Introduction to Internal Gravity Waves: Method of vertical normal modes

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OUTLINE

1. Introduction & Motivation
2. Mathematical Approach:
   - Recap: Surface Gravity Waves
   - Interfacial Waves
   - Internal Waves in Continuously Stratified Environment
3. Modeling Internal Tides
4. Highlights of Recent Research Efforts

Internal Gravity Waves: Governing Equations

Repeat: equations governing linear internal wave motions in a continuously stratified ambient:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_*}\frac{\partial p'}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_*}\frac{\partial p'}{\partial y} \tag{2}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_*}\frac{\partial p'}{\partial z} + b \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial b}{\partial t} + wN^2 = 0 \tag{5}$$

Combine to form a single equation in w and...

Solve

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_h^2 w = 0}, \tag{6}$$

using the method of vertical normal modes subject to the boundary conditions:

$$w = 0 \quad \text{at} \quad z = 0. \tag{7}$$

$$w = 0 \quad \text{at} \quad z = -H. \tag{8}$$

Consider waves propagating in the $x-$direction ($\partial/\partial y = 0$) since the problem is horizontally isotropic. We seek solutions of the form

$$w = W(z)e^{i(kx-\omega t)}, \tag{9}$$

where the frequency ω is taken to be positive. Substituting into (6) gives the ordinary differential equation for W:

$$W''(z) + k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W(z) = 0 \tag{10}$$

Employing the boundary conditions (7)-(8), we get the boundary conditions for W to be

$$W = 0 \quad \text{at} \quad z = 0, -H. \tag{11}$$

Thus, (10)-(11) forms a Sturm-Liouville problem, which, for a fixed frequency ω, has an infinite number of solutions W_n (called eigenfunctions or vertical modes) with corresponding eigenvalues k_n. From the governing equations (1)-(5), the other variables (u, v, p, b) with their $W(z)$ counterparts $(U(z), V(z), P(z), B(z))$ can be written in terms of W:

$$U = \frac{i}{k}W'; \quad V = \frac{f}{\omega k}W'; \quad P = i\rho_* \frac{\omega^2 - f^2}{\omega k^2}W'; \quad B = -\frac{iN^2}{\omega}W \tag{12}$$

Equations (10) and (11) imply that the vertically integrated horizontal velocities are zero:

$$\int_{-H}^{0} udz = 0; \quad \int_{-H}^{0} vdz = 0. \tag{13}$$

This feature is a distinguishing property of internal gravity waves compared to surface waves.

The general solution of w is given by a superposition of wave components such that

$$w = \sum_n W_n(z) \left[a_n^{\pm} \exp i(k_n^{\pm} x - \omega t) \right] \qquad (14)$$

where $a_n^{\pm}$ are arbitrary complex constants, and the $+$ and $-$ superscripts describe rightward and leftward propagating waves respectively. As usual, the real part of (13) is meant.

From (10), let

$$m^2 = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}. \tag{15}$$

Solutions to (10) may exhibit two kinds of behaviour, depending on the sign of m^2. In the parts of the water column where m is real ($m^2 \geq 0$), the waves are oscillatory. This then implies that one of the following inequalities must hold throughout this part of the water column:

$$N(z) \leq \omega \leq |f| \quad \text{or} \quad |f| \leq \omega \leq N(z).$$

$$N(z) \leq \omega \leq |f| \quad \text{or} \quad |f| \leq \omega \leq N(z). \tag{16}$$

In the ocean and atmosphere, the most common situation is $N > |f|$. The second condition is therefore consistent with our previous analysis in which $f = 0$. Thus, in the presence of rotation the internal wave frequency is additionally bounded below (or above) by $|f|$.

In the case when $m^2 < 0$ (i.e. outside the intervals in (16)), we get the so-called *evernescent waves*: an exponential-like decay of waves away from the source; there is a rapid decrease of the wave-amplitude.

It can be shown that the eigenfunctions in the Sturm-Liouville problem are orthogonal to each other. Let W_n be an eigenfunction of (10) with corresponding eigenvalue K_n such that

$$W_n''(z) + k_n^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W_n(z) = 0 \tag{17}$$

Suppose W_l is another eigenfunction with eigenvalue $k_l \neq k_n$. Multiply W_n'' by W_l, and integrate by part twice to get

$$\int_{-H}^{0} W_l W_n'' dz = W_l W_n' \Big|_{-H}^{0} - \int_{-H}^{0} W_l' W_n' dz$$

$$= -W_l' W_n \Big|_{-H}^{0} + \int_{-H}^{0} W_l'' W_n dz$$

$$\implies \int_{-H}^{0} W_l W_n'' dz = \int_{-H}^{0} W_l'' W_n dz$$

Thus, multiplying (17) by W_l and integrating yields

$$\int_{-H}^{0} W_l'' W_n dz + k_n^2 \int_{-H}^{0} \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W_n W_l dz = 0$$

Since W_l with corresponding k_l satisfies (17), we let

$$W_l'' = -k_l^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W_l$$

in the previous equation to get

$$(k_n^2 - k_l^2) \int_{-H}^{0} \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W_n W_l dz = 0$$

By assumption $k_l \neq k_n$ so we have

$$\boxed{\int_{-H}^{0} \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W_n W_l dz = 0} \qquad (18)$$

Solution Approach

$$W''(z) + k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2} W(z) = 0$$

- For special choices of N, say N =constant (uniform stratification), the equation may be solved analytically by the method of characteristics (which makes no assumptions about the boundaries) or the method of vertical normal modes (which requires the boundaries to be flat).
- For a general $N(z)$ profile with non-flat boundaries, the equation has to be solved numerically.

Solution: Uniform Stratification

The problem is to determine the eigvenvalues k_n and their relation to the frequency ω (i.e. the dispersion relation), and also determine the vertical structure of the eigenfunctions or modes, W_n. For N =constant, the equation and boundary conditions become

$$W''(z) + m^2 W(z) = 0$$
$$W = 0 \quad \text{at} \quad z = 0, -H$$

where

$$m^2 = k^2 \frac{N^2(z) - \omega^2}{\omega^2 - f^2}.$$

For wave-like solutions we assume $N > |f|$ and emply $|f| \leq \omega \leq N$.

The general solution is given by...

$$W(z) = C_1 \sin mz + C_2 \cos mz. \tag{19}$$

We can solve (19) straightaway using the boundary conditions, but for a systematic solution procedure in other cases where N is not constant, we recast (19) together with the boundary conditions $W = 0$ at $z = 0$ and $z = -H$ in the matrix form

$$\begin{pmatrix} 0 & 1 \\ -\sin mH & \cos mH \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{20}$$

For non-trivial solutions for the pair (C_1, C_2), the determinant of the matrix in (20) must be zero. Thus $\sin mH = 0$ so that

$$m_n = \pm\frac{n\pi}{H}, \quad \text{for} \quad n = 1, 2, 3, \cdots . \tag{21}$$

From equation (15): $m^2 = k^2 \frac{N^2(z)-\omega^2}{\omega^2-f^2}$
we get the dispersion relation

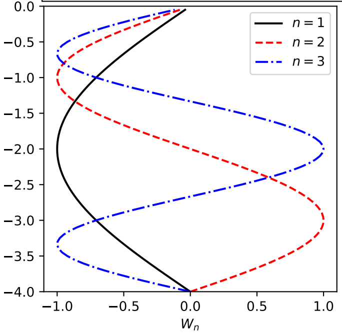
$$\boxed{k_n = \pm\frac{n\pi}{H}\left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}, \, n = 1, 2, 3, \cdots} \tag{22}$$

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2\left(\frac{n\pi}{H}\right)^2}{k^2 + \left(\frac{n\pi}{H}\right)^2}} \tag{23}$$

From (19): $W(z) = C_1 \sin mz + C_2 \cos mz$

and applying the boundary conditions ($W = 0$ at $z = 0, -H$) implies $C_2 = 0$ and C_1 is arbitrary. Let $C_1 = 1$ (i.e. normalizing the amplitude to 1). So the vertical modes become

$$\boxed{W_n(z) = \sin\left(\frac{n\pi z}{H}\right), \quad n = 1, 2, 3, \cdots} \tag{24}$$



Now consider (14):

$$w = \sum_n W_n(z) \left[a_n^{\pm} \exp i(k_n^{\pm} x - \omega t) \right]$$

We consider rightward propagating waves $k = k^{+}$, assume a_n to be real, and taking the real part gives

$$w = \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t) \tag{25}$$

From equation (12):

$$U = \frac{i}{k} W'; \quad V = \frac{f}{\omega k} W'; \quad P = i\rho_* \frac{\omega^2 - f^2}{\omega k^2} W'; \quad B = -\frac{iN^2}{\omega} W$$

we the structure of the other variables by considering the real parts of the complex expressions:

$$u = -\sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \tag{26}$$

$$v = \frac{f}{\omega} \sum_n a_n \frac{n\pi}{k_n H} \cos\left(\frac{n\pi z}{H}\right) \cos(k_n x - \omega t) \tag{27}$$

$$p = -\rho_* \frac{\omega^2 - f^2}{\omega} \sum_n a_n \frac{n\pi}{k_n^2 H} \cos\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \tag{28}$$

$$b = \frac{N^2}{\omega} \sum_n a_n \sin\left(\frac{n\pi z}{H}\right) \sin(k_n x - \omega t) \tag{29}$$

Next, we derive the isopycnal displacements (levels of constant density). Let the isopycnal at depth z_0 be represented by

$$z = z_0 + \zeta(t, x, z_0)$$

Thus, we have

Uniform Stratification: Modal Structure

$$w(t,x,z) = \frac{\partial \zeta}{\partial t}(t,x,z_0) + u(t,x,z)\frac{\partial \zeta}{\partial x}$$

Performing a Taylor expansion about $z = z_0$, and neglecting nonlinear terms, gives

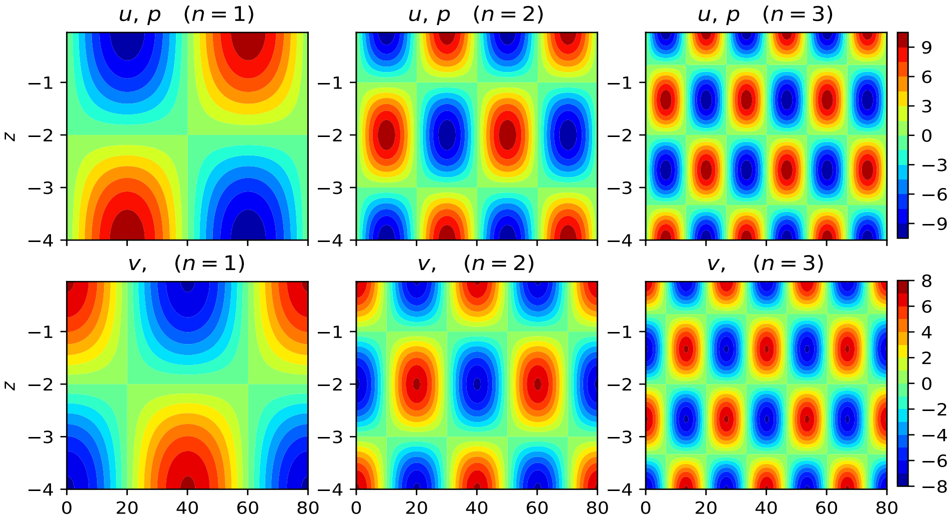
$$w(t,x,z_0) = \frac{\partial \zeta}{\partial t}(t,x,z_0)$$

Therefore, the isopycnal displacement ζ becomes

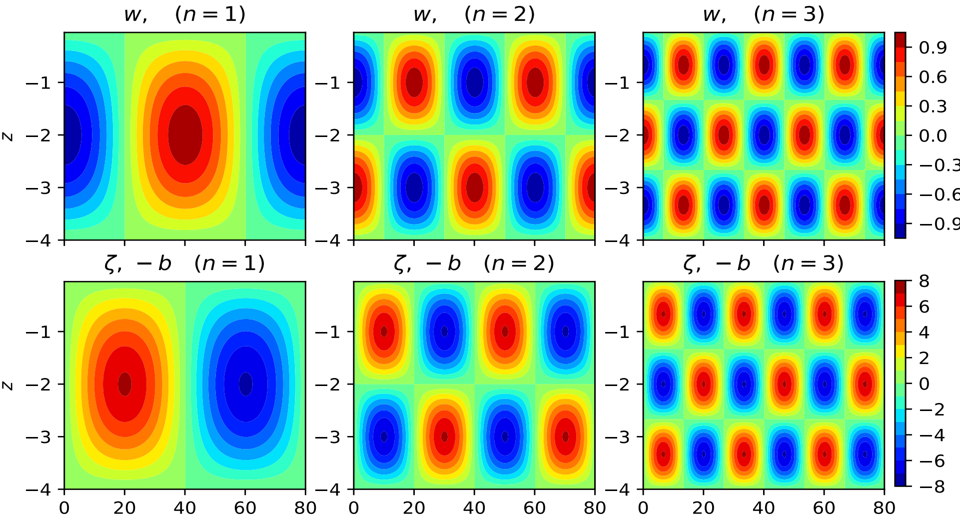
$$\zeta = -\frac{1}{\omega}\sum_n a_n \sin\left(\frac{n\pi z}{H}\right)\sin(k_n x - \omega t) \tag{30}$$

where we substitute z_0 by z. Equation (29) shows that $\zeta = -b/N^2$.

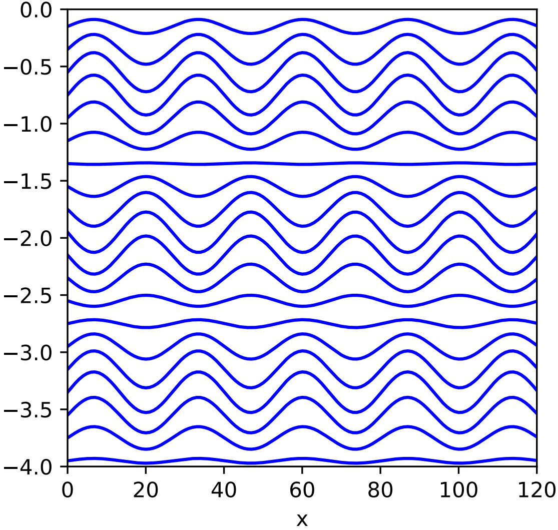
Uniform Stratification: Modal Structure



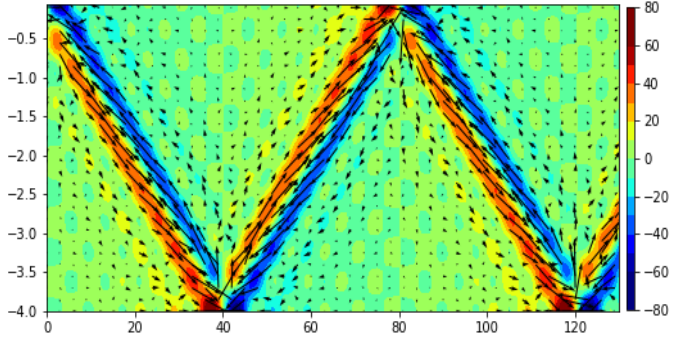
Uniform Stratification: Modal Structure



Uniform Stratification: Isopycnals for n=3



Uniform Stratification: Superpostion of modes



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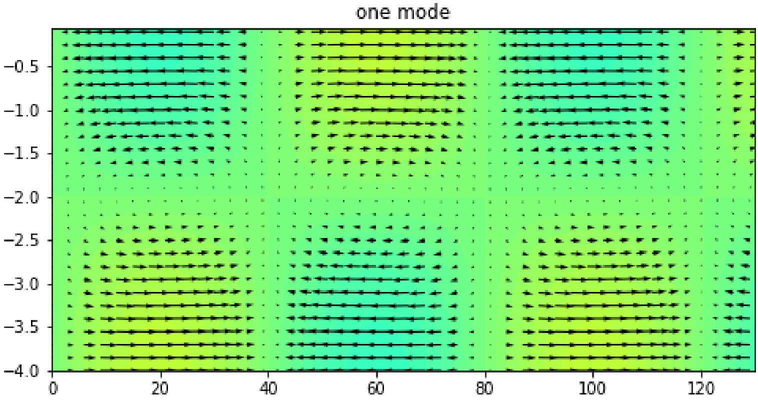
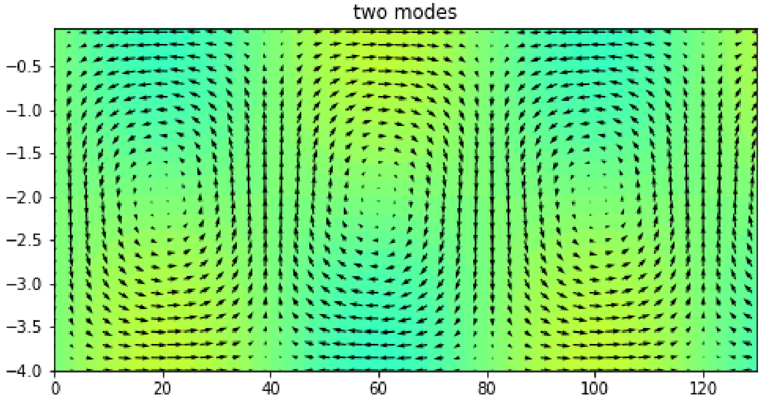
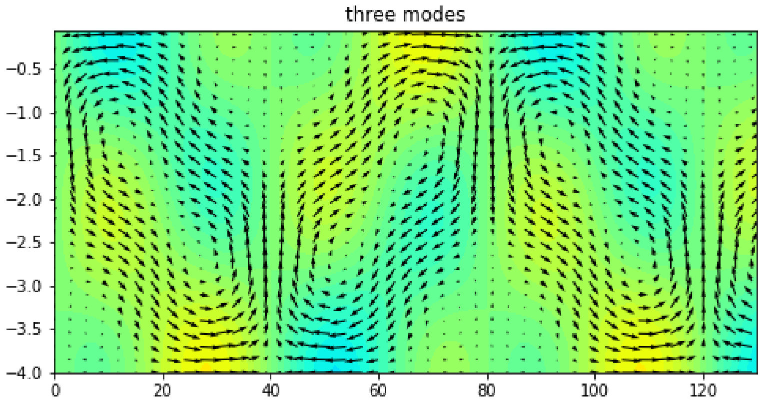
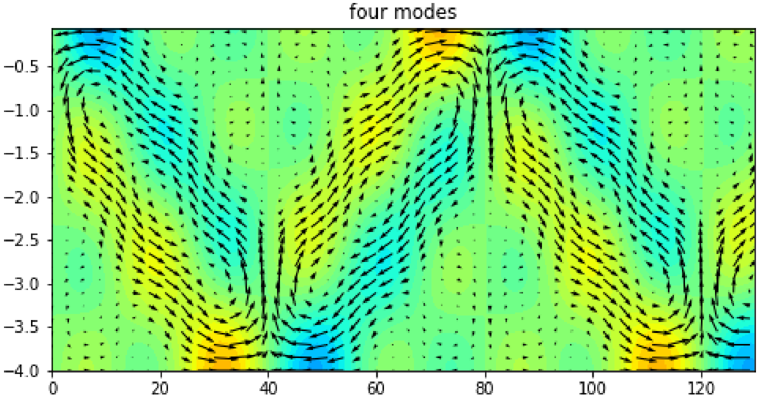
Uniform Stratification: Summing the U modal structur

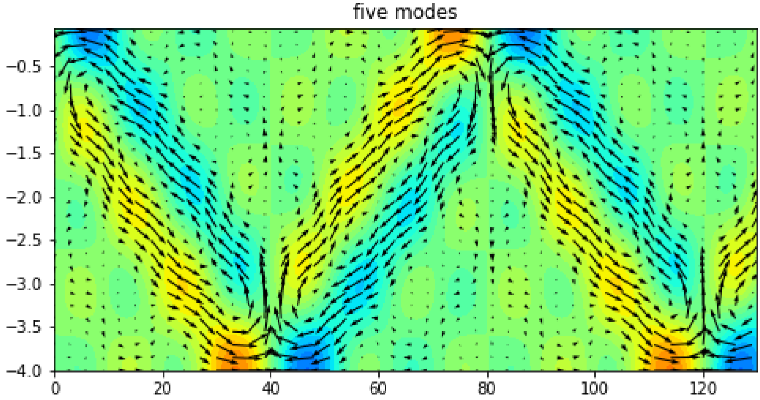



two modes
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
0
20
40
60
80
100
120

Uniform Stratification: Summing the U modal structur




four modes
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
0
20
40
60
80
100
120


five modes
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
0
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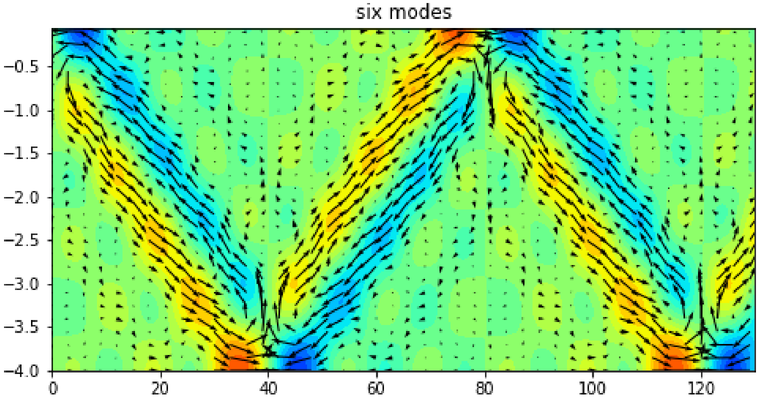
six modes
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
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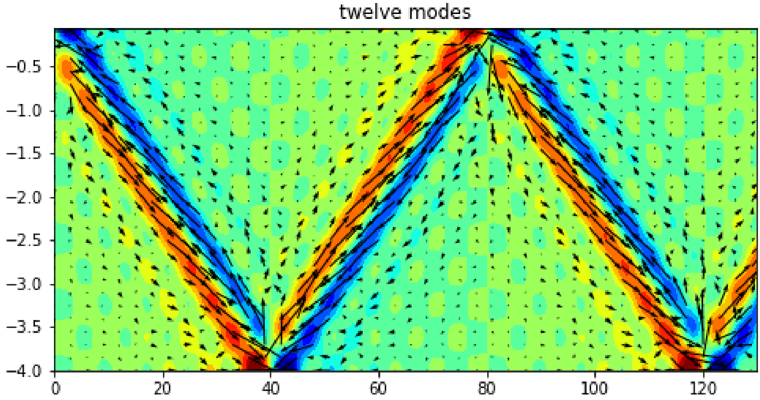
100
120

Uniform Stratification: Summing the U modal structure

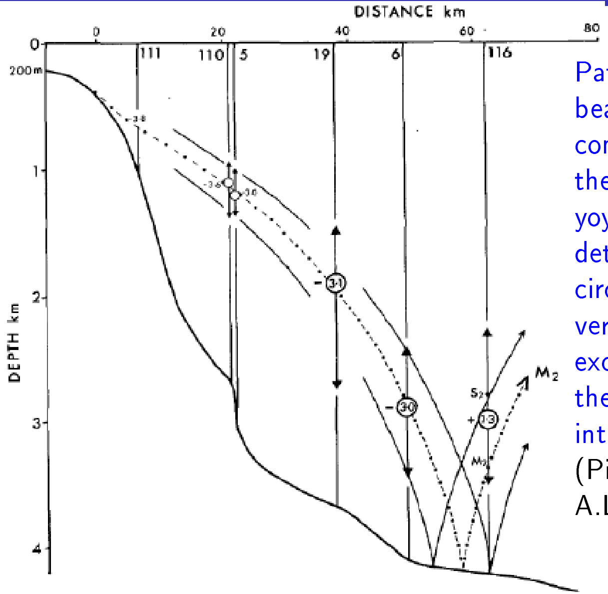


- In this case, the vertical modal structure of the solution is sinusoidal, $W_n(z) = \sin(n\pi z/H)$. We can interpret each eigenfunction as a standing wave in the vertical, i.e. a combination of up- and downward propagating waves with vertical wave number $m_n = n\pi/H$.
- From the dispersion relation, we can show that

$$\frac{m_n}{k_n} = \pm\frac{N^2 - \omega^2}{\omega^2 - f^2}$$

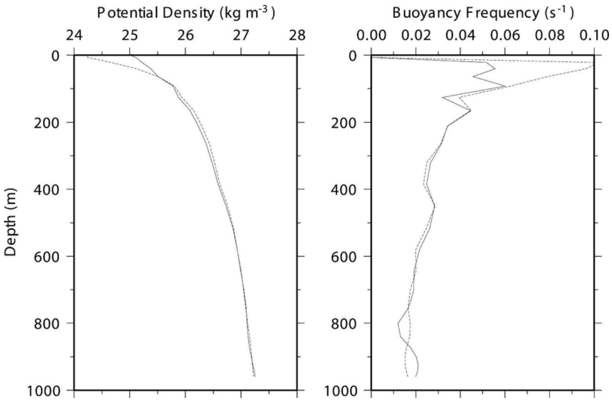
which is independent of modenumber n. Thus, one and the same angle, in the $x, z-$plane, is common for all modes.
- So it's not so surprising to find a well-defined pattern of diagonals when modes are superimposed. Note that m_n/k_n denotes the tangent of the angle that $\vec{k} = (k, m)$ makes with the horizontal.

Tidal beam in the Bay of Biscay



Path of an internal tide beam generated over the continental shelf break in the Bay of Biscay. CTD yoyoing is used to determine the depth (in circles) of maximum vertical isopycnal excursion. Depths follow theoretical path of internal tide propagation. (Pingree R.D. & New A.L. 1991)

Oceanic buoyancy frequency



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