# Introduction to Internal Gravity Waves

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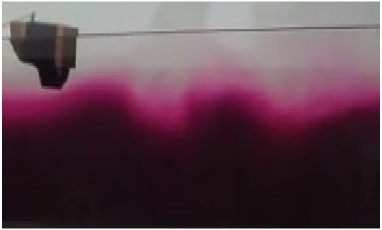
# OUTLINE

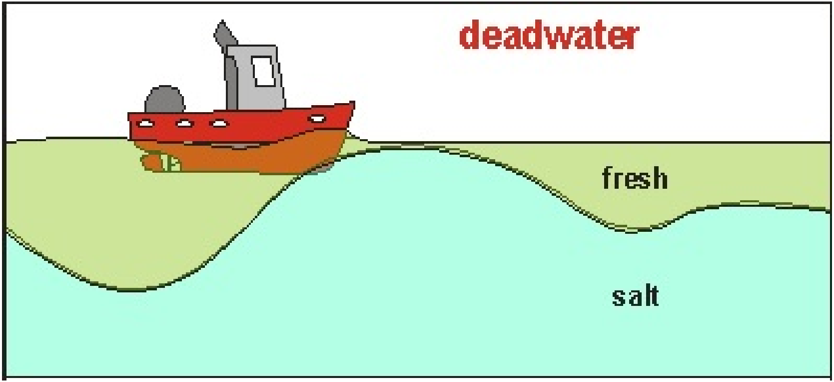
1. Introduction & Motivation
2. Mathematical Approach:
   - Surface Gravity Waves
   - Interfacial Waves
   - Internal Waves in Continuously Stratified Environment
3. Modeling Internal Tides
4. Highlights of Recent Research Efforts

# Introduction: what are internal waves?



- Internal gravity waves exist any time you have a lighter fluid above a heavier fluid with a mechanism to set them in motion.
- They have much larger amplitudes than waves we see on beaches (surface waves). This is because the density differences within the ocean are much smaller than on the surface.
- Internal waves with tidal frequency are called internal tides.

# Introduction: History of 'dead water' phenomenon




- Ships entering Norwegian fjords experienced increased drag
- It was a mystery for several years and attributed to 'dead water'
- First reported by Norwegian explorer Fridtjöf Nansen during his North poler expedition in 1893 on his ship Fram.

"When caught in dead water, *Fram* appeared to be held back, as if by some mysterious force, and she did not always answer the helm. In calm weather, with a light cargo, *Fram* was capable of 6 to 7 knots. When in dead water she was unable to make 1.5 knots. We made loops in our course, turned sometimes right around, tried all sorts of antics to get clear of it, but to very little purpose." (Walker J.M., 1991)

Nansen contacted an experienced physicist and meteorologies, Vilhelm Bjerknes, to study the problem scientifically.

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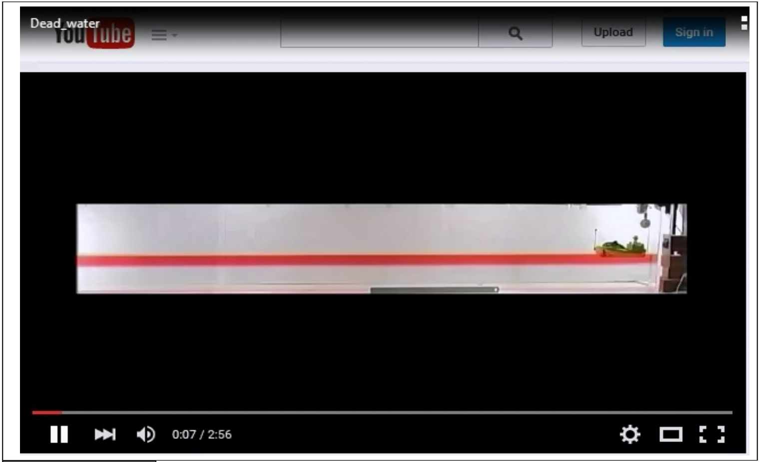
Vagn Walfrid Ekman

Vilhem Bjerknes then passed on the problem to his student, Vagn Walfrid Ekman. Ekman then discovered that the problem was due to interfacial/internal waves that are generated, propagate and produce a drag on the ship. Ekman performed the first laboratory experiment to generate the waves. The *Ekman spiral* is named after him.

# Introduction: Video of 'dead water' phenonmenon

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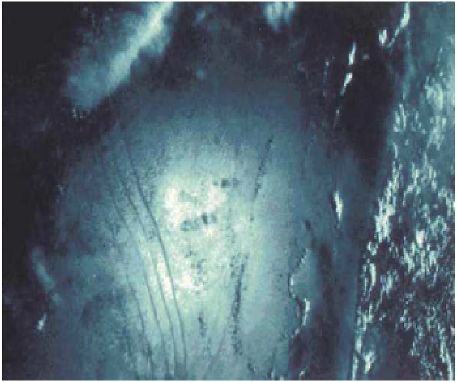
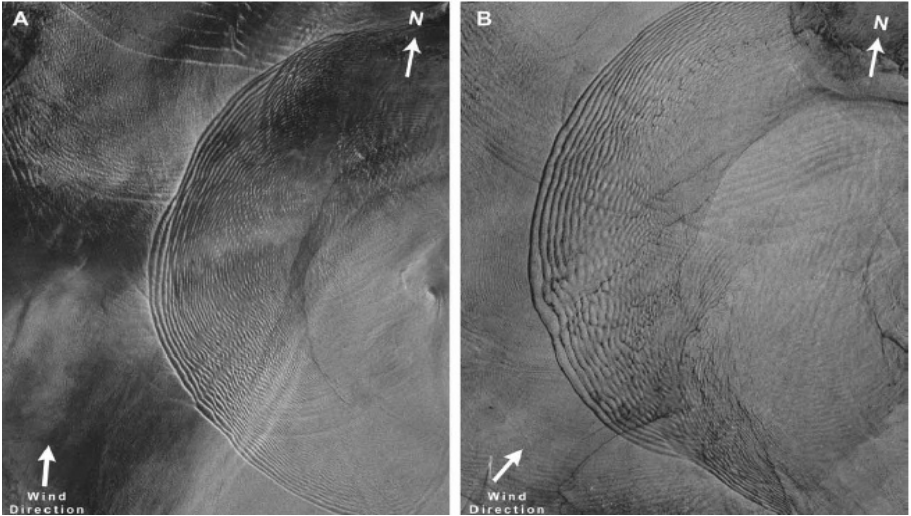
Credit: Matthieu Mercier (https://www.youtube.com/watch?v=bzcgAshAg2o)

# Introduction: Internal waves from spacecraft: 1975



Photograph from the Apollo-Soyuz spacecraft in 1975, made over the Andaman Sea, showing stripes due to internal waves. The stripes stretch over 100 km, and have a mutual distance of the order of a few tens of kilometers (Gerkeman & Zimmerman, 2008)

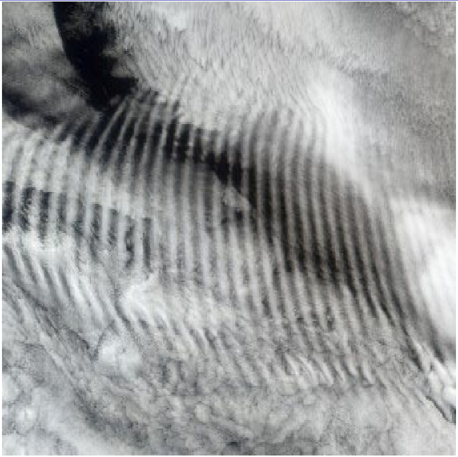
# Introduction: internal gravity waves in satellite imagery




Satellite image of internal waves over the Gulf of Maine west of Cape Cod on June 23, 2008 [Jackson et. al. (2013); Oceanography]

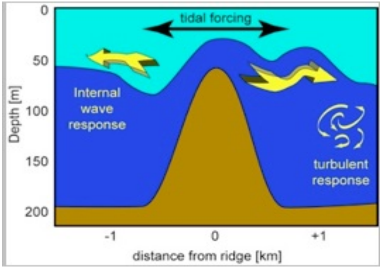
# Introduction: internal gravity waves in the atmosphere

Morning glory clouds
(https://en.wikipedia.org)

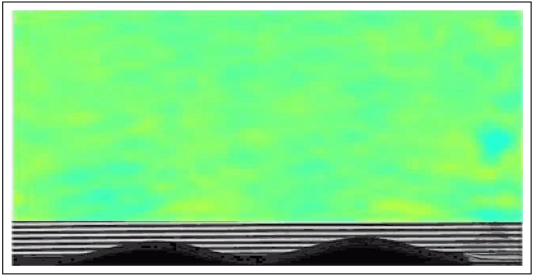


Turbulence waves (NASA)
(https://www.nasa.gov)

# Introduction: generation mechanisms




Generation by stratified flow over topographic bumps.
(Prof. Jonathan Nash)



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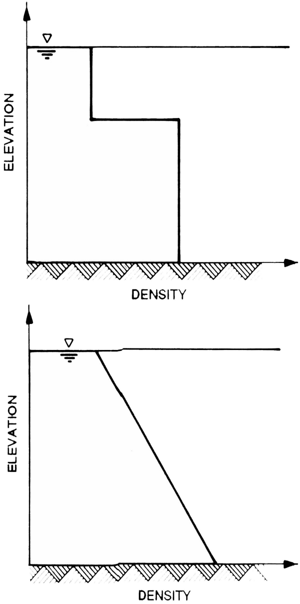
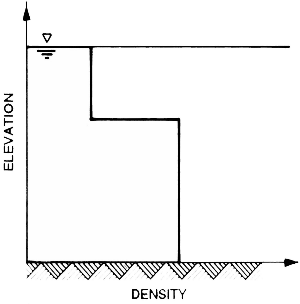
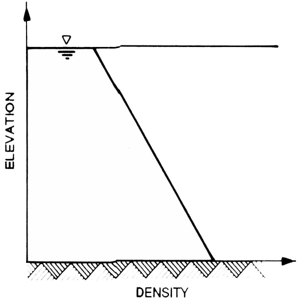
Generation in a laboratory tank by sinusoidal hills
(Prof. Bruce Sutherland)

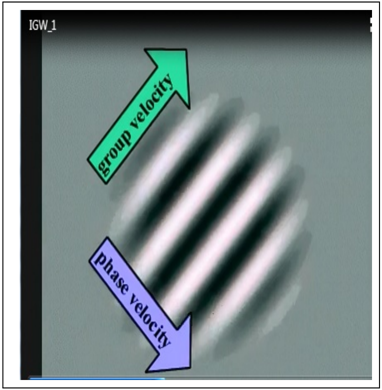
# Introduction: Continuously stratified fluid



# Introduction: Continuously stratified fluid

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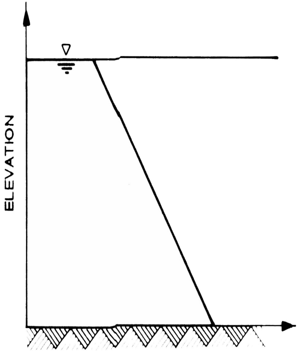
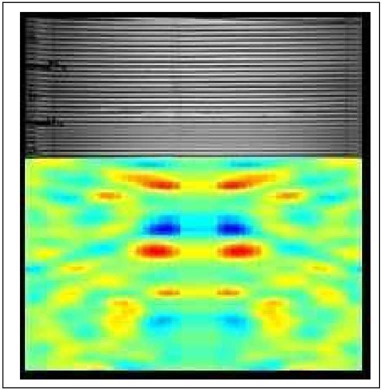
- Axisymmetric waves...
- Waves propagate down as conical beams

Ansong & Sutherland (2010) JFM, Vol. 648



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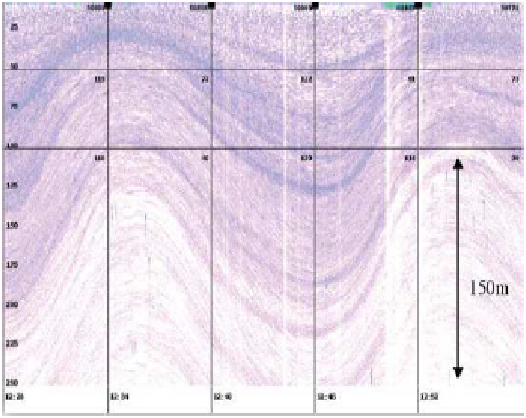
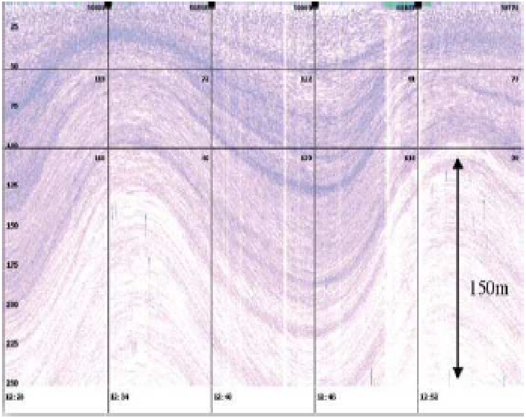
# WHY DO WE CARE ABOUT INTERNAL GRAVITY WAVES

# Motivation: Why do we care?



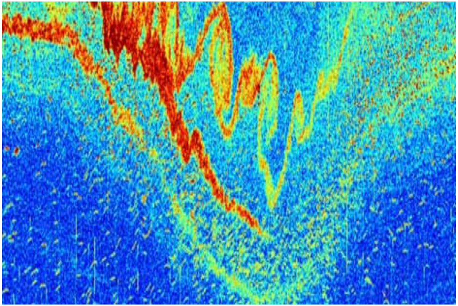

- Isopycnal (equal density surface) displacements due to the passage of an internal wave in Lombok Strait, covering the upper 250 m of the water column. Horizontal axis is time, and spacing between vertical lines is 6 minutes ( Susanto et. al., 2005).
- The picture gives a peak into the interior of the ocean.

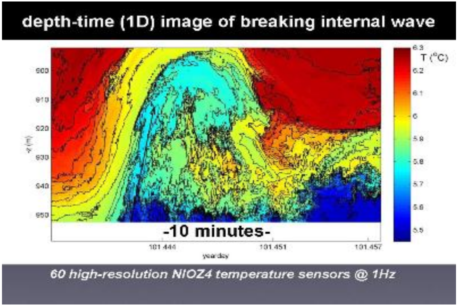
# Motivation: Why do we care?




- Larger displacements are in the interior and much smaller displacements are closer to the surface: a characteristic feature of internal waves
- Currents associated with them extend to the surface thereby changing the roughness of surface waves.

# Why do we care?






Hans van Haren

# Motivation: Why do we care?

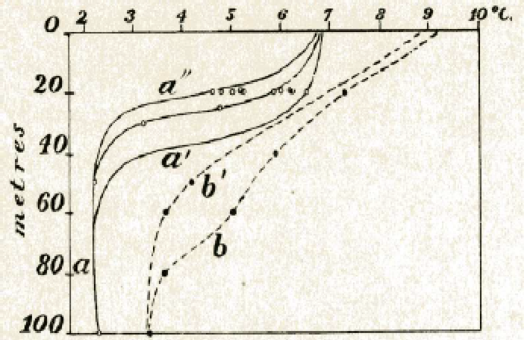
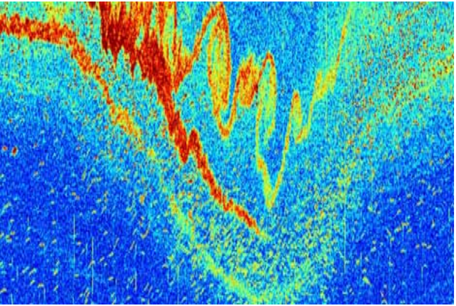


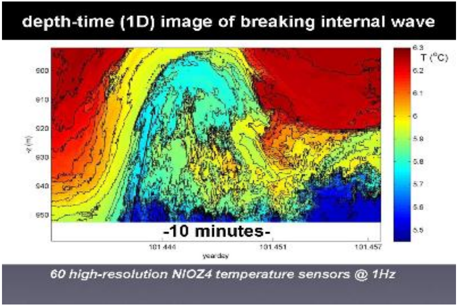

Fig. 14. Vertical Temperature-Curves at Stat. 15 (a) and Stat. 9 (b), 1900.

- Temperature profiles by Helland-Hansen & Nansen at two different locations represented by *a* (August, 1900) and *b* (July 1900). Profiles in 2.5 hours time at the same locations are $a'$ and $b'$. The measurements are shown in dots and the curves are constructed from them. (Helland-Hansen & Nansen, 1909).

# Why do we care?

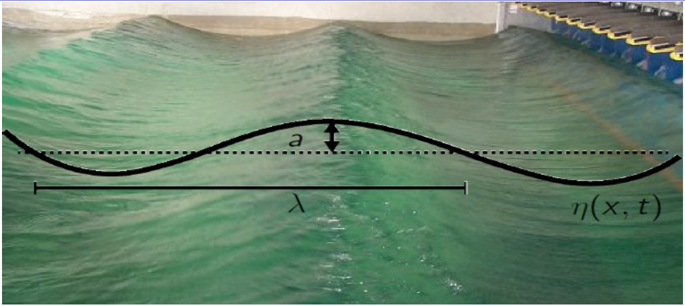
- Breaking internal waves affect the Meridional Overturning Circulation (thermohaline circulation)
- The spatial variability of mixing is important for accurate climate modeling
- The Navy care about internal gravity waves; submarines don't want to get caught up in IGWs.



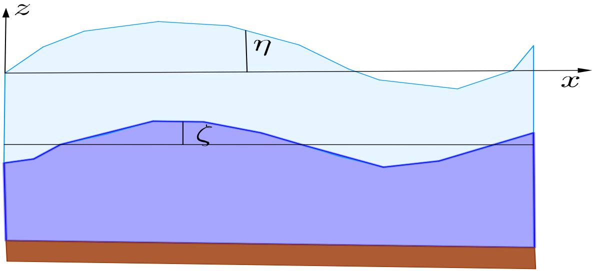



Hans van Haren

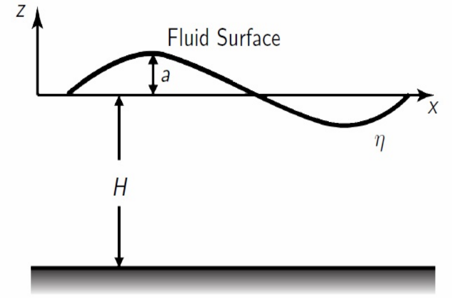
# Introduction: Types of Gravity Waves




Surface Gravity Waves

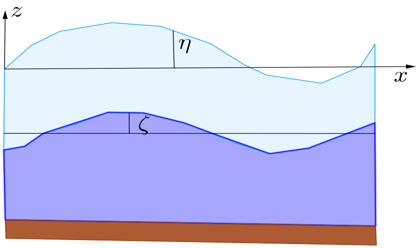


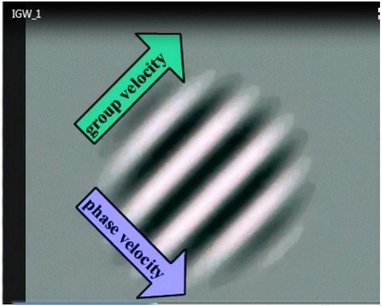

Interfacial Waves




THEORY OF....

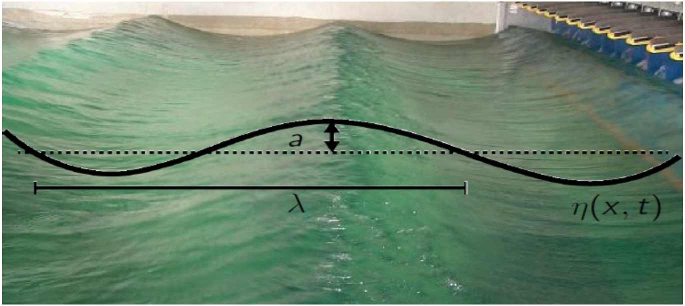
- Surface gravity waves
- Interfacial waves
- Internal waves in uniform stratification



# SURFACE GRAVITY WAVES

# Surface Gravity Waves: wave parameters

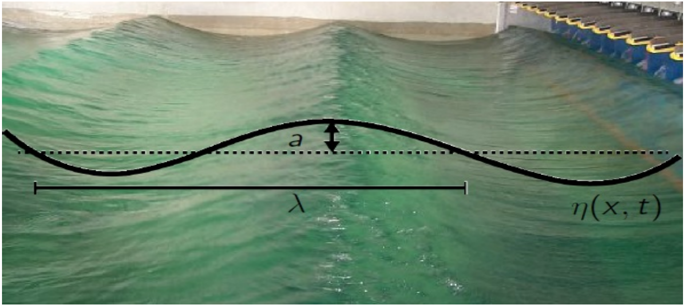



A simple way to describe a wave is

$$\eta(x,t) = a\cos(kx - \omega t)$$

- $a$ is the amplitude
- $k$ is the wavenumber ($k = 2\pi/\lambda$)
- $\omega$ is the frequency and $c = \omega/k$ is the phase speed

# Surface Gravity Waves: wave parameters




For the wave $\eta(x,t) = a\cos(kx - \omega t)$

The phase speed is $c = \omega/k$

**Important Fact**

Waves of different wavenumbers may travel at different speeds.

# Governing Equations of Fluid Mechanics

## Momentum Conservation:

The Navier-Stokes equations express the conservation of momentum of a fluid element ($ma = \sum F$):

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u} - 2\Omega \times \mathbf{u}} \quad (1)$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, $p$ pressure, $\rho$ density, $\mu$ viscosity, ˙ is Earth's angular velocity and the *total derivative* or *material derivative* and gradient are given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \implies \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

## Conservation of Mass:

The rate of mass inflow into a control volume must balance the rate of mass outflow, leading to the so-called continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$
$$\Longrightarrow \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$$

The total derivative $D\rho/Dt$ is the rate of change of density following a fluid particle. A fluid is called incompressible if its density does not change with pressure. Liquids are almost incompressible so we set $(1/\rho)D\rho/Dt = 0$ and the continuity equation becomes

$$\boxed{\frac{D\rho}{Dt} = 0} \quad \text{and so} \quad \boxed{\nabla \cdot \mathbf{u} = 0.}$$

# Surface Gravity Waves: Equations

The full governing equations for an incompressible fluid, non-viscous, with Coriolis force $f$ under Boussinesq approximation are:

$$
\begin{aligned}
\rho_* \left( \frac{Du}{Dt} - fv \right) &= -\frac{\partial p}{\partial x} \\
\rho_* \left( \frac{Dv}{Dt} + fu \right) &= -\frac{\partial p}{\partial y} \\
\rho_* \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} - g\rho \\
\rho_* \frac{D\rho}{Dt} &= -w\frac{d\rho_0}{dz} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

where $\rho_0(z)$ is the background density and $\rho_*$ is a constant density.

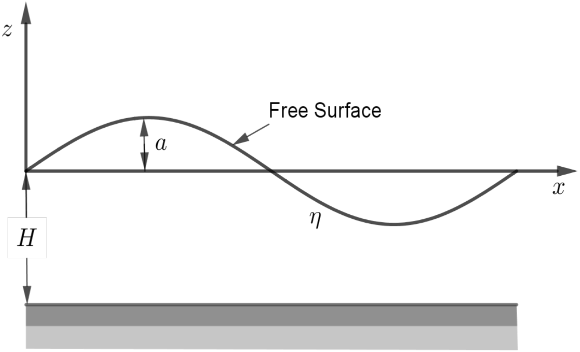
Rather than solve the previous equations, we first make simplifying assumptions about the flow field.

## Assumptions

We consider

- Small amplitude waves
- Unaffected by Earth's rotation
- Inviscid - viscosity is negligible
- Incompressible - no sound waves allowed ($\nabla \cdot \mathbf{u} = 0$)
- Irrotational ($\nabla \times \mathbf{u} = 0$)
- Two dimensional: $\mathbf{u} = (u, w)$

# Surface Gravity Waves: Setup




- Consider 2D flow in the $xz-$plane.
- No Coriolis frequency and
- Constant density ($\rho_* = \rho =$ constant).

# Surface Gravity Waves: Equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

- The Coriolis frequency is neglected by assuming that the frequency of the waves is large compared to the Coriolis frequency such that the waves are not affected by the Earth's rotation.
- The motion is generated from rest by wind action or dropping a stone in the water body.
- The resulting motion is irrotational, by the Kelvin's circulation theorem.

## Small Amplitude Assumption:

Assumed that the amplitude $a$ of oscillation of the free surface is small. That is, both $a/\lambda$ and $a/H$ are much smaller than one.

- $a/\lambda \ll 1$ implies that the slope of the sea/water surface is small.
- $a/H \ll 1$ implies that the instantaneous depth is not significantly different from the undisturbed depth.

These small amplitude assumptions allows for the problem to be linearized and the equations become...

# Surface Gravity Waves: Equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

# Surface Gravity Waves: Equations

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{w\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

The simplified equations become:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

# Surface Gravity Waves: Equations

## Definition (**Vorticity**)

The vorticity vector $\omega$ of a fluid element with velocity vector $\tilde{\mathbf{u}} = (u, v, w)$ is defined as

$$\omega = \nabla \times \mathbf{u} \tag{8}$$

with components:

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{9}$$

## Remark

Vorticity is a measure of the local rotation of a fluid element.

## Definition (Irrotational Flow)

A fluid motion is said to be irrotational if the vorticity is equal to zero

$$\omega = \nabla \times \mathbf{u} = 0, \tag{10}$$

which requires that

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \qquad i \neq j, \tag{11}$$

where $u_i$ and $u_j$ denote the velocity components in the $x_i$ and $x_j$ coordinates respectively.

## Remark

If the flow is irrotational, the velocity vector can be written as the gradient of a scalar function $\phi(\mathbf{x}, t)$. This is because

$$u_i = \frac{\partial \phi}{\partial x_i} \tag{12}$$

satisfies the condition of irrotationality in equation (10).

For example, in our $2D$ flow in the $xz-$plane (i.e., $x_1x_3-$plane), irrotationality implies that

$$\frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1} \quad \Longrightarrow \boxed{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,} \tag{13}$$

and the velocity field satisfies

$$u = \frac{\partial \phi}{\partial x}, \qquad w = \frac{\partial \phi}{\partial z} \tag{14}$$

Repeat: the simplified equations

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{15}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{17}$$

Substituting (14) into the continuity equation (17) results in the Laplace equation

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0.} \tag{18}$$

Also, from equation (16), we have

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

and inter-changing derivatives gives

$$\implies \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right) = -\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) - g,$$

$$\implies \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho}\right) = -g.$$

Integrating both sides with respect to $z$ to get the linearized Bernoulli equation:

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = 0.} \tag{19}$$

To solve the Laplace equation, we need to specify boundary conditions at the free surface and at the bottom.

## Bottom boundary condition:

At the bottom, we specify zero normal velocity such that

$$\boxed{w = \frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -H} \tag{20}$$

## Kinematic boundary condition:

The kinematic boundary condition at the free surface states that the fluid particle never leaves the surface, that is

$$\frac{D\eta}{Dt} = w_\eta, \quad \text{at} \quad z = \eta, \tag{21}$$

where the material derivative is $D/Dt = \partial/\partial t + u(\partial/\partial x)$, and $w_\eta$ is the vertical component of fluid velocity at the free surface. In other words,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta}, \quad \text{non-linear} \tag{22}$$

Recall that $\eta = \eta(x,t)$. For small amplitude waves $u\partial\eta/\partial x$ is one order smaller than the other terms in (22) so we neglect $u\partial\eta/\partial x$ and get

$$\frac{\partial \eta}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta}. \tag{23}$$

We can simplify this condition further by evaluating the right side at $z = 0$ rather than at the free surface. This can be justified by performing a Taylor series expansion of $\partial\phi/\partial z$ about $z = 0$:

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \eta\frac{\partial^2 \phi}{\partial^2 z} + \cdots \approx \left.\frac{\partial \phi}{\partial z}\right|_{z=0} \tag{24}$$

To first order of accuracy, (23) becomes

$$\boxed{\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0.} \tag{25}$$

## Dynamic boundary condition:

There is a dynamic boundary condition that the pressure just below the free surface is always equal to the ambient (atmospheric) pressure. Taking the ambient pressure to be zero results in

$$p = 0 \qquad \text{at} \quad z = \eta. \tag{26}$$

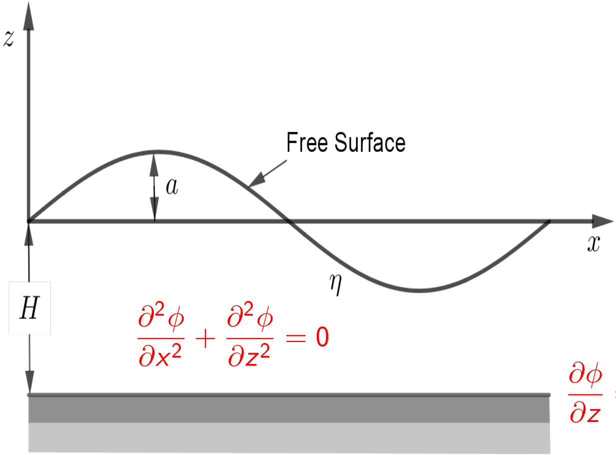
Substituting into the Bernoulli equation (19) yields

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = \eta \tag{27}$$

As in the case of the kinematic condition, for small amplitude waves, the term $\partial\phi/\partial t$ can be evaluated at $z = 0$ instead of at $z = \eta$ so that

$$\boxed{\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = 0} \tag{28}$$

# Surface Gravity Waves: Summary




$$\frac{\partial \phi}{\partial t} = -g\eta, \quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at } z = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ at } z = -H$$

## IMPORTANT

- The boundary conditions imply specifying a form for $\eta$.
- The kinematic condition imply separation of variables

Summarizing, we need to solve the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{29}$$

in the interior of the domain, subject to the conditions

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -H \tag{30}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \tag{31}$$

$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0 \tag{32}$$

To apply the boundary conditions, we need to assume a form for $\eta(x, t)$. We assume a sinusoidal component:

$$\eta = a\cos(kx - \omega t) \tag{33}$$

## Remark

A strong motivation for studying sinusoidal waves is that an arbitrary disturbance can be decomposed into various sinusoidal components by Fourier analysis, and the response of the system to an arbitrary small disturbance is the sum of the responses to the various sinusoidal components.

### Remark

A strong motivation for studying sinusoidal waves is that an arbitrary disturbance can be decomposed into various sinusoidal components by Fourier analysis, and the response of the system to an arbitrary small disturbance is the sum of the responses to the various sinusoidal components.

Assuming a separable solution: $\phi(x, z, t) = \psi(z)\Phi(x, t)$. The conditions in (31) and (32) imply that $\Phi(x, t)$ must be sine function of $kx - \omega t$. Thus, we assume a solution in the form

$$\phi = \psi(z) \sin(kx - \omega t), \tag{34}$$

where $\psi(z)$ and $\omega(k)$ are to be determined. Substituting (34) into the Laplace equation (29) gives the ODE:

$$\boxed{\frac{d^2\psi}{dz^2} - k^2\psi = 0.} \qquad (35)$$

A general solution is

$$\psi(z) = Ae^{kz} + Be^{-kz}.$$

Thus, the velocity potential $\phi$ in (34) is given by

$$\boxed{\phi = \left(Ae^{kz} + Be^{-kz}\right)\sin(kx - \omega t).} \qquad (36)$$

The constants $A$ and $B$ are determined from the conditions in (30) and (31).

Applying (30):

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = 0$$

gives

$$\left(kAe^{-kH} - kBe^{kH}\right)\sin(kx - \omega t) = 0,$$

$$\Longrightarrow k\left(Ae^{-kH} - Be^{kH}\right)\sin(kx - \omega t) = 0.$$

For a nontrivial solution,

$$\sin(kx - \omega t) \neq 0 \Longrightarrow Ae^{-kH} = Be^{kH}$$

$$\Longrightarrow \boxed{B = Ae^{-2kH}} \tag{37}$$

We next apply the kinematic condition (31). Before then...

## Remark

Suppose we applied condition (31) at $z = \eta$ instead of the linearized form at $z = 0$. Then from (36) we get

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = k\left(Ae^{k\eta} - Be^{-k\eta}\right)\sin(kx - \omega t)$$

For a small slope of the free surface, $k\eta << 1$, we can set $e^{k\eta} \approx e^{-k\eta} = 1$. This is effectively what we are doing by applying the surface boundary conditions at $z = 0$ instead of at $z = \eta$, which was justified using Taylor series expansions.

Substituting (33) and (36) into (31) gives

$$k\left(Ae^{kz} - Be^{-kz}\right)\sin(kx - \omega t) = a\omega \sin(kx - \omega t)$$

At $z = 0$ we have

$$k(A - B) = a\omega \qquad (38)$$

$$\Longrightarrow A - B = \frac{a\omega}{k} \quad \Longrightarrow B = A - \frac{a\omega}{k}$$

Employing (37) results in

$$A - \frac{a\omega}{k} = Ae^{-2kH} \quad \Longrightarrow A\left(1 - e^{-2kH}\right) = \frac{a\omega}{k}$$

$$\Longrightarrow \boxed{A = \frac{a\omega}{k\left(1 - e^{-2kH}\right)}} \qquad (39)$$

$$\boxed{\therefore B = \frac{a\omega e^{-2kH}}{k\left(1 - e^{-2kH}\right)}} \qquad (40)$$

From (36) we finally get the velocity potential

$$\phi = \left[\frac{a\omega e^{kz}}{k\left(1-e^{-2kH}\right)} + \frac{a\omega e^{-2kH}}{k\left(1-e^{-2kH}\right)}e^{-kz}\right]\sin(kx-\omega t).$$

$$\phi = \frac{a\omega}{k}\left[\frac{e^{kz}}{\left(1-e^{-2kH}\right)} + \frac{e^{-k(z+2H)}}{\left(1-e^{-2kH}\right)}\right]\sin(kx-\omega t). \tag{41}$$

Using the fact that

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$1-e^{-2kH} = 1-\frac{e^{-kH}}{e^{kH}} = \frac{e^{kH}-e^{-kH}}{e^{kH}}$$

$$\phi(x, z, t) = \frac{a\omega}{k} \frac{\cosh k(z + H)}{\sinh kH} \sin(kx - \omega t) \tag{42}$$

Recall that

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad w = \frac{\partial \phi}{\partial z}$$

Thus,

$$u(x, z, t) = a\omega \frac{\cosh k(z + H)}{\sinh kH} \cos(kx - \omega t) \tag{43}$$

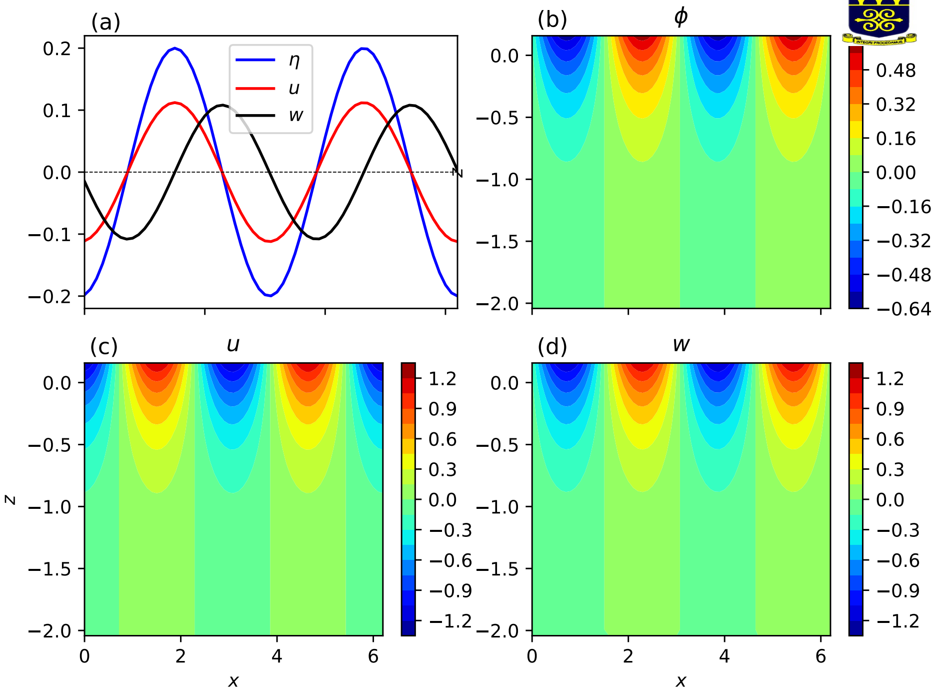
$$w(x, z, t) = a\omega \frac{\sinh k(z + H)}{\sinh kH} \sin(kx - \omega t) \tag{44}$$

$$u(x, z, t) = a\omega \frac{\cosh k(z + H)}{\sinh kH} \cos(kx - \omega t) \tag{45}$$

$$w(x, z, t) = a\omega \frac{\sinh k(z + H)}{\sinh kH} \sin(kx - \omega t) \tag{46}$$

**Remark**

Note that since $\eta = a\cos(kx - \omega t)$, we see that the $u$ velocity is in phase with the displacement while the $w$ velocity is $90^\circ$ out of phase with $\eta$.



(a)
η
u
w
(b)
ϕ
(c)
u
(d)
w
z
x
x

# Dispersion Relation

Recall:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

in the interior of the domain, subject to the conditions

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = -H \tag{47}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \tag{48}$$

$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0 \tag{49}$$

### Remark

Note that we solved the Laplace equation by using only the bottom and kinematic conditions (47) and (48); without employing the dynamic condition (49). Application of the dynamic condition (49) results in a relation between the wave number $k$ and frequency $\omega$.

Substituting (33) and (42) into dynamic condition (32) or (49), we have

$$\frac{-a\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)}\cos(kx-\omega t) = -ga\cos(kx-\omega t)$$

$$\implies \frac{\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)} = g$$

$$\implies \omega^2 = gk\tanh(kH)$$

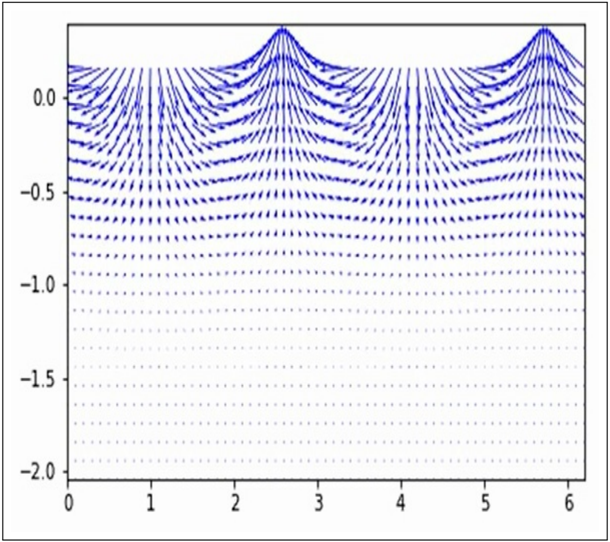
$$\boxed{\omega = \sqrt{gk\tanh(kH)}} \quad \text{or} \quad T = \sqrt{\frac{2\pi\lambda}{g}\coth\left(\frac{2\pi H}{\lambda}\right)}, \qquad (50)$$

where $T = 2\pi/\omega$ is the wave period. The wave speed $c = \omega/k$ is related to the wavelength by

$$\boxed{c = \sqrt{\frac{g}{k}\tanh(kH)} = \sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi H}{\lambda}}} \qquad (51)$$

These equations show that the speed of propagation of a wave component depends on its wavenumber. Waves for which $c$ is a function of $k$ are said to be dispersive since they separate into individual components. The relationship in (50) or (51) where $\omega$ is a function of $k$ is called a dispersion relation. This is because it expresses the nature of the dispersive process.

# Surface Waves: propagation animation



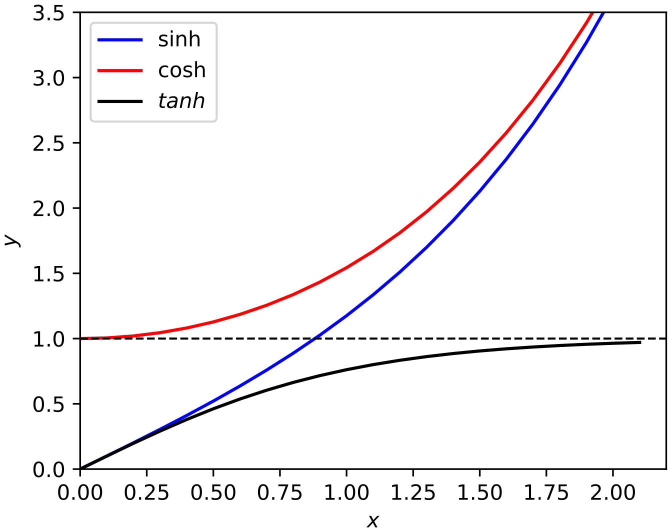

Play/Pause

# Approximations for Deep and Shallow Water Waves

## Remark

The analysis in the previous section is applicable irrespective of the relationship between the magnitude of $\lambda$ and the water depth $H$. Some interesting simplifications arise for shallow water ($H/\lambda \ll 1$) and deep water ($H/\lambda \gg 1$). We derive approximations for the phase speed for which (51) takes simple forms.

The behaviour of hyperbolic functions as $x \to \infty$ and as $x \to 0$ be employed for the simplifications.


sinh
cosh
tanh
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x

# Approximations

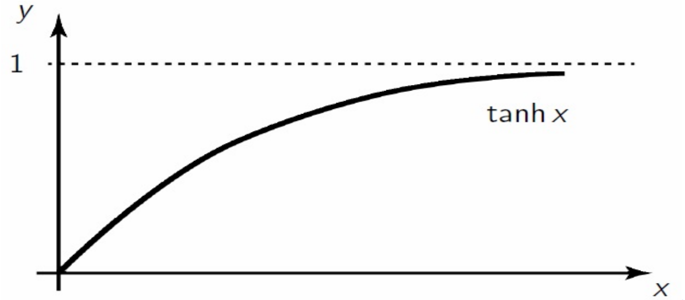
Using $\lambda = 2\pi/k$, we are interested in the cases when

- $H/\lambda \gg 1$ or $kH \gg 1$ (deep water)
- $H/\lambda \ll 1$ or $kH \ll 1$ (shallow water)

Note that

$$\tanh kH \to 1 \quad \text{when} \quad kH \gg 1$$

$$\tanh kH \to kH \quad \text{when} \quad kH \ll 1$$



## Deep Water Waves

$$kH \gg 1: \quad c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH} \rightarrow \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \tag{52}$$

## c is dependent on k

- The wave does not feel the bottom.
- Longer waves in deep water propagate faster.
- Deep water waves are dispersive: A wave "packet" separates or disperses.
- Within 2% accuracy, the approximation (52) is valied for $H > 0.32\lambda$ (i.e., $kH > 2$). Therefore, surface waves are classified as *deep water waves* if the depth is more than one-third of the wavelength. Deep water waves are dispersive since the phase speed depends on the wavelength.

## Shallow Water Waves

$$H/\lambda \ll 1 \text{ or } kH \ll 1: \quad c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kH} \rightarrow \sqrt{gH}$$

$$\therefore c = \sqrt{gH}. \tag{53}$$

## c is independent of k

- The wave does feel the bottom; phase speed increases with water depth.
- Shallow water waves are not dispersive: A wave "packet" stays together.
- The approximation gives better than 3% accuracy if $H < 0.07\lambda$. Thus, surface waves are regarded as *shallow-water waves* if the water depth is less than 7% of the wavelength.

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