# Introduction to Internal Gravity Waves: Waves in continuously stratified fluid

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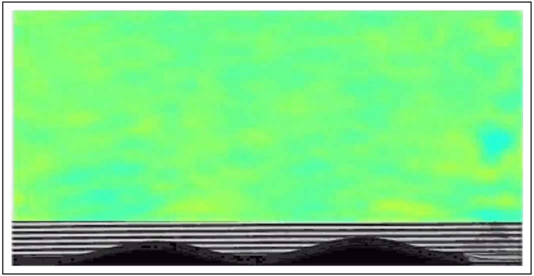
# OUTLINE

1. Introduction & Motivation
2. Mathematical Approach:
   - Recap: Surface Gravity Waves
   - Interfacial Waves
   - Internal Waves in Continuously Stratified Environment
3. Modeling Internal Tides
4. Highlights of Recent Research Efforts

# RECALL: generation mechanisms



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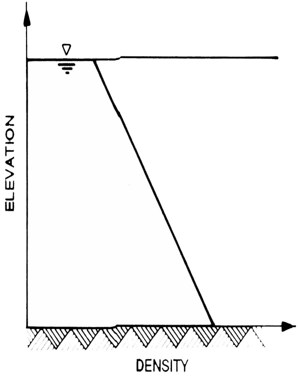
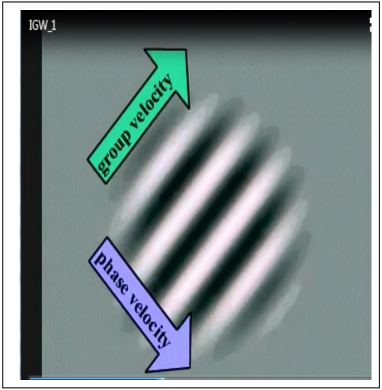
Generation in a laboratory tank by sinusoidal hills (Prof. Bruce Sutherland)

# Introduction: Continuously stratified fluid

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$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial y} \tag{2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_*} \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{D\rho}{Dt} = -w\frac{d\rho_0}{dz} \tag{5}$$

$f$ is the Coriolis frequency due to rotation of the Earth.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Considering small amplitude motions results in

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x} \quad (6)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial y} \quad (7)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_*} \quad (8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (9)$$

$$\frac{\partial \rho}{\partial t} = -w\frac{d\rho_0}{dz} \quad (10)$$

# Internal Gravity Waves: Governing Equations

To obtain equations governing the propagation of internal gravity waves, we assume that the flow is superimposed on a "background" state (i.e. it is a perturbation of a known static background) with only vertical dependences such that

$$p = p_0(z) + p'(x, y, z, t) \quad (11)$$

$$\rho = \rho_0(z) + \rho'(x, y, z, t) \quad (12)$$

and the static background density and pressure are in hydrostatic balance:

$$\frac{dp_0}{dz} = -g\rho_0 \quad (13)$$

Substituting (11)-(12) into (6)-(10), equations (6)-(7) become

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p'}{\partial x} \tag{14}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p'}{\partial y} \tag{15}$$

From equation (8) we have

$$\begin{aligned}\frac{\partial w}{\partial t} &= -\frac{1}{\rho_*}\frac{dp_0}{dz} - \frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g(\rho_0 + \rho')}{\rho_*} \\ &= -\frac{1}{\rho_*}\frac{dp_0}{dz} - \frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g\rho_0}{\rho_*} - \frac{g\rho'}{\rho_*}\end{aligned}$$

applying (13) results in

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} - \frac{g\rho'}{\rho_*} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} + b \tag{16}$$

where

$$b = -\frac{g\rho'}{\rho_*} \tag{17}$$

From (10) $[\partial\rho/\partial t = -w(d\rho_0/dz]$ we have

$$\begin{aligned} \frac{\partial\rho'}{\partial t} &= -w\frac{d\rho_0}{dz} \\ \Longrightarrow \frac{\partial\rho'}{\partial t} &= -\frac{\rho_* w}{g}\frac{g}{\rho_*}\frac{d\rho_0}{dz} = \frac{\rho_* w}{g}N^2 \end{aligned}$$

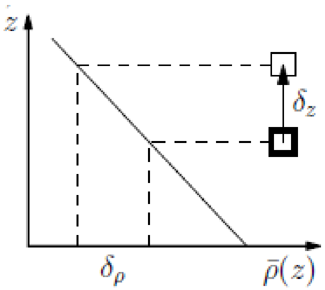
$$\boxed{\frac{\partial\rho'}{\partial t} - \frac{\rho_* w}{g}N^2 = 0} \tag{18}$$

$$\boxed{\therefore \frac{\partial b}{\partial t} + wN^2 = 0} \tag{19}$$

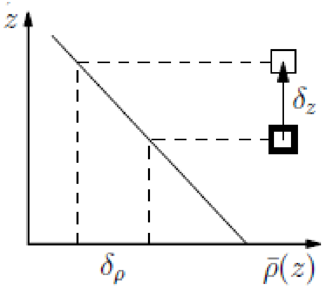
where we used (17) and the **buoyancy frequency** or **Brunt-Väisälä frequence**, $N$, is defined as

$$N^2 = -\frac{g}{\rho_*}\frac{d\rho_0}{dz}. \tag{20}$$

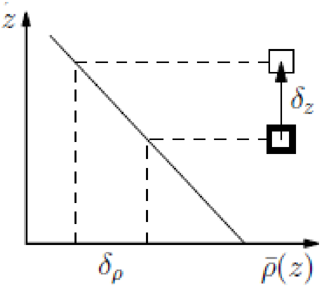
It describes the frequency that a vertically displaced fluid will oscillate at in an environment with background stratification given by $d\rho_0/dz$. This can be derived from first principles.



# Internal Gravity Waves: Buoyancy frequency




Consider fluid in a background density, $\rho_0(z)$, that is continuously decreasing with height. Consider a fluid parcel of density $\rho_* = \rho_0(z_0)$ situated initially at a vertical level $z_0$. If the parcel is displaced vertically by a small distance $\delta_z$, it will maintain its density which is different from that of the surrounding fluid (by neglecting thermodynamic effects). It therefore experiences a buoyancy force.




$$mass \times acceleration = Force_{buoyancy}$$

$$\implies \rho_* \frac{d^2 \delta_z}{dt^2} = -\delta_\rho g$$

where $\delta_\rho$ is the density difference between the fluid parcel and the surrounding fluid at its displaced position. Because the displacement $|\delta_z|$ is small, the density difference can be written in terms of $\delta_z$ to get $\delta_\rho \approx -\frac{d\rho_0}{dz}\delta_z$

Therefore, we get

$$\rho_* \frac{d^2 \delta_z}{dt^2} = g \frac{d\rho_0}{dz}\delta_z \quad \implies \frac{d^2 \delta_z}{dt^2} - \frac{g}{\rho_*}\frac{d\rho_0}{dz}\delta_z = 0.$$

This can be written as the “spring equation”:

$$\frac{d^2 \delta_z}{dt^2} + N^2 \delta_z = 0$$

# Internal Gravity Waves: Governing Equations

In summary, the equations governing linear internal wave motions in a continuously stratified ambient are given by

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p'}{\partial x} \tag{21}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p'}{\partial y} \tag{22}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} + b \tag{23}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{24}$$

$$\frac{\partial b}{\partial t} + wN^2 = 0 \tag{25}$$

Derive a single equation for the vertical velocity $w$, from (21)-(25).

Taking the derivative $\partial/\partial z$ of (22) and $\partial/\partial y$ of (23) gives

$$\frac{\partial}{\partial z}(22) \to \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}\right) + f\frac{\partial u}{\partial z} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right)$$
$$\frac{\partial}{\partial y}(23) \to \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}\right) = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial b}{\partial y}$$

Subtracting the first equation from the second gives

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - f\frac{\partial u}{\partial z} = \frac{\partial b}{\partial y}$$
$$\Longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = f\frac{\partial u}{\partial z} + \frac{\partial b}{\partial y} \qquad (26)$$

Similarly, taking $\partial/\partial z$ of (21) and $\partial/\partial x$ of (23) gives

$$\frac{\partial}{\partial z}(21) \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right) - f\frac{\partial v}{\partial z} = -\frac{1}{\rho_*}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\partial}{\partial x}(23) \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) = -\frac{1}{\rho_*}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial b}{\partial x}$$

Subtracting the second equation from the first gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - f\frac{\partial v}{\partial z} = -\frac{\partial b}{\partial x}$$

$$\Longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = f\frac{\partial v}{\partial z} - \frac{\partial b}{\partial x} \qquad (27)$$

Finally, taking $\partial/\partial y$ of (21) and $\partial/\partial x$ of (22) gives

$$\frac{\partial}{\partial y}(21) \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - f\frac{\partial v}{\partial y} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$$
$$\frac{\partial}{\partial x}(22) \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) + f\frac{\partial u}{\partial x} = -\frac{1}{\rho_*}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$$

Subtracting the second equation from the first gives

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$
$$\implies \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = f\frac{\partial w}{\partial z} \qquad (28)$$

where we used the continuity equation (24).

Next plan of attack is to eliminate the $u$ and $v$ terms. We're going to do: $\partial^2/\partial y \partial t$ of (26) and $\partial^2/\partial x \partial t$ of (27).

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = f\frac{\partial u}{\partial z} + \frac{\partial b}{\partial y} \cdots\cdots \quad [26]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = f\frac{\partial v}{\partial z} - \frac{\partial b}{\partial x} \cdots\cdots \quad [27]$$

$$\frac{\partial^2}{\partial y \partial t}(26) \rightarrow \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial z}\right] = f\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial}{\partial t}\frac{\partial^2 b}{\partial y^2} \quad (29)$$

$$\frac{\partial^2}{\partial x \partial t}(27) \rightarrow \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial^2 x^2}\right] = f\frac{\partial}{\partial t}\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial}{\partial t}\frac{\partial^2 b}{\partial x^2} \quad (30)$$

Equation (29) minus (30) gives

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial^2 y^2}-\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]=f\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial t}\left(\frac{\partial^2 b}{\partial x^2}+\frac{\partial^2 b}{\partial y^2}\right)$$

Thus,

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial^2 y^2}+\frac{\partial^2 w}{\partial z^2}\right]=f\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial t}\nabla_h^2 b$$

where we used (24) and

$$\nabla_h^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 w\right] = f\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial t}\nabla_h^2 b$$

The Coriolis term can be re-written using (28) and the last (buoyancy) term using (25).

$$\frac{\partial b}{\partial t} + wN^2 = 0\cdots\cdots \quad [25]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = f\frac{\partial w}{\partial z}\cdots\cdots \quad [28]$$

$$\implies \frac{\partial^2}{\partial t^2}\nabla^2 w + f\frac{\partial}{\partial z}\left(f\frac{\partial w}{\partial z}\right) + \nabla_h^2 wN^2 = 0$$

$$\therefore \boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_h^2 w = 0}, \qquad (31)$$

To solve (31), we need to specify boundary conditions at the surface and bottom of our domain. The simplest surface condition employs the so-called rigid-lid approximation in which the vertical velocity is taken to be zero at the mean level $z = 0$:

$$\boxed{w = 0 \quad \text{at} \quad z = 0.} \tag{32}$$

The simplest bottom boundary condition is obtained by assuming a flat bottom, with water depth $H$, such that

$$\boxed{w = 0 \quad \text{at} \quad z = -H.} \tag{33}$$

Details of the rigid-lid approximation and the bottom boundary condition are in the notes.

# Summary: Internal Gravity Waves

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_h^2 w = 0$$
$$w = 0 \quad \text{at} \quad z = 0$$
$$w = 0 \quad \text{at} \quad z = -H$$

WE WILL BE BACK!!

Basic Properties of Internal Waves from the dispersion relation:

- Direction of propagation
- Particle motion
- Phase and group velocity

# Internal Gravity Waves: Basic Properties

Consider waves propagating in the $x-$direction, taking $\partial/\partial y = 0$, and get the equation

$$\boxed{\frac{\partial^2}{\partial t^2} w_{xx} + f^2 w_{zz} + N^2 w_{xx} = 0}, \tag{34}$$

Consider a wave for which

$$w = w_0 e^{i(kx+mz-\omega t)} \tag{35}$$

where $w_0$ is the amplitude of fluctuations. Substituting (35) into (34) gives the dispersion relation

$$\omega^2 = \frac{f^2 m^2 + k^2 N^2}{k^2 + m^2}. \tag{36}$$

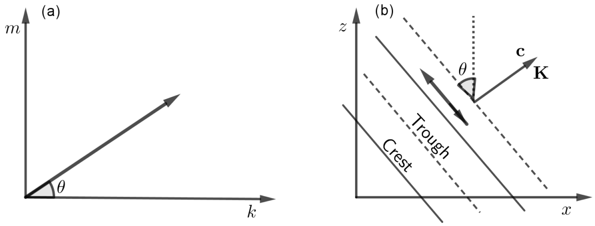
For simplicity consider waves in a non-rotating frame of reference, $f = 0$:

$$\boxed{\omega^2 = \frac{k^2}{k^2 + m^2} N^2}. \tag{37}$$

The wave frequency is taken to be positive, and the direction of propagation is determined by the wavenumber vector

$$\vec{K} = \mathbf{K} = (k,\ m) = |\mathbf{K}|(\cos\theta,\ \sin\theta), \tag{38}$$

where $\theta$ is the angle between the phase velocity vector $\mathbf{c}$ (and therefore $\mathbf{K}$) and the horizontal direction (see Figure in next slide).




From (37)

$$\omega^2 = \frac{k^2}{k^2 + m^2} N^2 = \frac{k^2}{|\vec{K}|^2} N^2$$

But $\cos\theta = \dfrac{k}{|\vec{K}|}$

$$\omega = N\cos\theta,$$

$$\boxed{\omega = N\cos\theta}, \quad (39)$$

$$\theta = \tan^{-1}(m/k), \quad -\pi/2 \leq \theta \leq \pi/2 \quad (40)$$

and $\theta$ is restricted to lie between $-\pi/2$ and $\pi/2$ so that the frequency in (39) is non-negative.

## Remark

Equation (39) shows that

- the frequency of an internal wave in a stratified fluid depends only on the *direction* of the wavenumber vector and not on the magnitude of the wavenumber. This is in contrast with surface and interfacial waves, for which frequency depends only on the magnitude.
- the frequency lies in the range $0 < \omega \leq N$. Therefore $N$ is the maximum possible frequency of internal waves in a stratified fluid.

Basic Properties of Internal Waves from the dispersion relation:

Direction of propagation

Particle motion

Phase and group velocity

Suppose the fluid motion is given by (35),

$$w = w_0 e^{i(kx+mz-\omega t)} \quad \text{and} \quad u = u_0 e^{i(kx+mz-\omega t)}$$

such that

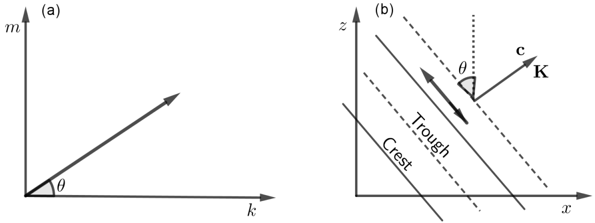
$$\frac{\partial w}{\partial z} = imw_0 e^{i(kx+mz-\omega t)} = imw \quad \text{and} \quad \frac{\partial u}{\partial x} = iku$$

Thus, from the continuity equation we have

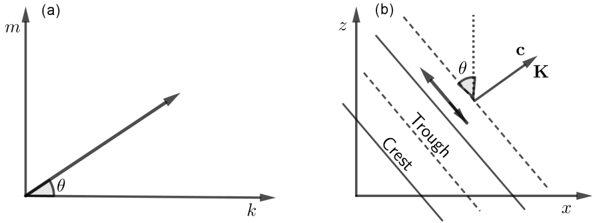
$$iku + imw = 0 \quad \Longrightarrow ku + mw = 0$$

$$\boxed{\mathbf{K} \cdot \mathbf{u} = 0} \qquad (41)$$

where $\mathbf{u} = (u, w)$, showing that the particle motion is perpendicular to the wavenumber vector (see Figure in next slide; in physical space).




- Fluid motion is parallel to lines of constant phase lines.
- So the angle $\theta$ in (39) can now be interpreted as the angle between the particle motion and the vertical direction (see Figure on right). In real space, it is measured counterclockwise from the vertical.
- In general, the sign of $\theta$ is determined by the sign of the ratio $m/k$ (see equation 40); for a wave propagating to the right ($k > 0$), $\theta$ is positive if crests move upward ($m > 0$) and negative if crests move downward ($m < 0$).




The angle $\theta$ is also a measure of the wave frequency relative to the buoyancy frequency (39). So if the frequency is known, we can compute

$$\theta = \pm \cos^{-1}(\omega/N), \tag{42}$$

where the sign is determined by the sign of $m/k$.

## Remark (**Limiting Cases**)

- The maximum frequency $\omega = N$ occurs when $\theta = 0$, that is when the particles move up and down vertically. This happens for $m = 0$ (see equation 37), showing that the motion is independent of the $z-$coordinate. So the fastest frequency waves have infinitely large vertical wave lengths (since $\lambda_m = 2\pi/m$), with lines of constant phase lying parallel to the z-axis.
- The frequency $\omega = 0$ for $\theta = \pi/2$, that is when the particle motion is purely horizontal. So waves with nearly zero frequency have crests that lie almost parallel to the horizontal axis.

Basic Properties of Internal Waves from the dispersion relation:

- Direction of propagation
- Particle motion
- Phase and group velocity

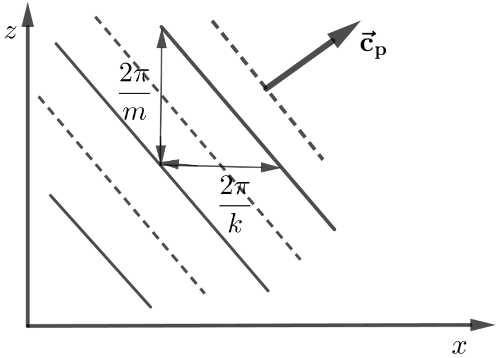
For a one-dimensional wave having structure in the x-direction alone, the phase speed is defined by

$$c_p = \frac{\omega}{k} = \frac{\lambda}{T}$$

which means the crest moves one wavelength $\lambda$ in the time of one wave period $T$. For waves having structure in two or three dimensions, the phase velocity can be described by imagining setting on a crest and moving in the direction of the wavenumber vector. Thus, the phase velocity is defined by

$$\mathbf{c} = \frac{\omega}{|\vec{k}|}\hat{\mathbf{k}} = \frac{\omega}{|\vec{k}|^2}\vec{k} = \frac{\omega}{|\vec{k}|^2}(k,\ m) \tag{43}$$

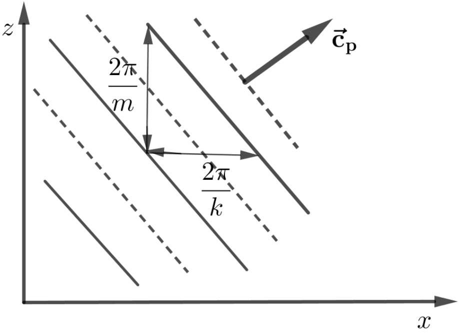
where $\hat{\mathbf{k}} = \vec{k}/|\vec{k}|$ is the unit vector in the direction of $\vec{k}$, (see Figure in next slide).




In two dimensions, the group velocity $c_g$ is defined as

$$\mathbf{c_g} = \frac{\partial \omega}{\partial k}\mathbf{i}_x + \frac{\partial \omega}{\partial m}\mathbf{i}_z = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right) \tag{44}$$

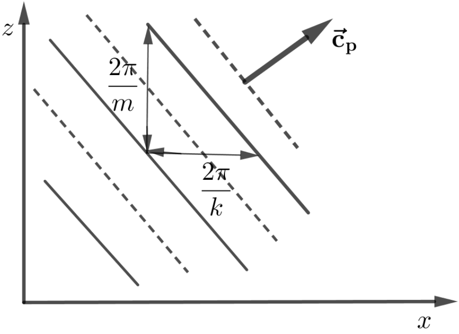
where $\mathbf{i}_x$ and $\mathbf{i}_z$ are unit vectors in the $x$ and $z$ directions respectively. From the dispersion relation (37) we have




$$\frac{\partial \omega}{\partial k} = \frac{Nm^2}{(k^2+m^2)^{3/2}} = \frac{Nm^2}{|\vec{k}|^3}$$

$$\frac{\partial \omega}{\partial m} = \frac{-Nmk}{(k^2+m^2)^{3/2}} = \frac{-Nmk}{|\vec{k}|^3}$$

$$\therefore \mathbf{c}_g = \frac{Nm}{|\vec{k}|^3}(m, -k) \tag{45}$$




Thus, from (43) and (45):

$$\mathbf{c} = \frac{\omega}{|\vec{k}|^2}(k, m), \quad \mathbf{c}_g = \frac{Nm}{|\vec{k}|^3}(m, -k)$$

we get

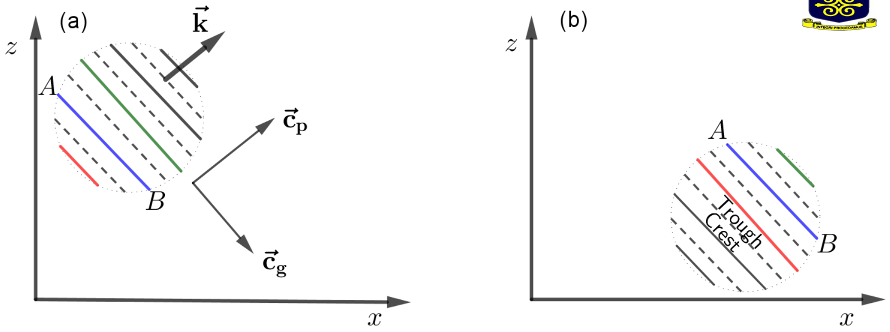
$$\boxed{\mathbf{c}_g \cdot \mathbf{c} = 0} \tag{46}$$

showing that the phase and group velocity vectors of internal waves are perpendicular. This means that while crests are moving in the direction of the wavenumber vector, the wave packet as a whole is moving in a direction that is parallel to the crests. From (38) and 39, the phase and group velocity may written in the form

$$\mathbf{c} = \frac{N}{|\vec{k}|}(\cos^2\theta, \sin\theta\cos\theta) = \frac{N\cos\theta}{|\vec{k}|}(\cos\theta, \sin\theta) \tag{47}$$

$$\mathbf{c_g} = \frac{N}{|\vec{k}|}(\sin^2\theta, -\sin\theta\cos\theta) = \frac{N\sin\theta}{|\vec{k}|}(\sin\theta, -\cos\theta) \tag{48}$$

This shows that the horizontal components of $\mathbf{c}$ and $\mathbf{c_g}$ are in the same direction, while their vertical components are opposite. Thus, if crests move upward the wavepacket as a whole moves downward and vice versa, as depicted in Figure

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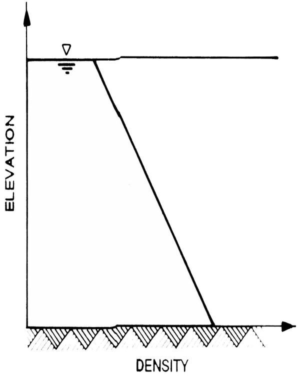


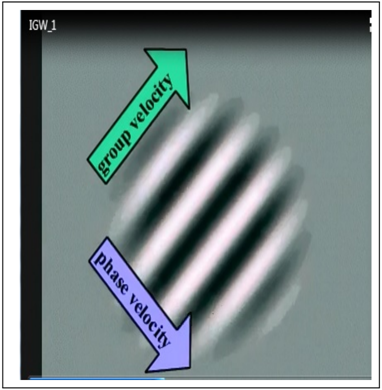


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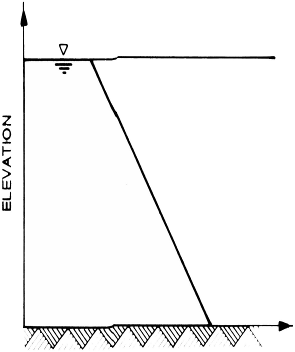
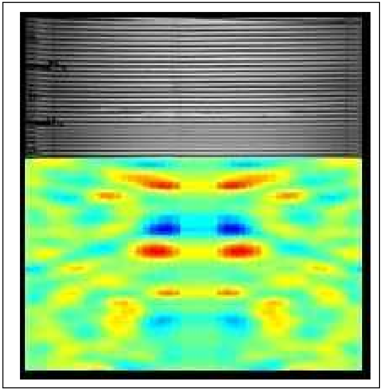
- Axisymmetric waves...
- Waves propagate down as conical beams

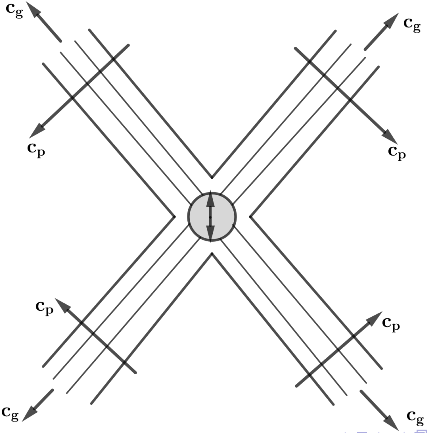
Ansong & Sutherland (2010) JFM, Vol. 648

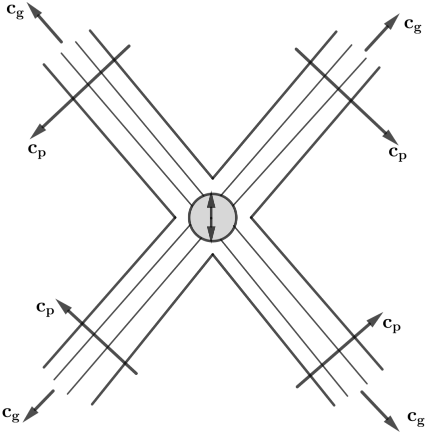


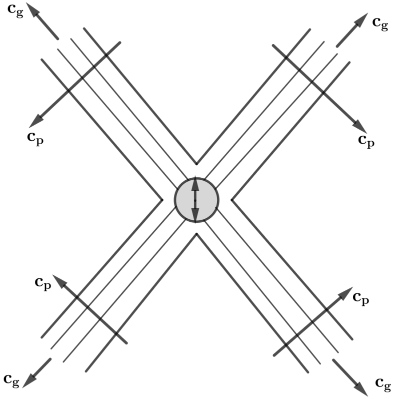
Play/Pause

The St. Andrews Cross






This was first illustrated by Mowbray & Rarity (1967) by oscillating a cylinder at frequeny $\omega$ in a tank filled with uniformly stratifed salt water. Energy radiated outward along four beams in a cross-shaped pattern: 'St. Andrews Cross', because of its resemblance to the Scottish flag.




Flag of Scotland

# REFERENCES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2. Bruce R. Sutherland, (2010), "Internal Gravity Waves", Cambridge University Press, UK
3. Gerkema, T. & Zimmerman, J.T.F., (2008), "An introduction to internal waves", Lecture notes, Royal NIOZ, Texel