Introduction to Internal Gravity Waves: Interfacial Waves

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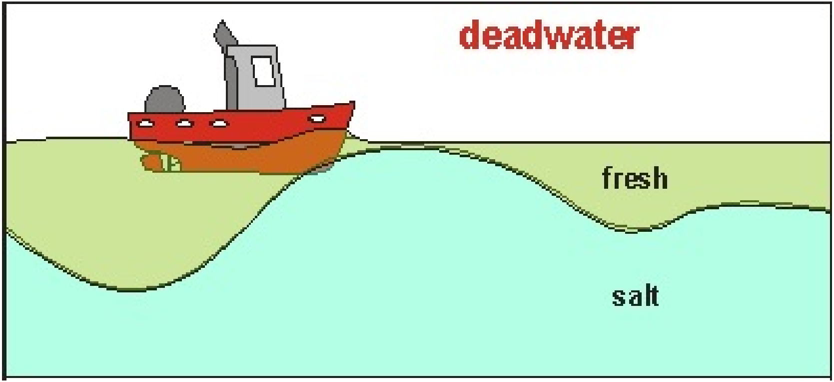
OUTLINE

1. Introduction & Motivation
2. Mathematical Approach:
   - Recap: Surface Gravity Waves
   - Interfacial Waves
   - Internal Waves in Continuously Stratified Environment
3. Modeling Internal Tides
4. Highlights of Recent Research Efforts

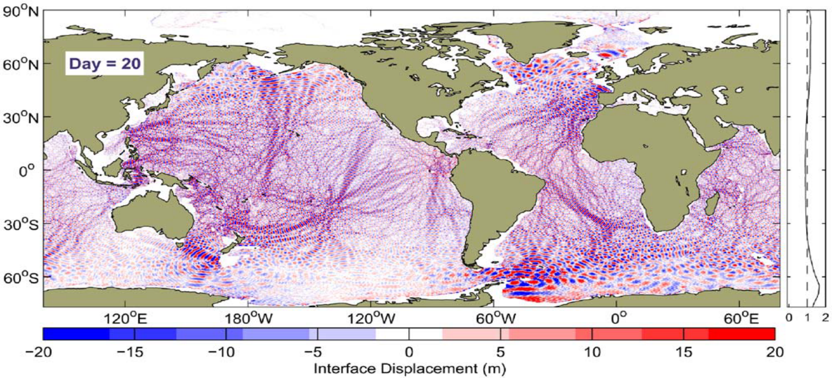
Interfacial Waves

Motivation: History of 'dead water' phenomenon

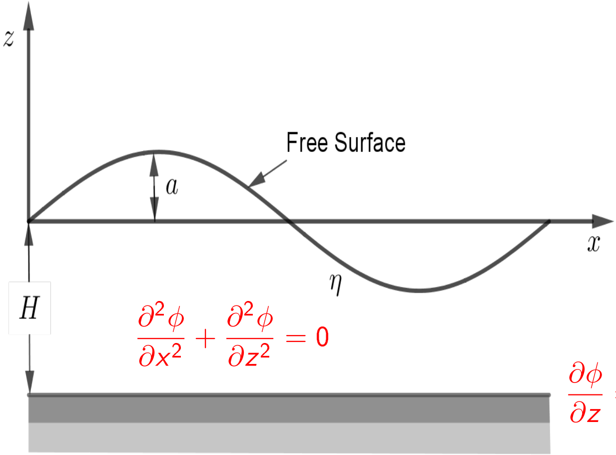


- Ships entering Norwegian fjords experienced increased drag
- It was a mystery for several years and attributed to 'dead water'
- First reported by Norwegian explorer Fridtjöf Nansen during his North poler expedition in 1893 on his ship Fram

Motivation: First global model of internal tides



- The first global simulation of internal tides constructed a 2-layer model and a 10-layer model with similar propagation features.
- There are hot-spots around the global ocean where internal tides are generated by tidal flow over topography.
- Harper L. Simmons, Robert W. Hallberg, & Brian K. Arbic (2004)

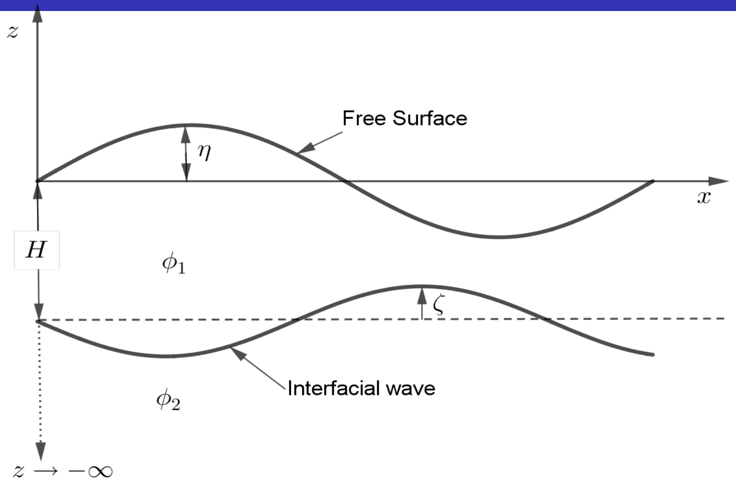


$$\frac{\partial \phi}{\partial t} = -g\eta, \quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at } z = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ at } z = -H$$

Interfacial Waves: Problem Setup



Plan of Attack

- Solve the Laplace equation in both layers
- Apply the continuity of p and w at the interface

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \tag{1}$$

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \tag{2}$$

with the conditions

$$\phi_2 \to 0 \qquad \text{at} \quad z \to -\infty \tag{3}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{at} \quad z = 0 \tag{4}$$

$$\frac{\partial \phi_1}{\partial t} = -g\eta \qquad \text{at} \quad z = 0 \tag{5}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad \text{at} \quad z = -H \tag{6}$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g\zeta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g\zeta \qquad \text{at} \quad z = -H \tag{7}$$

Assume displacements:

$$\eta = ae^{i(kx-\omega t)} \tag{8}$$

$$\zeta = be^{i(kx-\omega t)} \tag{9}$$

where a can be real but b should be left complex since η and ζ may not be in phase. The use of complex notation here is to simplify the algebra. Note that only the real part of the equation is meant.

Seek separable solutions

The boundary conditions imply separable solutions of the form

$$\phi_1 = \left(Ae^{kz} + Be^{-kz}\right) e^{i(kx-\omega t)} \tag{10}$$

$$\phi_2 = Ce^{kz} e^{i(kx-\omega t)} \tag{11}$$

We are left with the problem of finding A, B and C, as well as the relationship between amplitudes a and b. Applying (4) we have

$$\frac{\partial\phi_1}{\partial z} = k\left(Ae^{kz} - Be^{-kz}\right)e^{i\theta} = -ia\omega e^{i\theta},$$

and at $z = 0$, we get

$$k(A - B) = -ia\omega$$
$$\Longrightarrow A - B = -\frac{ia\omega}{k} \quad (12)$$

where $\theta = kx - \omega t$. Applying (5), we have

$$\frac{\partial\phi_1}{\partial t} = -i\omega\left(Ae^{kz} + Be^{-kz}\right)e^{i\theta} = -ga\omega e^{i\theta},$$

and at $z = 0$, we get

$$A + B = \frac{ga}{i\omega} = -i\frac{ag}{\omega} \quad (13)$$

Summing (12) and (13) results in

$$2A = i\left(-\frac{ag}{\omega} - \frac{a\omega}{k}\right)$$

$$\Longrightarrow A = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \qquad (14)$$

$$\Longrightarrow B = -\frac{iag}{\omega} - A = -\frac{iag}{\omega} + \frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right)$$

$$\Longrightarrow B = \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) \qquad (15)$$

From (6):

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \qquad z = -H$$

$$\Longrightarrow k\left(Ae^{kz} - Be^{-kz}\right)e^{i\theta} = -kCe^{kz}\omega e^{i\theta}$$

At $z = -H$, we have

$$Ae^{-kH} - Be^{kH} = Ce^{-kH}$$

$$\implies C = A - Be^{2kH}$$

$$C = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right)e^{2kH} \tag{16}$$

To determine the relationship between a and b, we may use either

$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\zeta}{\partial t} \quad \text{or} \quad \frac{\partial\phi_2}{\partial z} = \frac{\partial\zeta}{\partial t}$$

from equation (6). Employing the latter equation for simplicity, we have

$$kCe^{kz}e^{i\theta} = -i\omega be^{i\theta}$$

and at $z = -H$:

$$b = \frac{k}{-i\omega} Ce^{-kH} = \frac{ik}{\omega} Ce^{-kH}$$

$$\implies b = \frac{ik}{\omega} \left\{ -\frac{ai}{2} \left(\frac{\omega}{k} + \frac{g}{\omega} \right) - \frac{ai}{2} \left(\frac{\omega}{k} - \frac{g}{\omega} \right) e^{2kH} \right\} e^{-kH}$$

$$\therefore b = \frac{a}{2} \left(1 + \frac{gk}{\omega^2} \right) e^{-kH} + \frac{a}{2} \left(1 - \frac{gk}{\omega^2} \right) e^{kH} \qquad (17)$$

Interfacial Waves: Solution

Recapitulating, the velocity potentials are given by (10)-(11):

$$\phi_1 = \left(Ae^{kz} + Be^{-kz}\right) e^{i(kx-\omega t)}$$
$$\phi_2 = Ce^{kz} e^{i(kx-\omega t)}$$

with the constants given by

$$A = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \tag{18}$$
$$B = \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) \tag{19}$$
$$C = -\frac{ai}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ai}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) e^{2kH} \tag{20}$$

$$b = \frac{a}{2}\left(1 + \frac{gk}{\omega^2}\right) e^{-kH} + \frac{a}{2}\left(1 - \frac{gk}{\omega^2}\right) e^{kH}. \tag{21}$$

Solution

$$\phi_1 = -\frac{ia}{2}\left[\left(\frac{\omega}{k}+\frac{g}{\omega}\right)e^{kz} - \left(\frac{\omega}{k}-\frac{g}{\omega}\right)e^{-kz}\right]e^{i(kx-\omega t)}$$

$$\phi_2 = -\frac{ia}{2}\left[\left(\frac{\omega}{k}+\frac{g}{\omega}\right) + \left(\frac{\omega}{k}-\frac{g}{\omega}\right)e^{2kH}\right]e^{kz}e^{i(kx-\omega t)}$$

The velocities in the upper and lower layers can now be obtained via:

Velocities

$$(u_1, w_1) = \left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_1}{\partial z}\right)$$

$$(u_2, w_2) = \left(\frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_2}{\partial z}\right)$$

Dispersion relation

For the dispersion relation, we employ the dynamic conditions (7):

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \zeta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g \zeta \quad \text{at} \quad z = -H$$

After a few pages of algebra, the result can be written as (homework :))

Dispersion relation

$$\left(\frac{\omega^2}{gk} - 1\right)\left\{\frac{\omega^2}{gk}\left[\rho_1 \sinh(kH) + \rho_2 \cosh(kH)\right] - (\rho_2 - \rho_1)\sinh(kH)\right\} = 0. \tag{22}$$

Equation (22) shows that there are two possible roots, resulting in the barotropic or surface mode and the baroclinic or internal modes as discussed next.

Interfacial Waves: Barotropic mode

From (22), one of the roots is given by

$$\left(\frac{\omega^2}{gk} - 1\right) = 0$$

$$\therefore \omega^2 = gk. \tag{23}$$

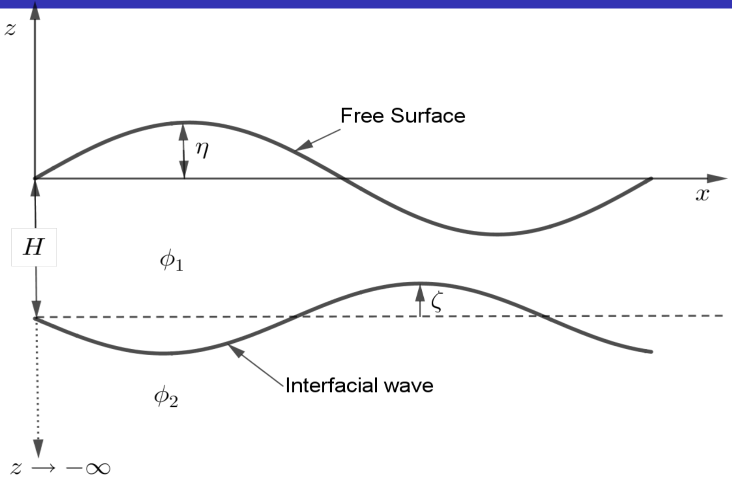
This is the same dispersion relation we obtained for a deep-water water wave. Equation (21) :

$$b = \frac{a}{2}\left(1 + \frac{gk}{\omega^2}\right)e^{-kH} + \frac{a}{2}\left(1 - \frac{gk}{\omega^2}\right)e^{kH}$$

shows that

$$\boxed{b = ae^{-kH}}$$

Interfacial Waves: Problem Setup



Displacement

- $\eta = ae^{i(kx-\omega t)}$
- $\zeta = be^{i(kx-\omega t)}$

$$\boxed{b = ae^{-kH}} \tag{24}$$

Remark

This implies that the amplitude at the interface is smaller than that at the surface by a factor e^{-kH}. Also, equation (24) together with (8)-(9) shows that the motions of the free surface and the interface are locked in phase. That is, they go up or down simultaneously. This mode is similar to a gravity wave propagating on the free surface of the upper liquid, in which the motion decays as e^{-kz} from the free surface. It is called the barotropic mode.

Summary: Barotropic Mode

The first expression gives the barotropic (surface) mode:

Barotropic Mode

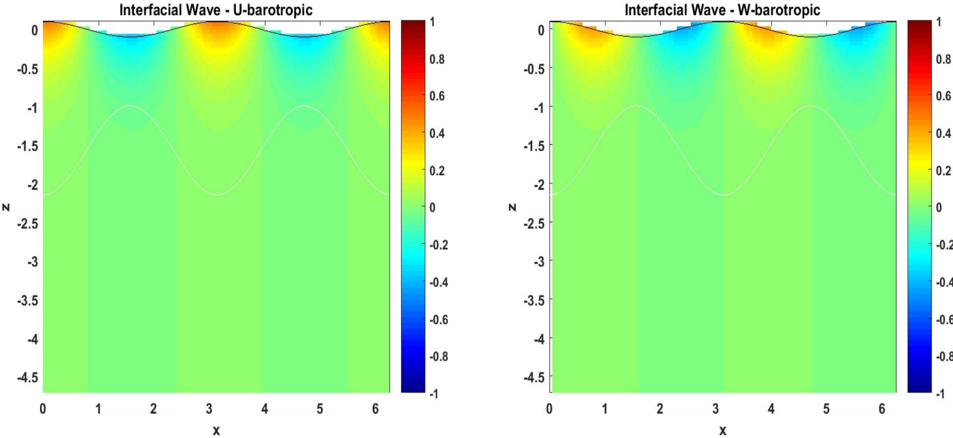
$$\omega = \sqrt{gk}$$

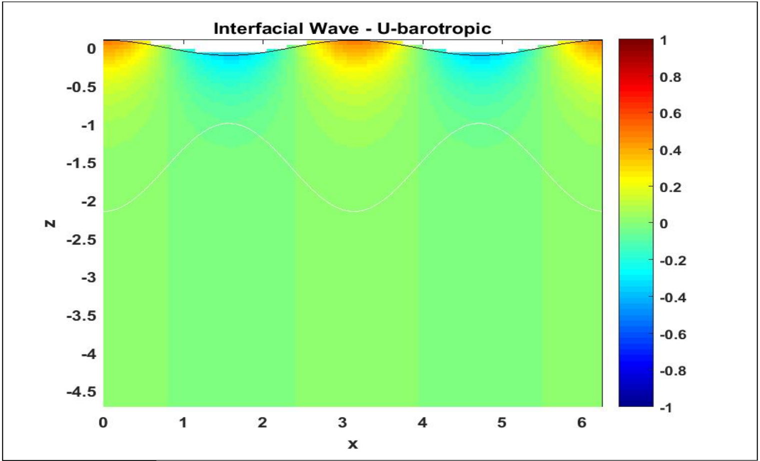
$$\text{and (15) implies} \quad b = ae^{-kH}$$

Important:

- The barotropic mode behaves like deep water waves
- The amplitude at the interface is smaller than that at the surface by the factor e^{-kH}
- The motions of the free surface and interface are locked in phase. They move up and down simultaneously.
- The barotropic mode is similar to surface waves propagating in the upper layer fluid.

Barotropic Mode




Interfacial Wave - U-barotropic
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
z
0
1
2
3
4
5
6
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Play/Pause

From (22), the second root is given by

$$\frac{\omega^2}{gk}\left[\rho_1 \sinh(kH) + \rho_2 \cosh(kH)\right] - (\rho_2 - \rho_1)\sinh(kH) = 0$$

$$\Longrightarrow \boxed{\omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh(kH)}{\rho_1 \sinh(kH) + \rho_2 \cosh(kH)}} \qquad (25)$$

We can determine the relation between a and b by substituting (25) into (21). From (25) we have

$$\frac{gk}{\omega^2} = \frac{\rho_2 \cosh kH + \rho_1 \sinh kH}{(\rho_2 - \rho_1)\sinh kH},$$

and so

$$1 - \frac{gk}{\omega^2} = 1 - \frac{\sinh kH + (\rho_2/\rho_1)\cosh kH}{[(\rho_2 - \rho_1)/\rho_1]\sinh kH} \qquad (26)$$

$$1+\frac{gk}{\omega^2}=1+\frac{\sinh kH+(\rho_2/\rho_1)\cosh kH}{[(\rho_2-\rho_1)/\rho_1]\sinh kH}.$$

Substituting the last two equations into (21) and after some algebra gives

$$b=-a\left(\frac{\rho_1}{\rho_2-\rho_1}\right)(\cosh kH+\sinh kH)=-a\left(\frac{\rho_1}{\rho_2-\rho_1}\right)e^{kH}$$

$$\implies a=-b\left(\frac{\rho_2-\rho_1}{\rho_1}\right)e^{-kH}. \quad (27)$$

From (8) and (9), we have

$$\eta=\frac{a}{b}\zeta$$

$$\boxed{\therefore \eta=-\zeta\left(\frac{\rho_2-\rho_1}{\rho_1}\right)e^{-kH}.} \quad (28)$$

$$\eta = -\zeta \left(\frac{\rho_2 - \rho_1}{\rho_1} \right) e^{-kH}$$

Remark

- Equation (28) shows that η and ζ have opposite signs, and if the density difference is small, the interface displacement is much larger than the surface displacement.
- This is one reason why it is important to study internal waves because under nonlinear conditions, the waves can break and mix the interior of the ocean thereby changing the local stratification.
- Secondly, large amplitude internal waves carry more energy as they propagate away from where they are generated.

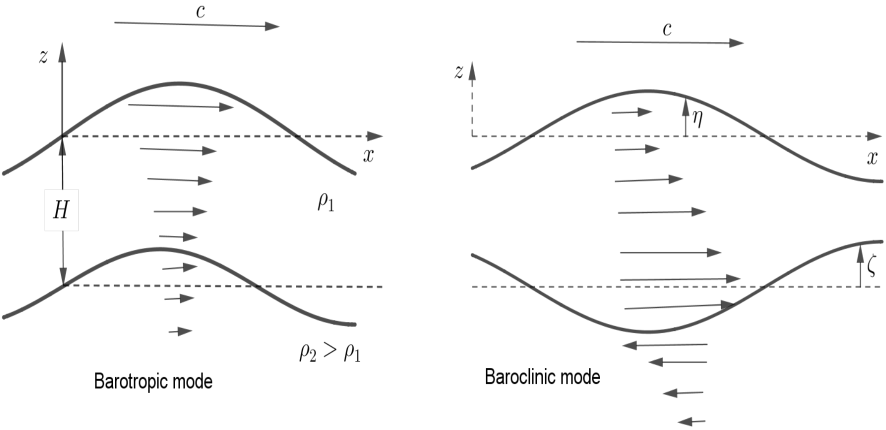
Interfacial Waves: Baroclinic or Internal mode

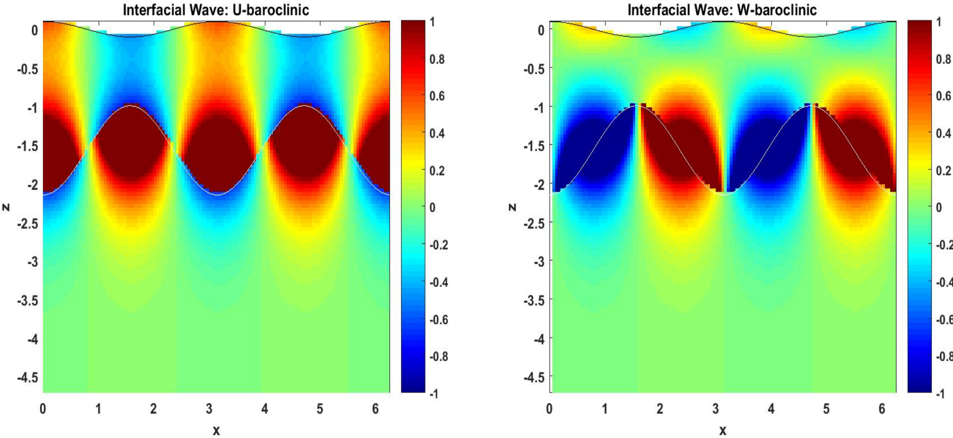
$$\eta = -\zeta \left(\frac{\rho_2 - \rho_1}{\rho_1} \right) e^{-kH}$$

Remark

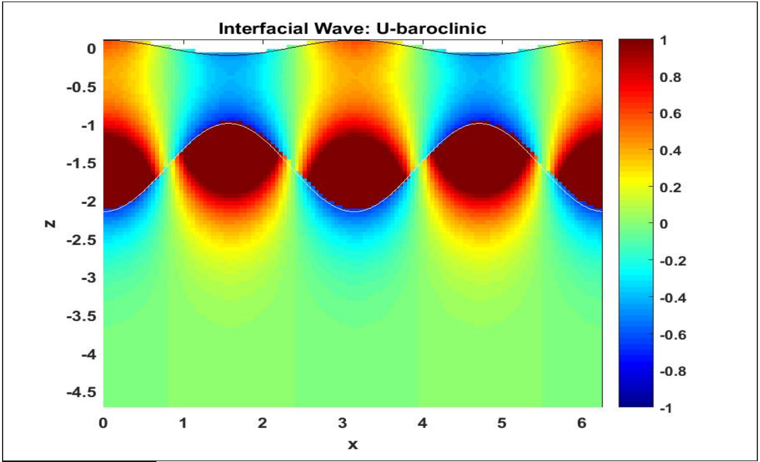
- It can also be shown that the horizontal velocity changes sign across the interface (see Figure on next slide).
- The analysis above shows that the presence of a density difference generates a motion that is quite different from the barotropic behaviour.

Barotropic vs Baroclinic Mode

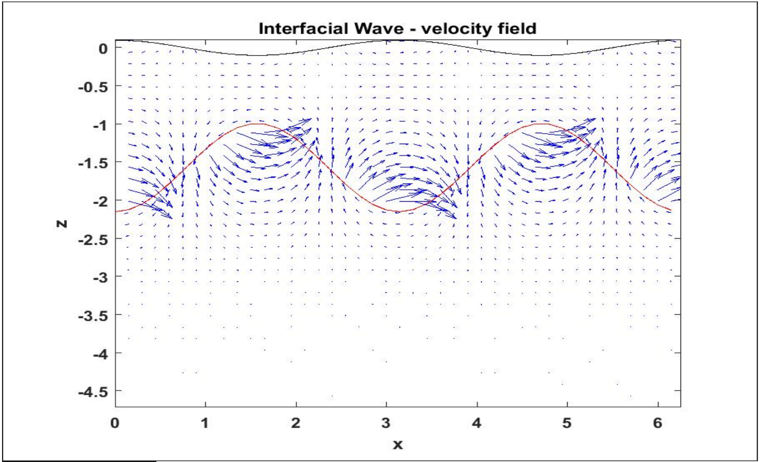



Interfacial Wave: U-baroclinic
Interfacial Wave: W-baroclinic
z
x
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
0 1 2 3 4 5 6
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Baroclinic Mode



Play/Pause


Interfacial Wave - velocity field
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
z
0
1
2
3
4
5
6
x

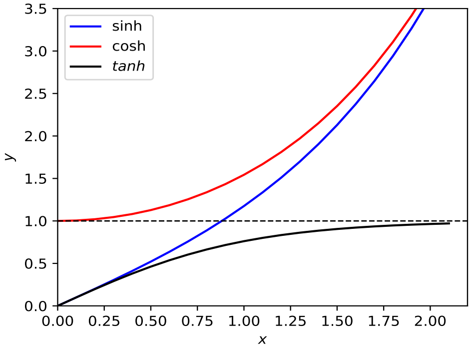
Play/Pause

Assume wavelengths are large compared to the upper layer depth.

- $H/\lambda \ll 1$ or $kH \ll 1$ (shallow water)

$$\sinh kH \to kH \quad \text{when} \quad kH \ll 1$$

$$\cosh kH \to 1 \quad \text{when} \quad kH \ll 1$$



Then the dispersion relation (25) becomes

$$\omega^2 = \frac{k^2 Hg(\rho_2 - \rho_1)}{\rho_2} = k^2 Hg'$$

where g' is called the **reduced gravity** and defined as

$$g' = \frac{g(\rho_2 - \rho_1)}{\rho_2}$$

Phase Speed & Displacements

Thus, we get

$$c = \sqrt{g'H}$$

$$\eta = -\zeta \left(\frac{\rho_2 - \rho_1}{\rho_1} \right)$$

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3. Gerkema, T. & Zimmerman, J.T.F., (2008), "An introduction to internal waves", Lecture notes, Royal NIOZ, Texel