = \frac{f}{\omega k}W'; \quad P = i\rho_* \frac{\omega^2 - f^2}{\omega k^2}W'; \quad B = -\frac{iN^2}{\omega}W \tag{3.7}$$

Equations (3.5) and (3.6) imply that the vertically integrated horizontal velocities are zero:

$$\int_{-H}^{0} u dz = 0; \quad \int_{-H}^{0} v dz = 0. \tag{3.8}$$

This feature is a distinguishing property of internal gravity waves compared to surface waves.

The general solution of w is given by a superposition of wave components such that

$$w = \sum_n W_n(z) \left[a_n^{\pm} \exp i(k_n^{\pm} x - \omega t)\right] \tag{3.9}$$

where $a_n^{\pm}$ are arbitrary complex constants, and the $+$ and $-$ superscripts describe rightward and leftward propagating waves respectively. As usual, the real part of (3.8) is meant.

3.1.1 Oscillatory versus exponential behaviour

From (3.5), let

$$m^2 = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}. \tag{3.10}$$

Solutions to (3.5) may exhibit two kinds of behaviour, depending on the sign of m^2. In the parts of the water column where m is real ($m^2 >\geq 0$), the waves are oscillatory. This then implies that one of the following inequalities must hold throughout this part of the water column:

$$N(z) \leq \omega \leq |f| \quad \text{or} \quad |f| \leq \omega \leq N(z). \tag{3.11}$$

In the ocean and atmosphere, the most common situation is $N > |f|$. However, in exceptionally circumstances, and in extremely weakly stratified regions, $|f|$ may exceed N; e.g. in convective layers. The second condition is therefore consistent with our previous analysis in which $f = 0$. Thus, in the presence of rotation the internal wave frequency is additionally bounded below (or above) by $|f|$.

In the case when $m^2 < 0$ (i.e. outside the intervals in (3.11)), we get the so-called *evernescent waves*: an exponential-like decay of waves away from the source; there is a rapid decrease of the wave-amplitude.

3.1.2 Orthogonality of eigenfunctions

It can be shown that the eigenfunctions in the Sturm-Liouville problem are orthogonal to each other. We skip the proof here.

3.2 Uniform stratification

We consider the case of a uniform stratification in which N is a constant. That is, density is linearly increasing from the top of the ocean to the bottom. The problem is to determine the eigvenvalues k_n and their relation to frequency ω (i.e. the dispersion relation), and also determine the vertical structure of the eigenfunctions or modes, W_n.

In the case of uniform stratification, (3.5) can easily be solved because we get

$$W''(z) + m^2 W(z) = 0 \tag{3.12}$$

where m is a constant (independent of z) in this case. For wave-like solutions we assume $N > |f|$ and employ the inequality

$$|f| \leq \omega \leq N.$$

The general solution to (3.12) is given by

$$W(z) = C_1 \sin mz + C_2 \cos mz. \tag{3.13}$$

We can solve (3.13) straightaway using the boundary conditions, but for a systematic solution procedure in other cases where N is not constant, we recast (3.13) together with the boundary conditions $W = 0$ at $z = 0$ and $z = -H$ in the matrix form

$$\begin{pmatrix} 0 & 1 \\ -\sin mH & \cos mH \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.14}$$

3.2.1 Dispersion relation

For non-trivial solutions for the pair (C_1, C_2), the determinant of the matrix in (3.14) must be zero. Thus $\sin mH = 0$ so that

$$m_n = \pm\frac{n\pi}{H}, \quad \text{for} \quad n = 1, 2, 3, \cdots . \tag{3.15}$$

From equation (3.10) we get the dispersion relation

$$\boxed{k_n = \pm\frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}, \; n = 1, 2, 3, \cdots} \tag{3.16}$$

This shows that there are an infinite number of eigenvalues k_n (which serve as horizontal wavenumbers) for a given frequency ω. As the mode number n increases $|k_n|$ increases, that is, the waves become shorter. Alternatively, for a given k and mode number n, we may express the frequency in terms of k and n:

$$\omega^2 = \frac{N^2k^2 + f^2\left(\frac{n\pi}{H}\right)^2}{k^2 + \left(\frac{n\pi}{H}\right)^2} \tag{3.17}$$

The lower bound of the frequency domain, $|f|$, is attained in the long-wave limit $|k| \to 0$ while the upper bound N is attained in the short-wave limit $|k| \to \infty$ as shown in Figure X.

Differentiating (3.17) yields the group velocity

$$c_g = \frac{k\left(\frac{n\pi}{H}\right)^2 (N^2 - f^2)}{\left[N^2k^2 + f^2\left(\frac{n\pi}{H}\right)^2\right]^{1/2}\left[k^2 + \left(\frac{n\pi}{H}\right)^2\right]^{3/2}} \tag{3.18}$$

$$\Longrightarrow c_g = \pm\frac{H}{n\pi}\frac{(\omega^2 - f^2)^{1/2}(N^2 - \omega^2)^{3/2}}{\omega(N^2 - f^2)} \tag{3.19}$$

where the $+$ sign applies for positive k and the $-$ sign if k is negative. The horizontal phase speed $c = \omega/k$ is given by

$$c_p = \frac{\left[N^2k^2 + f^2\left(\frac{n\pi}{H}\right)^2\right]^{1/2}}{k\left[k^2 + \left(\frac{n\pi}{H}\right)^2\right]^{1/2}} \tag{3.20}$$

$$\Longrightarrow c_p = \pm\frac{H\omega}{n\pi}\left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)^{1/2} \tag{3.21}$$

Equations (3.19) and (3.21) show that both the phase speed and group velocity are inverselyproportional to the modenumber n. Thus, higher modes propagate more slowly. Also, (3.19) shows that the group velocity goes to zero at the extreme limits of the frequency domain $\omega = |f|$, N. However, the phase speed tends to infinity at the lower bound $|f|$. See Figure..

