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# Course Outline

Department of Mathematics

University of Ghana

First Semester of 2014-2015 Academic Year

COURSE CODE : MATH 221

COURSE TITLE : Algebra

CREDITS : Three(3)

COURSE INSTRUCTORS :

|              |                                                      |                                                         |
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PREREQUISITES : Students taking this course are supposed to have passed MATH 121, MATH 122 and MATH 123. Hence it is expected that they have a good grasp of trigonometry. They must also be familiar with the basics of vectors and calculus. Students are required to revise these before coming to class.

LEARNING OUTCOMES : By the end of this course, students should be able to:

1. plot points using polar coordinates.
2. convert a point from polar coordinates to Cartesian coordinates and vice versa.
3. transform an equation from polar form to Cartesian form and conversely.
4. graph polar equations.
5. do calculations involving addition, subtraction, multiplication, division and equality of complex number.

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6. represent a complex number on an Argand diagram.
  7. calculate the modulus and an argument of a complex number.
  8. convert a complex number from rectangular form to polar form.
  9. find products and quotients of complex numbers in polar form.
  10. use De Moivre's theorem.
  11. find complex roots.
  12. find roots of a polynomial equation with real coefficients.
  13. find and/or sketch the locus of a complex number.
  14. shade the region on an Argand diagram that represents an inequality.
  15. analyse a conic with vertex or center at the origin.
  16. analyse a conic with vertex or center at  $(h, k)$ .
  17. find equations of tangents and normals to a conic.
  18. use rotation of axes to transform equations.
  19. identify conics without rotation of axes.
  20. solve problems involving some special matrices.
  21. use the elementary row operations.
  22. compute the determinant of a matrix.
  23. evaluate the inverse of a matrix.
  24. solve a system of linear equations.
  25. evaluate scalar and vector products.
  26. evaluate scalar and vector triple products.
  27. do conversions involving cylindrical and spherical coordinates.
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ASSIGNMENTS AND EVALUATION: There will be five(5) exercises, each covering one of the five major divisions of the course content. Selected questions from the exercises will constitute assignments. There will be two class tests and an end-of-semester examination. Assessment will be conducted as follows:

1. Assignments and Tutorials: 10%
2. Two class tests: 20%
3. End of semester exam: 70%

INSTRUCTIONAL METHOD : In each week, there will be three hours of lectures and one hour of tutorials. During tutorials, solutions to the assignments and class tests will be provided and discussed. Students are expected to solve questions in the exercises not given as assignments as they may be called to provide such solutions on the board for class discussion. There will be bonus marks for students who actively participate in tutorials.

#### RECOMMENDED BOOKS

1. J.A. Beacher, J.A. Penna, M.L. Bittinger, *College Algebra*, Addison Wesley, 2nd ed., 2005.
2. M. Sullivan, *College Algebra*, Prentice Hall, 5th ed., 2005.
3. M. Sullivan, *Algebra and Trigonometry*, Prentice Hall, 9th ed., 2012.
4. J.K. Backhouse, S.P.T. Houldsworth, B.E.D. Cooper, *Pure Mathematics 2*, Longman, New ed., 2010.

#### COURSE CONTENTS :

##### 1. **POLAR COORDINATES**(2 WEEKS)

- (a) Plotting polar coordinates
- (b) Converting a point from polar coordinates to rectangular coordinates and vice versa
- (c) Transforming an equation from polar to rectangular form and conversely



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(d) Graphing polar equations

- i. by plotting points and making use of symmetry
- ii. by first sketching  $r = f(\theta)$  in Cartesian coordinates
- iii. by converting  $r = f(\theta)$  to a rectangular equation

(e) Tangents parallel and at right angles to the polar axis(initial line)

## 2. **COMPLEX NUMBERS**(3 WEEKS)

(a) Conjugate of a complex number

(b) Algebra of complex numbers(addition, subtraction, division, multiplication)

(c) Equality of complex numbers

(d) Argand diagram

i. Modulus of a complex number

ii. Argument of complex number

(e) Multiplication and division of complex numbers in polar form

(f) De Moivre's theorem

i. Expressing,  $\cos(n\theta)$ ,  $\sin(n\theta)$  and  $\tan(n\theta)$  in terms of  $\cos \theta$ ,  $\sin \theta$  and  $\tan \theta$

ii. Expressing  $\cos^n \theta$  or  $\sin^n \theta$  in terms of the cosines or sines of multiples of  $\theta$

(g) Exponential form of a complex number

(h) Roots of a complex number

i.  $n$ th roots of a complex number

ii. cube roots of unity

iii.  $n$ th roots of unity

(i) Polynomial equations with real coefficients

(j) Loci in the complex plane

(k) Modulus inequality

(l) Elementary transformations from the  $z$ -plane to the  $w$ -plane

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**3. CONICS SECTIONS: Parabola, Ellipse, Hyperbola, Rectangular hyperbola)(3 WEEKS)**

- (a) Conic with vertex or centre at the origin
- (b) Conic with vertex or centre at  $(h, k) \neq (0, 0)$
- (c) Tangents and normals to a conic
- (d) The general form of the equation of a conic
  - i. Identifying conics with rotation of axis
  - ii. Identifying conics without rotation of axis

**4. MATRICES AND SYSTEMS OF LINEAR EQUATIONS(3 WEEKS)**

- (a) Matrix algebra (Addition, Subtraction, Multiplication , Scalar Multiplication and equality of matrices)
- (b) Determinant of a matrix and its properties
- (c) Inverse of a matrix(using minors and cofactors)
- (d) Elementary row operations (EROS)
  - i. Row echelon form
  - ii. Reduced row echelon form
  - iii. Finding the inverse of a matrix using EROS
- (e) Solving systems of linear equations
  - i. Gaussian elimination
  - ii. Gauss - Jordan reduction(elimination)
  - iii. Using an inverse matrix
  - iv. Cramer's rule

**5. VECTORS IN 3D(2 WEEKS)**

- (a) Scalar product and its properties
- (b) Vector product and its properties

- 
- (c) Area of a parallelogram
  - (d) Scalar triple product and its properties
  - (e) Volume of a parallelepiped
  - (f) Vector triple product and its properties
  - (g) Cylindrical and spherical coordinates



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# Chapter 1

## Polar Coordinates

A point  $P$  in a plane whose rectangular(Cartesian) coordinates are  $(x, y)$  can be described in terms of polar coordinates. The polar coordinate system consists of a point  $O$  called the pole and a horizontal half-line emanating from the pole called the polar axis or initial line.

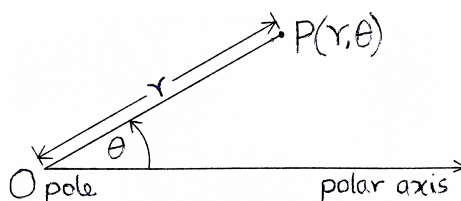


Figure 1.1: The polar coordinate system

The origin in rectangular coordinates coincide with the pole in polar coordinates and the positive  $x$ -axis in rectangular coordinates coincide with the polar axis in polar coordinates.

### Definition 1.1 (Polar coordinates)

If  $r \geq 0$  is the distance from the pole to  $P$  and  $\theta$  is the angle of inclination measured from the polar axis to the line  $OP$ , then the point  $P$  can be described by the ordered pair  $(r, \theta)$  called the polar coordinates of  $P$ .

### Remark 1.1

When  $r < 0$ , the point  $(r, \theta)$  is on the ray from the pole extending in the direction opposite the terminal side of  $\theta$  at a distance  $|r|$  units from the pole.

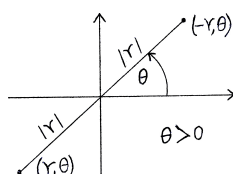


Figure 1.2: The point  $(r, \theta)$  with  $r < 0$

### Example 1.1

Plot the points with the following polar coordinates:

- (a)  $(5, 0)$                       (c)  $\left(3, -\frac{\pi}{4}\right)$                       (e)  $\left(-3, -\frac{\pi}{4}\right)$   
 (b)  $(-2, 0)$                       (d)  $\left(-3, \frac{\pi}{4}\right)$

SOLUTION

(a)

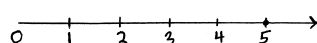


Figure 1.3: The point with polar coordinates  $(5, 0)$

(b)

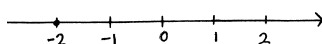


Figure 1.4: The point with polar coordinates  $(-2, 0)$

(c)

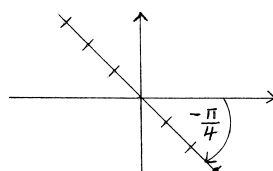


Figure 1.5: The point with polar coordinates  $\left(3, -\frac{\pi}{4}\right)$

(d)

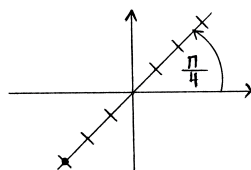


Figure 1.6: The point with polar coordinates  $\left(-3, \frac{\pi}{4}\right)$

(e)

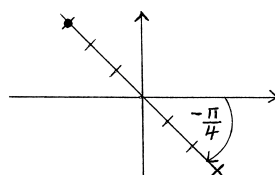


Figure 1.7: The point with polar coordinates  $\left(-3, -\frac{\pi}{4}\right)$

### Remark 1.2

- (a) The polar coordinates of the pole are  $(0, \theta)$ , where  $\theta$  can be any angle.
- (b) A point with polar coordinates  $(r, \theta)$  can also be represented by either of the following:  
 $(r, \theta + 2k\pi)$  or  $(-r, \theta + \pi + 2k\pi)$  where  $k \in \mathbb{Z}$ .

### Example 1.2

Consider the point  $\left(2, \frac{\pi}{2}\right)$ .  $P$  can also be located using the polar coordinates:  $\left(2, \frac{5\pi}{2}\right)$ ,  $\left(2, -\frac{3\pi}{2}\right)$ ,  $\left(-2, \frac{7\pi}{2}\right)$ ,  $\left(-2, -\frac{\pi}{2}\right)$ , etc

### Converting from polar coordinates to rectangular coordinates and conversely

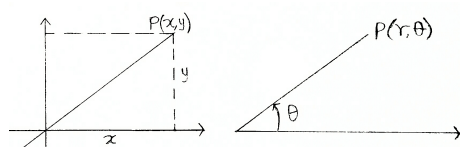


Figure 1.8: Rectangular and polar coordinate systems

**Theorem 1.1**

Let  $P$  be a point in a plane. If  $P$  has polar coordinates  $(r, \theta)$  and Cartesian coordinates  $(x, y)$ , then  $x = r \cos \theta$  and  $y = r \sin \theta$ .

PROOF

Let  $P$  be a point in a plane. Let  $(r, \theta)$  and  $(x, y)$  be the polar and Cartesian coordinates of  $P$  respectively.

Let  $r = 0$ . Then regardless of  $\theta$ , the point  $P$  is the pole, whose rectangular coordinates are  $(0, 0)$  and the theorem holds.

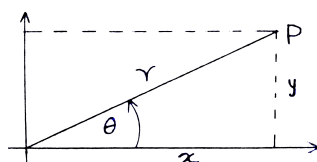


Figure 1.9: Rectangular and Polar coordinate systems together, where  $r > 0$

Let  $r > 0$ . Then the point  $P$  is on the terminal side of  $\theta$  and we have  $\cos \theta = \frac{x}{r}$  and  $\sin \theta = \frac{y}{r}$ . That is,  $x = r \cos \theta$  and  $y = r \sin \theta$ .

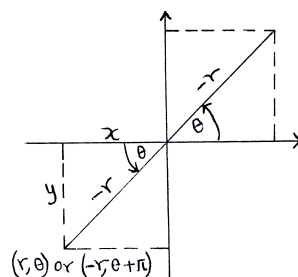


Figure 1.10: Rectangular and Polar coordinate systems together, where  $r < 0$

Let  $r < 0$ . Then  $(r, \theta)$  can be represented as  $(-r, \theta + \pi)$ , where  $-r > 0$ . We have

$$\begin{aligned} \cos(\pi + \theta) &= -\cos \theta = \frac{x}{-r} \\ \sin(\pi + \theta) &= -\sin \theta = \frac{y}{-r} \end{aligned}$$

Hence  $x = r \cos \theta$  and  $y = r \sin \theta$ .



**Example 1.3**

Find the rectangular coordinates of the points with the following polar coordinates

- (a)  $\left(6, \frac{\pi}{6}\right)$   
 (b)  $\left(-4, -\frac{\pi}{4}\right)$

SOLUTION

(a) We have that  $r = 6$  and  $\theta = \frac{\pi}{6}$ . Thus,

$$\begin{aligned} x &= r \cos \theta = 6 \cos \frac{\pi}{6} = 6 \left( \frac{\sqrt{3}}{2} \right) = 3\sqrt{3} \\ y &= r \sin \theta = 6 \sin \frac{\pi}{6} = 6 \left( \frac{1}{2} \right) = 3 \end{aligned}$$

Hence the rectangular coordinates of  $\left(6, \frac{\pi}{6}\right)$  are  $(3\sqrt{3}, 3)$ , in the first quadrant.

(b) We have that  $r = -4$  and  $\theta = -\frac{\pi}{4}$ . Thus,

$$\begin{aligned} x &= r \cos \theta = -4 \cos \left( -\frac{\pi}{4} \right) = -2\sqrt{2} \\ y &= r \sin \theta = -4 \sin \left( -\frac{\pi}{4} \right) = 2\sqrt{2} \end{aligned}$$

Hence the rectangular coordinates of  $\left(-4, -\frac{\pi}{4}\right)$  are  $(-2\sqrt{2}, 2\sqrt{2})$ , in the second quadrant.

**Corollary 1.1**

If  $P$  is a point in a plane with polar coordinates  $(r, \theta)$  and rectangular coordinates  $(x, y)$ , then

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}, x \neq 0.$$

PROOF

Let  $P$  be a point in a plane with polar coordinates  $(r, \theta)$  and Cartesian coordinates  $(x, y)$ .

Then by Theorem(1.1)  $x = r \cos \theta$  and  $y = r \sin \theta$ . Now

$$\begin{aligned} x^2 + y^2 &= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\ &= r^2 (\cos^2 \theta + \sin^2 \theta) \\ &= r^2 \end{aligned}$$

Hence  $r^2 = x^2 + y^2$ . Now, we have that

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

Hence  $\tan \theta = \frac{y}{x}$ ,  $x \neq 0$ .

### Remark 1.3

The value of  $\theta$  is usually computed such that it satisfies  $-\pi < \theta \leq \pi$  or  $0 \leq \theta < 2\pi$ . Because  $\tan$  is  $\pi$ -periodic and  $-\frac{\pi}{2} < \tan^{-1} \frac{y}{x} < \frac{\pi}{2}$ , care must be exercised in using  $\tan \theta = \frac{y}{x}$ . We have to choose  $\theta$  such that it is consistent with the quadrant in which the point  $P$  is located. This may require adding  $\pi$  to or subtracting  $\pi$  from  $\tan^{-1} \frac{y}{x}$ , when appropriate. Whenever a range is not given for  $\theta$ , the default shall be  $-\pi < \theta \leq \pi$ , which is the range usually used.

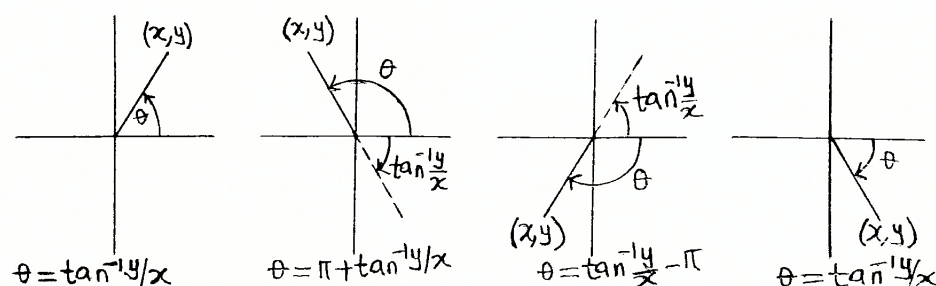


Figure 1.11: Choosing  $\theta$  consistent with a quadrant, where  $-\pi < \theta \leq \pi$ .

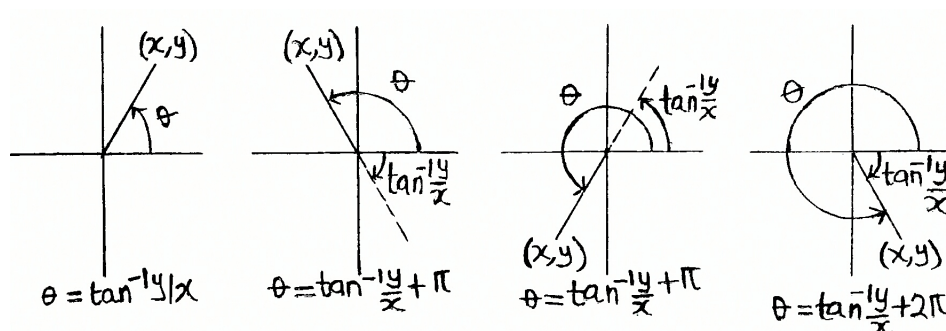


Figure 1.12: Choosing  $\theta$  consistent with a quadrant, where  $0 \leq \theta < 2\pi$ .

## CHAPTER 1. POLAR COORDINATES

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In order to convert from rectangular to polar coordinates:

1. Plot the point  $(x, y)$ .
2. If  $x = 0$  or  $y = 0$ , use your illustration to find  $(r, \theta)$ .
3. If  $x \neq 0$  and  $y \neq 0$ , then  $r = \sqrt{x^2 + y^2}$ .
4. Find  $\theta$  as described above.

### Example 1.4

Find the polar coordinates  $(r, \theta)$  of the points with the following rectangular coordinates, where  $-\pi < \theta \leq \pi$ .

- (a)  $(0, -2)$       (b)  $(-3, 0)$       (c)  $(-1, \sqrt{3})$       (d)  $(-2, 2)$

### SOLUTION

(a)

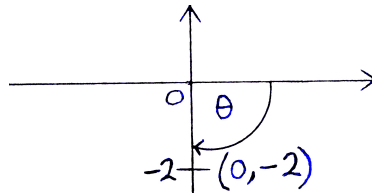


Figure 1.13:

The polar coordinates of  $(0, -2)$  are  $(2, -\frac{\pi}{2})$ .

(b)

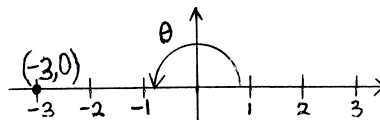


Figure 1.14:

The polar coordinates of  $(-3, 0)$  are  $(3, \pi)$ .

(c) We have that  $r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1 + 3} = 2$  and

$$\tan^{-1} \left( \frac{-\sqrt{3}}{-1} \right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

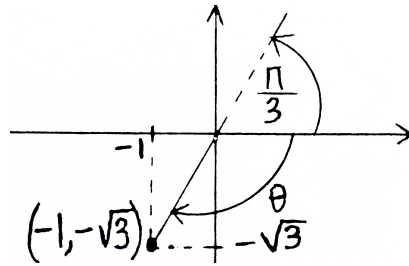


Figure 1.15:

Hence  $\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$ . The polar coordinates of  $(-1, -\sqrt{3})$  are  $(2, -\frac{2\pi}{3})$ .

(d) We have that  $r = \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = 2\sqrt{2}$  and  $\tan^{-1}\left(\frac{2}{-2}\right) = -\frac{\pi}{4}$ .

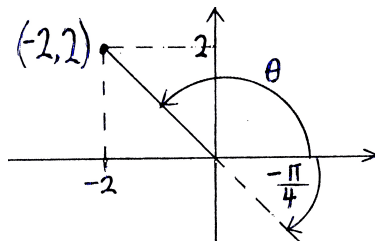


Figure 1.16:

We get  $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ . The polar coordinates of  $(-2, 2)$  are  $(2\sqrt{2}, \frac{3\pi}{4})$ .

### Example 1.5

Transform the following polar equations into Cartesian form.

(i)  $r^2 = 4 \cos 2\theta$  (ii)  $r = \frac{4}{1 - \cos \theta}$  (iii)  $r = a \tan \theta \sec \theta, a > 0$  (iv)  $r = a(1 - \cos \theta), a > 0$

### SOLUTION

(i)  $r^2 = 4 \cos 2\theta = 4(\cos^2 \theta - \sin^2 \theta)$ . Multiplying through by  $r^2$ , we have

$(r^2)^2 = 4[(r \cos \theta)^2 - (r \sin \theta)^2]$ . This yields  $(x^2 + y^2)^2 = 4(x^2 - y^2)$ .

(ii) From  $r = \frac{4}{1 - \cos \theta}$ , we get

$$r(1 - \cos \theta) = 4$$

$$r - r \cos \theta = 4$$

$$\begin{aligned}
 \sqrt{x^2 + y^2} - x &= 4 \\
 \sqrt{x^2 + y^2} &= x + 4 \\
 x^2 + y^2 &= (x + 4)^2 \\
 x^2 + y^2 &= x^2 + 8x + 16 \\
 y^2 &= 8(x + 2)
 \end{aligned}$$

(iii) We have that

$$\begin{aligned}
 r &= a \tan \theta \sec \theta \\
 &= \frac{a \sin \theta}{\cos \theta} \times \frac{1}{\cos \theta} \\
 &= \frac{a \sin \theta}{\cos^2 \theta} \\
 r \cos^2 \theta &= a \sin \theta
 \end{aligned}$$

Multiplying through by  $r$ , we obtain

$$\begin{aligned}
 (r \cos \theta)^2 &= ar \sin \theta \\
 x^2 &= ay
 \end{aligned}$$

(iv) We are given that  $r = a(1 - \cos \theta)$ . Multiplying through by  $r$ , we get

$$\begin{aligned}
 r^2 &= a(r - r \cos \theta) \\
 x^2 + y^2 &= a \left[ \sqrt{x^2 + y^2} - x \right] \\
 \left( \frac{x^2 + y^2}{a} + x \right)^2 &= x^2 + y^2
 \end{aligned}$$

### Example 1.6

Obtain the polar form of the following Cartesian equations .

(a)  $2x^2 + 2y^2 = 3$     (b)  $4x^2y = 1$     (c)  $y^2 = x(a - x)$     (d)  $x^2 + 3y^2 = 3$

SOLUTION

(a) We have that

$$\begin{aligned}
 2x^2 + 2y^2 = 2(x^2 + y^2) &= 3 \\
 2r^2 &= 3
 \end{aligned}$$

Hence  $r^2 = \frac{3}{2}$ .

(b)

$$\begin{aligned}
 4x^2y &= 1 \\
 4(r \cos \theta)^2(r \sin \theta) &= 1 \\
 4r^3 \cos^2 \theta \sin \theta &= 1 \\
 2r^3 \cos \theta (2 \cos \theta \sin \theta) &= 1 \\
 2r^3 \cos \theta \sin 2\theta &= 1 \\
 r^3 &= \frac{1}{2 \cos \theta \sin 2\theta}
 \end{aligned}$$

(c)

$$\begin{aligned}
 y^2 &= x(a - x) \\
 r^2 \sin^2 \theta &= r \cos(a - r \cos \theta) \\
 r \sin^2 \theta &= a \cos \theta - r \cos^2 \theta \\
 r(\sin^2 \theta + \cos^2 \theta) &= a \cos \theta \\
 r &= a \cos \theta
 \end{aligned}$$

(d)

$$\begin{aligned}
 x^2 + 3y^2 &= 3 \\
 r^2 \cos^2 \theta + 3r^2 \sin^2 \theta &= 3 \\
 r^2(\cos^2 \theta + 3 \sin^2 \theta) &= 3
 \end{aligned}$$

We know that  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$  and  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ , and so we get

$$\begin{aligned}
 r^2[1 + \cos 2\theta + 3(1 - \cos 2\theta)] &= 6 \\
 r^2(4 - 2 \cos 2\theta) &= 6 \\
 r^2(2 - \cos 2\theta) &= 3 \\
 r^2 &= \frac{3}{2 - \cos 2\theta}
 \end{aligned}$$

### Graphing polar equations

We shall employ the following methods:

1. By plotting points:

We may need to check for symmetry in order to reduce the number of points needed to

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## CHAPTER 1. POLAR COORDINATES

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draw the graph. We consider the following tests for symmetry:

- (i) If a polar equation is unchanged when  $\theta$  is replaced by  $-\theta$ , then the curve is symmetric about the polar axis.

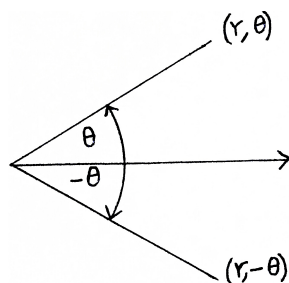


Figure 1.17: Symmetry about the polar axis.

- (ii) If the equation is unchanged when  $r$  is replaced by  $-r$  or when  $\theta$  is replaced by  $\theta + \pi$ , then the curve is symmetric about the pole. That is the curve remains unchanged if rotated through  $180^\circ$  about the origin.

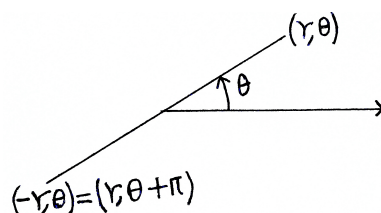


Figure 1.18: Symmetry about the pole.

- (iii) If a polar equation is unchanged when  $\theta$  is replaced by  $\pi - \theta$ , then the curve is symmetric about the vertical line  $\theta = \frac{\pi}{2}$ .

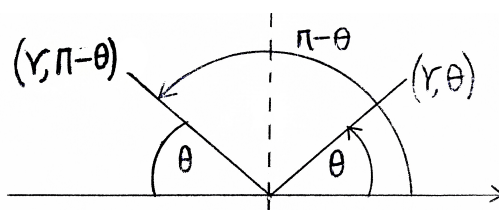


Figure 1.19: Symmetry about the line  $\theta = \frac{\pi}{2}$ .

**Remark 1.4**

The three tests for symmetry given here are sufficient conditions for symmetry, but they are not necessary conditions. In other words, an equation may fail these tests and still have graph that is symmetric with respect to the polar axis, the line  $\theta = \frac{\pi}{2}$  or the pole. For example, the graph of  $r = \sin 2\theta$  turns out to be symmetric with respect to the polar axis, the pole and the line  $\theta = \frac{\pi}{2}$ , but all three tests fail.

**Example 1.7**

Graph the equation  $r = 2 \cos 2\theta$ .

**SOLUTION**

Replacing  $\theta$  by  $-\theta$ , we have  $r = 2 \cos(2(-\theta)) = 2 \cos(-2\theta) = 2 \cos 2\theta$ . Hence the graph is symmetric about the polar axis.

Replacing  $\theta$  by  $\pi - \theta$ , we get  $r = 2 \cos[2(\pi - \theta)] = 2 \cos(2\pi - 2\theta) = 2 \cos 2\theta$ . Hence the graph is symmetric about the line  $\theta = \frac{\pi}{2}$ .

Replacing  $\theta$  by  $\theta + \pi$ , we get  $r = 2 \cos[2(\theta + \pi)] = 2 \cos(2\theta + 2\pi) = 2 \cos 2\theta$ . Hence the graph is symmetric about the pole.

| $\theta$            | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---------------------|---|-----------------|-----------------|-----------------|-----------------|
| $r = 2 \cos \theta$ | 2 | 1               | 0               | -1              | -2              |

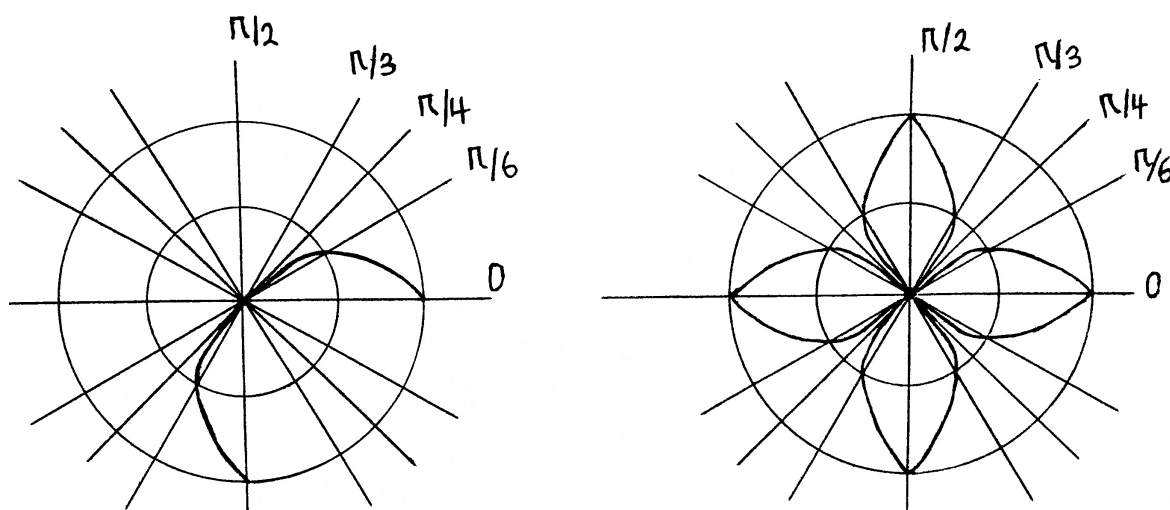


Figure 1.20: The graph of  $r = 2 \cos \theta$



**Example 1.8**

Graph the equation  $r = 1 + 2 \cos \theta$ .

**SOLUTION**

Replace  $\theta$  by  $\pi - \theta$ . Then  $r = 1 + 2 \cos(\pi - \theta) = 1 - 2 \cos \theta$ . The test fails. The graph may not be symmetric about the line  $\theta = \frac{\pi}{2}$ .

Replace  $r$  by  $-r$ . Then  $-r = 1 + 2 \cos \theta$ , that is,  $r = -1 - 2 \cos \theta$ . The test fails. Replace  $\theta$  by  $\theta + \pi$ . Then  $r = 1 + 2 \cos(\theta + \pi) = 1 - 2 \cos \theta$ . The test fails. The graph may not be symmetric about the pole.

Replace  $\theta$  by  $-\theta$ . Then  $r = 1 + 2 \cos(-\theta) = 1 + 2 \cos \theta$ . The graph is symmetric about the polar axis.

| $\theta$                | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|-------------------------|---|-----------------|-----------------|-----------------|------------------|------------------|-------|
| $r = 1 + 2 \cos \theta$ | 3 | 2.7             | 2               | 1               | 0                | -0.7             | -1    |

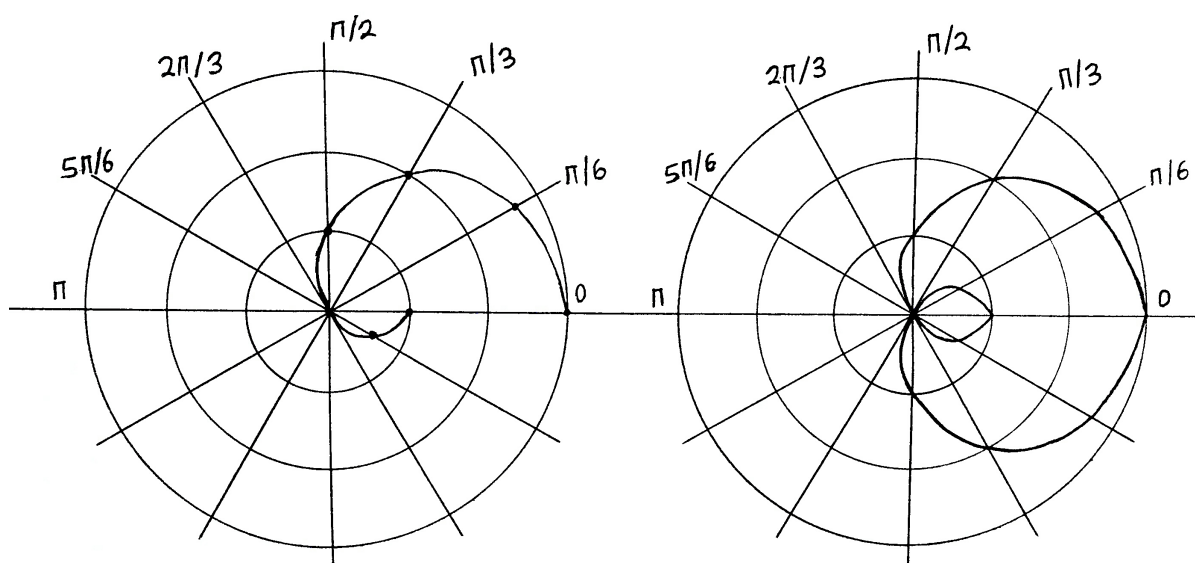


Figure 1.21: The graph of  $r = 1 + 2 \cos \theta$

2. By first sketching  $r = f(\theta)$  in Cartesian coordinates

**Example 1.9**

Graph the equation  $r = 2 \cos 2\theta$ .

SOLUTION

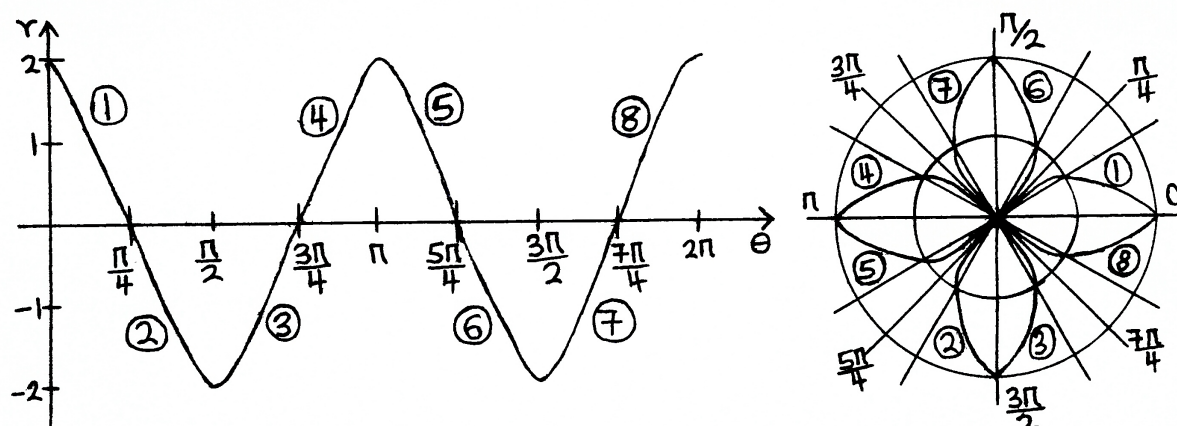


Figure 1.22: The graph of  $r = 2 \cos 2\theta$

**Example 1.10**

Sketch the graph of  $r = 1 + 2 \cos \theta$ .

SOLUTION

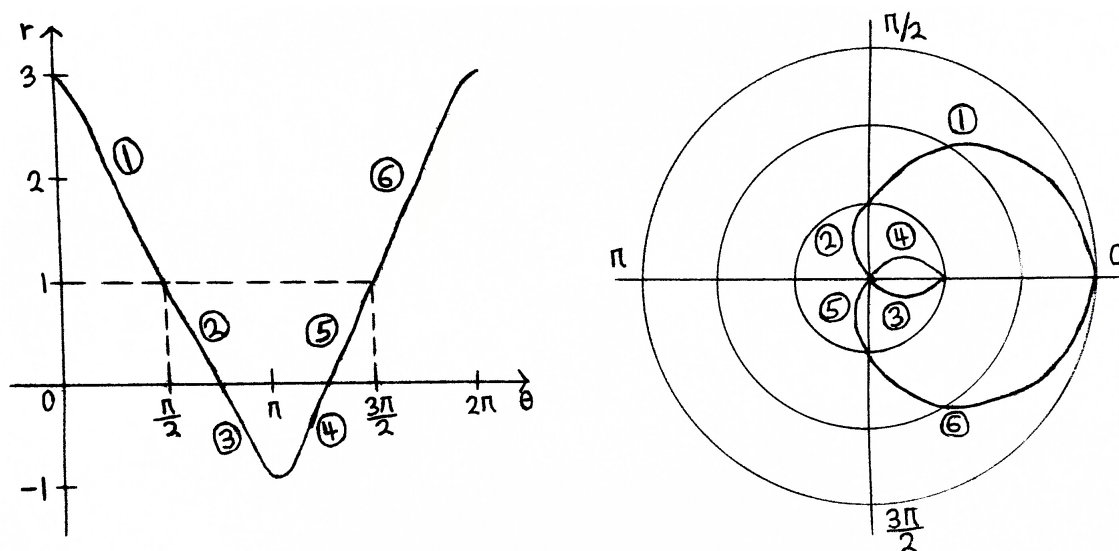


Figure 1.23: The graph of  $r = 1 + 2 \cos \theta$

3. By converting to rectangular equation

**Example 1.11**

Identify and sketch the graph of the equation  $r = 4 \sin \theta$ .

SOLUTION

$$r = 4 \sin \theta$$

$$r^2 = 4r \sin \theta$$

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + (y - 2)^2 = 4, \text{ by completing the square}$$

This is the equation of the circle with centre  $(0, 2)$  and radius 2.

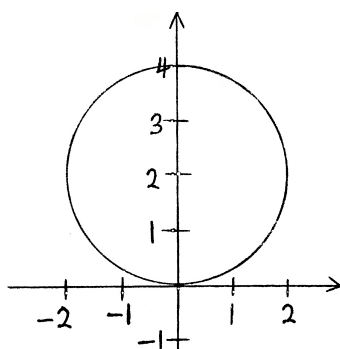


Figure 1.24:

**Example 1.12**

Identify and sketch the graph of  $\theta = \frac{\pi}{4}$ .

SOLUTION

From  $\theta = \frac{\pi}{4}$ , we have that  $\tan \theta = \tan \frac{\pi}{4} = 1$ . This gives us  $\frac{y}{x} = 1$ , that is,  $y = x$  in the first quadrant.

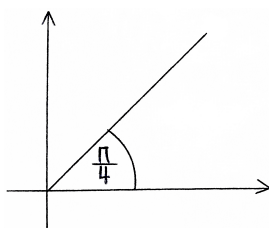


Figure 1.25:

### Common polar coordinate graphs

#### (A) Lines

1.)  $\theta = \beta$ . By converting to Cartesian coordinates, we have that  $\tan \theta = \tan \beta$ , giving us  $\frac{y}{x} = \tan \beta$ . That is  $y = (\tan \beta)x$ . This is a line that goes through the origin and makes an angle  $\beta$  with the positive  $x$ -axis, that is, a straight line through the origin with slope of  $\tan \beta$ .

2.)  $r \cos \theta = a$ . In Cartesian coordinates,  $x = a$ . This is a vertical line through  $(a, 0)$ .

3.)  $r \sin \theta = b$ . In Cartesian coordinates,  $y = b$ . This is a horizontal line through  $(0, b)$ .

#### (B) Circles

1.)  $r = a$ ,  $a > 0$ . In Cartesian coordinates  $x^2 + y^2 = a^2$ . This is a circle of radius  $a$  centered at the origin.

2.)  $r = 2a \cos \theta$ . In rectangular coordinates,  $(x - a)^2 + y^2 = a^2$ . This is a circle of radius  $|a|$  and centre  $(a, 0)$ .

3.)  $r = 2b \sin \theta$ . In rectangular coordinates  $x^2 + (y - b)^2 = b^2$ . This is a circle of radius  $|b|$  and centre  $(0, b)$ .

4.)  $r = 2a \cos \theta + 2b \sin \theta$ . In Cartesian coordinates  $(x - a)^2 + (y - b)^2 = a^2 + b^2$ . This is a circle of radius  $\sqrt{a^2 + b^2}$  and centre  $(a, b)$  [General equation of a circle].

#### (C) Cardioids and limacons

1.) Cardioids:  $r = a \pm a \cos \theta$  and  $r = a \pm a \sin \theta$  with  $a > 0$ . These have a graph that is vaguely heart shaped and always contain the origin.

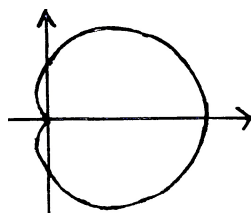


Figure 1.26:

2.) Limaçons with an inner loop:  $r = a \pm b \cos \theta$  and  $r = a \pm b \sin \theta$  with  $0 < a < b$ . These will have an inner loop and will always contain the origin. The graph in Example(1.8) is that of a limaçon with an inner loop.

3.) Limaçons without an inner loop:  $r = a \pm b \cos \theta$  and  $r = a \pm b \sin \theta$  with  $a > b > 0$ . These do not have an inner loop and do not contain the origin.

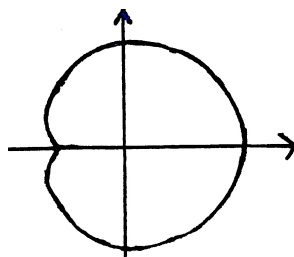


Figure 1.27:

(D) Lemniscates and Roses

1.) Lemniscates:  $r^2 = a^2 \cos 2\theta$  and  $r^2 = a^2 \sin 2\theta$  with  $a > 0$ .

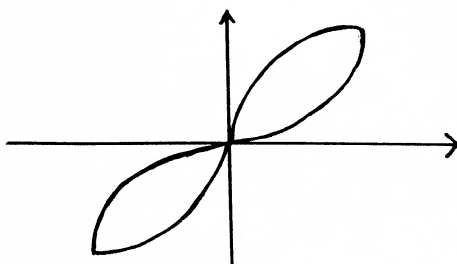


Figure 1.28:

2.) Roses:  $r = a \cos(n\theta)$ ,  $r = a \sin(n\theta)$ ,  $a \neq 0$ ,  $n \geq 2$ . If  $n$  is even, the rose has  $2n$  petals. If  $n$  is odd, the rose has  $n$  petals. In Example(1.9), the graph is a rose with 4 petals,  $n = 2$ .

### Tangents to Polar Curves

Consider polar equations of the form  $r = f(\theta)$ .  $x = r \cos \theta$  and  $y = r \sin \theta$  represent the parametric equations of the curve, with  $\theta$  as the parameter. We have that

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta.$$

Thus we get

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta} / \frac{dx}{d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

#### Example 1.13

Determine the equation of the tangent line to  $r = 3 + 8 \sin \theta$  at  $\theta = \frac{\pi}{6}$ .

#### SOLUTION

When  $\theta = \frac{\pi}{6}$ ,  $r = 3 + 8 \sin \frac{\pi}{6} = 7$ ,  $x = 7 \cos \frac{\pi}{6} = \frac{7\sqrt{3}}{2}$  and  $y = 7 \sin \frac{\pi}{6} = \frac{7}{2}$ .

$$\frac{dr}{d\theta} = 8 \cos \theta \quad \text{and} \quad \frac{dy}{dx} = \frac{8 \cos \theta \sin \theta + (3 + 8 \sin \theta) \cos \theta}{8 \cos \theta - (3 + 8 \sin \theta) \sin \theta}$$

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{6}} = \frac{8 \cos \frac{\pi}{6} \sin \frac{\pi}{6} + 7 \cos \frac{\pi}{6}}{8 \cos \frac{\pi}{6} - 7 \sin \frac{\pi}{6}} = \frac{11\sqrt{3}}{5}$$

Equation of tangent line at  $\theta = \frac{\pi}{6}$  is  $y = \frac{7}{2} + \frac{11\sqrt{3}}{5} \left( x - \frac{7\sqrt{3}}{2} \right)$ .

### Tangents parallel to and at right angles to the polar axis

Tangents to a polar curve that are parallel to the polar axis occur at points where  $y = r \sin \theta$  takes a maximum or a minimum value. That is when  $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$ , giving us  $\frac{dy}{d\theta} = \frac{d}{d\theta}(r \sin \theta) = 0$ . Tangents to the polar curve that are at right angles to the polar axis occur when  $x$  takes a maximum or a minimum value. That occurs when  $\frac{dy}{dx}$  is undefined. That is when  $\frac{dx}{d\theta} = \frac{d}{d\theta}(r \cos \theta) = 0$ .

**Example 1.14**

Find, for the cardioid with equation  $r = 2a(1 + \cos \theta)$ , those points on the curve where

- (a) tangents to the curve are parallel to the polar axis,
- (b) tangents to the curve are at right angles to the polar axis.

SOLUTION

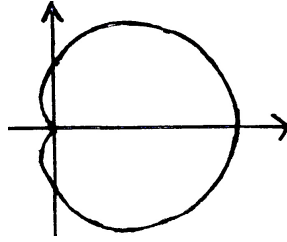


Figure 1.29:

- (a) Tangents are parallel to the polar axis when  $\frac{d}{d\theta}(r \sin \theta) = 0$ .

$$\begin{aligned} \frac{d}{d\theta}[2a(\sin \theta + \sin \theta \cos \theta)] &= 0 \\ 2a(\cos \theta + \cos^2 \theta - \sin^2 \theta) &= 0 \\ \cos \theta + \cos^2 \theta - \sin^2 \theta &= 0, \text{ since } a \neq 0 \\ 2\cos^2 \theta + \cos \theta - 1 &= 0 \\ (2\cos \theta - 1)(\cos \theta + 1) &= 0 \\ \cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = -1 \end{aligned}$$

We have that  $\theta = \pm\frac{\pi}{3}, \pi$  and  $r = 3a, 3a, 0$ . Tangents are parallel to the polar axis at  $(3a, \frac{\pi}{3})$ ,  $(3a, -\frac{\pi}{3})$  and  $(0, \pi)$ .

- (b) Tangents are at right angles to the polar axis when  $\frac{d}{d\theta}(r \cos \theta) = 0$ .

$$\begin{aligned} \frac{d}{d\theta}[2a(\cos \theta + \cos^2 \theta)] &= 0 \\ 2a(-\sin \theta - 2\cos \theta \sin \theta) &= 0 \\ \sin \theta(1 + 2\cos \theta) &= 0, \text{ since } a \neq 0 \\ \sin \theta = 0 \quad \text{or} \quad \cos \theta = -\frac{1}{2} \end{aligned}$$

We have that  $\theta = 0, \pi$  or  $\theta = \pm \frac{2\pi}{3}$  and  $r = 4a, 0$  or  $r = a, a$ . Tangents are perpendicular to the polar axis at  $(4a, 0)$ ,  $\left(a, \frac{2\pi}{3}\right)$ ,  $\left(a, -\frac{2\pi}{3}\right)$ .



Exercise One

- Plot each of the given points in polar coordinates, and find other polar coordinates  $(r, \theta)$  of each point for which:
  - $r > 0, -2\pi \leq \theta < 0$
  - $r < 0, 0 \leq \theta < 2\pi$
  - $r > 0, 2\pi \leq \theta < 4\pi$

(a)  $\left(-3, -\frac{3\pi}{4}\right)$     (b)  $(-3, \pi)$     (c)  $\left(6, \frac{\pi}{6}\right)$     (d)  $\left(2, -\frac{2\pi}{3}\right)$     (e)  $\left(1, \frac{\pi}{2}\right)$
- Find the rectangular coordinates of the points whose polar coordinates have been provide.
 

(a)  $\left(3, \frac{\pi}{2}\right)$     (b)  $\left(-2, \frac{2\pi}{3}\right)$     (c)  $\left(-1, -\frac{\pi}{3}\right)$     (d)  $(-2, 0)$     (e)  $(-3, \pi)$
- The rectangular coordinates of a point are given. Find polar coordinates  $(r, \theta)$  of each point where
  - $-\pi < \theta \leq \pi$
  - $0 \leq \theta < 2\pi$

(a)  $(-3, 3)$     (b)  $(\sqrt{3}, 1)$     (c)  $(-2, -2\sqrt{3})$     (d)  $(0, -1)$     (e)  $(1, -1)$
- Transform to polar form the equations given in Cartesian form by
  - $2x^2 + 2y^2 = 3$
  - $4x^2y = 1$
  - $(x^2 + y^2)^2 = 2a^2xy, a > 0$
- Transform to Cartesian form the equations given in polar form by
  - $r = \frac{3}{3 - \cos \theta}$
  - $r \sec \theta = 4$
  - $r = 2a \tan \theta, a > 0$
- Identify and graph the following polar equations
  - $r = 3 - 3 \sin \theta$
  - $r = 2 - \cos \theta$
  - $r = 1 + 2 \sin \theta$
  - $r^2 = 9 \cos(2\theta)$
  - $r = 4 \sin(5\theta)$
  - $r = 3 \cos(4\theta)$



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# Chapter 2

## Complex Numbers

### Definition 2.1 (Complex Numbers)

A complex number is any number of the form  $a + bi$  where  $a, b \in \mathbb{R}$  and  $i$ , called the imaginary unit, is defined by  $i^2 = -1$ .

### Definition 2.2 (Real and Imaginary parts)

Let  $z$  be a complex number such that  $z = a + bi$  where  $a, b \in \mathbb{R}$ . The real number  $a$  is called the real part of  $z$ , abbreviated  $\text{Re}(z)$ . The real number  $b$  is called the imaginary part of  $z$ , abbreviated  $\text{Im}(z)$ .

### Example 2.1

Let  $z = 3 - 7i$ . Then  $\text{Re}(z) = 3$  and  $\text{Im}(z) = -7$ .

### Remark 2.1

- (i) The notations  $a + bi$  and  $a + ib$  can be used interchangeably.
- (ii) The set of complex is denoted by  $\mathbb{C}$ . Since any real number  $x$  can be written as  $z = x + 0i$ , we see that the set of real numbers is a subset of  $\mathbb{C}$ , that is  $\mathbb{R} \subset \mathbb{C}$ .

### Definition 2.3 (Conjugate of a complex number)

If  $z$  is a complex number, then the number obtained by changing the sign of its imaginary part is called the complex conjugate, or simply the conjugate, of  $z$  and it is denoted by  $\bar{z}$ . In other words, if  $z = a + bi$ , then its conjugate is  $\bar{z} = a - bi$ .

### Example 2.2

Find  $\bar{z}$  and  $\bar{w}$  if  $z = -2 - 3i$  and  $w = 1 + i$ .

SOLUTION

$\bar{z} = -2 + 3i$  and  $\bar{w} = 1 - i$ .

**Definition 2.4 (Addition and Multiplication)**

If  $a + bi, c + di \in \mathbb{C}$ , then we add and multiply  $a + bi$  and  $c + di$  as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \text{and}$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

**Proposition 2.1**

The following properties hold in  $\mathbb{C}$ .

- (A1) Additive closure: For all  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 + z_2 \in \mathbb{C}$ .
- (A2) Additive associativity: For all  $z_1, z_2, z_3 \in \mathbb{C}$ ,  $z_1 + (z_2 + z_3) = (z_2 + z_3) + z_1$ .
- (A3) Existence of additive identity: There is a complex number  $0 = 0 + 0i$ , such that for all  $z \in \mathbb{C}$ ,  $z + 0 = 0 + z = z$ .
- (A4) Existence of additive inverses: For all  $z \in \mathbb{C}$ , there is a complex number  $-z$  such that  $z + (-z) = -z + z = 0$ .
- (A5) Additive Commutativity: For all  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 + z_2 = z_2 + z_1$ .
- (M1) Multiplicative closure: For all  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 z_2 \in \mathbb{C}$ .
- (M2) Multiplicative associativity: For all  $z_1, z_2, z_3 \in \mathbb{C}$ ,  $z_1(z_2 z_3) = (z_1 z_2)z_3$ .
- (M3) Existence of multiplicative identity: There is a complex number  $1 = 1 + 0i$ , such that for all  $z \in \mathbb{C}$ ,  $z(1) = (1)z = z$ .
- (M4) Existence of multiplicative inverses: For all  $\mathbb{C} \ni z \neq 0$ , there is a complex number  $w$  such that  $zw = wz = 1$ .
- (M5) Multiplicative Commutativity: For all  $z_1, z_2 \in \mathbb{C}$ ,  $z_1 z_2 = z_2 z_1$ .
- (D1) Left distributivity: For all  $z_1, z_2, z_3 \in \mathbb{C}$ ,  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ .
- (D2) Right distributivity: For all  $z_1, z_2, z_3 \in \mathbb{C}$ ,  $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$ .

(A1) Let  $z_1, z_2 \in \mathbb{C}$ . Then  $z_1 = a + bi$  and  $z_2 = c + di$  where  $a, b, c, d \in \mathbb{R}$ . Now,  $z_1 + z_2 = (a + c) + (b + d)i$ . Since  $a + c, b + d \in \mathbb{R}$ ,  $z_1 + z_2 \in \mathbb{C}$ . Hence  $\mathbb{C}$  is additively closed, that is,  $\mathbb{C}$  is closed under addition.

(A2) Let  $z_1, z_2, z_3 \in \mathbb{C}$ . Then  $z_1 = a + bi$ ,  $z_2 = c + di$ ,  $z_3 = e + fi$  with  $a, b, c, d, e, f \in \mathbb{R}$ .

$$\begin{aligned}
 z_1 + (z_2 + z_3) &= (a + bi) + [(c + di) + (e + fi)] \\
 &= (a + bi) + [(c + e) + (d + f)i] \\
 &= [a + (c + e)] + [b + (d + f)]i \\
 &= [(a + c) + e] + [(b + d) + f]i, \text{ using additive associativity in } \mathbb{R} \\
 &= [(a + c) + (b + d)i] + [e + fi] \\
 &= [(a + bi) + (c + di)] + [e + fi] \\
 &= (z_1 + z_2) + z_3
 \end{aligned}$$

Thus  $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ . Therefore, addition is associative in  $\mathbb{C}$ .

(M4) Let  $z \in \mathbb{C}$ . Then  $z = a + bi$  where  $a, b \in \mathbb{R}$ . Suppose that  $z \neq 0$ . Then either  $a \neq 0$  or  $b \neq 0$ , and so  $a^2 + b^2 \neq 0$ . Let  $w = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$ , then  $w$  is a well-defined complex number.

$$\begin{aligned}
 zw &= (a + bi) \left( \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i \right) \\
 &= \left( \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} \right) + \left( \frac{-ab}{a^2 + b^2} + \frac{ba}{a^2 + b^2} \right) i \\
 &= \frac{a^2 + b^2}{a^2 + b^2} + \frac{-ab + ab}{a^2 + b^2}i \\
 &= 1 + 0i = 1 \\
 wz &= \left( \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i \right) (a + bi) \\
 &= \left( \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} \right) + \left( \frac{ab}{a^2 + b^2} - \frac{ba}{a^2 + b^2} \right) i \\
 &= \frac{a^2 + b^2}{a^2 + b^2} + \frac{ab - ab}{a^2 + b^2}i \\
 &= 1 + 0 = 1
 \end{aligned}$$

Hence the multiplicative inverse of  $a + bi$  is  $\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$ .

(D1) Let  $z_1, z_2, z_3 \in \mathbb{C}$ . Then  $z_1 = a + bi$ ,  $z_2 = c + di$ ,  $z_3 = e + fi$  with  $a, b, c, d, e, f \in \mathbb{R}$ .

$$\begin{aligned}
 z_1(z_2 + z_3) &= (a + bi)[(c + di) + (e + fi)] \\
 &= (a + bi)[(c + e) + (d + f)i] \\
 &= [a(c + e) - b(d + f)] + [a(d + f) + b(c + e)]i \\
 &= [(ac - bd) + (ae - bf)] + [(ad + bc) + (af + be)]i \\
 &= [(ac - bd) + (ad + bc)i] + [(ae - bf) + (af + be)]i \\
 &= (a + bi)(c + di) + (a + bi)(e + fi) \\
 &= z_1z_2 + z_1z_3
 \end{aligned}$$

**Remark 2.2**

(a.) Let  $z$  and  $w$  be complex numbers such that  $z = a + bi$  and  $w = c + di$  where  $a, b, c, d \in \mathbb{R}$ . Then

$$\begin{aligned}
 z - w &= z + (-w) \\
 &= (a + bi) + (-c - di) \\
 &= (a - c) + (b - d)i
 \end{aligned}$$

(b.) Let  $z$  and  $w$  be complex numbers such that  $z = a + bi$  and  $w = c + di$  where  $a, b, c, d \in \mathbb{R}$ . Then

$$\begin{aligned}
 \frac{z}{w} &= ww^{-1} \\
 &= (a + bi) \left( \frac{c}{c^2 + d^2} + \frac{-d}{c^2 + d^2}i \right) \\
 &= \frac{(a + bi)(c - di)}{c^2 + d^2} \\
 &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i
 \end{aligned}$$

This can also be viewed as multiplying both numerator and denominator of  $\frac{z}{w}$  by the conjugate  $\bar{w}$  of  $w$ . That is

$$\begin{aligned}
 \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\
 &= \frac{(a + bi)(c - di)}{c^2 + d^2} \\
 &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i
 \end{aligned}$$

**Example 2.3**

Write the given numbers in the form  $a + bi$  where  $a, b \in \mathbb{R}$ .

- (i)  $(-2 + 2i)^5$   
 (ii)  $\frac{(5 - 4i) - (3 + 7i)}{(4 + 2i) + (2 - 3i)}$

SOLUTION

(i)

$$\begin{aligned} (-2 + 2i)^5 &= -32(1 - i)^5 \\ &= -32[(1 - i)^2]^2(1 - i) \\ &= -32(1 - 2i - 1)^2(1 - i) \\ &= -32(2i)^2(1 - i) \\ &= -32(-4)(1 - i) \\ &= 128(1 - i) \end{aligned}$$

(ii)

$$\begin{aligned} \frac{(5 - 4i) - (3 + 7i)}{(4 + 2i) + (2 - 3i)} &= \frac{2 - 11i}{6 - i} \times \frac{6 + i}{6 + i} \\ &= \frac{1}{37}(12 + 11 + 2i - 66i) \\ &= \frac{1}{37}(23 - 64i) \end{aligned}$$

**Proposition 2.2**

Let  $z_1, z_2 \in \mathbb{C}$ . Then

- (1)  $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$   
 (2)  $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$   
 (3)  $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$   
 (4)  $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$   
 (5)  $\overline{\overline{z}} = z$   
 (6)  $\operatorname{Re}(z) = \frac{z + \overline{z}}{2}$  and  $\operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$

PROOF

(1) Let  $z_1 = a + bi$  and  $z_2 = c + di$  where  $a, b, c, d \in \mathbb{R}$ . We have that

$z_1 + z_2 = (a + c) + (b + d)i$ . Thus

$$\begin{aligned}\overline{z_1 + z_2} &= (a + c) - (b + d)i \\ &= (a - bi) + (c - di) \\ &= \overline{z_1} + \overline{z_2}\end{aligned}$$

Therefore  $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ .

(3) Let  $z_1 = a + bi$  and  $z_2 = c + di$  where  $a, b, c, d \in \mathbb{R}$ . We have that

$z_1 z_2 = (ac - bd) + (ad + bc)i$ . Consequently

$$\begin{aligned}\overline{z_1 z_2} &= (ac - bd) - (ad + bc)i \\ &= (ac - adi) - (bd + bci) \\ &= a(c - di) - b(d + ci) \\ &= a(c - di) - bi(c - di) \\ &= (a - bi)(c - di) \\ &= \overline{z_1} \overline{z_2}\end{aligned}$$

Therefore  $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$

(4) Let  $z_1 = a + bi$  and  $z_2 = c + di$  where  $a, b, c, d \in \mathbb{R}$ . Then

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + bi}{c + di} \\ &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

Consequently, we have that

$$\begin{aligned}\overline{z_1/z_2} &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i \\ &= \frac{(ac - bci) + (bd + adi)}{c^2 + d^2}\end{aligned}$$



$$\begin{aligned}
 \overline{z_1/z_2} &= \frac{c(a-bi) + d(b+ai)}{c^2 + d^2} \\
 &= \frac{c(a-bi) + di(a-bi)}{c^2 + d^2} \\
 &= \frac{(a-bi)(c+di)}{c^2 + d^2} \\
 &= \frac{(a-bi)(c+di)}{(c-di)(c+di)} \\
 &= \frac{a-bi}{c-di} \\
 &= \overline{z_1}/\overline{z_2}
 \end{aligned}$$

**Definition 2.5 (Equality of complex numbers)**

Complex numbers  $z_1 = a_1 + b_1i$  and  $z_2 = a_2 + b_2i$  are equal if  $a_1 = a_2$  and  $b_1 = b_2$ . That is, if  $z_1 = z_2$  then  $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$  and  $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$ .

**Example 2.4**

Solve the following equations for  $z = a + bi$  where  $a, b \in \mathbb{R}$ .

- (a)  $z - 2\bar{z} + 7 - 6i = 0$
- (b)  $z^2 = i$

SOLUTION

(a)

$$\begin{aligned}
 a + bi - 2(a - bi) + 7 - 6i &= 0 \\
 (a - 2a + 7) + (b + 2b - 6)i &= 0 \\
 (7 - a) + (3b - 6)i &= 0
 \end{aligned}$$

By equality of complex numbers, we get  $7 - a = 0$  and  $3b - 6 = 0$ . Hence  $a = 7$  and  $b = 2$ . Therefore  $z = 7 + 2i$ .

(b)

$$\begin{aligned}
 (a + bi)^2 &= i \\
 (a^2 - b^2) + 2abi &= i
 \end{aligned}$$

By equality of complex numbers,

$$a^2 - b^2 = 0 \dots\dots\dots(1) \text{ and}$$

$$2ab = 1 \dots\dots\dots(2).$$

From (1),  $(a - b)(a + b) = 0$  and so  $b = a$  or  $b = -a$ .

When  $b = a$ , equation (2) becomes  $2a^2 = 1$ , giving us  $a = \pm \frac{\sqrt{2}}{2}$ . Hence  $z = \pm \frac{\sqrt{2}}{2}(1 + i)$ .

When  $b = -a$ , equation (2) becomes  $-2a^2 = 1$ , giving us  $a^2 = -\frac{1}{2}$ . This is impossible since  $a \in \mathbb{R}$ . Therefore,  $z = \pm \frac{\sqrt{2}}{2}(1 + i)$ .

### Example 2.5

Find the square roots of  $3 + 4i$ .

### SOLUTION

Let  $\sqrt{3 + 4i} = a + bi$ , where  $a, b \in \mathbb{R}$ . Then we have that  $3 + 4i = a^2 - b^2 + 2abi$ . By equality of complex numbers, we obtain

$$a^2 - b^2 = 3 \dots\dots\dots(1)$$

$$ab = 2 \dots\dots\dots(2)$$

From (2)  $b = \frac{2}{a}$ . Putting  $b = \frac{2}{a}$  into (1) yields

$$a^2 - \frac{4}{a^2} = 3$$

$$(a^2)^2 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

Thus  $a^2 = 4$  or  $a^2 = -1$ . Since  $a \in \mathbb{R}$ ,  $a^2 = -1$  does not hold and so  $a = \pm 2$ .

When  $a = 2$ ,  $b = 1$  and when  $a = -2$ ,  $b = -1$ . Therefore the square roots of  $3 + 4i$  are  $2 + i$  and  $-2 - i$ .

### The Argand Diagram

A complex number  $z = x + yi$  is uniquely determined by an ordered pair of real numbers  $(x, y)$ . The first and second entries of the ordered pair correspond with  $\text{Re}(z)$  and

## CHAPTER 2. COMPLEX NUMBERS

$\operatorname{Im}(z)$  respectively. This allows us to associate a complex number  $z = x + yi$  with a point  $(x, y)$  in a coordinate plane in a diagram called the Argand diagram.

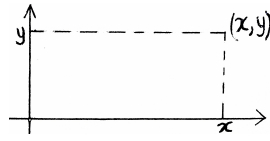


Figure 2.1: The complex plane

The coordinate plane shown above is called the complex plane or the  $z$ -plane. The horizontal axis is called the real axis because each point on that axis represents a real number. The vertical axis is called the imaginary axis because a point on that axis represents a pure imaginary number. A complex number  $z = x + yi$  can be viewed as a two-dimensional position vector.

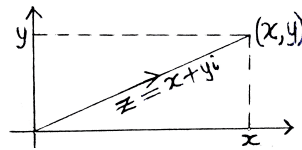


Figure 2.2: A complex number as a vector

### Definition 2.6 (Modulus of a complex number)

The modulus of a complex number  $z = x + yi$  is the real number  $|z| = \sqrt{x^2 + y^2}$ .

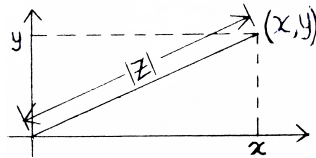


Figure 2.3: The modulus of a complex number

### Example 2.6

Find the modulus of  $z = 2 - 3i$  and represent  $z$  on an Argand diagram.

SOLUTION

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}.$$

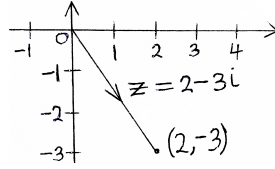


Figure 2.4:

**Proposition 2.3**

Let  $z$  and  $w$  be complex numbers. Then the following hold.

- (1)  $|z|^2 = z\bar{z}$ , and so  $|z| = \sqrt{z\bar{z}}$
- (2)  $|zw| = |z||w|$
- (3)  $|z/w| = |z|/|w|$
- (4)  $|z|^2 = |z^2|$
- (5)  $|z| = |-z|$  and  $|z| = |\bar{z}|$

PROOF

Let  $z = a + bi$  and  $w = x + yi$  where  $a, b, x, y \in \mathbb{R}$ .

(1)

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

(2)

$$\begin{aligned} |zw|^2 &= |(ax - by) + (ay + bx)i|^2 \\ &= (ax - by)^2 + (ay + bx)^2 \\ &= a^2x^2 + b^2y^2 - 2abxy + a^2y^2 + b^2x^2 + 2abxy \\ &= a^2(x^2 + y^2) + b^2(x^2 + y^2) \\ &= (a^2 + b^2)(x^2 + y^2) \\ &= |z|^2|w|^2 \end{aligned}$$

That is  $|zw|^2 = |z|^2|w|^2 = (|z||w|)^2$ , giving us  $|zw| = |z||w|$ .

(3)

$$\begin{aligned}
 |z/w|^2 &= \left| \frac{ax + by}{x^2 + y^2} + \frac{bx - ay}{x^2 + y^2}i \right|^2 \\
 &= \left( \frac{ax + by}{x^2 + y^2} \right)^2 + \left( \frac{bx - ay}{x^2 + y^2} \right)^2 \\
 &= \frac{a^2x^2 + b^2y^2 + 2abxy + b^2x^2 + a^2y^2 - 2abxy}{(x^2 + y^2)^2} \\
 &= \frac{a^2(x^2 + y^2) + b^2(x^2 + y^2)}{(x^2 + y^2)^2} \\
 &= \frac{(a^2 + b^2)(x^2 + y^2)}{(x^2 + y^2)^2} \\
 &= \frac{a^2 + b^2}{x^2 + y^2} = |z|^2/|w|^2
 \end{aligned}$$

That is  $\left| \frac{z}{w} \right|^2 = \frac{|z|^2}{|w|^2} = \left( \frac{|z|}{|w|} \right)^2$ , giving us  $\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$ .

(4)

$$\begin{aligned}
 |z^2| &= |(a^2 - b^2) + 2abi| \\
 &= \left( (a^2 - b^2)^2 + 4a^2b^2 \right)^{\frac{1}{2}} \\
 &= (a^4 + b^4 - 2a^2b^2 + 4a^2b^2)^{\frac{1}{2}} \\
 &= (a^4 + b^4 + 2a^2b^2)^{\frac{1}{2}} \\
 &= [(a^2 + b^2)^2]^{\frac{1}{2}} \\
 &= a^2 + b^2 = |z|^2
 \end{aligned}$$

**Definition 2.7 (Argument of a complex number)**

Let  $0 \neq z \in \mathbb{C}$ . The angle  $\theta$  of inclination of the vector  $z$  measured from the positive real axis is called an argument of  $z$  and is denoted by  $\arg(z)$ .

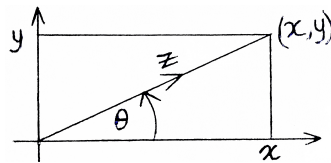


Figure 2.5: Argument of a complex number

An argument  $\theta$  of a complex number  $z = x + yi$  must satisfy  $\cos \theta = \frac{x}{|z|}$  and  $\sin \theta = \frac{y}{|z|}$ .

Since cosine and sine are  $2\pi$ -periodic, an argument of a complex number is not unique.

We usually use  $\tan \theta = \frac{y}{x}$  when finding argument of a complex number.

**Definition 2.8 (Principal argument)**

Let  $0 \neq z \in \mathbb{C}$ . The symbol  $\arg(z)$  represents a set of values. The argument  $\theta$  of a complex number that lies in the interval  $-\pi < \theta \leq \pi$  (or  $0 \leq \theta < 2\pi$ ) is called the principal value of  $\arg(z)$  or the principal argument of  $z$ . The principal argument of  $z$  is unique and is represented by the symbol  $\text{Arg}(z)$ , that is  $-\pi < \text{Arg}(z) \leq \pi$  or  $0 \leq \text{Arg}(z) < 2\pi$  as the case may be.

**Remark 2.3**

(i) In general,  $\arg(z)$  and  $\text{Arg}(z)$  are related by  $\arg(z) = \text{Arg}(z) + 2n\pi$  where  $n \in \mathbb{Z}$ .

(ii) An argument can be assigned to any non-zero complex number  $z$ . If  $z = 0$ , then  $\arg(z)$  is indeterminate.

**Example 2.7**

If  $z = i$ , find the values of  $\text{Arg}(z)$  and  $\arg(z)$ , where  $-\pi < \text{Arg}(z) \leq \pi$ .

SOLUTION

We have that  $\text{Arg}(z) = \frac{\pi}{2}$ , and so  $\arg(z) = \frac{\pi}{2} + 2n\pi$  where  $n \in \mathbb{Z}$ .

**Polar form of a complex number**

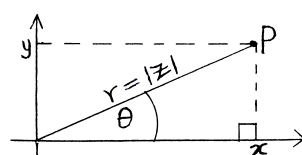


Figure 2.6:

From  $x = r \cos \theta$  and  $y = r \sin \theta$ , we get

$$\begin{aligned} z &= x + yi \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

**Definition 2.9 (Polar form of a complex number)**

Let  $0 \neq z \in \mathbb{C}$ . Let  $\theta = \arg(z)$  and  $r = |z|$ . Then the polar form of the complex number  $z$  is given by  $z = r(\cos \theta + i \sin \theta)$ .

**Example 2.8**

Express the complex number  $-\sqrt{3} - i$  in the form  $r(\cos \theta + i \sin \theta)$ , where  $-\pi < \theta \leq \pi$ .

SOLUTION

Let  $z = -\sqrt{3} - i$ . Then  $r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$  and  $\tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \frac{\pi}{6}$ .

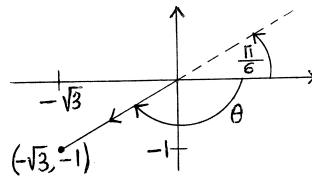


Figure 2.7:

Hence  $\theta = \text{Arg}(z) = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$ . Therefore  $z = 2\left(\cos\left(\frac{5\pi}{6}\right) - i \sin\left(\frac{5\pi}{6}\right)\right)$ .

**Multiplication and Division of complex numbers in polar form**

Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  where  $\theta_1$  and  $\theta_2$  are any arguments of  $z_1$  and  $z_2$  respectively.  $r_1$  and  $r_2$  are their respective moduli.

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos \theta_1 + i \sin \theta_1] [\cos \theta_2 + i \sin \theta_2] \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\ &= \frac{r_1}{r_2} \left[ \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \right] \\ &= \frac{r_1}{r_2} \left[ \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \right] \\ &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \end{aligned}$$

**Example 2.9**

Find arguments of  $z_1 z_2$  and  $\frac{z_1}{z_2}$  if  $z_1 = i$  and  $z_2 = -\sqrt{3} - i$ .

SOLUTION

From Example(2.7) and Example(2.8),  $\text{Arg}(z_1) = \frac{\pi}{2}$  and  $\text{Arg}(z_2) = -\frac{5\pi}{6}$ .  
 $\arg(z_1 z_2) = \frac{\pi}{2} - \frac{5\pi}{6} = -\frac{2\pi}{6} = -\frac{\pi}{3}$  and  $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} + \frac{5\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$ .

**Remark 2.4**

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$  and  $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ . However, it is not true in general that  $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$  and  $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$ . For instance, in Example(2.9),  $\text{Arg}(z_1) - \text{Arg}(z_2) = \frac{4\pi}{3} \neq -\frac{2\pi}{3} = \text{Arg}\left(\frac{z_1}{z_2}\right)$ .

**Integer powers of  $z$**

Let  $z = r(\cos \theta + i \sin \theta)$ . Then

$$\begin{aligned} z^2 = z z &= r^2[\cos(2\theta) + i \sin(2\theta)] \\ z^3 = z^2 z &= r^3[\cos(2\theta + \theta) + i \sin(2\theta + \theta)] \\ &= r^3[\cos(3\theta) + i \sin(3\theta)] \\ z^4 = z^3 z &= r^4[\cos(4\theta) + i \sin(4\theta)] \end{aligned}$$

We also have that

$$\begin{aligned} z^{-1} = \frac{1}{z} &= \frac{1}{r}[\cos(0 - \theta) + i \sin(0 - \theta)] \\ &= r^{-1}[\cos(-\theta) + i \sin(-\theta)] \\ z^{-2} = \frac{1}{z^2} &= \frac{1}{r^2}[\cos(0 - 2\theta) + i \sin(0 - 2\theta)] \\ &= r^{-2}[\cos(-2\theta) + i \sin(-2\theta)] \\ z^{-3} = \frac{1}{z^3} &= \frac{1}{r^3}[\cos(0 - 3\theta) + i \sin(0 - 3\theta)] \\ &= r^{-3}[\cos(-3\theta) + i \sin(-3\theta)] \end{aligned}$$

Continuing in this manner, we obtain a formula for the  $n$ th power of  $z$  where  $n \in \mathbb{Z}$ :

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$



**Theorem 2.1 (De Moivre's theorem)**

Let  $z = r(\cos \theta + i \sin \theta)$  be a complex number. Then

$$z^n = r^n [\cos(n\theta) + i \sin(n\theta)], \text{ where } n \in \mathbb{Z}.$$

**Remark 2.5**

From the De Moivre's theorem, it is clear that  $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$ . In some books, this is the statement of the De Moivre's theorem. The De Moivre's theorem is actually true for  $n \in \mathbb{Q}$ .

**Example 2.10**

Compute the following:

(i)  $z^3$  if  $z = -\sqrt{3} - i$

(ii)  $\left[ \cos\left(\frac{\pi}{9}\right) + i \sin\left(\frac{\pi}{9}\right) \right]^{12} \left[ 2 \left( \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) \right]^5$

SOLUTION

(i) From Example(2.8),  $z = 2 \left[ \cos\left(\frac{5\pi}{6}\right) - i \sin\left(\frac{5\pi}{6}\right) \right]$ .

$$\begin{aligned} z^3 &= 2^3 \left[ \cos\left(\frac{5\pi}{2}\right) - i \sin\left(\frac{5\pi}{2}\right) \right] \\ &= 8 \left[ \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) \right] \\ &= 8(0 - i) \\ &= -8i \end{aligned}$$

(ii)

$$\begin{aligned} &\left[ \cos\left(\frac{\pi}{9}\right) + i \sin\left(\frac{\pi}{9}\right) \right]^{12} \left[ 2 \left( \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) \right]^5 \\ &= \left[ \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right] \left[ 32 \left( \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right) \right] \\ &= 32 \left[ \cos\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right) + i \sin\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right) \right] \\ &= 32 \left[ \cos\left(\frac{13\pi}{6}\right) + i \sin\left(\frac{13\pi}{6}\right) \right] \\ &= 32 \left[ \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right] \\ &= 32 \left[ \frac{\sqrt{3}}{2} + \frac{1}{2}i \right] \\ &= 16(\sqrt{3} + i) \end{aligned}$$

Trigonometric applications of De Moivre's theorem

(1) Expressing  $\cos(n\theta)$ ,  $\sin(n\theta)$  and  $\tan(n\theta)$  in terms of  $\cos \theta$ ,  $\sin \theta$  and  $\tan \theta$ :

From De Moivre's theorem,  $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$ . We expand the LHS using binomial theorem and then equate corresponding real parts or imaginary parts as may be required.

**Example 2.11**

Prove the following trigonometric identities using methods based on De Moivre's theorem.

- (i)  $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$   
 (ii)  $\tan 6\theta = 2t \left( \frac{3 - 10t^2 + 3t^4}{1 - 15t^2 + 15t^4 - t^6} \right)$  where  $t = \tan \theta$ .

SOLUTION

(i)

$$\begin{aligned} \cos 4\theta + i \sin 4\theta &= (\cos \theta + i \sin \theta)^4 \\ &= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 \\ &\quad + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4 \\ &= (\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) + i(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta) \end{aligned}$$

Equating real parts we get

$$\begin{aligned} \cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \\ &= \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2 \\ &= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 + \cos^4 \theta - 2 \cos^2 \theta \\ &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \end{aligned}$$

(ii)

$$\begin{aligned} \cos 6\theta + i \sin 6\theta &= (\cos \theta + i \sin \theta)^6 \\ &= \cos^6 \theta + 6 \cos^5 \theta (i \sin \theta) + 15 \cos^4 \theta (i \sin \theta)^2 + 20 \cos^3 \theta (i \sin \theta)^3 \\ &\quad + 15 \cos^2 \theta (i \sin \theta)^4 + 6 \cos \theta (i \sin \theta)^5 + (i \sin \theta)^6 \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned}\cos 6\theta + i \sin 6\theta &= (\cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta) \\ &\quad i(6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta)\end{aligned}$$

Equating real and imaginary parts, we get

$$\begin{aligned}\sin 6\theta &= 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta \\ \cos 6\theta &= \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta\end{aligned}$$

We have that

$$\tan 6\theta = \frac{6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta}{\cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta}$$

Dividing both numerator and denominator by  $\cos^6 \theta$ , we get

$$\begin{aligned}\tan 6\theta &= \frac{6 \tan \theta - 20 \tan^3 \theta + 6 \tan^5 \theta}{1 - 15 \tan^2 \theta + 15 \tan^4 \theta - \tan^6 \theta} \\ &= \frac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6} \\ &= 2t \left( \frac{3 - 10t^2 + 3t^4}{1 - 15t^2 + 15t^4 - t^6} \right)\end{aligned}$$

(2) Expressing  $\cos^n \theta$  and  $\sin^n \theta$  in terms of the cosines or sines of multiples of  $\theta$ :

Let  $z = \cos \theta + i \sin \theta$ . Then

$$\begin{aligned}\frac{1}{z} = z^{-1} &= (\cos \theta + i \sin \theta)^{-1} \\ &= \cos(-\theta) + i \sin(-\theta) \\ &= \cos \theta - i \sin \theta\end{aligned}$$

Consequently, we get

$$\begin{aligned}z + \frac{1}{z} &= 2 \cos \theta \\ z - \frac{1}{z} &= 2i \sin \theta\end{aligned}$$

Now

$$\begin{aligned}z^n &= \cos(n\theta) + i \sin(n\theta) \\ \frac{1}{z^n} &= \cos(n\theta) - i \sin(n\theta)\end{aligned}$$

Thus, we have that

$$\begin{aligned} z^n + \frac{1}{z^n} &= 2 \cos(n\theta) \\ z^n - \frac{1}{z^n} &= 2i \sin(n\theta) \end{aligned}$$

**Example 2.12**

Prove the following trigonometric identities using methods based on De Moivre's theorem.

- (i)  $\cos^4 \theta + \sin^4 \theta = \frac{1}{4} (\cos(4\theta) + 3)$   
(ii)  $\sin^6 \theta = \frac{1}{32} (10 - 15 \cos(2\theta) + 6 \cos(4\theta) - \cos(6\theta))$

SOLUTION

(i)

$$\begin{aligned} \cos^4 \theta &= \frac{1}{16} \left( z + \frac{1}{z} \right)^4 \\ &= \frac{1}{16} \left[ z^4 + 4z^3 \left( \frac{1}{z} \right) + 6z^2 \left( \frac{1}{z} \right)^2 + 4z \left( \frac{1}{z} \right)^3 + \left( \frac{1}{z} \right)^4 \right] \\ &= \frac{1}{16} \left[ \left( z^4 + \frac{1}{z^4} \right) + 4 \left( z^2 + \frac{1}{z^2} \right) + 6 \right] \\ &= \frac{1}{16} (2 \cos(4\theta) + 8 \cos(2\theta) + 6) \end{aligned}$$

$$\begin{aligned} \sin^4 \theta &= \left[ \frac{1}{2i} \left( z - \frac{1}{z} \right) \right]^4 \\ &= \frac{1}{16} \left( z - \frac{1}{z} \right)^4 \\ &= \frac{1}{16} \left[ z^4 + 4z^3 \left( -\frac{1}{z} \right) + 6z^2 \left( -\frac{1}{z} \right)^2 + 4z \left( -\frac{1}{z} \right)^3 + \left( -\frac{1}{z} \right)^4 \right] \\ &= \frac{1}{16} \left[ \left( z^4 + \frac{1}{z^4} \right) - 4 \left( z^2 + \frac{1}{z^2} \right) + 6 \right] \\ &= \frac{1}{16} (2 \cos(4\theta) - 8 \cos(2\theta) + 6) \end{aligned}$$

Therefore,  $\cos^4 \theta + \sin^4 \theta = \frac{1}{16} (4 \cos(4\theta) + 12) = \frac{1}{4} (\cos(4\theta) + 3)$ .

Exponential form of a complex number

The formula below is called Euler's formula.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

The polar form of a complex number is  $z = r(\cos \theta + i \sin \theta)$ . Therefore  $z = re^{i\theta}$ .

**Definition 2.10**

Let  $z$  be a non-zero complex number with an argument  $\theta$  and modulus  $r$ . Then the exponential form of  $z$  is given by  $z = re^{i\theta}$ .

**Remark 2.6**

Let  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ . Then it is easily seen that  $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$  and  $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$ . This is in agreement with the observations made in Remark(2.4).

**Example 2.13**

Express  $\cos \theta$  and  $\sin \theta$  each in terms of  $e^{i\theta}$  and  $e^{-i\theta}$ . Use your result to prove that  $16 \cos^3 \theta \sin^2 \theta = 2 \cos \theta - \cos 3\theta - \cos 5\theta$

SOLUTION

$$e^{i\theta} = \cos \theta + i \sin \theta \dots \dots \dots (1)$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \dots \dots \dots (2)$$

Adding equations (1) and (2), we get  $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ . Hence  $\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$ .  
Subtracting equation (2) from equation (1), we get  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ . This gives us  $\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$ .

$$\begin{aligned} \cos^3 \theta &= \frac{1}{8} (e^{i\theta} + e^{-i\theta})^3 \\ &= \frac{1}{8} [e^{3i\theta} + 3e^{2i\theta} e^{-i\theta} + 3e^{i\theta} e^{-2i\theta} + e^{-3i\theta}] \\ &= [(e^{i(3\theta)} + e^{-i(3\theta)}) + 3(e^{i\theta} + e^{-i\theta})] \\ &= \frac{1}{8} (2 \cos 3\theta + 6 \cos \theta) \end{aligned}$$

$$\begin{aligned}
 \sin^2 \theta &= -\frac{1}{4} (e^{i\theta} - e^{-i\theta})^2 \\
 &= -\frac{1}{4} (e^{2i\theta} + e^{-2i\theta} - 2e^{i\theta}e^{-i\theta}) \\
 &= -\frac{1}{4} (2 \cos 2\theta - 2)
 \end{aligned}$$

$$\begin{aligned}
 16 \cos^3 \theta \sin^2 \theta &= 16 \times \frac{1}{8} (2 \cos 3\theta + 6 \cos \theta) \times -\frac{1}{4} (2 \cos 2\theta - 2) \\
 &= -2(\cos 3\theta + 3 \cos \theta)(\cos 2\theta - 1) \\
 &= -2(\cos 3\theta \cos 2\theta - \cos 3\theta + 3 \cos \theta \cos 2\theta - 3 \cos \theta)
 \end{aligned}$$

From trigonometric identities,

$$\begin{aligned}
 \cos(A + B) &= \cos A \cos B - \sin A \sin B \quad \text{and} \\
 \cos(A - B) &= \cos A \cos B + \sin A \sin B.
 \end{aligned}$$

Thus  $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$ .

$$\begin{aligned}
 16 \cos^3 \theta \sin^2 \theta &= -2 \cos 3\theta \cos 2\theta + 2 \cos 3\theta - 6 \cos \theta \cos 2\theta + 6 \cos \theta \\
 &= -(\cos 5\theta + \cos \theta) + 2 \cos 3\theta - 3(\cos 3\theta + \cos \theta) + 6 \cos \theta \\
 &= 2 \cos \theta - \cos 3\theta - \cos 5\theta
 \end{aligned}$$

### ***n*th roots of a complex number**

#### **Definition 2.11**

A number  $w$  is an  $n$ th root of a non-zero complex number  $z$  if  $w^n = z$ , where  $n \in \mathbb{Z}^+$ .

Now, suppose that  $z = r(\cos \theta + i \sin \theta)$  and  $w = \rho(\cos \phi + i \sin \phi)$  are the polar forms of the complex numbers  $z$  and  $w$ . Then  $w^n = z$  becomes  $[\rho(\cos \phi + i \sin \phi)]^n = r(\cos \theta + i \sin \theta)$ , giving us  $\rho^n[\cos(n\phi) + i \sin(n\phi)] = r(\cos \theta + i \sin \theta)$  by De Moivre's theorem. Equating corresponding real and imaginary parts we get

$$\begin{aligned}
 \rho^n \cos(n\phi) &= r \cos \theta \\
 \rho^n \sin(n\phi) &= r \sin \theta
 \end{aligned}$$

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Squaring both sides of the two equations and then adding the results we get

$\rho^{2n}(\cos^2(n\phi) + \sin^2(n\phi)) = r^2(\cos^2\theta + \sin^2\theta)$ . Thus  $(\rho^n)^2 = r^2$ . Since  $\rho$  and  $r$  are both positive,  $\rho^n = r$ . Hence  $\rho = \sqrt[n]{r}$ . Consequently,  $\cos(n\phi) = \cos\theta$  and  $\sin(n\phi) = \sin\theta$ . We have that  $n\phi = \theta + 2k\pi$  where  $k \in \mathbb{Z}$ . Hence  $\phi = \frac{\theta + 2k\pi}{n}, k \in \mathbb{Z}$ . Therefore,

$$w = \sqrt[n]{r} \left[ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right].$$

When  $k$  takes values  $k = 0, 1, 2, \dots, n-1$  we obtain  $n$  distinct  $n$ th roots of  $z$ . These roots have the same modulus  $\sqrt[n]{r}$  but different arguments.

For  $k \geq n$  we obtain the same roots as we had before. This is because sine and cosine are  $2\pi$ -periodic. To see this: Suppose  $k = n + m$  where  $m = 0, 1, 2, \dots$ , that is  $k \geq n$ . Then

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi.$$

This gives us

$$\sin\phi = \sin\left(\frac{\theta + 2m\pi}{n}\right) \quad \text{and} \quad \cos\phi = \cos\left(\frac{\theta + 2m\pi}{n}\right).$$

Therefore, the  $n$   $n$ th roots of a non-zero complex number  $z = r(\cos\theta + i\sin\theta)$  are give by

$$w_k = \sqrt[n]{r} \left[ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right] \quad \text{where } k = 0, 1, 2, \dots, n-1.$$

### Example 2.14

Find the three cube roots of  $z = i$ .

### SOLUTION

We have that  $r = |z| = \sqrt{0^2 + 1^2} = 1$  and  $\theta = \text{Arg}(z) = \frac{\pi}{2}$ . The three cube roots of  $z = i$  are

$$w_k = \sqrt[3]{1} \left( \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) \right) \quad \text{where } k = 0, 1, 2.$$

$$w_0 = \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)$$

$$w_1 = \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)$$

$$w_2 = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right)$$

**Remark 2.7**

Since the roots have the same modulus, the  $n$ th roots of a non-zero complex number lie on a circle of radius  $\sqrt[n]{r}$  centered at the origin in the complex plane. The difference between the arguments of any two successive roots  $w_k$  and  $w_{k+1}$  is  $\frac{2\pi}{n}$ . Hence the  $n$ th roots are equally spaced on this circle, beginning with the root whose argument is  $\frac{\theta}{n}$ . The vectors representing the  $n$ th roots form a regular  $n$ -sided polygon. Hence the sum of the  $n$ th roots is zero.

**Example 2.15**

Find the fifth roots of  $-1$  and hence prove that  $\cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) = \frac{1}{2}$ .

SOLUTION

We have that  $-1 = \cos \pi + i \sin \pi$ . The fifth roots of  $-1$  are

$$w_k = \sqrt[5]{1} \left[ \cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right) \right] \quad \text{where } k = 0, 1, 2, 3, 4.$$

$$w_0 = \cos\left(\frac{\pi}{5}\right) + i \sin\left(\frac{\pi}{5}\right)$$

$$w_1 = \cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)$$

$$w_2 = \cos \pi + i \sin \pi = -1$$

$$w_3 = \cos\left(\frac{7\pi}{5}\right) + i \sin\left(\frac{7\pi}{5}\right) = \cos\left(\frac{3\pi}{5}\right) - i \sin\left(\frac{3\pi}{5}\right)$$

$$w_4 = \cos\left(\frac{9\pi}{5}\right) + i \sin\left(\frac{9\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right) - i \sin\left(\frac{\pi}{5}\right)$$

Now, we have that  $w_0 + w_1 + w_2 + w_3 + w_4 = 2 \cos\left(\frac{\pi}{5}\right) + 2 \cos\left(\frac{3\pi}{5}\right) - 1 = 0$ .

Hence,  $\cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) = \frac{1}{2}$ .

**Cube roots of unity**

We have that  $1 = \cos 0 + i \sin 0$ . The cube roots of unity are

$$z_k = \cos\left(\frac{0 + 2k\pi}{3}\right) + i \sin\left(\frac{0 + 2k\pi}{3}\right) \quad \text{where } k = 0, 1, 2.$$

$$z_0 = \cos 0 + i \sin 0 = 1$$

$$z_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

$$z_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right)$$



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On an Argand diagram, we have

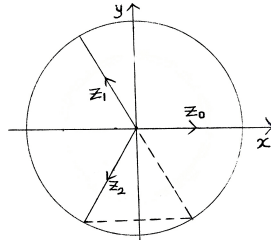


Figure 2.8:

We note that

(a) The points representing  $z_0$ ,  $z_1$  and  $z_2$  lie on a circle centred at the origin and has radius 1.

(b)  $z_0$ ,  $z_1$  and  $z_2$  are separated by  $\frac{2\pi}{3}$ .

(c)  $z_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)^2$ . That is  $z_2 = z_1^2$ . Also, we have

$z_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) - i \sin\left(\frac{4\pi}{3}\right) = \left(\cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right)\right)^2$ . That is  $z_1 = z_2^2$ . Therefore, if one of the complex roots is denoted by  $w$ , then the other is  $w^2$ . Thus the three cube roots of unity can be denoted by  $1, w, w^2$ .

(d) The vectors  $z_0$ ,  $z_1$ ,  $z_2$  form an equilateral triangle.

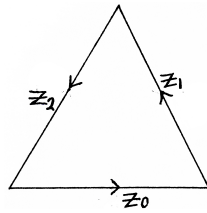


Figure 2.9:

We have that  $z_0 = -z_2 - z_1$ , which yields  $z_0 + z_1 + z_2 = 0$ . That is  $1 + w + w^2 = 0$ .

### **$n$ th roots of unity**

The  $n$ th roots of unity are

$$z_k = \sqrt[n]{1} \left[ \cos\left(\frac{0 + 2k\pi}{n}\right) + i \sin\left(\frac{0 + 2k\pi}{n}\right) \right] \quad \text{where } k = 0, 1, 2, \dots, n-1.$$

We note that

(a)

$$\begin{aligned}\arg(z_2) &= 2\arg(z_1) \\ \arg(z_3) &= 3\arg(z_1) \\ &\vdots \\ \arg(z_{n-1}) &= (n-1)\arg(z_1)\end{aligned}$$

Thus,  $z_2 = z_1^2$ ,  $z_3 = z_1^3, \dots, z_{n-1} = z_1^{n-1}$ . If the complex root with the smallest positive argument is denoted by  $w$ , then the other complex roots are  $w^2, w^3, \dots, w^{n-1}$ .

(b)  $z_0, z_1, \dots, z_{n-1}$  form an  $n$ -sided regular polygon. Thus  $z_1 + z_2 + \dots + z_{n-1} = 0$

(c) The complex  $n$ th roots of unity occur in conjugate pairs,

(d) When  $n$  is even, one  $n$ th root of unity is  $-1$ . This is not the case when  $n$  is odd.

### Example 2.16

Find the fifth roots of unity. If  $w$  is the root with the smallest positive argument and if  $u = w + w^4$  and  $v = w^2 + w^3$ , show that  $u + v = -1$  and  $u - v = \sqrt{5}$ . Hence find  $\cos \frac{2\pi}{5}$ .

### SOLUTION

The fifth roots of unity are

$$z_k = \cos\left(\frac{2k\pi}{5}\right) + i \sin\left(\frac{2k\pi}{5}\right) \quad \text{where } k = 0, 1, 2, 3, 4.$$

$$z_0 = \cos 0 + i \sin 0 = 1$$

$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) = w$$

$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) = w^2$$

$$z_3 = \cos\left(\frac{6\pi}{5}\right) + i \sin\left(\frac{6\pi}{5}\right) = w^3 = \cos\left(\frac{4\pi}{5}\right) - i \sin\left(\frac{4\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{8\pi}{5}\right) + i \sin\left(\frac{8\pi}{5}\right) = w^4 = \cos\left(\frac{2\pi}{5}\right) - i \sin\left(\frac{2\pi}{5}\right)$$

$$z_0 + z_1 + z_2 + z_3 + z_4 = 0$$

$$1 + w + w^2 + w^3 + w^4 = 0$$

$$(w + w^4) + (w^2 + w^3) = -1$$

Thus  $u + v = -1$ .

$$\begin{aligned}
 (u - v)^2 &= u^2 + v^2 - 2uv \\
 &= w^2 + w^8 + 2w^5 + w^4 + w^6 + 2w^5 - 2(w^3 + w^4 + w^6 + w^7) \\
 &= w^2 - 2w^3 - w^4 + 4w^5 - w^6 - 2w^7 + w^8
 \end{aligned}$$

However,  $w^5 = 1$ ,  $w^6 = w$ ,  $w^7 = w^2$  and  $w^8 = w^3$ .

$$\begin{aligned}
 (u - v)^2 &= w^2 - 2w^3 - w^4 + 4 - w - 2w^2 + w^3 \\
 &= -w - w^2 - w^3 - w^4 + 4 \\
 &= -(w + w^2 + w^3 + w^4) + 4 \\
 &= 1 + 4 \\
 &= 5
 \end{aligned}$$

Thus  $u - v = \sqrt{5}$ . Adding  $u + v = -1$  and  $u - v = \sqrt{5}$ , we get

$$\begin{aligned}
 2u &= \sqrt{5} - 1 \\
 2 \left( 2 \cos \frac{2\pi}{5} \right) &= \sqrt{5} - 1 \\
 \cos \frac{2\pi}{5} &= \frac{1}{4}(\sqrt{5} - 1)
 \end{aligned}$$

### Polynomial equations with real coefficients

#### Proposition 2.4

If a polynomial equation with real coefficients has complex roots, they occur in conjugate pairs.

#### PROOF

Let  $P(x)$  be a polynomial with real coefficients. Let  $x = a + bi$  where  $a, b \in \mathbb{R}$ . Then  $P(x) = P(a + bi) = c + di$  for some  $c, d \in \mathbb{R}$ .  $(bi)^n$  is real if  $n$  is even, it is imaginary if  $n$  is odd.  $c$  contains only even powers of  $b$  and  $d$  contains only odd powers of  $b$ . Thus if the sign of  $b$  changes,  $c$  does not change sign but  $d$  does change sign. Thus  $P(a + bi) = c + di \implies P(a - bi) = c - di$ . Suppose that  $a + bi$  is a root of the equation  $P(x) = 0$ . Then  $P(a + bi) = c + di = 0$ . Thus  $c = d = 0$  and so  $c - di = 0$ . Hence  $P(a - bi) = 0$ . Therefore  $a - bi$  is also a root of the equation  $P(x) = 0$ .

**Example 2.17**

One root of the equation  $2z^3 - 5z^2 + 12z - 5 = 0$  is  $1 - 2i$ . Find the other two roots.

SOLUTION

Let  $P(z) = 2z^3 - 5z^2 + 12z - 5$ . Since  $1 - 2i$  is a root of  $P(z) = 0$ ,  $1 + 2i$  is also a root of  $P(z) = 0$ .

$$\begin{aligned} [z - (1 - 2i)][z - (1 + 2i)] &= z^2 - z(1 + 2i) - z(1 - 2i) + (1 - 2i)(1 + 2i) \\ &= z^2 - z(1 + 2i + 1 - 2i) + 1 + 4 \\ &= z^2 - 2z + 5 \end{aligned}$$

$$\begin{array}{r} 2z - 1 \\ z^2 - 2z + 5 \overline{) 2z^3 - 5z^2 + 12z - 5} \\ \underline{2z^3 - 4z^2 + 10z + 0} \phantom{0} \\ -z^2 + 2z - 5 \\ \underline{-z^2 + 2z - 5} \\ 0 \end{array}$$

Hence  $P(z) = (z^2 - 2z + 5)(2z - 1)$ . The other two roots are  $1 + 2i$  and  $\frac{1}{2}$ .

**Example 2.18**

Prove that  $(z^n - e^{i\theta})(z^n - e^{-i\theta}) = z^{2n} - 2z^n \cos \theta + 1$ . Hence, or otherwise, find the roots of the polynomial  $z^6 - z^3\sqrt{2} + 1$  in the form  $\cos \alpha + i \sin \alpha$ , where  $-\pi < \alpha \leq \pi$ .

SOLUTION

$$\begin{aligned} (z^n - e^{i\theta})(z^n - e^{-i\theta}) &= (z^n)^2 - z^n e^{-i\theta} - z^n e^{i\theta} + e^{i\theta} e^{-i\theta} \\ &= z^{2n} - z^n(e^{i\theta} + e^{-i\theta}) + e^0 \\ &= z^{2n} - z^n(2 \cos \theta) + 1 \\ &= z^{2n} - 2z^n \cos \theta + 1 \end{aligned}$$

Comparing  $z^6 - z^3\sqrt{2} + 1$  to  $z^{2n} - 2z^n \cos \theta + 1$ , we see that  $n = 3$  and  $\cos \theta = \frac{\sqrt{2}}{2}$ . Thus  $\theta = \frac{\pi}{4} + 2k\pi$ ,  $k \in \mathbb{Z}$ .

$$z^6 - z^3\sqrt{2} + 1 = \left(z^3 - e^{i(\frac{\pi}{4}+2k\pi)}\right) \left(z^3 - e^{-i(\frac{\pi}{4}+2k\pi)}\right) = 0.$$

If  $z^3 - e^{i(\frac{\pi}{4}+2k\pi)} = 0$ , we get  $z_k = \cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)$ ,  $k = 0, 1, 2$ .

$$\begin{aligned} z_0 &= \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \\ z_1 &= \cos\left(\frac{9\pi}{12}\right) + i \sin\left(\frac{9\pi}{12}\right) \\ z_2 &= \cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right) = \cos\left(\frac{7\pi}{12}\right) - i \sin\left(\frac{7\pi}{12}\right) \end{aligned}$$

If  $z^3 - e^{-i(\frac{\pi}{4}+2k\pi)} = 0$ , we get  $z_k = \cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) - i \sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)$ ,  $k = 0, 1, 2$ .

$$\begin{aligned} z_0 &= \cos\left(\frac{\pi}{12}\right) - i \sin\left(\frac{\pi}{12}\right) \\ z_1 &= \cos\left(\frac{9\pi}{12}\right) - i \sin\left(\frac{9\pi}{12}\right) \\ z_2 &= \cos\left(\frac{17\pi}{12}\right) - i \sin\left(\frac{17\pi}{12}\right) = \cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right) \end{aligned}$$

Therefore the roots of  $z^6 - z^3\sqrt{2} + 1$  are  $\cos\left(\frac{\pi}{12}\right) \pm i \sin\left(\frac{\pi}{12}\right)$ ,  $\cos\left(\frac{9\pi}{12}\right) \pm i \sin\left(\frac{9\pi}{12}\right)$  and  $\cos\left(\frac{7\pi}{12}\right) \pm i \sin\left(\frac{7\pi}{12}\right)$ .

### Loci in the complex plane

#### Definition 2.12 (The locus of a complex number)

Let  $z$  be a complex number. The path which represents all possible values of  $z$  satisfying a given condition is called the locus of  $z$ .

#### Remark 2.8

Let  $\mathbf{a}$  and  $\mathbf{b}$  be position vectors of the points  $A$  and  $B$  respectively. We know that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}.$$

That is  $\mathbf{b} - \mathbf{a}$  is the vector with initial point  $A$  and terminal point  $B$ .  $|\mathbf{b} - \mathbf{a}|$  is the distance from  $A(\mathbf{a})$  to the point  $B(\mathbf{b})$ . Since complex numbers are viewed as vectors in 2D, this concept applies to them.

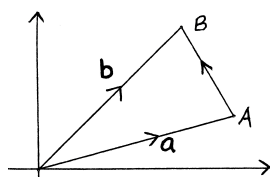


Figure 2.10:

### Example 2.19

Sketch the locus of  $z$  given that

- (a)  $|z| = 2$                       (b)  $|z - 1 + 2i|$                       (c)  $|z + 3| = 3$   
 (d)  $|z - i| = 3$                       (e)  $|2z + 3 - 2i| = 4$

SOLUTION

(a)  $|z - (0 + 0i)| = 2$

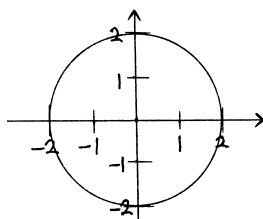


Figure 2.11:

(b)  $|z - (1 - 2i)| = 3$

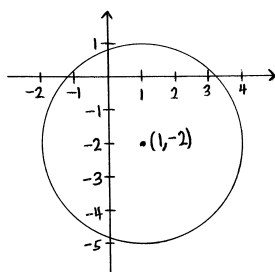


Figure 2.12:

(c)  $|z - (-3 + 0i)| = 2$

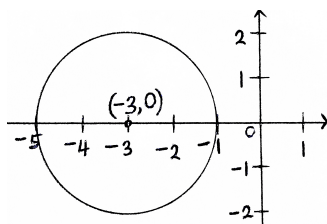


Figure 2.13:

(d)  $|z - (0 + i)| = 3$

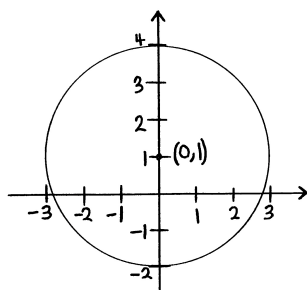


Figure 2.14:

(e)

$$\begin{aligned} |2z + 3 - 2i| &= 4 \\ |2| \left| z + \frac{3}{2} - i \right| &= 4 \\ \left| z - \left( -\frac{3}{2} + i \right) \right| &= 2 \end{aligned}$$

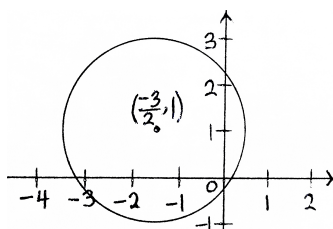


Figure 2.15:

**Example 2.20**

Sketch the locus of  $z$  and obtain the Cartesian equation of the locus of  $z$  where

(a)  $|z + 2| = |z - i|$

(b)  $\frac{|z - 3 + 2i|}{|z + 2 - 3i|} = 1$

SOLUTION

(a)  $|z - (-2 + 0i)| = |z - (0 + i)|$

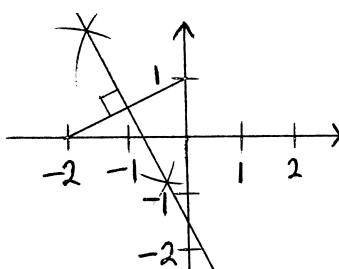


Figure 2.16: The perpendicular bisector of the line joining  $(-2, 0)$  and  $(0, 1)$

Let  $z = x + iy$ . Then

$$|(x + 2) + yi| = |x + (y - 1)i|$$

$$(x + 2)^2 + y^2 = x^2 + (y - 1)^2$$

$$x^2 + 4x + 4 + y^2 = x^2 + y^2 - 2y + 1$$

$$2y + 4x + 3 = 0$$

(b)  $|z - (3 - 2i)| = |z - (-2 + 3i)|$

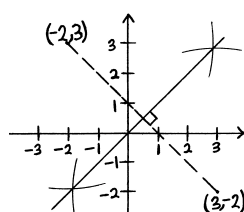


Figure 2.17: The perpendicular bisector of the line joining  $(-2, 3)$  and  $(3, -2)$



Let  $z = x + iy$ . Then we have

$$\begin{aligned} |(x-3) + (y+2)i| &= |(x+2) + (y-3)i| \\ (x-3)^2 + (y+2)^2 &= (x+2)^2 + (y-3)^2 \\ x^2 - 6x + 9 + y^2 + 4y + 4 &= x^2 + 4x + 4 + y^2 - 6y + 9 \\ y &= x \end{aligned}$$

**Example 2.21**

Identify the locus of  $z$  given that  $|z + 2| = 3|z - 2 - 4i|$ .

SOLUTION

Let  $z = x + iy$ . Then

$$\begin{aligned} |(x+2) + yi| &= 3|(x-2) + (y-4)i| \\ (x+2)^2 + y^2 &= 9[(x-2)^2 + (y-4)^2] \\ x^2 + 4x + 4 + y^2 &= 9(x^2 - 4x + 4 + y^2 - 8y + 16) \\ 8x^2 + 8y^2 - 40x - 72y + 176 &= 0 \\ x^2 + y^2 - 5x - 9y + 22 &= 0 \\ \left(x - \frac{5}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 &= -22 + \frac{25}{4} + \frac{81}{4} = \frac{9}{2} \end{aligned}$$

The locus of  $z$  is a circle of radius  $\frac{3}{\sqrt{2}}$  and centre  $\left(\frac{5}{2}, \frac{9}{2}\right)$

**Remark 2.9**

The angles subtended at the circumference of a circle by a chord of that circle, in a segment, are all equal.

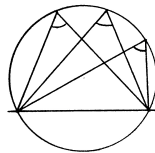


Figure 2.18:

Consider the figure below:

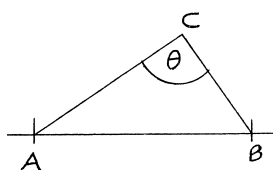


Figure 2.19:

If the point  $C$  moves such that  $\theta = \angle ACB$  remains constant, then  $C$  is moving along an arc of a circle which passes through  $A$  and  $B$ . Thus the locus of  $C$  is as shown below:

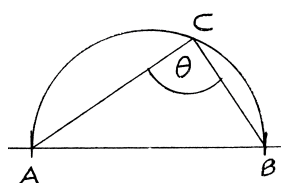


Figure 2.20:

### Example 2.22

Sketch the locus of  $z$  where  $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{3}$

SOLUTION

$\arg(z-i) - \arg(z+2) = \frac{\pi}{3}$ . Let  $\arg(z-i) = \alpha$  and  $\arg(z+2) = \beta$ . Then  $\alpha - \beta = \frac{\pi}{3}$ .

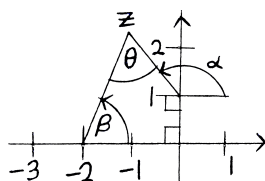


Figure 2.21:

$$\theta + \beta + \frac{\pi}{2} + \frac{\pi}{2} + \pi - \alpha = 2\pi$$

$$\theta + \beta - \alpha + 2\pi = 2\pi$$

$$\theta = \alpha - \beta = \frac{\pi}{3}$$

$z$  moves on an arc of a circle which passes through  $(0, 1)$  and  $(-2, 0)$  as shown below:

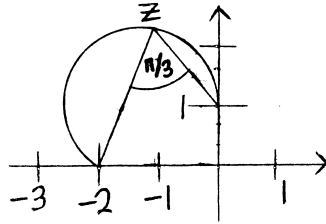


Figure 2.22:

**Remark 2.10**

We may be asked to find the centre and radius of the circle on whose arc lies  $z$ . We proceed as follows for Example(2.23): Let  $z = x + iy$ . Then

$$\frac{z - i}{z + 2} = \frac{x(x + 2) + (y - 1)y}{(x + 2)^2 + y^2} + \frac{(x + 2)(y - 1) - xy}{(x + 2)^2 + y^2}i$$

We have that

$$\begin{aligned} \frac{(x + 2)(y - 1) - xy}{x(x + 2) + y(y - 1)} &= \sqrt{3} \\ \sqrt{3}x^2 + \sqrt{3}y^2 + (2\sqrt{3} + 1)x + (-\sqrt{3} - 2)y + 2 &= 0 \end{aligned}$$

Multiplying through by  $\frac{\sqrt{3}}{3}$ , we get

$$\begin{aligned} x^2 + y^2 + \frac{(6 + \sqrt{3})}{3}x - \frac{(3 + 2\sqrt{3})}{3} + \frac{2\sqrt{3}}{3} &= 0 \\ \left[ x + \frac{(6 + \sqrt{3})}{6} \right]^2 + \left[ y - \frac{(3 + 2\sqrt{3})}{6} \right]^2 &= \frac{5}{3}, \text{ by completing squares.} \end{aligned}$$

We have that centre =  $\left( -\frac{(6 + \sqrt{3})}{6}, \frac{(3 + 2\sqrt{3})}{6} \right)$ , and radius =  $\sqrt{\frac{5}{3}}$ . Try with

$$\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{4}$$

**Example 2.23**

Sketch the locus of  $z$  such that

(a)  $\arg(z) = \frac{\pi}{4}$

(b)  $\arg(z - 2 + 3i) = \frac{2\pi}{3}$

SOLUTION

$$(a) \arg(z - (0 + 0i)) = \frac{\pi}{4}$$

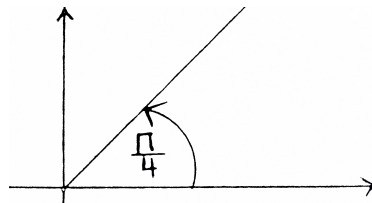


Figure 2.23:

$$(b) \arg(z - (2 - 3i)) = \frac{2\pi}{3}$$

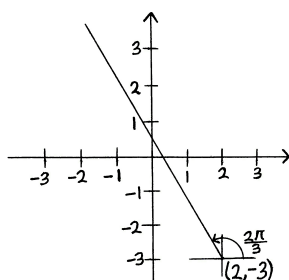


Figure 2.24:

### Modulus Inequalities

#### Example 2.24

Shade the region on an Argand diagram that represents

$$(a) |z + i| \leq 4$$

$$(b) |z - 2| < 2$$

$$(c) |z - 3| < |z - i|$$

$$(d) 2 < |z - 3 - 2i| < 3$$

(a)

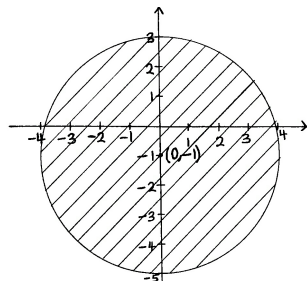


Figure 2.25:

(b)

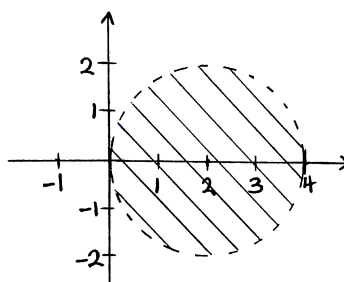


Figure 2.26:

(c)

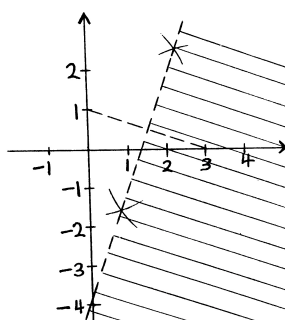


Figure 2.27:

(d)

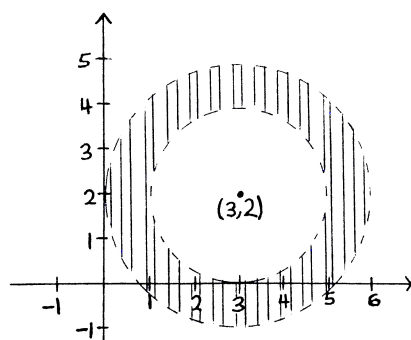


Figure 2.28:

### Elementary transformations from the $z$ -plane to the $w$ -plane

Let  $z$  and  $w$  be two complex variables such that  $w = f(z)$ . As  $z$  describes a locus in the  $z$ -plane,  $w$  describes a related locus in the  $w$ -plane in accordance with the relation  $w = f(z)$  that exists between  $w$  and  $z$ .

#### Example 2.25

For the transformation  $\frac{2}{i+2z}$ ,  $z \neq \frac{-i}{2}$ , find the locus of  $w$  as  $z$  moves on the circle with equation  $|z| = 1$ .

#### SOLUTION

By making  $z$  the subject, we get  $z = \frac{2 - wi}{2w}$ . Since  $|z| = 1$ , we have that  $\frac{2 - wi}{2w} = 1$ , thus  $|2 - wi| = 2|w|$ . Let  $w = u + iv$ . Then we get

$$\begin{aligned} |(2 + v) - ui| &= 2|u + iv| \\ (2 + v)^2 + u^2 &= 4(u^2 + v^2) \\ u^2 + v^2 - \frac{4}{3}v &= \frac{4}{3} \\ u^2 + \left(v - \frac{2}{3}\right)^2 &= \frac{16}{9} \end{aligned}$$

The locus of  $w$  is a circle with radius  $\frac{4}{3}$  and centre  $\left(0, \frac{2}{3}\right)$ .

**Example 2.26**

Find the locus of  $w$  given that  $w = z + \frac{1}{z}$  and that  $z$  moves on the circle with equation  $|z| = 3$ .

SOLUTION

We have that  $z = 3(\cos \theta + i \sin \theta)$ , and so  $\frac{1}{z} = \frac{1}{3}(\cos \theta - i \sin \theta)$ . Hence  $w = \frac{10}{3} \cos \theta + \frac{8}{3}i \sin \theta$ . Let  $u = \frac{10}{3} \cos \theta$  and  $v = \frac{8}{3} \sin \theta$ . Then we have

$$\begin{aligned} \frac{9u^2}{100} &= \cos^2 \theta & \text{and} & & \frac{9v^2}{64} &= \sin^2 \theta \\ \frac{9u^2}{100} + \frac{9v^2}{64} &= \cos^2 \theta + \sin^2 \theta = 1 \\ \frac{9u^2}{100} + \frac{9v^2}{64} &= 1 \end{aligned}$$

The locus of  $w$  is an ellipse.

Exercise Two

1. Solve the following equations for  $z = a + bi$  where  $a, b \in \mathbb{R}$ .

(a)  $\frac{z}{1 + \bar{z}} = 3 + 4i$

(b)  $z + 2\bar{z} = \frac{2 - i}{1 + 3i}$

(c)  $(\bar{z})^2 = 4z$

2. Write the given numbers in the form  $a + bi$  where  $a, b \in \mathbb{R}$ .

(a)  $2i^6 + \left(\frac{2}{-i}\right)^3 + 5i^{-5} - 12i$

(b)  $2i^3 - 3i^2 + 5i$

(c)  $(2 + 3i) \left(\frac{2 - i}{1 + 2i}\right)^2$

(d)  $\left(\frac{1}{2} - \frac{1}{4}i\right) \left(\frac{2}{3} + \frac{5}{3}i\right)$

3. If  $z$  is a complex number such that  $|z| = 1$ , compute

$$|1 + z|^2 + |1 - z|^2.$$

4. Suppose that  $z = a + bi$ ,  $w = u + vi$  where

$$a = \left(\frac{|w| + u}{2}\right)^{\frac{1}{2}}, \quad b = \left(\frac{|w| - u}{2}\right)^{\frac{1}{2}}.$$

Prove That  $z^2 = w$  if  $v \geq 0$  and that  $(\bar{z})^2 = w$  if  $v \leq 0$ .

5. Let  $F : \mathbb{C} \rightarrow \mathbb{C}$  be defined by  $F(a + bi) = a - bi$ . Show that:

(a)  $F(xy) = F(x)F(y)$  for all  $x, y \in \mathbb{C}$ .

(b)  $F(x\bar{x}) = |x|^2$ .

(c) Using parts (a) and (b) show that  $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$ .

6. Express the following in polar form  $r(\cos \theta + i \sin \theta)$  where  $-\pi < \theta \leq \pi$ .

(a)  $5 - 5i$

(b)  $-3i$



(c)  $-\sqrt{3} + i$

7. (a) Use De Moivre's theorem to prove that

$$(\cos p\theta + i \sin p\theta)(\cos q\theta + i \sin q\theta) = \cos(p + q)\theta + i \sin(p + q)\theta.$$

Hence simplify

$$\frac{\cos 3\theta + i \sin 3\theta}{\cos 5\theta - i \sin 5\theta}$$

(b) Use De Moivre's theorem to simplify the following:

- i.  $\frac{(\cos 4\theta + i \sin 4\theta)(\cos 3\theta + i \sin 3\theta)}{\cos 5\theta + i \sin 5\theta}$
- ii.  $\frac{\cos \theta - i \sin \theta}{\cos 4\theta - i \sin 4\theta}$
- iii.  $\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)^2 \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)^4$

8. Use de Moivre's theorem to prove that, if  $\theta$  is not a multiple of  $\pi$ ,

$$\frac{\sin 5\theta}{\sin \theta} = 16 \cos^4 \theta - 12 \cos^2 \theta + 1.$$

9. Express the following in terms of cosines of multiples of  $\theta$

(a)  $\cos^7 \theta$

(b)  $\sin^6 \theta$

10. Find the square roots of

(a)  $-7 + 24i$

(b)  $8i$

(c)  $16 - 30i$

11. Find the four fourth roots of  $z = 1 + i$ .

12. One root of the equation  $2z^4 - 11z^3 + 27z^2 - 25z + 7 = 0$  is  $2 - i\sqrt{3}$ . Find the other three roots.

13. Sketch the locus of  $z$  where

(a)  $\arg\left(\frac{z - 2i}{z - 3}\right) = \frac{\pi}{6}$

(b)  $\arg\left(\frac{z}{z - 1 + 2i}\right) = \frac{\pi}{3}$

(c)  $\arg(z + 1 - i) = \pi$

14. Shade on an Argand diagram the region represented by

(a)  $1 < |z + 4 - i| \leq 5$

(b)  $|z + 3 - 2i| > 5$

15. Shade and label on an Argand diagram the region  $R$  for which

$$|z| \leq |z - i|.$$

Given that

$$w = \frac{2z - 3i}{2}$$

shade and label on another Argand diagram the region  $S$  of the  $z$ -plane for which

$$|w| \leq |w - i|.$$

16. Show that the transformation given by

$$\frac{w + i}{w - i} = \frac{z + 1}{z - 1}, \quad w \neq i, \quad z \neq 1$$

transforms the real axis of the  $z$ -plane into the imaginary axis of the  $w$ -plane.

17. The transformation

$$w = \frac{z + 2i}{iz + 2}$$

where  $z \neq 2i$ , maps the complex number  $z = x + yi$  onto the complex number  $w = u + vi$ .

- (a) Find the complex numbers representing the two points in the complex plane which are invariant under the transformation, giving your answers in the form  $a + bi$ , where  $a, b \in \mathbb{R}$ .
- (b) Show that if the point representing  $z$  lies on the imaginary axis, then the point representing  $w$  also lies on the imaginary axis.
- (c) Show that if the point representing  $z$  lies on the real axis, then the point representing  $w$  lies on the circle with equation  $u^2 + v^2 = 1$ . Sketch this circle, and indicate clearly onto which part of the  $w$ -plane the positive real axis of the  $z$ -plane is mapped.



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## Chapter 3

# Conic Sections(Conics)

### Definition 3.1 (A conic)

Let  $D$  denote a fixed line called the directrix. Let  $F$  denote a fixed point called the focus, which is not on  $D$ . Let  $e$  be a fixed positive number called the eccentricity. A conic is the set of points  $P$  in the plane such that the ratio of the distance from  $F$  to  $P$  to the distance from  $D$  to  $P$  equals  $e$ , That is,  $\frac{|FP|}{|DP|} = e$ .

If  $e = 1$ , the conic is a parabola

If  $e < 1$ , the conic is an ellipse

If  $e > 1$ , the conic is a hyperbola

### The Parabola ( $e = 1$ )

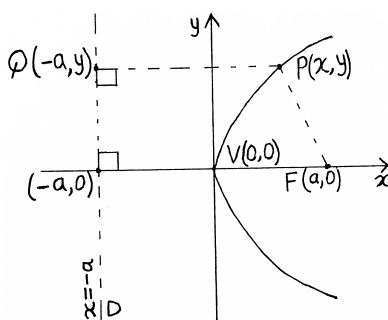


Figure 3.1: The parabola  $y^2 = 4ax$

The line through the focus  $F$  and perpendicular to the directrix  $D$  is called the axis of symmetry of the parabola. The point of intersection of the parabola with its axis of symmetry is called the vertex  $V$ .  $a > 0$  is the distance from  $F$  to  $V$ .

From Definition(3.1),

$$\begin{aligned}
 |FP|^2 &= |QP|^2 \\
 (x-a)^2 + (y-0)^2 &= (x+a)^2 + (y-y)^2 \\
 x^2 + a^2 - 2ax + y^2 &= x^2 + a^2 + 2ax \\
 \boxed{y^2 = 4ax}
 \end{aligned}$$

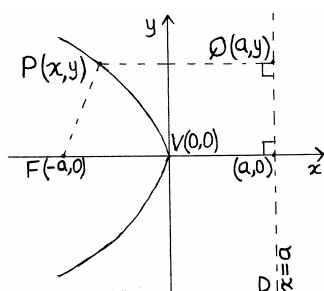


Figure 3.2: The parabola  $y^2 = -4ax$

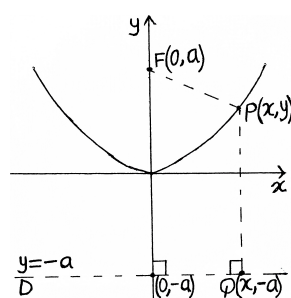


Figure 3.3: The parabola  $x^2 = 4ay$

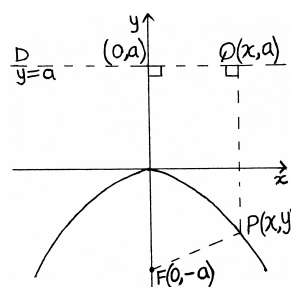


Figure 3.4: The parabola  $x^2 = -4ay$

**Definition 3.2 (A Parabola)**

A parabola is the collection of all points  $P$  in the plane that are the same distance from a fixed point  $F$  as they are from a fixed line  $D$ .

**Definition 3.3 (Latus Rectum)**

A chord to a parabola, that passes through the focus is called a focal chord. A focal chord perpendicular to the axis of symmetry is called the Latus Rectum.

Consider the parabola  $y^2 = 4ax$ . When  $x = a$ , we have that  $y = \pm 2a$ . Thus the latus rectum meets the parabola at the points  $(a, 2a)$  and  $(a, -2a)$ , and it is of length  $4a$ .

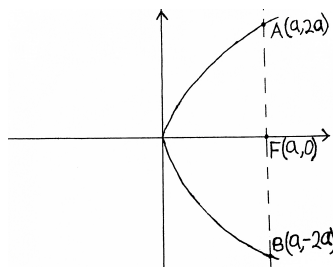


Figure 3.5: The latus rectum

**Example 3.1**

Analyse the equation  $x^2 = -12y$ .

**SOLUTION**

We have that  $x^2 = -12y = -4(3)y$ . Thus  $a = 3$ , Vertex is  $(0, 0)$ , Focus =  $(0, -3)$  and directrix is  $y = 3$ . The latus rectum is  $y = -3$ . It meets the parabola at  $(6, -3)$  and  $(-6, -3)$  and it is of length 12.

**Parabolas with Vertex at  $(h, k)$**

Consider a parabola with vertex at the origin and axis of symmetry being a coordinate axis. Suppose that the parabola is shifted  $h$  units horizontally and  $k$  units vertically. The resulting parabola has vertex at  $(h, k)$  with axis of symmetry parallel to a coordinate axis. The equations of such parabolas have the same forms as those with vertex at the origin, except that  $x$  is replaced by  $x - h$  (the horizontal shift) and  $y$  is replaced by  $y - k$  (the vertical shift).

**Remark 3.1**

A polynomial equation involving two variables such that it is quadratic in one variable and linear in the other defines a parabola.

**Example 3.2**

Analyse the equation  $x^2 + 4x - 4y = 0$

**SOLUTION**

By completing the squares, we get  $(x + 2)^2 = 4y + 4 = 4(y + 1)$ . It is easily seen that Vertex =  $(-2, -1)$  and  $a = 1$ . In order to obtain other pieces of information about the parabola, it is advisable to sketch it.

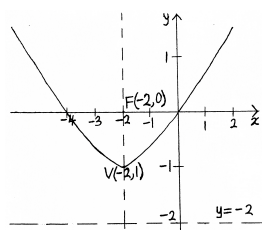


Figure 3.6:

We have that the focus is  $(-2, 0)$ , the directrix is  $y = -2$ , the latus rectum is  $y = 0$  and it is defined by  $(-4, 0)$  and  $(0, 0)$ .

**Alternatively:**

Let  $X = x + 2$  and  $Y = y + 1$ . Then

$$(x + 2)^2 = 4(y + 1) \text{ --- (*)}$$

$$X^2 = 4Y \text{ --- (**)}$$

For equation(\*\*): Vertex =  $(0, 0)$ ,  $a = 1$ , Focus =  $(0, 1)$  and directrix  $Y = -1$

For equation\*:

$X = 0$  and  $Y = 0 \Rightarrow x = -2$  and  $y = -1$ . Equation (\*) has vertex  $(-2, -1)$

$X = 0$  and  $Y = 1 \Rightarrow x = -2$  and  $y = 0$ . Equation (\*) has focus  $(-2, 0)$

$Y = -1 \Rightarrow y = -2$ . Equation (\*) has directrix  $y = -2$

The latus rectum passes through  $(-2, 0)$  and it is parallel to the  $x$ -axis. Thus its equation is  $y = 0$  and it meets the parabola at  $(-4, 0)$  and  $(0, 0)$ .



**Example 3.3**

Find the equation of the parabola with vertex at  $(-2, 3)$  and focus at  $(0, 3)$ .

SOLUTION

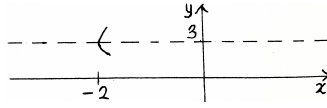


Figure 3.7:

$y = 3$  is the axis of symmetry,  $a = 2$ . The parabola opens to the right. Equation of the parabola is  $(y - 3)^2 = 4(2)(x + 2)$ , that is  $(y - 3)^2 = 8(x + 2)$ .

**Parametric Equations of the Parabola**

The parabola  $y^2 = 4ax$  has parametric equations

$$x = at^2 \quad \text{and} \quad y = 2at$$

where  $a > 0$  and  $t \in \mathbb{R}$ . Thus  $(at^2, 2at)$  is an arbitrary point on the parabola  $y^2 = 4ax$ .

We can verify this as follows:  $x = at^2 = a \left( \frac{y}{2a} \right)^2 = a \left( \frac{y^2}{4a^2} \right) = \frac{y^2}{4a}$ , giving us  $y^2 = 4ax$ .

Similarly, the parabola  $x^2 = 4ay$  has parametric equations

$$x = 2at \quad \text{and} \quad y = at^2$$

where  $a > 0$  and  $t \in \mathbb{R}$ . Hence  $(2at, at^2)$  is an arbitrary point on the parabola  $x^2 = 4ay$ .

**Tangents and Normals to the Parabola  $y^2 = 4ax$  at  $(at^2, 2at)$**

$x = at^2 \Rightarrow \frac{dy}{dt} = 2at$  and  $y = 2at \Rightarrow \frac{dy}{dt} = 2a$ . Consequently we have that

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$ . Alternatively, from  $y^2 = 4ax$ , we have

$2y \frac{dy}{dx} = 4a$ , giving us,  $\frac{dy}{dx} = \frac{2a}{y}$ . Thus  $\frac{dy}{dx} \Big|_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$ . The equation of the tangent

to the parabola  $y^2 = 4ax$  at  $(at^2, 2at)$  is  $y - 2at = \frac{1}{t}(x - at^2)$ , that is  $ty = x + at^2$ .

Gradient of normal at  $(at^2, 2at) = \frac{-1}{\left(\frac{1}{t}\right)} = -t$ . The equation of the normal to the parabola

$y^2 = 4ax$  at  $(at^2, 2at)$  is  $y - 2at = -t(x - at^2)$ , that is,  $y + tx = 2at + at^3$ .

**Tangents and Normals to the Parabola  $y^2 = 4ax$  at  $(h, k)$**

$\left. \frac{dy}{dx} \right|_{(h,k)} = \frac{2a}{k}$ . Equation of the tangent to the parabola  $y^2 = 4ax$  at  $(h, k)$  is  $y - k = \frac{2a}{k}(x - h)$ , that is  $ky - k^2 = 2ax - 2ah$ . Since  $(h, k)$  lies on  $y^2 = 4ax$ , we get  $k^2 = 4ah$ . Thus  $ky = 4ah + 2ax - 2ah$ , that is  $ky = 2a(h + x)$ . Gradient of normal at  $(h, k)$  is  $-\frac{k}{2a}$ . Equation of normal to the parabola  $y^2 = 4ax$  at  $(h, k)$  is  $y - k = -\frac{k}{2a}(x - h)$ , that is  $2a(y - k) = k(h - x)$ .

**Example 3.4**

The normal to the parabola with equation  $y^2 = 4ax$  at the point  $P(at^2, 2at)$  meets the curve again at the point  $Q(as^2, 2as)$ . Show that  $s = -t - \frac{2}{t}$ . Show that as  $t$  varies, the locus of the mid-point of  $PQ$  has the equation  $y^4 + 2a(2a - x)y^2 + 8a^4 = 0$ .

**SOLUTION**

We have that  $2y \frac{dy}{dx} = 4a$ , giving us  $\frac{dy}{dx} = \frac{2a}{y}$ . Hence  $\left. \frac{dy}{dx} \right|_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$ . Equation of normal to the parabola  $y^2 = 4ax$  at  $P(at^2, 2at)$  is  $y - 2at = -t(x - at^2)$ . The point  $Q(as^2, 2as)$  lies on the normal to  $y^2 = 4ax$  at  $P(at^2, 2at)$  and so we get

$$\begin{aligned} 2as - 2at &= -t(as^2 - at^2) \\ 2a(s - t) &= -at(s^2 - t^2) \\ 2 &= -t(s + t) \\ s &= -t - \frac{2}{t} \end{aligned}$$

The mid-point of  $PQ$  is  $\left( \frac{a(t^2 + s^2)}{2}, a(t + s) \right)$ . Parametric equations of the locus of the mid-point of  $PQ$  are  $x = \frac{a(t^2 + s^2)}{2}$  and  $y = a(t + s)$ . From  $y = a(t + s)$  we obtain

$$\begin{aligned} \frac{y}{a} &= t + s \\ &= t - t - \frac{2}{t}, \text{ since } s = -t - \frac{2}{t} \\ &= \frac{-2}{t} \end{aligned}$$

Hence  $t = \frac{-2a}{y}$ .

From  $x = \frac{a(t^2 + s^2)}{2}$  we obtain

$$\begin{aligned}\frac{2x}{a} &= t^2 + s^2 \\ &= t^2 + \left(t + \frac{2}{t}\right)^2 \\ &= \frac{1}{t^2} [2t^4 + 4t^2 + 4]\end{aligned}$$

Multiplying through by  $t^2$  and making the substitution  $t = \frac{-2a}{y}$ , we get

$$\begin{aligned}\frac{2x}{a} \left[ \frac{4a^2}{y^2} \right] &= 2 \left( \frac{16a^4}{y^4} \right) + 4 \left( \frac{4a^2}{y^2} \right) + 4 \\ \frac{8ax}{y^2} &= \frac{32a^4}{y^4} + \frac{16a^2}{y^2} + 4\end{aligned}$$

Multiplying through by  $\frac{y^4}{4}$ , we get

$$\begin{aligned}2axy^2 &= 8a^4 + 4a^2y^2 + y^4 \\ y^4 + 4a^2y^2 - 2axy^2 + 8a^4 &= 0 \\ y^4 + 2a(2a - x)y^2 + 8a^4 &= 0\end{aligned}$$

### Example 3.5

Find an equation of the normal at  $P(4t, 4t^2)$  on the parabola with equation  $x^2 = 4y$ . The normal at  $P$  meets the  $y$ -axis at the point  $Q$ . Find, in cartesian form an equation of the locus of the point  $M$ , the mid-point of  $PQ$ , as  $t$  varies.

### SOLUTION

From  $4y = x^2$ , we obtain  $4\frac{dy}{dx} = 2x$ , which then yields  $\frac{dy}{dx} = \frac{x}{2}$ . Hence

$\frac{dy}{dx} \Big|_{(4t, 4t^2)} = \frac{4t}{2} = 2t$ . Equation of normal to  $x^2 = 4y$  at  $P$  is  $y - 4t^2 = -\frac{1}{2t}(x - 4t)$ . At  $Q$ ,  $x = 0$ , and so  $y = 4t^2 - \frac{1}{2t}(-4t) = 4t^2 + 2$ . That is  $Q$  is  $(0, 4t^2 + 2)$ .

$M = \left( \frac{0 + 4t}{2}, \frac{4t^2 + 2 + 4t^2}{2} \right) = (2t, 4t^2 + 1)$ . Parametric equations of the locus of  $M$  are  $x = 2t$  and  $y = 4t^2 + 1$ . Eliminating  $t$ , we get  $x^2 = 4t^2 = y - 1$ , that is  $y = x^2 + 1$ . The Cartesian equation of the locus of  $M$  is  $y = x^2 + 1$ .

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The Ellipse ( $e < 1$ )

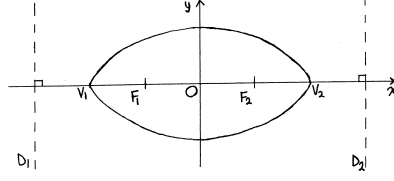


Figure 3.8: The ellipse

The line containing the foci is called the major axis. The mid-point of the line joining the foci is the centre of the ellipse. The perpendicular bisector of the line  $F_1F_2$  is called the minor axis. The major axis and the minor axis meet at the centre. The two points of intersection of the ellipse and the major axis are the vertices  $V_1$  and  $V_2$ . The distance from one vertex to the other is the length of the major axis.

Take the points  $V_1$  and  $V_2$  to be  $(-a, 0)$  and  $(a, 0)$  respectively, where  $a \in \mathbb{R}^+$ . Then the point  $(ae, 0)$  lies between  $O$  and  $V_2(a, 0)$ . We take  $(ae, 0)$  as the focus  $F_2$ . The point  $(\frac{a}{e}, 0)$  lies to the right of  $V_2(a, 0)$ . We take the line  $x = \frac{a}{e}$  as the directrix  $D_2$ .

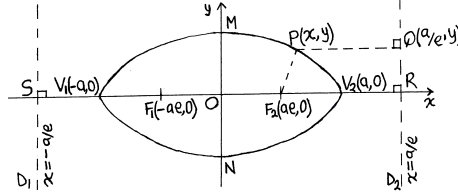


Figure 3.9:

Let  $P(x, y)$  be any point on the ellipse. From Definition(3.1),  $\frac{|F_2P|}{|QP|} = e$ , that is,  $|F_2P|^2 = e^2|QP|^2$ . This gives us

$$\begin{aligned} (x - ae)^2 + (y - 0)^2 &= e^2 \left[ \left( x - \frac{a}{e} \right)^2 + (y - y)^2 \right] \\ x^2 + a^2e^2 - 2aex + y^2 &= e^2 \left( x^2 + \frac{a^2}{e^2} - \frac{2ax}{e} \right) \\ x^2 + a^2e^2 - 2aex + y^2 &= e^2x^2 + a^2 - 2aex \\ (1 - e^2)x^2 + y^2 &= (1 - e^2)a^2 \\ \frac{x^2}{a^2} + \frac{y^2}{(1 - e^2)a^2} &= 1 \end{aligned}$$

### CHAPTER 3. CONIC SECTIONS(CONICS)

Let  $(1 - e^2)a^2 = b^2$ , where  $b > 0$ . Then we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From the equation above we obtain  $x^2 = \left(1 - \frac{y^2}{b^2}\right)a^2$ . Since  $x^2 \geq 0$  and  $a^2 > 0$ , we have that  $1 - \frac{y^2}{b^2} \geq 0$ , which then yields  $y^2 \leq b^2$ . Thus  $|y| \leq b$ , that is,  $-b \leq y \leq b$ . This means that the points  $M$  and  $N$  are  $(0, b)$  and  $(0, -b)$  respectively.

Similarly,  $y^2 = \left(1 - \frac{x^2}{a^2}\right)b^2$  implies that  $1 - \frac{x^2}{a^2} \geq 0$ , and so  $x^2 \leq a^2$ . Thus  $|x| \leq a$ , that is  $-a \leq x \leq a$ . This confirms that  $V_1(-a, 0)$  and  $V_2(a, 0)$  are actually lying on the ellipse. Now,  $b^2 = a^2(1 - e^2) = a^2 - a^2e^2$ , and so  $a^2 - b^2 = a^2e^2 > 0$ . Thus  $a^2 > b^2$ , hence  $a > b > 0$ .

$\therefore$  The Cartesian equation of the ellipse with centre  $(0, 0)$ , foci at  $(-ae, 0)$  and  $(ae, 0)$ , vertices at  $(-a, 0)$  and  $(a, 0)$  is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where  $a > b > 0$  and  $b^2 = (1 - e^2)a^2$

Similarly:

The Cartesian equation of the ellipse with centre  $(0, 0)$ , foci at  $(0, -ae)$  and  $(0, ae)$ , vertices at  $(0, -a)$  and  $(0, a)$  is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

where  $a > b > 0$  and  $b^2 = (1 - e^2)a^2$

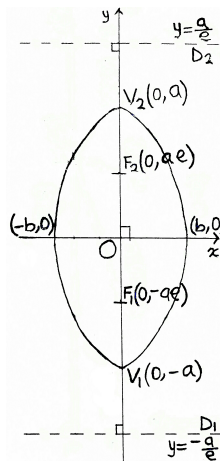


Figure 3.10:

**Example 3.6**

Analyse the equation

(a)  $9x^2 + y^2 = 9$

(b)  $9x^2 + 25y^2 = 225$

SOLUTION

(a) Dividing through the given equation by 9, we get  $\frac{x^2}{1^2} + \frac{y^2}{3^2} = 1$ . By inspection,  $a = 3, b = 1$ , centre =  $(0, 0)$ . Since the  $y$ -axis is the major axis, vertices are  $(0, 3)$  and  $(0, -3)$ .

$$\begin{aligned} b^2 &= a^2 - (ae)^2 \\ (ae)^2 &= a^2 - b^2 = 9 - 1 = 8 \\ ae &= 2\sqrt{2} \end{aligned}$$

The foci are  $F_1 (0, -2\sqrt{2})$  and  $F_2 (0, 2\sqrt{2})$  and the eccentricity is  $e = \frac{2}{3}\sqrt{2}$ . Intercepts on the  $x$ -axis(the minor axis) are  $(-1, 0)$  and  $(1, 0)$ . The directrices are  $y = \pm \frac{9}{4}\sqrt{2}$ .

(b) Dividing through the given equation by 225, we get  $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ . By inspection  $a = 5, b = 3$ , centre =  $(0, 0)$ . The major axis is the  $x$ -axis, so the vertices are  $(-5, 0)$  and  $(5, 0)$ . Intercepts on the minor axis(the  $y$ -axis) are  $(0, -3)$  and  $(0, 3)$ . Now,

$(ae)^2 = a^2 - b^2 = 25 - 9 = 16$ , and so  $ae = 4$  and  $e = \frac{4}{5}$ . The foci are  $(-4, 0)$  and  $(4, 0)$ . The directrices are  $x = \pm \frac{25}{4}$ .

**Example 3.7**

Find the equation of the ellipse having one focus at  $(0, 2)$  and vertices at  $(0, -3)$  and  $(0, 3)$ .

SOLUTION

The major axis is the  $y$ -axis and  $(0, 0)$  is the centre.  $ae = 2$  and  $a = 3$ , hence  $b^2 = a^2 - (ae)^2 = 9 - 4 = 5$ . Equation of the ellipse is  $\frac{x^2}{5} + \frac{y^2}{9} = 1$ , that is,  $9x^2 + 5y^2 = 45$ .

**Example 3.8**

Find the Cartesian equation of the ellipse with foci  $\left(\pm \frac{8}{9}, 0\right)$  and directrices  $x = \pm 2$ .

We have

$$ae = \frac{8}{9} \text{------(1)}$$

$$\frac{a}{e} = 2 \text{----- (2)}$$

From (2)  $e = \frac{a}{2}$ . Putting  $e = \frac{a}{2}$  into (1), we get  $a \left(\frac{a}{2}\right) = \frac{8}{9}$ , and so  $a^2 = \frac{16}{9}$ . We also have that  $b^2 = a^2 - (ae)^2 = \frac{16}{9} - \frac{64}{81} = \frac{80}{81}$ . Equation of ellipse is  $\frac{x^2}{\left(\frac{16}{9}\right)} + \frac{y^2}{\left(\frac{80}{81}\right)} = 1$ .

**Remark 3.2**

When  $a = b$ ,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  becomes  $x^2 + y^2 = a^2$ , which is the equation of a circle centred at  $(0, 0)$  with radius  $a$ .

**The focal distances property of the ellipse**

Consider the ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $a > b > 0$  and  $b^2 = a^2 - (ae)^2 = a^2(1 - e^2)$ .

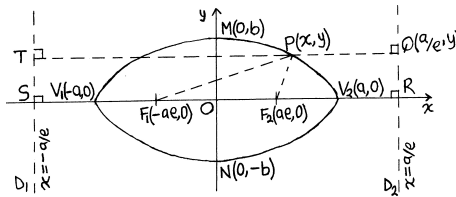


Figure 3.11:

Applying Definition(3.1) to the focus  $F_2$  and directrix  $x = \frac{a}{e}$ , we get

$$|F_2P| = e|QP| \text{----- (1)}$$

Applying Definition(3.1) to the focus  $F_1$  and directrix  $x = -\frac{a}{e}$ , we get

$$|F_1P| = e|TP| \text{----- (2)}$$

Adding (1) and (2), we get

$$\begin{aligned} |F_1P| + |F_2P| &= e(|TP| + |QP|) \\ &= e|TQ| \\ &= e(|SO| + |OR|) \\ &= e\left(\frac{a}{e} + \frac{a}{e}\right) \\ &= 2a \end{aligned}$$

**Definition 3.4 (The Ellipse)**

An ellipse is the collection of all points in the plane, the sum of whose distances from two fixed points, called the foci, is a constant.

[Try to obtain the cartesian equation of the ellipse using Definition(3.4) ]

**Ellipses with centre at  $(h, k)$**

The equations of ellipses with centre at  $(h, k)$  have the same forms as a standard ellipse except that  $x$  is replaced by  $x - h$  (the horizontal shift) and  $y$  is replaced by  $y - k$  (the vertical shift).

**Example 3.9**

Analyse the equation  $4x^2 + y^2 - 8x + 4y + 4 = 0$

**SOLUTION**

By completing the squares, we have

$$\begin{aligned} 4[x^2 - 2x] + [y^2 + 4y] &= -4 \\ 4[x^2 - 2x + (-1)^2 - (-1)^2] + [y^2 + 4y + 2^2 - 2^2] &= -4 \\ 4[(x - 1)^2 - 1] + (y + 2)^2 - 4 &= -4 \\ 4(x - 1)^2 - 4 + (y + 2)^2 &= 0 \\ 4(x - 1)^2 + (y + 2)^2 &= 4 \\ \frac{(x - 1)^2}{1^2} + \frac{(y + 2)^2}{2^2} &= 1 \end{aligned}$$

Thus  $a = 2$ ,  $b = 1$ , centre =  $(1, -2)$ . Also,  $(ae)^2 = 4 - 1 = 3$ , and so  $ae = \sqrt{3}$  and  $e = \frac{\sqrt{3}}{2}$ .

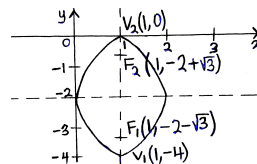


Figure 3.12:



### CHAPTER 3. CONIC SECTIONS(CONICS)

The major axis is  $x = 1$  and the minor axis is  $y = -2$ . The vertices are  $(1, 0)$  and  $(1, -4)$ . Intercepts on minor axis are  $(0, -2)$  and  $(2, -2)$ . From  $\frac{a}{e} = \frac{4\sqrt{3}}{3}$ , we see that the directrices are  $y = -2 + \frac{4}{3}\sqrt{3}$  and  $y = -2 - \frac{4}{3}\sqrt{3}$ .

**Alternatively:**

$$\frac{(x-1)^2}{1} + \frac{(y+2)^2}{4} = 1 \text{ --- } (E_1)$$

Let  $X = x - 1$  and  $Y = y + 2$ . Then

$$\frac{X^2}{1} + \frac{Y^2}{4} = 1 \text{ --- } (E_2)$$

For equation  $(E_2)$ , we have that  $a = 2$ ,  $b = 1$  and centre  $= (0, 0)$ .  $(ae)^2 = a^2 - b^2 = 3$ , and so  $ae = \sqrt{3}$  and  $e = \frac{\sqrt{3}}{2}$ . The major axis is  $X = 0$  and the minor axis is  $Y = 0$ .

The vertices are  $(0, 2)$  and  $(0, -2)$ . Intercepts on the minor axis are  $(1, 0)$  and  $(-1, 0)$ .  $\frac{a}{e} = \frac{4}{3}\sqrt{3}$ , and so the directrices are  $Y = \pm \frac{4}{3}\sqrt{3}$

For equation  $(E_1)$ , we make use of  $x = X + 1$  and  $y = Y - 2$ . Centre  $= (1, -2)$ . The major axis is  $x = 1$  and the minor axis is  $y = -2$ . The vertices are  $(-1, 0)$  and  $(1, -4)$ . Intercepts on the minor axis are  $(2, -2)$  and  $(0, -2)$ . The directrices are  $y = -2 \pm \frac{4}{3}\sqrt{3}$ .

#### Example 3.10

Find the equation of the ellipse with centre  $(2, -3)$ , a focus at  $(3, -3)$  and a vertex at  $(5, -3)$ .

#### SOLUTION

The major axis is  $y = -3$ . We have that  $ae = 3 - 2 = 1$ ,  $a = 5 - 2 = 3$  and  $b^2 = a^2 - (ae)^2 = 9 - 1 = 8$ . The equation of the given ellipse is  $\frac{(x-2)^2}{8} + \frac{(y+3)^2}{9} = 1$ .

#### Parametric equations of the ellipse

The ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  has parametric equations

$$x = a \cos t \quad \text{and} \quad y = b \sin t$$

where  $0 \leq t < 2\pi$  and  $a > b > 0$ . To see this, observe that  $\frac{x}{a} = \cos t$ ,  $\frac{y}{b} = \sin t$ , and so

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

Similarly, the ellipse  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$  has parametric equations

$$x = b \cos t \quad \text{and} \quad y = a \sin t$$

where  $0 \leq t < 2\pi$  and  $a > b > 0$ .

**Tangents and Normals to the ellipse**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  **at**  $(a \cos t, b \sin t)$

$$x = a \cos t \Rightarrow \frac{dx}{dt} = -a \sin t \text{ and } y = b \sin t \Rightarrow \frac{dy}{dt} = b \cos t. \text{ Hence } \frac{dy}{dx} = -\frac{b \cos t}{a \sin t}.$$

Alternatively, differentiating implicitly we get,  $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$ , and so

$$\frac{dy}{dx} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y} = -\frac{b^2 x}{y a^2}. \text{ Hence } \left. \frac{dy}{dx} \right|_{(a \cos t, b \sin t)} = -\frac{b^2 a \cos t}{a^2 b \sin t} = -\frac{b \cos t}{a \sin t}. \text{ Equation of the tangent to the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a \cos t, b \sin t) \text{ is}$$

$$\begin{aligned} y - b \sin t &= -\frac{b \cos t}{a \sin t}(x - a \cos t) \\ ya \sin t - ab \sin^2 t &= -xb \cos t + ab \cos^2 t \\ ya \sin t + xb \cos t &= ab(\cos^2 t + \sin^2 t) = ab \\ \frac{x}{a} \cos t + \frac{y}{b} \sin t &= 1 \end{aligned}$$

Equation of normal to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at  $(a \cos t, b \sin t)$  is

$$y - b \sin t = \frac{a \sin t}{b \cos t}(x - a \cos t), \text{ that is, } ax \sin t - by \cos t = (a^2 - b^2) \sin t \cos t.$$

**Tangents and normals to the ellipse**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  **at**  $(h, k)$

$$\frac{dy}{dx} = -\frac{b^2 x}{y a^2}, \text{ hence } \left. \frac{dy}{dx} \right|_{(h, k)} = -\frac{b^2 h}{a^2 k}. \text{ Equation of tangent to the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (h, k) \text{ is}$$

$$\begin{aligned} y - k &= -\frac{b^2 h}{a^2 k}(x - h) \\ a^2 k y - a^2 k^2 &= -b^2 h x + b^2 h^2 \\ b^2 h x + a^2 k y &= b^2 h^2 + a^2 k^2 \\ \frac{h}{a^2} x + \frac{k}{b^2} y &= \frac{h^2}{a^2} + \frac{k^2}{b^2}, \text{ by dividing through by } a^2 b^2 \end{aligned}$$

Since  $(h, k)$  lies on  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , we have that  $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$ . Thus  $\frac{h}{a^2} x + \frac{k}{b^2} y = 1$ .

Equation of normal is  $y - k = \frac{a^2 k}{b^2 h}(x - h)$ , that is,  $a^2 k x - b^2 h y = h k(a^2 - b^2)$ .

### Example 3.11

The tangents to the ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at the points  $P(a \cos \theta, b \sin \theta)$  and  $Q(-a \sin \theta, b \cos \theta)$  intersect at the point  $R$ . As  $\theta$  varies, show that  $R$  lies on the curve with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ .

Differentiating  $b^2x^2 + a^2y^2 = a^2b^2$  implicitly, we get  $2b^2x + 2a^2y \frac{dy}{dx} = 0$ . Hence  $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$ .

Thus  $\frac{dy}{dx}\bigg|_P = -\frac{b^2a \cos \theta}{a^2b \sin \theta} = -\frac{b \cos \theta}{a \sin \theta}$  and  $\frac{dy}{dx}\bigg|_Q = -\frac{b^2(-a \sin \theta)}{a^2b \cos \theta} = \frac{b \sin \theta}{a \cos \theta}$ .

Equation of the tangent to the ellipse at  $P$  is

$$y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta}(x - a \cos \theta) \text{ --- (E}_1\text{)}$$

Equation of the tangent to the ellipse at  $Q$  is

$$y - b \cos \theta = \frac{b \sin \theta}{a \cos \theta}(x + a \sin \theta) \text{ --- (E}_2\text{)}$$

Subtracting  $(E_1)$  from  $(E_2)$ , we get

$$\begin{aligned} b(\sin \theta - \cos \theta) &= \frac{b \cos \theta}{a \sin \theta}(x - a \cos \theta) + \frac{b \sin \theta}{a \cos \theta}(x + a \sin \theta) \\ \sin \theta - \cos \theta &= \frac{x \cos \theta}{a \sin \theta} + \frac{x \sin \theta}{a \cos \theta} - \frac{\cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta}{\cos \theta} \\ \frac{x}{a} \left[ \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} \right] &= \sin \theta - \cos \theta + \frac{\cos^2 \theta}{\sin \theta} - \frac{\sin^2 \theta}{\cos \theta} \\ \frac{x}{a} \left[ \frac{1}{\sin \theta \cos \theta} \right] &= \frac{1}{\sin \theta}(\cos^2 \theta + \sin^2 \theta) - \frac{1}{\cos \theta}(\sin^2 \theta + \cos^2 \theta) = \frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta} \\ \frac{x}{a} &= \cos \theta - \sin \theta \\ x &= a(\cos \theta - \sin \theta) \end{aligned}$$

Putting  $x = a \cos \theta - a \sin \theta$  into equation  $(E_2)$ , we obtain

$y = b \cos \theta + \frac{b \sin \theta}{a \cos \theta}(a \cos \theta) = b(\cos \theta + \sin \theta)$ . Hence the point  $R$  is given by  $(a(\cos \theta - \sin \theta), b(\cos \theta + \sin \theta))$ . Parametric equations of  $R$  are

$$x = a(\cos \theta - \sin \theta) \text{ and } y = b(\cos \theta + \sin \theta)$$

Eliminating  $\theta$  from the parametric equations we get

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= (\cos \theta - \sin \theta)^2 + (\cos \theta + \sin \theta)^2 \\ &= \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta \\ &= 2 \end{aligned}$$

Therefore The locus of  $R$  is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ .

**The Hyperbola ( $e > 1$ )**

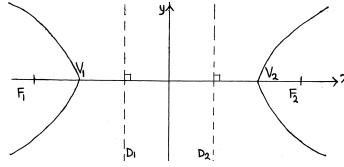


Figure 3.13: The hyperbola

The line through the foci is called the transverse axis. The mid-point of the line segment joining the foci is the centre of the hyperbola. The line through the centre and perpendicular to the transverse axis is the conjugate axis. The hyperbola consists of two separate curves, called branches. The transverse axis meets the hyperbola at two points called the vertices of the hyperbola.

Take the points  $V_1$  and  $V_2$  to be  $(-a, 0)$  and  $(a, 0)$  respectively, where  $a \in \mathbb{R}^+$ . Then the point  $(ae, 0)$  lies to the right of  $V_2$ . We take  $(ae, 0)$  as the focus  $F_2$ . The point  $(\frac{a}{e}, 0)$  lies between  $O$  and  $V_2$ . We take the line  $x = \frac{a}{e}$  as the directrix  $D_2$ .

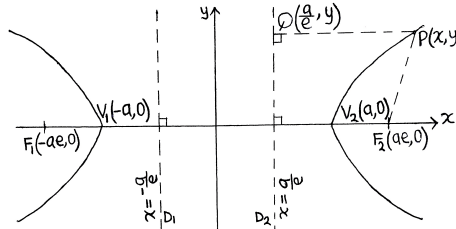


Figure 3.14:

Let  $P(x, y)$  be any point on the hyperbola. From Definition(3.1),  $\frac{|F_2P|}{|QP|} = e$ , that is,  $|F_2P|^2 = e^2|QP|^2$ . So we have that

$$\begin{aligned} (x - ae)^2 + (y - 0)^2 &= e^2 \left[ \left( x - \frac{a}{e} \right)^2 + (y - y)^2 \right] \\ x^2 + a^2e^2 - 2aex + y^2 &= e^2 \left( x^2 + \frac{a^2}{e^2} - \frac{2ax}{e} \right) \\ x^2 + a^2e^2 - 2aex + y^2 &= e^2x^2 + a^2 - 2aex \\ a^2(e^2 - 1) &= (e^2 - 1)x^2 - y^2 \\ \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} &= 1 \end{aligned}$$

Let  $b^2 = a^2(e^2 - 1)$ , where  $b > 0$ . Then we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

From the equation above, we get  $y^2 = b^2 \left( \frac{x^2}{a^2} - 1 \right)$ . Since  $y^2 \geq 0$  and  $b^2 > 0$ , we have that  $\frac{x^2}{a^2} - 1 \geq 0$ . Thus  $x \leq -a$  or  $x \geq a$ , that is,  $|x| \geq a$ . No part of the curve lies in the region where  $-a < x < a$ , that is where  $|x| < a$ .

$\therefore$  The Cartesian equation of the hyperbola with centre at  $(0,0)$ , foci at  $(-ae, 0)$  and  $(ae, 0)$ , and vertices  $(-a, 0), (a, 0)$  is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  where  $b^2 = (ae)^2 - a^2 = a^2(e^2 - 1)$

Similarly,

The Cartesian equation of the hyperbola with centre  $(0,0)$ , foci  $(0, -ae)$  and  $(0, ae)$ , vertices  $(0, -a)$  and  $(0, a)$  is  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  where  $b^2 = a^2(e^2 - 1)$

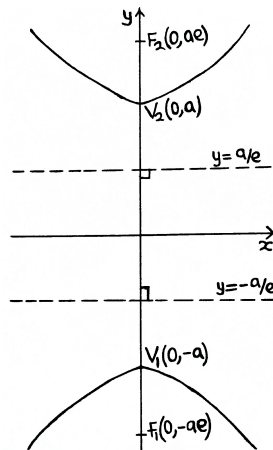


Figure 3.15:

### Finding asymptotes of a hyperbola

Consider the hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . We have that

$$y^2 = b^2 \left( \frac{x^2}{a^2} - 1 \right) = \frac{b^2 x^2}{a^2} \left( 1 - \frac{a^2}{x^2} \right), \text{ where } x \neq 0$$

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}, \text{ where } 1 - \frac{a^2}{x^2} \geq 0$$

It is easily seen that  $y \rightarrow \pm \frac{b}{a}x$  as  $x \rightarrow \pm\infty$ . The lines  $y = \frac{b}{a}x$  and  $y = -\frac{b}{a}x$  are oblique asymptotes to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . Similarly, the hyperbola  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  has two oblique asymptotes  $y = \frac{a}{b}x$  and  $y = -\frac{a}{b}x$ .

### Example 3.12

Analyse the equation  $y^2 - 4x^2 = 4$ .

#### SOLUTION

By dividing through the given equation by 4, we get  $\frac{y^2}{4} - x^2 = 1$ . By inspection  $a = 2$ ,  $b = 1$ , centre =  $(0, 0)$  and the transverse axis is the  $y$ -axis. The vertices  $(0, -2)$  and  $(0, 2)$ . From  $b^2 = (ae)^2 - a^2$ , we get  $(ae)^2 = b^2 + a^2 = 1 + 4 = 5$  and so  $ae = \sqrt{5}$  and  $e = \frac{\sqrt{5}}{2}$ . The foci are  $(0, -5)$  and  $(0, \sqrt{5})$ , the directrices are  $y = \pm \frac{a}{e} = \pm \frac{4\sqrt{5}}{5}$  and the asymptotes are  $y = \pm 2x$ .

### Example 3.13

Find the equation of the hyperbola with centre at the origin, one focus at  $(3, 0)$  and one vertex at  $(-2, 0)$ .

#### SOLUTION

The  $x$ -axis is the transverse axis,  $ae = 3$  and  $a = 2$ . Thus  $b^2 = (ae)^2 - a^2 = 9 - 4 = 5$ . Equation of the ellipse is  $\frac{x^2}{4} - \frac{y^2}{5} = 1$ .

### Example 3.14

Find the Cartesian equation of the hyperbola with foci at  $(0, \pm\sqrt{5})$  and directrices  $y = \pm \frac{4}{\sqrt{5}}$ .

The transverse axis is the  $y$ -axis, the centre is  $(0,0)$ ,  $ae = \sqrt{5}$  and  $\frac{a}{e} = \frac{4}{\sqrt{5}}$ . We have that  $e = \frac{\sqrt{5}}{a}$  and so  $a \left( \frac{a}{\sqrt{5}} \right) = \frac{4}{\sqrt{5}}$ . Thus  $a^2 = 4$  and  $b^2 = (ae)^2 - a^2 = 5 - 4 = 1$ . The equation of the ellipse is  $\frac{y^2}{4} - \frac{x^2}{1} = 1$ .

### The focal distances property of the hyperbola

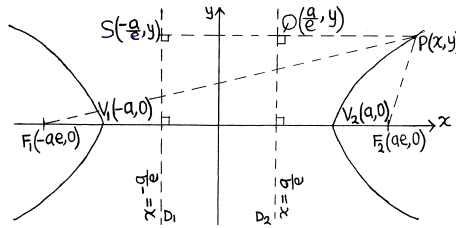


Figure 3.16:

Applying Definition(3.1) to the focus  $F_2$  and directrix  $x = \frac{a}{e}$ , we get

$$|F_2P| = e|QP| \text{-----} (E_1)$$

Applying Definition(3.1) to the focus  $F_1$  and directrix  $x = -\frac{a}{e}$ , we get

$$|F_1P| = e|SP| \text{-----} (E_2)$$

Subtracting  $(E_1)$  from  $(E_2)$ , we get

$$\begin{aligned} |F_1P| - |F_2P| &= e(|SP| - |QP|) \\ &= e|SQ| \\ &= e \left( \frac{2a}{e} \right) \\ &= 2a \end{aligned}$$

### Definition 3.5 (The Hyperbola)

A hyperbola is the collection of all points in the plane, the difference of whose distances from two fixed points, called the foci, is a constant.

[ Try to obtain the cartesian equation of a hyperbola using Definition(3.5)]

### Hyperbolas with centre at $(h, k)$

The equation of hyperbolas with centre at  $(h, k)$  have the same forms as a standard hyperbola, except that  $x$  is replaced by  $x - h$  and  $y$  is replaced by  $y - k$ .

**Example 3.15**

Analyse the equation  $-x^2 + 4y^2 - 2x - 16y + 11 = 0$

SOLUTION

By completing the squares, we have

$$\begin{aligned} -[x^2 + 2x] + 4[y^2 - 4y] &= -11 \\ -[(x + 1)^2 - 1] + 4[(y - 2)^2 - 4] &= -11 \\ (y - 2)^2 - \frac{(x + 1)^2}{4} &= 1 \end{aligned}$$

$$(y - 2)^2 - \frac{(x + 1)^2}{4} = 1 \text{ --- (E}_1\text{)}$$

Let  $Y = y - 2$  and  $X = x + 1$ . Then

$$Y^2 - \frac{X^2}{4} = 1 \text{ --- (E}_2\text{)}$$

For  $(E_2)$ :

Transverse axis is  $X = 0$ , conjugate axis is  $Y = 0$ ,  $a = 1$ ,  $b = 2$  and centre is  $(0, 0)$ . From  $b^2 = (ae)^2 - a^2$ , we have  $(ae)^2 = a^2 + b^2 = 5$ . Thus  $(ae) = \sqrt{5}$  and  $e = \sqrt{5}$ . The vertices are  $(0, \pm 1)$ , the foci are  $(0, \pm\sqrt{5})$ , the directrices are  $Y = \pm\frac{\sqrt{5}}{5}$  and the asymptotes are  $Y = \pm\frac{1}{2}X$ .

For  $(E_1)$ :

We make use of  $x = X - 1$  and  $y = Y + 2$ . The transverse axis is  $x = -1$ , the conjugate axis is  $y = 2$ ,  $a = 1$ ,  $b = 2$  and the centre is  $(-1, 2)$ ,  $ae = \sqrt{5}$  and  $e = \sqrt{5}$ . The vertices are  $(-1, 2 \pm 1)$ , the foci are  $(-1, 2 \pm \sqrt{5})$  and the directrices are  $y = 2 \pm \frac{\sqrt{5}}{5}$ . The asymptotes are given by

$$\begin{aligned} y - 2 &= \pm\frac{1}{2}(x + 1) \\ y &= 2 \pm \frac{1}{2}(x + 1) \\ y &= \frac{1}{2}x + \frac{5}{2} \quad \text{and} \quad y = -\frac{1}{2}x + \frac{3}{2} \end{aligned}$$

**Parametric equations of the hyperbola**

The hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  has parametric equations

$$x = a \sec t \quad \text{and} \quad y = b \tan t$$

where  $0 \leq t < 2\pi$ ,  $a, b > 0$  and  $t \neq \frac{\pi}{2}$ ,  $t \neq \frac{3\pi}{2}$ .



Similarly, the hyperbola  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$  has parametric equations

$$y = a \sec t \quad \text{and} \quad x = b \tan t$$

where  $0 \leq t < 2\pi$ ,  $a, b > 0$  and  $t \neq \frac{\pi}{2}$ ,  $t \neq \frac{3\pi}{2}$ .

**Remark 3.3**

Consider the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . When  $a = b$ , we get  $x^2 - y^2 = a^2$ . This is called rectangular hyperbola, since the asymptotes are  $y = \pm x$  which are at right angles to each other. In this case,  $(ae)^2 = a^2 + a^2 = 2a^2$ , which gives us  $e = \sqrt{2}$ .

A special case of the rectangular hyperbola is  $xy = c^2$ , where the  $x$ -axis and  $y$ -axis are the asymptotes. It has parametric equations  $x = ct$  and  $y = \frac{c}{t}$ , where  $t \neq 0$ .

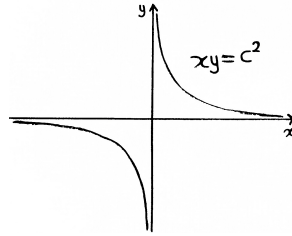


Figure 3.17: The rectangular hyperbola  $xy = c^2$

**Tangents and normals to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at  $P(a \sec t, b \tan t)$**

From the parametric equations  $x = a \sec t$  and  $y = b \tan t$ , we have that  $\frac{dx}{dt} = a \sec t \tan t$

and  $\frac{dy}{dt} = b \sec^2 t$ . Thus  $\frac{dy}{dx} = \frac{b \sec^2 t}{a \sec t \tan t} = \frac{b \sec t}{a \tan t}$ . Alternatively, by

differentiating  $b^2 x^2 - a^2 y^2 = a^2 b^2$  implicitly, we have that  $2b^2 x - 2a^2 y \frac{dy}{dx} = 0$ . As a result,

$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{b^2 (a \sec t)}{a^2 (b \tan t)} = \frac{b \sec t}{a \tan t}$ . Equation of tangent is given by

$$\begin{aligned} y - b \tan t &= \frac{b \sec t}{a \tan t} (x - a \sec t) \\ y \tan t - b \tan^2 t &= \frac{bx}{a} \sec t - b \sec^2 t \\ b(\sec^2 t - \tan^2 t) &= \frac{bx}{a} \sec t - y \tan t \\ b &= \frac{bx}{a} \sec t - y \tan t \\ \frac{x}{a} \sec t - \frac{y}{b} \tan t &= 1 \end{aligned}$$

Equation of normal is given by

$$\begin{aligned} y - b \tan t &= -\frac{a \tan t}{b \sec t}(x - a \sec t) \\ by \sec t - b^2 \tan t \sec t &= -ax \tan t + a^2 \tan t \sec t \\ by \sec t + ax \tan t &= (a^2 + b^2) \tan t \sec t \\ by + ax \sin t &= (a^2 + b^2) \tan t, \text{ by multiplying through by } \cos t \end{aligned}$$

**Tangents and normals to the hyperbola**  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  **at**  $(h, k)$

We have that  $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ , and so  $\frac{dy}{dx}\bigg|_{(h,k)} = \frac{b^2 h}{a^2 k}$ . Equation of the tangent is given by

$$\begin{aligned} y - k &= \frac{b^2 h}{a^2 k}(x - h) \\ a^2 ky - a^2 k^2 &= b^2 hx - b^2 h^2 \\ b^2 h^2 - a^2 k^2 &= b^2 hx - a^2 ky \\ \frac{h^2}{a^2} - \frac{k^2}{b^2} &= \frac{hx}{a^2} - \frac{ky}{b^2}, \text{ by dividing through by } a^2 b^2 \end{aligned}$$

Since  $(h, k)$  lies on the hyperbola, we have that  $\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$ . Thus  $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$ .

Equation of normal is given by

$$\begin{aligned} y - k &= \frac{-a^2 k}{b^2 h}(x - h) \\ b^2 hy - b^2 kh &= -a^2 kx + a^2 kh \\ a^2 kx + b^2 hy &= (a^2 + b^2)kh \end{aligned}$$

### Example 3.16

Show that an equation of the tangent to the rectangular hyperbola with equation  $xy = c^2$  (with  $c > 0$ ) at the point  $\left(ct, \frac{c}{t}\right)$  is  $t^2 + x = 2ct$ . Find also the equation of the normal to the rectangular hyperbola at the point  $\left(ct, \frac{c}{t}\right)$ .

Tangents are drawn from the point  $(-3, 3)$  to the rectangular hyperbola with equation  $xy = 16$ . Find the coordinates of the points of contact of these tangents with the hyperbola.

Differentiating  $xy = c^2$ , we get  $x \frac{dy}{dx} + y = 0$ , and so  $\frac{dy}{dx} = -\frac{y}{x}$ . Thus  $\frac{dy}{dx} \Big|_{(ct, \frac{c}{t})} = -\frac{1}{t^2}$ .

Equation of the tangent to  $xy = c^2$  at  $(ct, \frac{c}{t})$  is given by

$$\begin{aligned} y - \frac{c}{t} &= -\frac{1}{t^2}(x - ct) \\ t^2y - ct &= -x + ct \\ t^2y + x &= 2ct \end{aligned}$$

Equation of normal to  $xy = c^2$  at  $(ct, \frac{c}{t})$  is given by

$$\begin{aligned} y - \frac{c}{t} &= t^2(x - ct) \\ ty - c &= t^3x - ct^4 \\ t^3x - ty &= c(t^4 - 1) \end{aligned}$$

Comparing  $xy = 16$  to  $xy = c^2$ , we get  $c = 4$ . Thus  $(4t, \frac{4}{t})$  is a point on  $xy = 16$ .

Equation of tangent to  $xy = 16$  at  $(4t, \frac{4}{t})$  is  $t^2y + x = 8t$ . The point  $(-3, 3)$  lies on  $t^2y + x = 8t$ . Thus  $3t^2 - 8t - 3 = 0$ , giving us  $t = -\frac{1}{3}, 3$ . The tangents from  $(-3, 3)$  meet  $xy = 16$  at the points  $(-\frac{4}{3}, -12)$  and  $(12, \frac{4}{3})$ .

### General form of the equation of a conic

Consider the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ --- (G1)}$$

where  $A$ ,  $B$  and  $C$  are not all zero. Suppose that  $B = 0$ , then  $(G_1)$  becomes

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \text{ --- (G2)}$$

where either  $A \neq 0$  or  $C \neq 0$ .

### Theorem 3.1

Excluding degenerate cases, the equation

$Ax^2 + Cy^2 + Dx + Ey + F = 0$ , where  $A$  and  $C$  cannot both equal zero

- (a) defines a parabola if  $AC = 0$ ,
- (b) defines an ellipse if  $AC > 0$ ,
- (c) defines a hyperbola if  $AC < 0$ .

Consider the following cases:

Case I

Let  $AC = 0$ . Then  $A = 0$  or  $B = 0$ , but not both. That is  $A = 0$  and  $C \neq 0$  or  $A \neq 0$  and  $C = 0$ . When  $C = 0$ ,  $G2$  becomes

$$Ax^2 + Dx + Ey + F = 0, A \neq 0 \text{ --- (G3)}$$

Completing the squares, we get

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{Ey}{A} + \frac{D^2 - 4AF}{4A^2} \text{ --- (G4)}$$

If  $E \neq 0$ , equation (G4) can be written as

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A} \left[y - \frac{(D^2 - 4AF)}{4AE}\right]$$

which is an equation of a parabola.

If  $E = 0$  and  $D^2 - 4AF = 0$ , (G4) becomes  $x = -\frac{D}{2A}$  which is a vertical line.

If  $E = 0$  and  $D^2 - 4AF > 0$ , then

$$x = -\frac{D}{2A} \pm \sqrt{\frac{D^2 - 4AF}{4A^2}}$$

which gives two vertical lines.

If  $E = 0$  and  $D^2 - 4AF < 0$ , then  $\left(x + \frac{D}{2A}\right)^2 < 0$  which is impossible. The graph of (G3) contains no points.

Similarly, when  $A = 0$ , (G2) becomes  $Cy^2 + Dx + Ey + F = 0$ ,  $C \neq 0$  the graph of which

(a) is a parabola if  $D \neq 0$ ,

(b) is a horizontal line if  $D = 0$  and  $E^2 - 4CF = 0$ ,

(c) is two horizontal lines if  $D = 0$  and  $E^2 - 4CF > 0$ ,

(d) contains no points if  $D = 0$  and  $E^2 - 4CF < 0$ .

Case II

Let  $AC > 0$ , then  $A$  and  $C$  are of the same sign. Completing the squares, (G2) becomes

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F \text{ --- (G5)}$$

If  $\frac{D^2}{4A} + \frac{E^2}{4C} - F$  has the same sign as  $A$ , then the graph of  $(G5)$  is an ellipse.

If  $\frac{D^2}{4A} + \frac{E^2}{4C} - F = 0$ , then  $x = -\frac{D}{2A}$ ,  $y = -\frac{E}{2C}$ . Thus the graph of  $(G5)$  is the point  $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$ .

If  $\frac{D^2}{4A} + \frac{E^2}{4C} - F$  is of opposite sign to  $A$ , the graph of  $(G5)$  contains no points.

### Case III

Let  $AC < 0$ , then  $A$  and  $C$  are of opposite signs.

If  $\frac{D^2}{4A} + \frac{E^2}{4C} - F \neq 0$ , the graph of  $(G5)$  is a hyperbola.

If  $\frac{D^2}{4A} + \frac{E^2}{4C} - F = 0$ , the graph of  $(G5)$  is two intersecting lines. For instance, let  $A = m^2$  and  $C = -n^2$ , where  $n, m \in \mathbb{R} \setminus \{0\}$ , then  $A > 0$ ,  $C < 0$ . The two lines are

$$y = \frac{E}{2n^2} \pm \frac{m}{n} \left(x + \frac{D}{2m^2}\right) \text{ and they intersect at } \left(-\frac{D}{2m^2}, \frac{E}{2n^2}\right).$$

### Rotation of axes

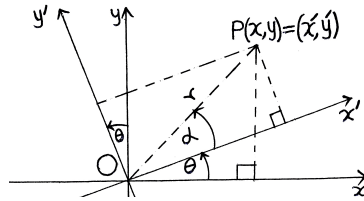


Figure 3.18: Rotation of axes through  $\theta$  radians

The origin is fixed. The  $x$ -axis and  $y$ -axis are rotated counterclockwise through an angle  $\theta$  to new positions denoted by  $x'$  and  $y'$  respectively.  $\alpha$  is the angle  $OP$  makes with the positive  $x'$ -axis.

It is easily seen from the diagram that  $x' = r \cos \alpha$  and  $y' = r \sin \alpha$ . We have

$$\begin{aligned} x &= r \cos(\alpha + \theta) \\ &= (r \cos \alpha) \cos \theta - (r \sin \alpha) \sin \theta \\ &= x' \cos \theta - y' \sin \theta \\ y &= r \sin(\alpha + \theta) \\ &= (r \sin \alpha) \cos \theta + (r \cos \alpha) \sin \theta \\ &= x' \sin \theta + y' \cos \theta \end{aligned}$$

**Theorem 3.2** To transform the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad B \neq 0$$

into an equation in  $x'$  and  $y'$  without an  $x'y'$  term, rotate the axes through an angle  $\theta$  satisfying

$$\cot(2\theta) = \frac{A - C}{B}$$

PROOF

Let us rotate the  $x$  and  $y$  axes through an angle  $\theta$  to obtain the  $x'$  and  $y'$  axes. Let  $P = (x, y) = (x', y')$  be a point in the plane.

Then  $x = x' \cos \theta - y' \sin \theta$

$$y = x' \sin \theta + y' \cos \theta.$$

Equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  becomes

$$\begin{aligned} & (A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta)x'^2 \\ & + [B(\cos^2 \theta - \sin^2 \theta) + 2(C - A) \sin \theta \cos \theta]x'y' \\ & + (A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta)y'^2 + (D \cos \theta + E \sin \theta)x' \\ & + (-D \sin \theta + E \cos \theta)y' + F \\ & = 0 \end{aligned}$$

To eliminate the  $x'y'$ -term, we have

$$\begin{aligned} B(\cos^2 \theta - \sin^2 \theta) + 2(C - A) \sin \theta \cos \theta &= 0 \\ B \cos(2\theta) + (C - A) \sin(2\theta) &= 0 \\ B \cos(2\theta) &= (A - C) \sin(2\theta) \end{aligned}$$

$$\text{Thus } \cot(2\theta) = \frac{A - C}{B}.$$

**Remark 3.4**

We choose the acute angle  $\theta$  satisfying  $\cot(2\theta) = \frac{A - C}{B}$

**Example 3.17**

Analyse the equation  $xy = 1$

$A = 0$ ,  $B = 1$  and  $C = 0$  and so  $\cot(2\theta) = \frac{0-0}{1} = 0$ . Hence  $2\theta = \frac{\pi}{2}$ , giving us  $\theta = \frac{\pi}{4}$ .

$$\begin{aligned}x &= \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' - y') \\y &= \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' + y')\end{aligned}$$

Putting the above into  $xy = 1$ , we have that

$$\begin{aligned}\left[ \frac{\sqrt{2}}{2}(x' - y') \right] \left[ \frac{\sqrt{2}}{2}(x' + y') \right] &= 1 \\ \frac{1}{2}(x' - y')(x' + y') &= 1 \\ \frac{x'^2}{2} - \frac{y'^2}{2} &= 1\end{aligned}$$

This is a rectangular hyperbola with centre  $(0, 0)$ , transverse axis is  $x'$ -axis, conjugate axis is  $y'$ -axis, vertices  $(\pm 2, 0)$ , foci  $(\pm 2\sqrt{2}, 0)$ ,  $e = \sqrt{2}$ , directrices  $x' = \pm\sqrt{2}$ , asymptotes  $y' = \pm 1x'$ , in the  $x'y'$ -plane.

### Example 3.18

Analyse the equation  $4x^2 - 4xy + y^2 + 5\sqrt{5}x + 5 = 0$

### SOLUTION

$A = 4$ ,  $B = -4$  and  $C = 1$ . Thus  $\tan(2\theta) = -\frac{4}{3}$ , from which we get  $\cos(2\theta) = -\frac{3}{5}$ .

$$\begin{aligned}\sin \theta &= \sqrt{\frac{1 - \cos(2\theta)}{2}} = \frac{2\sqrt{5}}{5} \\ \cos \theta &= \sqrt{\frac{1 + \cos(2\theta)}{2}} = \frac{\sqrt{5}}{5}\end{aligned}$$

So we have that  $x = \frac{\sqrt{5}}{5}(x' - 2y')$  and  $y = \frac{\sqrt{5}}{5}(2x' + y')$ . Substituting these into  $4x^2 - 4xy + y^2 + 5\sqrt{5}x + 5 = 0$ , and then simplifying, we get  $(y' - 1)^2 = -x'$ . This is the equation of a parabola with vertex  $(0, 1)$  in the  $x'y'$ -plane.

**Theorem 3.3**

Except for degenerate cases, the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

defines

- (a) a parabola if  $B^2 - 4AC = 0$
- (b) an ellipse (or a circle) if  $B^2 - 4AC < 0$
- (c) a hyperbola if  $B^2 - 4AC > 0$ .

PROOF

We have dealt with the case  $B = 0$ . So suppose that  $B \neq 0$ . By rotating axes through an angle  $\theta$ , we get

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta \\ y &= x' \sin \theta + y' \cos \theta \end{aligned}$$

Then we have

$$\begin{aligned} &A[x'^2 \cos^2 \theta + y'^2 \sin^2 \theta - x'y' \sin 2\theta] + B \left[ \frac{1}{2}x'^2 \sin 2\theta - \frac{1}{2}y'^2 \sin 2\theta + x'y' \cos 2\theta \right] \\ &+ C[x'^2 \sin^2 \theta + y'^2 \cos^2 \theta + x'y' \sin 2\theta] + D[x' \cos \theta - y' \sin \theta] \\ &+ E[x' \sin \theta + y' \cos \theta] + F = 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} &[A \cos^2 \theta + \frac{B}{2} \sin 2\theta + C \sin^2 \theta]x'^2 + [-A \sin 2\theta + B \cos 2\theta + C \sin 2\theta]x'y' \\ &+ [A \sin^2 \theta - \frac{B}{2} \sin 2\theta + C \cos^2 \theta]y'^2 + [D \cos \theta + E \sin \theta]x' \\ &+ [-D \sin \theta + E \cos \theta]y' + F = 0. \end{aligned}$$

We now have  $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$  where

$$\begin{aligned} A' &= A \cos^2 \theta + \frac{B}{2} \sin 2\theta + C \sin^2 \theta \\ B' &= (C - A) \sin 2\theta + B \cos 2\theta \\ C' &= A \sin^2 \theta - \frac{B}{2} \sin 2\theta + C \cos^2 \theta \\ D' &= D \cos \theta + E \sin \theta \\ E' &= -D \sin \theta + E \cos \theta \\ F' &= F \end{aligned}$$



$$\begin{aligned}
 A'C' &= \frac{1}{4}(A^2 + C^2) \sin^2(2\theta) + \frac{AB}{2} \sin(2\theta)(\sin^2 \theta - \cos^2 \theta) + AC(\cos^4 \theta + \sin^4 \theta) \\
 &\quad + \frac{BC}{2} \sin(2\theta)(\cos^2 \theta - \sin^2 \theta) - \frac{B^2}{4} \sin^2(2\theta) \\
 &= \frac{1}{4}[A^2 + C^2 - B^2] \sin^2(2\theta) + \left[ \frac{BC}{2} - \frac{AB}{2} \right] \sin 2\theta \cos 2\theta + AC [\cos^4 \theta + \sin^4 \theta] \\
 &= \frac{1}{4}[A^2 + C^2 - B^2] \sin^2(2\theta) + \frac{B}{2}(C - A) \sin 2\theta \cos 2\theta + AC[1 - \frac{1}{2} \sin^2 2\theta]
 \end{aligned}$$

Now,

$$B' = (C - A) \sin 2\theta + B \cos 2\theta$$

$$B'^2 = (C - A)^2 \sin^2(2\theta) + B^2 \cos^2(2\theta) + 2B(C - A) \sin 2\theta \cos 2\theta$$

$$2B(C - A) \sin(2\theta) \cos(2\theta) = B'^2 - (C - A)^2 \sin^2(2\theta) - B^2 \cos^2(2\theta)$$

$$\begin{aligned}
 4A'C' &= (A^2 + C^2 - B^2) \sin^2(2\theta) + B'^2 - (C - A)^2 \sin^2(2\theta) - B^2 \cos^2(2\theta) \\
 &\quad + AC[4 - 2 \sin^2(2\theta)] \\
 &= B'^2 + [A^2 + C^2 - B^2 - C^2 - A^2 + 2AC - 2AC] \sin^2(2\theta) + 4AC \\
 &\quad - B^2 \cos^2(2\theta) \\
 &= B'^2 - B^2 \sin^2(2\theta) - B^2 \cos^2(2\theta) + 4AC \\
 &= B'^2 - B^2 + 4AC
 \end{aligned}$$

$$\Rightarrow B^2 - 4AC = B'^2 - 4A'C'$$

Suppose that  $\cot(2\theta) = \frac{A-C}{B}$ ,  $B \neq 0$ . Then  $B \cos(2\theta) = (A - C) \sin(2\theta)$ . Thus

$B' = (C - A) \sin(2\theta) + B \cos \theta = 0$ . So we have  $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$  and

$B^2 - 4AC = -4A'C'$ . By applying Theorem(3.1),

(a) We get a parabola if  $A'C' = 0$ . That is  $B^2 - 4AC = 0$ .

(b) We get an ellipse (or circle) if  $A'C' > 0$ . That is  $B^2 - 4AC < 0$ .

(c) We get a hyperbola if  $A'C' < 0$ . That is  $B^2 - 4AC > 0$ .

Exercise Three

1. Find the Cartesian equation of the parabola whose focus and directrix are respectively:
  - (a)  $(3, 0)$ ,  $x + 5 = 0$
  - (b)  $(0, 4)$ ,  $y + 4 = 0$
  - (c)  $(1, 1)$ ,  $y + 1 = 0$
2. Find the equation of the tangent and normal to the given parabola, at the given point.
  - (a)  $y = 4(x + 1)^2$  at  $(-3, 16)$
  - (b)  $y^2 = x$  at  $(1, 1)$
3. The tangent with equation  $ty = x + t^2$  to the parabola with equation  $y^2 = 4x$  passes through the point  $(5, -6)$ . Find the possible values of  $t$ , and the coordinates of the point of contact of the tangent with the parabola for each value of  $t$ .
4. The points  $A$  and  $B$  lie on the parabola with equation  $y^2 = 4x$  and have coordinates  $(t^2, 2t)$  and  $(4t^2, 4t)$  respectively. The tangents to the parabola at  $A$  and  $B$  meet at the point  $C$  and  $M$  is the midpoint of the chord  $AB$ . Show that the line  $CM$  is parallel to the  $x$ -axis for all values of  $t$ , where  $t \in \mathbb{R} \setminus \{0\}$ .
5. Show that the tangents to the parabola  $y^2 = 4ax$  at the points  $P(at_1^2, 2at_1)$  and  $Q(at_2^2, 2at_2)$  meet at the point  $R(at_1t_2, a(t_1 + t_2))$ .
  - (a) If  $PQ$  passes through the focus  $(a, 0)$  show that
    - i.  $t_1t_2 = -1$
    - ii.  $R$  lies on the directrix  $x + a = 0$ .
  - (b) Show that the area of  $\triangle PQR$  is  $\frac{1}{2}a^2|t_1 - t_2|^3$ .
6. A line with gradient 4 touches the parabola with equation  $y^2 = 4x$ . Find the coordinates of the point of contact of this tangent with the parabola and an equation of the normal to the parabola at that point.

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7. The variable chord  $PQ$  on the parabola with equation  $y^2 = 4x$  subtends a right angle at the origin  $O$ . Given that  $P$  is  $(p^2, 2p)$  and  $Q$  is  $(q^2, 2q)$ , find the fixed point on the  $x$ -axis through which  $PQ$  passes.
8. Find the cartesian equation of the ellipse having
- (a) foci  $\left(\pm\frac{8}{9}, 0\right)$  and directrices  $x = \pm 2$
  - (b) center at  $(-3, 1)$ , vertex at  $(-3, 3)$ , focus at  $(-3, 0)$
  - (c) center at  $(1, 2)$ , focus at  $(1, 4)$  and contains the point  $(2, 2)$
  - (d) foci at  $(5, 1)$  and  $(-1, 1)$  with length of major axis being 8
9. Analyse the given equation:
- (a)  $4y^2 + 9x^2 = 36$
  - (b)  $2x^2 + 3y^2 - 8x + 6y + 5 = 0$
  - (c)  $4x^2 + y^2 + 4y = 0$
10. The tangents at the points  $(a \cos \alpha, b \sin \alpha)$  and  $(a \cos \beta, b \sin \beta)$  on the ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  are perpendicular. Prove that
- $$\tan \alpha \tan \beta = -\frac{b^2}{a^2}$$
11. The point  $P(7 \cos t, 5 \sin t)$  is on the ellipse  $\frac{x^2}{49} + \frac{y^2}{25} = 1$ . The line through  $P$  parallel to the  $y$ -axis meets the  $x$ -axis at  $X$ . The point  $Q$  is the line  $XP$  produced so that  $XQ = 2XP$ . Find the Cartesian equation of the locus of  $Q$  as  $t$  varies.
12. Find an equation for the hyperbola described:
- (a) Foci at  $(0, \pm\sqrt{5})$ ; directrices  $y = \pm\frac{4}{\sqrt{5}}$
  - (b) Center at  $(4, -1)$ ; focus at  $(7, -1)$ ; vertex at  $(6, -1)$
  - (c) Vertices at  $(0, \pm 6)$ ; asymptote the line  $y = 2x$
  - (d) Foci at  $(3, 7)$  and  $(7, 7)$ ; vertex at  $(6, 7)$

13. Analyse and sketch the following equations.

(a)  $4y^2 - x^2 = 16$

(b)  $4x^2 - y^2 - 24x - 4y + 16 = 0$

(c)  $x^2 - 3y^2 - 8x - 6y + 22 = 0$

14. Find an equation of the tangent and an equation of the normal to the hyperbola with equation

(a)  $x^2 - 2y^2 = 1$  at  $(3, 2)$

(b)  $3y^2 - 5x^2 = 7$  at  $(1, -2)$

15. The tangents at  $P\left(ct_1, \frac{c}{t_1}\right)$  and  $Q\left(ct_2, \frac{c}{t_2}\right)$  to the rectangular hyperbola with equation  $xy = c^2$  meet on the rectangular hyperbola with equation  $xy = \frac{c^2}{4}$ . Prove that  $PQ$  is a tangent to the curve with equation  $xy = 4c^2$ .

16. The variable point  $P(a \sec t, b \tan t)$  is on the hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and  $N$  is the point  $(3a, 3b)$ . The point  $Q$  lies on the line  $PN$  so that  $PQ = 2QN$ . As  $t$  varies, find in cartesian form the locus of  $Q$ .

17. Show that the chord joining  $P\left(ct_1, \frac{c}{t_1}\right)$  and  $Q\left(ct_2, \frac{c}{t_2}\right)$  where  $P$  and  $Q$  are on the curve with equation  $xy = c^2$  has equation

$$x + t_1 t_2 y = c(t_1 + t_2)$$

Given that  $t_1$  and  $t_2$  vary but that  $PQ$  always passes through the point  $(2c, 2c)$ , show that the locus of the mid-point of  $PQ$  is the curve with equation

$$xy = c(x + y)$$

18. By rotating the axes obtain new equations containing no  $xy$ -term.

(a)  $x^2 + 4xy + y^2 - 3 = 0$

(b)  $5x^2 + 6xy + 5y^2 - 8 = 0$

(c)  $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0$

(d)  $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

(e)  $x^2 + 4xy + 4y^2 + 5\sqrt{5}y + 5 = 0$

19. Identify each equation without applying a rotation of axes.

(a)  $2x^2 - 3xy + 4y^2 + 2x + 3y - 5 = 0$

(b)  $9x^2 + 12xy + 4y^2 - x - y = 0$

(c)  $x^2 + 3xy - 2y^2 + 3x + 2y + 5 = 0$

20. Show that distance is invariant under a rotation of axes. That is, show that the distance from

$P_1 = (x_1, y_1)$  to  $P_2 = (x_2, y_2)$  in the  $xy$ -plane equals the distance from  $P_1 = (x'_1, y'_1)$  to  $P_2 = (x'_2, y'_2)$  in the  $x'y'$ -plane.



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## Chapter 4

# Matrices and Systems of Linear Equations

### Definition 4.1 (Matrix)

An  $m \times n$  matrix  $A$  is a rectangular array of  $mn$  numbers arranged in  $m$  horizontal rows and  $n$  vertical columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

If  $m = n$ , we say that  $A$  is a square matrix of order  $n$  and that the numbers  $a_{11}, a_{22}, \dots, a_{nn}$  form the main diagonal of  $A$ . We write  $A = [a_{ij}]$  or  $A = (a_{ij})$ .

### Definition 4.2 (Equality of Matrices)

Two matrices  $A$  and  $B$  are equal if they have the same size and have equal corresponding entries. That is  $A = B$  if  $a_{ij} = b_{ij} \forall i, j$ .

### Definition 4.3 (Addition of Matrices)

Let  $A$  and  $B$  be  $m \times n$  matrices. The sum of  $A$  and  $B$  is the  $m \times n$  matrix,  $A + B$  whose  $i, j$ th entry is equal to  $a_{ij} + b_{ij}$ . That is  $A + B$  is the  $m \times n$  matrix obtained by adding the corresponding entries of  $A$  and  $B$ .

**Definition 4.4 (Scalar Multiplication)**

Let  $A$  be an  $m \times n$  matrix and let  $\lambda$  be a scalar. Then the matrix  $\lambda A$ , the scalar multiplication of  $\lambda$  and  $A$ , is the  $m \times n$  matrix whose  $i, j$ th entry is equal to  $\lambda a_{ij}$ .

**Definition 4.5 (Zero Matrix)**

A zero matrix is any matrix all of whose entries are zero. It is denoted by  $O$ , if its size is obvious from the context. More specifically,  $O_{mn}$  denotes the  $m \times n$  zero matrix and  $O_n$  denotes the  $n \times n$  zero matrix.

**Theorem 4.1**

Let  $A, B$  and  $C$  be  $m \times n$  matrices and let  $s, t$  be scalars. Then

1.  $A + B = B + A$
2.  $A + (B + C) = (A + B) + C$
3.  $O + A = A + O = A$
4.  $A + (-A) = (-A) + A = O$
5.  $t(A + B) = tA + tB$
6.  $(s + t)A = sA + tA$
7.  $(st)A = s(tA)$
8.  $1(A) = A$

**Definition 4.6 (Matrix Multiplication)**

If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, their matrix product  $C = AB$  is the  $m \times p$  matrix whose  $i, j$ th entry is the dot product of the  $i$ th row of  $A$  with the  $j$ th column of  $B$ . That is

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj} \\ &= \sum_{k=1}^n a_{ik}b_{kj} \end{aligned}$$

**Example 4.1**

Let  $A = \begin{pmatrix} 2 & -1 & 3 \\ -3 & 4 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 7 & 4 & 4 & 2 \\ 1 & 0 & -1 & 0 \\ -2 & 4 & 3 & -1 \end{pmatrix}$ , then



$$AB = \begin{pmatrix} 7 & 20 & 2 & 1 \\ -17 & -12 & 8 & -6 \end{pmatrix}$$

**Theorem 4.2**

Let  $A$ ,  $B$  and  $C$  be  $m \times n$  matrices and let  $s, t$  be scalars. Then

1.  $A(BC) = (AB)C$
2.  $A(B + C) = AB + AC$
3.  $(A + B)C = AC + BC$
4.  $t(AB) = (tA)B = A(tB)$

**Definition 4.7**

A square matrix  $A = (a_{ij})$  is called upper triangular if  $a_{ij} = 0$  for  $i > j$ . It is called lower triangular if  $a_{ij} = 0$  for  $i < j$ .

Upper triangular

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Lower triangular

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

**Definition 4.8 (Transpose of a Matrix)**

If  $A = [a_{ij}]$  is an  $m \times n$  matrix, then the  $n \times m$  matrix  $A^T = [a_{ij}^T]$ , where  $a_{ij}^T = a_{ji}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , is called the transpose of  $A$ . That is, the entries in each row of  $A^T$  are the entries in the corresponding column of  $A$ .

**Example 4.2**

Let  $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$ , then  $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ .

**Theorem 4.3 (Properties of the transpose)**

Let  $A$  and  $B$  be  $m \times n$  matrices and let  $t$  be a scalar. Then

1.  $(A^T)^T = A$
2.  $(A + B)^T = A^T + B^T$
3.  $(tA)^T = tA^T$
4.  $(AB)^T = B^T A^T$

**Definition 4.9 (Symmetric Matrix)**

A matrix  $A = [a_{ij}]$  is called symmetric if  $A^T = A$ . That is,  $A$  is symmetric if it is a square matrix for which  $a_{ij} = a_{ji}$ . The entries of  $A$  are symmetric with respect to the main diagonal.

**Example 4.3**

Let  $M = \begin{pmatrix} -4 & 3 & 8 \\ 3 & 0 & 2 \\ 8 & 2 & -5 \end{pmatrix}$ . Then  $M^T = M$ . Hence  $M$  is symmetric.

**Definition 4.10 (Skew-Symmetric Matrix)**

A matrix  $A = (a_{ij})$  is called skew symmetric or anti-symmetric if  $A^T = -A$ .

**Example 4.4**

Let  $B = \begin{pmatrix} 0 & -1 & 3 & 6 \\ 1 & 0 & 2 & -5 \\ -3 & -2 & 0 & 4 \\ -6 & 5 & -4 & 0 \end{pmatrix}$ . Then  $B^T = -B$

**Definition 4.11 (Trace)**

If  $A = (a_{ij})$  is an  $n \times n$  matrix, then the trace of  $A$ , denoted by  $\text{Tr}(A)$ , is defined as the sum of all elements on the main diagonal of  $A$ . That is  $\text{Tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$ .

**Example 4.5**

Let  $B = \begin{pmatrix} 2 & 12 & 14 \\ 20 & -1 & 5 \\ -11 & 0 & 2 \end{pmatrix}$ , then  $\text{Tr}(B) = 2 + (-1) + 2 = 3$ .

**Theorem 4.4 (Properties of the trace)**

Let  $A$  and  $B$  be square matrices of order  $n$ . Then

1.  $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
2.  $\text{Tr}(tA) = t\text{Tr}(A)$
3.  $\text{Tr}(AB) = \text{Tr}(BA)$
4.  $\text{Tr}(A) = \text{Tr}(A^T)$

**Definition 4.12 (Diagonal Matrix)**

A square matrix  $A = (a_{ij})$  for which every entry off the main diagonal is zero (that is,  $a_{ij} = 0$  for  $i \neq j$ ) is called a diagonal matrix.

**Definition 4.13 (Scalar Matrix)**

A diagonal matrix  $A = (a_{ij})$  for which all entries on the main diagonal are equal (that is,  $a_{ij} = c$  for  $i = j$  and  $a_{ij} = 0$  for  $i \neq j$ ) is called a scalar matrix.

**Definition 4.14 (Identity Matrix)**

The  $n \times n$  scalar matrix all of whose diagonal entries are 1 is called the identity matrix of order  $n$  denoted by  $I_n$ . Thus,  $I_n = (\delta_{ij})$  where  $\delta_{ij}$  is the Kröneckel Delta function, given

$$\text{by } \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

**Definition 4.15 (Non-Singular Matrix)**

An  $n \times n$  matrix  $A$  is called non-singular or invertible or regular if  $\exists$  an  $n \times n$  matrix  $B$  such that  $AB = BA = I_n$ . The matrix  $B$  is called the inverse of  $A$ , denoted by  $A^{-1}$ . If  $\nexists$  no such matrix  $B$ , then  $A$  is called singular or non-invertible or irregular.

**Theorem 4.5 (Properties of the Inverse)**

- (a) If  $A$  is a non-singular matrix, then  $A^{-1}$  is non-singular and  $(A^{-1})^{-1} = A$ .
- (b) If  $A$  and  $B$  are non-singular matrices, then  $AB$  is non-singular and  $(AB)^{-1} = B^{-1}A^{-1}$ .
- (c) If  $A$  is a non-singular matrix, then  $(A^T)^{-1} = (A^{-1})^T$ .

**Definition 4.16 (Elementary Row Operations)**

The following operations are called elementary row operations (EROS).

1. Interchange two rows.
2. Multiply a row by a non-zero constant.

3. Add a multiple of a row to another row.

**Remark 4.1**

Dividing a row by a non-zero constant is implied in the definition. Dividing a row by  $k \in \mathbb{R} \setminus \{0\}$  is the same as multiplying by the row by  $\frac{1}{k}$ . Subtracting a multiple of one row from another row is the same as adding the negative of one row to another row.

**Theorem 4.6**

Let  $A$  be a square matrix. If a sequence of EROS reduces  $A$  to  $I$ , then the same sequence of EROS transforms  $I$  to  $A^{-1}$ . Hence to find the inverse,  $A^{-1}$  of  $A$ :

- (a) Construct the matrix  $[A|I]$
- (b) Use EROS to obtain  $[A|I] \rightarrow [I|A^{-1}]$

If  $A$  cannot be reduced to  $I$ , then  $A$  does not have an inverse.

**Example 4.6**

Find the inverse of the given matrices using EROS:

$$(a) \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & -5 & 4 \end{pmatrix} \quad (b) \begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & 5 \\ -1 & 1 & 7 \end{pmatrix}$$

SOLUTION

$$(a) \left( \begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 1 & -5 & 4 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{aligned} -R_1 &\longrightarrow R_1 \\ R_1 + R_3 &\longrightarrow R_3 \end{aligned} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & -3 & 5 & 1 & 0 & 1 \end{array} \right)$$

$$R_2 \leftrightarrow R_3 \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & -3 & 5 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 \end{array} \right)$$

$$\begin{aligned} -\frac{2}{3}R_2 + R_1 &\longrightarrow R_1 \\ -\frac{1}{3}R_2 &\longrightarrow R_2 \\ -\frac{1}{2}R_3 &\longrightarrow R_3 \end{aligned} \left( \begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{3} & -\frac{5}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{array} \right)$$

$$\begin{aligned} \frac{13}{3}R_3 + R_1 &\longrightarrow R_1 \\ \frac{5}{3}R_3 + R_2 &\longrightarrow R_2 \end{aligned} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & -\frac{13}{6} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{array} \right)$$

**Definition 4.17 (Reduced Row Echelon Form)**

An  $m \times n$  matrix  $A$  is said to be in reduced row echelon form if it satisfies the following properties:

- (a) All zeros rows, if there are any, appear at the bottom of the matrix.
- (b) The first non-zero entry from the left of a non-zero row is 1, called the leading one of its row.
- (c) For each non-zero row, the leading one appears to the right and below any leading ones in the preceding rows.
- (d) If a column contains a leading one, then all other entries in that column are zero.

An  $m \times n$  matrix satisfying properties (a),(b) and (c) is said to be in row echelon form.

**Example 4.7**

Reduce the following matrices to rref.

$$\begin{aligned} \text{(a)} & \left( \begin{array}{ccccc} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array} \right) & \text{(b)} & \left( \begin{array}{ccccc} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{array} \right) & \text{(c)} & \left( \begin{array}{ccc} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{array} \right) \end{aligned}$$

SOLUTION

$$\begin{aligned} \text{(c)} & \left( \begin{array}{ccc} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{array} \right) & -R_2 + R_1 &\longrightarrow R_1 & \left( \begin{array}{ccc} 1 & -1 & 0 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{array} \right) \\ & & -2R_2 + R_3 &\longrightarrow R_3 & \left( \begin{array}{ccc} 1 & -1 & 0 \\ 2 & -1 & -1 \\ 0 & -1 & 1 \end{array} \right) \end{aligned}$$

$$-2R_1 + R_2 \longrightarrow R_2 \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{matrix} R_2 + R_1 \longrightarrow R_1 \\ R_2 + R_3 \longrightarrow R_3 \end{matrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

### Determinant of a Matrix

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then determinant of  $A$ ,  $|A|$  or  $\det A$ , is given by  $|A| = ad - cb$ .

Let  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ . If the row and column through a particular entry are crossed

out, four entries are left. The determinant of the resulting  $2 \times 2$  matrix is called the **minor** of that element. Thus Minor of  $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ , Minor of  $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and so on. There is a sign associated with each minor according to the alternating law of signs  $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ . The minor of an element together with its associated sign is called the **cofactor** of that element.

#### Example 4.8

Consider the matrix  $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 0 & 1 & 5 \end{pmatrix}$ . The minor of 2 =  $\begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} = -5$ . The cofactor of

2 = 5. The minor of 3 =  $\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} = 5$ . The cofactor of 3 = 5 and so on.

To find the determinant of  $A$ , we add the products of the entries of a row (or column) of the matrix with their respective cofactors. Thus, using the first row, we get

$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ . We can make use of any other row or column.

#### Example 4.9

Evaluate the determinant of the given matrix

$$(a) \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & -4 \\ -4 & 2 & 1 \end{pmatrix} \quad (b) \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{pmatrix} \quad (c) \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

SOLUTION

(a)

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & -4 \\ -4 & 2 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 2 & -4 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -4 \\ -4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix} \\ &= 1[2 - (-8)] - 2[2 - 16] - 3[4 - (-8)] \\ &= 10 + 28 - 36 \\ &= 2 \end{aligned}$$

### Some properties of the determinant

1. If any row (or column) is added to or subtracted from any other row (or column) the value of the determinant does not change.
2. The value of the determinant is unchanged if a multiple of any row (or column) is added of any other row (or column).

3. If two rows (or columns) are interchanged the determinant changes sign. That is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. The value of the determinant is unchanged when the rows and columns are

$$\text{completely interchanged. That is, } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \text{ Thus } |A| = |A^T|.$$

5. If all the entries in one row (or column) of a determinant are zero, the determinant is zero.

Finding the inverse of a matrix using determinants

$2 \times 2$  Matrix:

Let  $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ . Then  $|A| = sp - rq$  and  $A^{-1} = \frac{1}{|A|} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$ .

$3 \times 3$  Matrix:

To find the inverse of a  $3 \times 3$  matrix  $A$ ,

1. Obtain the matrix of minors.
  2. Obtain the matrix of cofactors.
  3. Find the transpose of the matrix of cofactors to obtain the adjoint or the adjugate matrix denoted by  $\text{adj}A$ .
  4. The inverse of  $A$  is then obtained as  $A^{-1} = \frac{1}{|A|}\text{adj}A$ .
- Note that a matrix  $A$  has no inverse if  $|A| = 0$ .

**Example 4.10**

Find the inverse of the matrix

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & -5 & 4 \end{pmatrix}$$

SOLUTION

$$\text{Let } A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & -5 & 4 \end{pmatrix}. \text{ The matrix of minors is}$$

$$\begin{pmatrix} \begin{vmatrix} 0 & -2 \\ -5 & 4 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & -5 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -5 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -10 & 2 & 0 \\ 13 & -5 & 3 \\ -4 & 2 & 0 \end{pmatrix}.$$



The matrix of cofactors is  $\begin{pmatrix} -10 & -2 & 0 \\ -13 & -5 & -3 \\ -4 & -2 & 0 \end{pmatrix}$  and so  $\text{Adj}A = \begin{pmatrix} -10 & -13 & -4 \\ -2 & -5 & -2 \\ 0 & -3 & 0 \end{pmatrix}$ . We get

$$|A| = 2 \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix} = 6 \text{ and so } A^{-1} = \frac{1}{6} \begin{pmatrix} -10 & -13 & -4 \\ -2 & -5 & -2 \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & -\frac{13}{6} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{5}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

**Definition 4.18 (Linear Equation)**

An equation of the form  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ , expressing  $b$  (called the constant term of the equation) in terms of the variables  $x_1, x_2, \dots, x_n$  (called the unknowns) and the known constants  $a_1, a_2, \dots, a_n$  (called the coefficients) is called a linear equation.

**Definition 4.19 (System of Linear Equations)**

A system of  $m$  linear equations in  $n$  unknowns  $x_1, x_2, \dots, x_n$  is a set of  $m$  linear equations each in  $n$  unknowns. A linear system can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Thus  $A\mathbf{x} = \mathbf{b}$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ ,  $\mathbf{b} = (b_1, b_2, \dots, b_m)^T$  and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**Definition 4.20**

Linear systems with at least one solution are called consistent and linear systems with no solutions are called inconsistent.

**Solving Linear Systems**

1. The Gauss-Jordan reduction procedure for solving the linear system  $A\mathbf{x} = \mathbf{b}$  is as follows.

- (a) Form the augmented matrix  $[A|\mathbf{b}]$
- (b) Obtain the rref  $[C|\mathbf{d}]$  of the augmented matrix  $[A|\mathbf{b}]$  by using EROS
- (c) For each non-zero row of the matrix  $[C|\mathbf{d}]$ , solve the corresponding equation for the unknown associated with the leading one in that row.

2. The Gaussian elimination procedure for solving the linear system  $A\mathbf{x} = \mathbf{b}$  is as follows:

- (a) Form the augmented matrix  $[A|\mathbf{b}]$
- (b) Obtain a ref  $[C|\mathbf{d}]$  of the augmented matrix  $[A|\mathbf{b}]$  by using EROS
- (c) solve the linear system corresponding to  $[C|\mathbf{d}]$  by back substitution

**Remark 4.2**

If  $[C|\mathbf{d}]$  has a row in which the only non-zero entry lies in the last column, then  $A\mathbf{x} = \mathbf{b}$  is inconsistent.

**Example 4.11**

Solve the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned}$$

SOLUTION

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array} \right) \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array} \right)$$

$$\begin{array}{l} -R_2 + R_1 \rightarrow R_1 \\ 2R_2 + R_3 \rightarrow R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array} \right) \begin{array}{l} \frac{2}{5}R_3 + R_1 \rightarrow R_1 \\ -\frac{1}{5}R_3 + R_2 \rightarrow R_2 \\ -\frac{1}{5}R_3 \rightarrow R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Therefore  $x_1 = 0$ ,  $x_2 = 1$ ,  $x_3 = 2$ .

**Example 4.12**

Solve the system

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 1 \\2x_1 - 2x_2 - x_3 &= 3 \\-x_1 + x_2 - x_3 &= -3\end{aligned}$$

SOLUTION

$$\begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 2 & -2 & -1 & | & 3 \\ -1 & 1 & -1 & | & -3 \end{pmatrix} \xrightarrow[-R_1 + R_3 \rightarrow R_3]{-2R_1 + R_2 \rightarrow R_2} \begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 0 & 0 & -5 & | & 1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}$$

$$\xrightarrow[-5R_3 + R_2 \rightarrow R_2]{-2R_3 + R_1 \rightarrow R_1} \begin{pmatrix} 1 & -1 & 0 & | & 5 \\ 0 & 0 & 0 & | & -9 \\ 0 & 0 & 1 & | & -2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -1 & 0 & | & 5 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & -9 \end{pmatrix}$$

The last row suggests that  $0 = 9$ , which is false. The system is inconsistent.

**Example 4.13**

Solve the system

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 1 \\x_1 - x_2 - x_3 &= 0 \\-2x_2 - 3x_3 &= -1\end{aligned}$$

SOLUTION

$$\begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 1 & -1 & -1 & | & 0 \\ 0 & -2 & -3 & | & -1 \end{pmatrix} \xrightarrow{-R_1 + R_2 \rightarrow R_2} \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 0 & -2 & -3 & | & -1 \\ 0 & -2 & -3 & | & -1 \end{pmatrix}$$

$$\xrightarrow[-R_2 + R_3 \rightarrow R_3]{\frac{1}{2}R_2 + R_1 \rightarrow R_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} & | & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & | & \frac{1}{2} \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & | & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & | & \frac{1}{2} \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Therefore,  $x_1 = \frac{1}{2} - \frac{1}{2}t$ ,  $x_2 = \frac{1}{2} - \frac{3}{2}t$ ,  $x_3 = t$ , where  $t \in \mathbb{R}$ .

Therefore, the system has infinitely many solutions. The variables corresponding to leading ones are called basic variables. The others are called free variables.

**Definition 4.21 (Homogeneous System )**

A linear system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n} &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n} &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn} &= 0 \end{aligned}$$

is called a homogeneous system. In matrix form ,  $A\mathbf{x} = \mathbf{0}$ .

The solution  $x_1 = x_2 = \dots = x_n = 0$  to the homogeneous system is called the trivial solution. A solution  $(x_1, x_2, \dots, x_n)$  to a homogeneous system in which not all  $x_i$  are zero is called a non-trivial solution. The homogeneous system is always consistent since it always has the trivial solution.

**Theorem 4.7**

If  $A$  is an  $n \times n$  matrix, the homogeneous system  $A\mathbf{x} = \mathbf{0}$  has a non-trivial solution if and only if  $A$  is singular, that is  $|A| = 0$ .

**Definition 4.22 (Non-Homogeneous Systems)**

A linear system  $A\mathbf{x} = \mathbf{b}$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ ,  $\mathbf{b} = (b_1, b_2, \dots, b_m)^T$  and  $A$  is  $m \times n$  is called non-homogeneous if  $\mathbf{b} \neq \mathbf{0}$ . That is at least one of the constant terms is not zero.

**Theorem 4.8**

A linear system  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if  $\text{rank } A = \text{rank } [A|\mathbf{b}]$ . That is the ranks of the coefficient and augmented matrices are equal.

**Example 4.14**

For what values of  $k$ , if any , will the system

$$\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= -2 \end{aligned}$$

have

- (a) a unique solution
- (b) infinitely many solutions
- (c) no solution

SOLUTION

The augmented matrix for the system is given by

$$\left[ \begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array} \right]$$

For a unique solution

$$\begin{vmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{vmatrix} \neq 0$$

We have that

$$\begin{aligned} \begin{vmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & k \\ 0 & k-1 & 1-k \\ 0 & 1-k & 1-k^2 \end{vmatrix} = \begin{vmatrix} k-1 & 1-k \\ 1-k & 1-k^2 \end{vmatrix} \\ &= (k-1)(1-k^2) - (1-k)^2 \\ &= -(1-k)^2(1+k) - (1-k)^2 \\ &= (1-k)^2(1+k+1) \\ &= (1-k)^2(2+k) \end{aligned}$$

Therefore, the values of  $k$  for which the system has a unique solution constitute the set  $\mathbb{R} \setminus \{-2, 1\}$ . When  $k = -2$ , the augmented matrix becomes

$$\left[ \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \end{array} \right]$$

Applying elementary row operations, we get

$$\begin{array}{l} -R_1 + R_2 \longrightarrow R_2 \\ 2R_1 + R_3 \longrightarrow R_3 \end{array} \begin{pmatrix} 1 & 1 & -2 & | & 1 \\ 0 & -3 & 3 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{3}R_2 + R_1 \longrightarrow R_1 \\ -\frac{1}{3}R_2 \longrightarrow R_2 \\ R_2 + R_3 \longrightarrow R_3 \end{array} \begin{pmatrix} 1 & 0 & -1 & | & 1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The system has infinitely many solutions given by  $x = 1 + t$ ,  $y = t$ ,  $z = t$ .

When  $x = 1$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & -2 \end{pmatrix} \begin{array}{l} -R_1 + R_2 \longrightarrow R_2 \\ -R_1 + R_3 \longrightarrow R_3 \end{array} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

The system has no solutions, since the last row indicates that  $0 = -3$ .

Let a system of linear equations be given by  $A\mathbf{x} = \mathbf{b}$ . If  $|A| \neq 0$ , then  $A^{-1}$  exists. Hence

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

[Try this for Example (4.11)]

### Cramer's Rule

If  $A\mathbf{x} = \mathbf{b}$  is a linear system of  $n$  equations in  $n$  unknowns, and if  $\det A$  is not zero, then the unique solution of the linear system can be written in the form

$$x_i = \frac{\det(M_i)}{\det(A)}, \quad i = 1, 2, \dots, n$$

where  $M_i$  is the matrix obtained from  $A$  when column  $i$  is replaced by  $\mathbf{b}$ .

#### Example 4.15

Use Cramer's rule to solve the system

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y + z &= 1 \\ 2x + y + z &= -2 \end{aligned}$$

#### SOLUTION

$$\text{We have that } A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1 & -3 \end{vmatrix} = -4$$

$$\det(M_1) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = 8$$

$$\det(M_2) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & -4 & -3 \end{vmatrix} = -4$$

$$\det(M_3) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -4 \end{vmatrix} = -4$$

$$x = \frac{\det(M_1)}{\det(A)} = \frac{8}{-4} = -2, y = \frac{\det(M_2)}{\det(A)} = \frac{-4}{-4} = 1, z = \frac{\det(M_3)}{\det(A)} = \frac{-4}{-4} = 1$$

Exercise Four

1. Show that

(a) if  $A$  and  $B$  are  $2 \times 2$  matrices then the trace of  $AB - BA$  is zero.

(b) if  $E$  is a  $2 \times 2$  matrix and the trace of  $E$  is zero then  $E^2 = \lambda I_2$  for some scalar  $\lambda$ .

2. If  $A$  is a square matrix such that  $A^2 = A$  and  $(A - A^T)^2 = O$  prove that  $(AA^T)^2 = AA^T$ .

3. Let  $A, B$  be square matrices with  $A$  symmetric and  $B$  skew-symmetric. Determine which of the following are symmetric or skew-symmetric.

(a)  $AB + BA$     (b)  $AB - BA$     (c)  $A^p B^q A^p$

4. Solve the following systems of linear equations using Gaussian elimination or Gauss-Jordan elimination.

$$\begin{array}{rcl} x & + & y + z + w = 4 \\ & & 2y - z + w = 0 \\ \text{(a)} \quad & & 3x + 2y + z - w = 6 \\ & & x - 2y - 2z + 2w = -1 \end{array}$$

$$\begin{array}{rcl} & & 3x - 2y + 2z = 6 \\ \text{(b)} \quad & & 7x - 3y + 2z = -1 \\ & & 2x - 3y + 4z = 0 \end{array}$$

$$\begin{array}{rcl} & & 2x - 3y - z = 0 \\ \text{(c)} \quad & & 3x + 2y + 2z = 2 \\ & & x + 5y + 3z = 2 \end{array}$$

5. Solve for  $x$

$$\text{(a)} \quad \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = 0$$



$$(b) \begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix} = -4x$$

6. Determine conditions for which the linear system

$$3x - y + z = 3$$

$$ax + y + z = 4$$

$$8x - y + 3z = b$$

has

(a) a unique solution

(b) infinitely many solutions

(c) no solution

7. Calculate the inverse of the matrix  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 6 \\ 1 & 2 & 3 \end{pmatrix}$  using

(a) elementary row operations

(b) cofactors

Hence, solve the system of linear equations

$$x + y + z = 1$$

$$x + 3y + 6z = 2$$

$$x + 2y + 3z = 1$$

8. Using Cramer's rule, solve the system

$$x + 2y - z = -3$$

$$2x - 4y + z = -7$$

$$-2x + 2y - 3z = 4$$



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## Chapter 5

### Vectors in three dimensional space

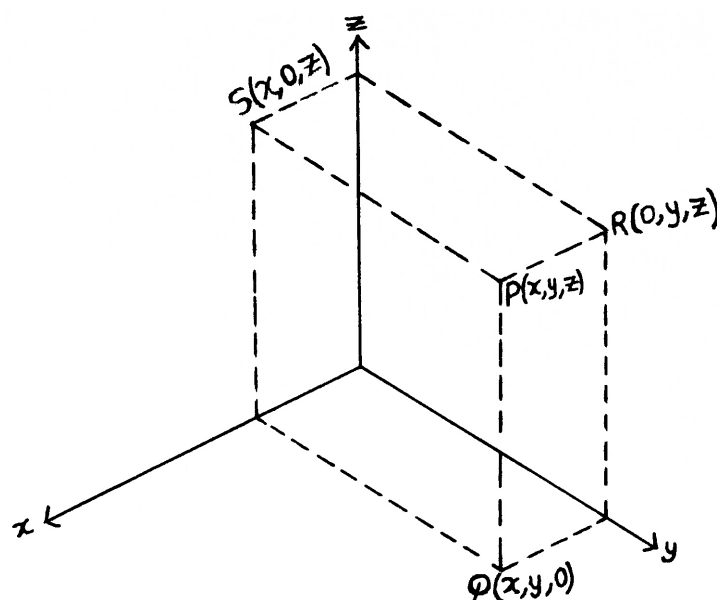


Figure 5.1:

$P(x, y, z)$  is the general point in 3-D space.  $Q(x, y, 0)$  is in the  $xy$ -plane,  $R(0, y, z)$  is in the  $yz$ -plane and  $S(x, 0, z)$  is in the  $xz$ -plane. The position vector of  $P$  is  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ .

**Definition 5.1 (Scalar Product or Inner Product or Dot product)**

Let  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  be two vectors in 3-D. The inner product of  $\mathbf{a}$  and  $\mathbf{b}$ , written  $\mathbf{a} \cdot \mathbf{b}$  or  $\langle \mathbf{a}, \mathbf{b} \rangle$ , is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

**Example 5.1**

Evaluate the scalar product of the two vectors  $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  and  $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ .

SOLUTION

$$\mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + 2\mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 2(2) + 2(-1) + 1(-2) = 0$$

**Theorem 5.1 (Properties of the scalar product)**

Let  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be vectors in  $\mathbb{R}^3$  and  $\alpha \in \mathbb{R}$ . Then

- (i)  $\mathbf{a} \cdot \mathbf{a} \geq 0$  ( $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ ) such that  $\mathbf{a} \cdot \mathbf{a} = 0$  if and only if  $\mathbf{a} = \mathbf{0}$ .
- (ii)  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- (iii)  $\alpha \mathbf{a} \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \alpha \mathbf{b}$
- (iv)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$  and  $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$

PROOF

Let  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ ,  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  and  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$  where  $a_k, b_k, c_k \in \mathbb{R}$  with  $1 \leq k \leq 3$ . Let  $\alpha \in \mathbb{R}$ .

(i) From Definition(5.1)  $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$ . Since  $a_1^2 \geq 0$ ,  $a_2^2 \geq 0$ ,  $a_3^2 \geq 0$ , we have that  $a_1^2 + a_2^2 + a_3^2 \geq 0$ . Hence  $\mathbf{a} \cdot \mathbf{a} \geq 0$ .

Let  $\mathbf{a} \cdot \mathbf{a} = 0$ . Then Thus  $a_1 = a_2 = a_3 = 0$  and so  $\mathbf{a} = \mathbf{0}$ . Conversely, suppose that  $\mathbf{a} = \mathbf{0}$ . Then  $a_1 = a_2 = a_3 = 0$ , and so we get  $\mathbf{a} \cdot \mathbf{a} = 0^2 + 0^2 + 0^2 = 0$ .

(ii)

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= b_1a_1 + b_2a_2 + b_3a_3, \text{ by commutativity of multiplication in } \mathbb{R} \\ &= \mathbf{b} \cdot \mathbf{a} \end{aligned}$$

(iii) We have that

$$\begin{aligned} \alpha(\mathbf{a} \cdot \mathbf{b}) &= \alpha(a_1b_1 + a_2b_2 + a_3b_3) \\ &= \alpha(a_1b_1) + \alpha(a_2b_2) + \alpha(a_3b_3) \\ &= (\alpha a_1)b_1 + (\alpha a_2)b_2 + (\alpha a_3)b_3 \\ &= \alpha \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

and similarly,

$$\begin{aligned}
 \alpha(\mathbf{a} \cdot \mathbf{b}) &= \alpha(a_1b_1 + a_2b_2 + a_3b_3) \\
 &= \alpha(a_1b_1) + \alpha(a_2b_2) + \alpha(a_3b_3) \\
 &= a_1(\alpha b_1) + a_2(\alpha b_2) + a_3(\alpha b_3) \\
 &= \mathbf{a} \cdot (\alpha \mathbf{b})
 \end{aligned}$$

Hence  $\alpha \mathbf{a} \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \alpha \mathbf{b}$ .

(iv)

$$\begin{aligned}
 \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot ((b_1 + c_1)\mathbf{i} + (b_2 + c_2)\mathbf{j} + (b_3 + c_3)\mathbf{k}) \\
 &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
 &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
 &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
 \end{aligned}$$

Similarly,  $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$ .

### Theorem 5.2

Let  $\mathbf{a}$ ,  $\mathbf{b}$  be two vectors in  $\mathbb{R}^3$  and let  $\theta$  be the angle between them, where  $0 \leq \theta \leq \pi$ .

Then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

Thus we can express the angle between  $\mathbf{a}$  and  $\mathbf{b}$  as

$$\theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right)$$

if  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors.

### PROOF

Let  $\theta = 0$ . Then  $\mathbf{a} = \lambda \mathbf{b}$  where  $\lambda > 0$ . We have that  $\mathbf{a} \cdot \mathbf{b} = \lambda \mathbf{b} \cdot \mathbf{b} = \lambda(\mathbf{b} \cdot \mathbf{b}) = \lambda|\mathbf{b}|^2$ .

Now, we have

$$\begin{aligned}
 |\mathbf{a}||\mathbf{b}| \cos \theta &= |\lambda \mathbf{b}||\mathbf{b}| \cos 0 \\
 &= |\lambda||\mathbf{b}||\mathbf{b}|(1) \\
 &= \lambda|\mathbf{b}|^2, \text{ since } |\lambda| = \lambda \text{ for } \lambda > 0
 \end{aligned}$$

Thus  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ .

Let  $\theta = \pi$ . Then  $\mathbf{a} = \lambda \mathbf{b}$  where  $\lambda < 0$ . We have that  $\mathbf{a} \cdot \mathbf{b} = \lambda \mathbf{b} \cdot \mathbf{b} = \lambda(\mathbf{b} \cdot \mathbf{b}) = \lambda|\mathbf{b}|^2$ .

Now, we have

$$\begin{aligned} |\mathbf{a}||\mathbf{b}| \cos \theta &= |\lambda \mathbf{b}||\mathbf{b}| \cos \pi \\ &= |\lambda||\mathbf{b}||\mathbf{b}|(-1) \\ &= -\lambda|\mathbf{b}|^2(-1), \text{ since } |\lambda| = -\lambda \text{ for } \lambda < 0 \\ &= \lambda|\mathbf{b}|^2 \end{aligned}$$

Thus  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ .

Consider  $0 < \theta < \pi$ .

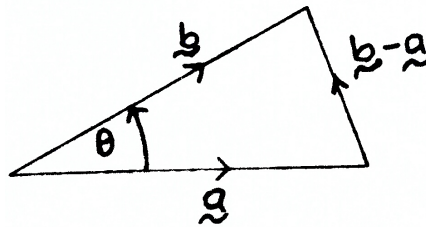


Figure 5.2:

Using the cosine rule, we get  $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$ . Also, we have

$$\begin{aligned} |\mathbf{b} - \mathbf{a}|^2 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} \end{aligned}$$

As a result, we obtain

$$\begin{aligned} |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta \\ -2\mathbf{a} \cdot \mathbf{b} &= -2|\mathbf{a}||\mathbf{b}| \cos \theta \\ \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \theta \end{aligned}$$

Therefore,  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$  where  $0 \leq \theta \leq \pi$ .

### Corollary 5.1 (Cauchy-Schwartz inequality)

For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , we have  $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ , with equality if and only if  $\mathbf{a}$  is a scalar multiple of  $\mathbf{b}$  or one of them is zero.

Suppose that  $\mathbf{a}$  is not a scalar multiple of  $\mathbf{b}$  and neither  $\mathbf{a}$  nor  $\mathbf{b}$  is zero. Then  $0 < \theta < \pi$ , and so  $|\cos \theta| < 1$ . We have

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \theta \\ |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| |\cos \theta| \\ |\mathbf{a} \cdot \mathbf{b}| &< |\mathbf{a}||\mathbf{b}|\end{aligned}$$

When  $\mathbf{a}$  is a scalar multiple of  $\mathbf{b}$ ,  $\theta = 0$  or  $\theta = \pi$  and so  $|\cos \theta| = 1$ . Thus  $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ . Let  $\mathbf{a} = \mathbf{0}$ . Then  $|\mathbf{a}| = 0$ , so  $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{0} \cdot \mathbf{b}| = |0| = 0$  and  $|\mathbf{a}||\mathbf{b}| = 0|\mathbf{b}| = 0$ . Thus  $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ .

**Definition 5.2 (Vector product or Cross product)**

Let  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ ,  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  and  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$  be vectors in  $\mathbb{R}^3$ . The vector product of  $\mathbf{a}$  and  $\mathbf{b}$ , denoted by  $\mathbf{a} \times \mathbf{b}$ , is defined to be the vector

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\end{aligned}$$

**Example 5.2**

Find  $\mathbf{a} \times \mathbf{b}$  given that  $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .

SOLUTION

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} &= \mathbf{i}(1 - 2) - \mathbf{j}(-3 - 1) + \mathbf{k}(6 + 1) \\ &= -\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

**Theorem 5.3 (Properties of the vector product)**

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors in  $\mathbb{R}^3$  and let  $\alpha$ ,  $\beta$  and  $\gamma$  be scalars. Then

- (i)  $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- (ii)  $\mathbf{a} \times (\beta\mathbf{b} + \gamma\mathbf{c}) = \beta(\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{c})$  and  $(\alpha\mathbf{a} + \beta\mathbf{b}) \times \mathbf{c} = \alpha(\mathbf{a} \times \mathbf{c}) + \beta(\mathbf{b} \times \mathbf{c})$

Let  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ ,  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  and  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$  and let  $\alpha$ ,  $\beta$  and  $\gamma$  be scalars.

$$(i) \mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{i}(a_3b_2 - a_2b_3) - \mathbf{j}(a_3b_1 - a_1b_3) + \mathbf{k}(a_2b_1 - a_1b_2)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

$$= -[\mathbf{i}(a_3b_2 - a_2b_3) - \mathbf{j}(a_3b_1 - a_1b_3) + \mathbf{k}(a_2b_1 - a_1b_2)]$$

$$= -(\mathbf{b} \times \mathbf{a})$$

$$(ii) \text{ We have that } \mathbf{a} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}(a_2c_3 - a_3c_2) - \mathbf{j}(a_1c_3 - a_3c_1) + \mathbf{k}(a_1c_2 - a_2c_1).$$

And also

$$\beta\mathbf{b} + \gamma\mathbf{c} = (\beta b_1 + \gamma c_1)\mathbf{i} + (\beta b_2 + \gamma c_2)\mathbf{j} + (\beta b_3 + \gamma c_3)\mathbf{k}$$

$$\mathbf{a} \times (\beta\mathbf{b} + \gamma\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ \beta b_1 + \gamma c_1 & \beta b_2 + \gamma c_2 & \beta b_3 + \gamma c_3 \end{vmatrix}$$

$$= \mathbf{i}[a_2(\beta b_3 + \gamma c_3) - a_3(\beta b_2 + \gamma c_2)] - \mathbf{j}[a_1(\beta b_3 + \gamma c_3) - a_3(\beta b_1 + \gamma c_1)] + \mathbf{k}[a_1(\beta b_2 + \gamma c_2) - a_2(\beta b_1 + \gamma c_1)]$$

$$= \mathbf{i}[\beta(a_2b_3 - a_3b_2) + \gamma(a_2c_3 - a_3c_2)] - \mathbf{j}[\beta(a_1b_3 - a_3b_1) + \gamma(a_1c_3 - a_3c_1)] + \mathbf{k}[\beta(a_1b_2 - a_2b_1) + \gamma(a_1c_2 - a_2c_1)]$$

$$= \beta[(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}]$$

$$+ \gamma[(a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}]$$

$$= \beta(\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{c})$$



**Remark 5.1**

(i) From Theorem(5.3),  $\mathbf{a} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{a})$ . This gives us  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ . In particular, we have that  $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$ .

(ii) We also have that  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ,  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$  and  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ . To see this, we can use Definition(5.2). For instance,

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}(0 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(1 - 0) = \mathbf{k}$$

Figure(1.3) provides a summary of these results. The rules are:

- (1) The vector product of any two consecutive vectors taken in an anti-clockwise direction is the third vector.
- (2) The vector product of any two consecutive vectors taken in a clockwise direction is the negative of the third vector.

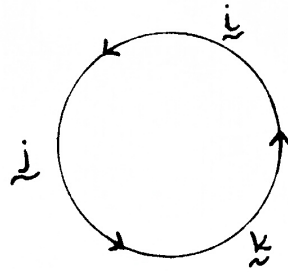


Figure 5.3: The vector products of the unit vector  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$

(iii) Let  $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$  be any vector in the plane determined(spanned) by  $\mathbf{a}$  and  $\mathbf{b}$ , where  $\mathbf{a}$  is not parallel to  $\mathbf{b}$ , that is the angle  $\theta$  between  $\mathbf{a}$  and  $\mathbf{b}$  satisfies  $0 < \theta < \pi$ . Then  $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} = (\lambda a_1 + \mu b_1)\mathbf{i} + (\lambda a_2 + \mu b_2)\mathbf{j} + (\lambda a_3 + \mu b_3)\mathbf{k}$ . So we have

$$\begin{aligned} \mathbf{d} \cdot (\mathbf{a} \times \mathbf{b}) &= (\lambda a_1 + \mu b_1)(a_2 b_3 - a_3 b_2) - (\lambda a_2 + \mu b_2)(a_1 b_3 - a_3 b_1) \\ &\quad + (\lambda a_3 + \mu b_3)(a_1 b_2 - a_2 b_1) \\ &= \lambda(a_1 a_2 b_3 - a_1 a_3 b_2 - a_1 a_2 b_3 + a_2 a_3 b_1 + a_1 a_3 b_2 - a_2 a_3 b_1) \\ &\quad + \mu(a_2 b_1 b_3 - a_3 b_1 b_2 - a_1 b_2 b_3 + a_3 b_1 b_2 + a_1 b_2 b_3 - a_2 b_1 b_3) \\ &= \lambda(0) + \mu(0) \\ &= 0 \end{aligned}$$

Therefore,  $\mathbf{a} \times \mathbf{b}$  is perpendicular to the plane determined by  $\mathbf{a}$  and  $\mathbf{b}$ , and it has the same direction as that in which the screw advances when turned from  $\mathbf{a}$  to  $\mathbf{b}$ . This is illustrated in Figure(5.4).

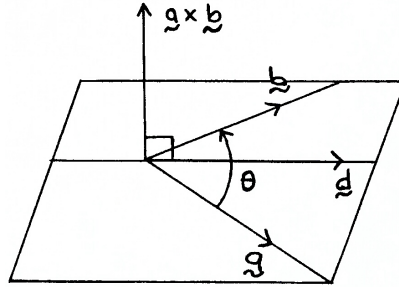


Figure 5.4:

Another way to specify the direction of a vector product is illustrated in Figure(5.5). It is called the right-hand rule.

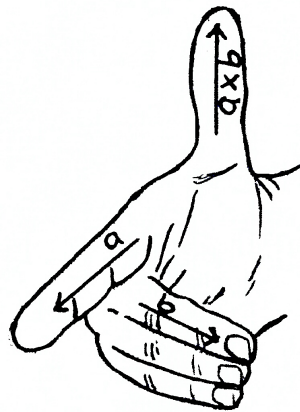


Figure 5.5: The right-hand rule

#### Theorem 5.4

Let  $\mathbf{a}$ ,  $\mathbf{b}$  be two vectors in  $\mathbb{R}^3$  and let  $\theta$  be the angle between them, where  $0 \leq \theta \leq \pi$ . Then

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}}$$

where  $\hat{\mathbf{n}}$  is a unit vector perpendicular to the plane of  $\mathbf{a}$  and  $\mathbf{b}$ .

Let  $\mathbf{a}$ ,  $\mathbf{b}$  be two vectors in  $\mathbb{R}^3$  and let  $\theta$  be the angle between them, where  $0 \leq \theta \leq \pi$ .

When  $\theta = 0$ , we have  $\mathbf{b} = \lambda \mathbf{a}$  where  $\lambda > 0$ , and so  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{a}) = \lambda(\mathbf{a} \times \mathbf{a}) = \lambda \mathbf{0} = \mathbf{0}$ .

Also,  $|\mathbf{a}||\mathbf{b}| \sin(\theta) \hat{\mathbf{n}} = \lambda |\mathbf{a}|^2 \sin(0) \hat{\mathbf{n}} = 0 \hat{\mathbf{n}} = \mathbf{0}$ . Hence the result holds.

When  $\theta = \pi$ , we have  $\mathbf{b} = \lambda \mathbf{a}$  where  $\lambda < 0$ , and so  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{a}) = \lambda(\mathbf{a} \times \mathbf{a}) = \lambda \mathbf{0} = \mathbf{0}$ .

Also,  $|\mathbf{a}||\mathbf{b}| \sin(\theta) \hat{\mathbf{n}} = -\lambda |\mathbf{a}|^2 \sin(0) \hat{\mathbf{n}} = 0 \hat{\mathbf{n}} = \mathbf{0}$ . Hence the result holds.

Now let us consider the case where  $0 < \theta < \pi$ . We have

$$\begin{aligned}
 |\mathbf{a} \times \mathbf{b}|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\
 &= (a_2 b_3 - a_3 b_2)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_1 b_2 - a_2 b_1)^2 \\
 &= a_2^2 b_3^2 + a_3^2 b_2^2 - 2a_2 a_3 b_2 b_3 + a_1^2 b_3^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 a_2 b_1 b_2 \\
 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
 &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
 &= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2 \theta \\
 &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \theta) \\
 &= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta
 \end{aligned}$$

Thus  $|\mathbf{a} \times \mathbf{b}|^2 = (|\mathbf{a}||\mathbf{b}| \sin \theta)^2$ . Since  $0 < \theta < \pi$  and so  $\sin \theta > 0$ , we have that

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$ . Let  $\hat{\mathbf{n}}$  be a unit vector perpendicular to the plane of  $\mathbf{a}$  and  $\mathbf{b}$ , then

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta) \hat{\mathbf{n}}$ .

### The area of a parallelogram

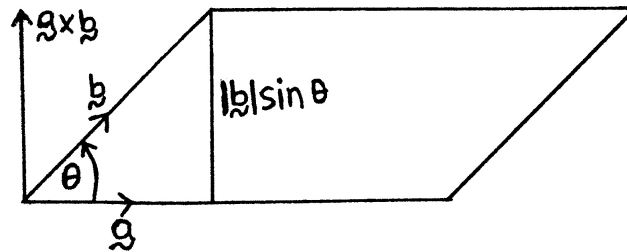


Figure 5.6:

Let  $\mathbf{a}$  and  $\mathbf{b}$  be displacements along adjacent edges of a parallelogram as shown in Figure(5.6). Then

$$\begin{aligned}\text{Area of parallelogram} &= \text{Length of base} \times \text{Perpendicular height} \\ &= |\mathbf{a}||\mathbf{b}| \sin \theta \\ &= |\mathbf{a} \times \mathbf{b}|\end{aligned}$$

If  $\mathbf{a}$  and  $\mathbf{b}$  are displacements along adjacent edges of a triangle, then the area of the triangle is  $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ .

**Example 5.3**

Compute the area of the triangle with vertices  $A(1, -2, 1)$ ,  $B(2, 1, 1)$  and  $C(-1, 1, -2)$ .

SOLUTION

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} = \mathbf{i} + 3\mathbf{j} \text{ and } \vec{AC} = \vec{OC} - \vec{OA} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}. \text{ Thus} \\ \vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}. \text{ Hence, } |\vec{AB} \times \vec{AC}| = 3|-3\mathbf{i} + \mathbf{j} + 3\mathbf{k}| = 3\sqrt{19}.\end{aligned}$$

Therefore, the area of the triangle  $ABC$  is  $\frac{3}{2}\sqrt{19}$  square units.

**Definition 5.3 (Scalar triple product)**

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors. The scalar triple product of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  (in that order) is defined to be the real number  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ .

**Example 5.4**

Compute the scalar triple product of the vectors  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ .

SOLUTION

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}. \text{ Thus } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -3 - 6 + 3 = -6.$$

**Remark 5.2**

Now let  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ ,  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  and  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ . Then

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This presents us with another way to compute the scalar triple product of three vectors.

**Example 5.5**

Compute the scalar triple product of the vectors  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ . [*This is the question in Example(5.4)*]

SOLUTION

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = -3 - 6 + 3 = -6.$$

**Theorem 5.5** (Properties of scalar triple product)

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors in  $\mathbb{R}^3$ . Then

- (i)  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$  (cyclic permutation)
- (ii)  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$  (interchange of dot and cross)

PROOF

(i) Since  $\mathbf{b} \times \mathbf{c}$  is orthogonal to both  $\mathbf{b}$  and  $\mathbf{c}$  we have that  $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ .

Replacing  $\mathbf{b}$  by  $\mathbf{a} + \mathbf{b}$  in  $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ , we get

$$(\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} + \mathbf{b}) \times \mathbf{c}] = 0$$

$$(\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})] = 0$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

Thus  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})] = \mathbf{b} \cdot -(\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ .

Replacing  $\mathbf{c}$  by  $\mathbf{a} + \mathbf{c}$  in  $\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ , we get

$$(\mathbf{a} + \mathbf{c}) \cdot [\mathbf{b} \times (\mathbf{a} + \mathbf{c})] = 0$$

$$(\mathbf{a} + \mathbf{c}) \cdot [(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c})] = 0$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

Thus  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -[\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})] = \mathbf{c} \cdot -(\mathbf{b} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ .

Therefore,  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ .

(ii) From part (i), we know that  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ . By commutativity of the scalar product, we have that  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ .

### The volume of a parallelepiped

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be displacements along adjacent edges of a parallelepiped, as shown in Figure(5.7).

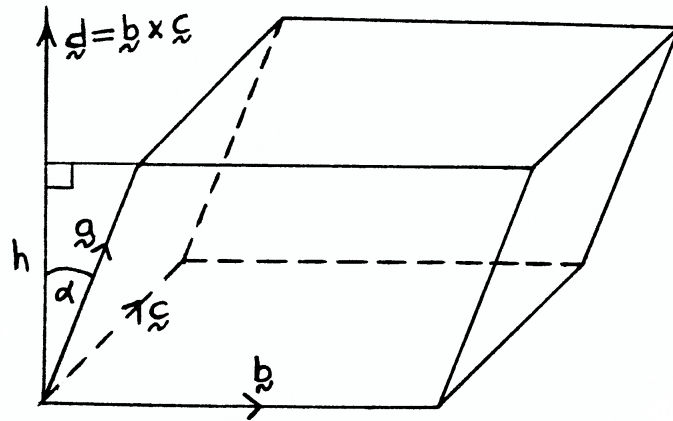


Figure 5.7:

The vector  $\mathbf{d} = \mathbf{b} \times \mathbf{c}$  is perpendicular to the base of the parallelepiped and  $|\mathbf{d}| = |\mathbf{b} \times \mathbf{c}|$  is equal to the area of the base. Let  $\alpha$  be the angle between  $\mathbf{a}$  and  $\mathbf{d}$ . Then we have  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{d} = |\mathbf{a}||\mathbf{d}| \cos \alpha$ . But  $|\mathbf{a} \cos \alpha| = h$ , the height of the parallelepiped. Thus

$$\begin{aligned} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| &= |\mathbf{a}||\mathbf{d}| \cos \alpha| \\ &= |\mathbf{d}|h \\ &= \text{area of base} \times \text{height} \end{aligned}$$

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Therefore, volume of parallelepiped =  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ . That is, volume of parallelepiped is the

absolute value of the determinant  $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ . If  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are displacements along adjacent edges of a tetrahedron, then the volume of the tetrahedron is  $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ .

**Example 5.6**

What is the volume of the parallelepiped with sides  $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ ,  $5\mathbf{i} - 3\mathbf{k}$  and  $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ?

SOLUTION

Let  $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = 5\mathbf{i} - 3\mathbf{k}$  and  $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ . Then we have that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 5 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 5 & -3 \\ 5 & -1 \end{vmatrix} = -(-5 + 15) = -10$$

Therefore, the volume of the parallelepiped with sides  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 10$  cubic units.

**Definition 5.4 (Vector triple product)**

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors in  $\mathbb{R}^3$ . The vector triple product of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  is defined to be the vector  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ .

**Example 5.7**

Compute the vector triple product of the vectors  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ .

SOLUTION

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{vmatrix} = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}. \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} = -9\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}.$$

**Theorem 5.6**

Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors in  $\mathbb{R}^3$ . Then

(i)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(ii)  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  is orthogonal to  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ . Since  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  is perpendicular to  $\mathbf{b} \times \mathbf{c}$ , it must lie in the plane of  $\mathbf{b}$  and  $\mathbf{c}$ . Hence  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{b} + \mu \mathbf{c}$ , for some  $\lambda, \mu \in \mathbb{R}$ . Now define the triad  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  such that  $\mathbf{i} = \frac{1}{b} \mathbf{b}$  and  $\mathbf{j}$  lies in the plane of  $\mathbf{b}$  and  $\mathbf{c}$ . Then  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are of the form  $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$ ,  $\mathbf{b} = b \mathbf{i}$  and  $\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j}$ . We have that

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b & 0 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = bc_2 \mathbf{k}. \text{ And so we obtain}$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & bc_2 \end{vmatrix} \\ &= a_2 bc_2 \mathbf{i} - a_1 bc_2 \mathbf{j} \\ &= a_2 bc_2 \mathbf{i} - a_1 bc_2 \mathbf{j} + a_1 bc_1 \mathbf{i} - a_1 bc_1 \mathbf{i} \\ &= (a_1 c_1 + a_2 c_2) b \mathbf{i} - a_1 b (c_1 \mathbf{i} + c_2 \mathbf{j}) \\ &= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \end{aligned}$$

Therefore,  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$ .

(ii)

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= -[\mathbf{c} \times (\mathbf{a} \times \mathbf{b})], \text{ since the vector product is anti-commutative} \\ &= -[(\mathbf{c} \cdot \mathbf{b}) \mathbf{a} - (\mathbf{c} \cdot \mathbf{a}) \mathbf{b}], \text{ using part (i)} \\ &= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}, \text{ since the scalar product is commutative} \end{aligned}$$

### Example 5.8

Compute the vector triple product of the vectors  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ .

### SOLUTION

We have that  $\mathbf{a} \cdot \mathbf{c} = -1 - 2 - 2 = -5$  and  $\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 1 = 1$ . Hence we get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -5(2\mathbf{i} + \mathbf{j} + \mathbf{k}) - 1(-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -9\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}.$$



**Example 5.9**

Show that  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ .

SOLUTION

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \cdot \mathbf{d}, \text{ by interchanging dot and cross} \\ &= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] \cdot \mathbf{d} \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \end{aligned}$$

Alternatively, we can proceed as follows:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})], \text{ by interchanging dot and cross} \\ &= \mathbf{a} \cdot [(\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}] \\ &= (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \end{aligned}$$

**Remark 5.3**

Consider the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  in 3-D.

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})]\mathbf{c} - [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{d} \text{ ----- (1)}$$

$$\text{Also, } (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{b} - [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{a} \text{ ----- (2)}$$

(1)-(2) gives us

$$\mathbf{0} = [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{a} - [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{b} - [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})]\mathbf{c} - [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{d}$$

Any four vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  in 3-D are linearly dependent and the above equation is the relationship connecting them.

**Proposition 5.1 (The Jacobi Identity)**

Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  be vectors in 3-D. Then  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ .

PROOF

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \text{ ----- (1)}$$

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \text{ ----- (2)}$$

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \quad (3)$$

Adding (1), (2) and (3), we get  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ .

### Cylindrical and spherical coordinates

#### Definition 5.5 (Cylindrical coordinates)

The cylindrical coordinates  $(r, \theta, z)$  of a point  $(x, y, z)$  are defined by

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$\text{where } r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1} \left( \frac{y}{x} \right)$$

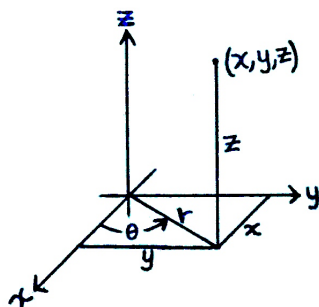


Figure 5.8: The cylindrical coordinate system

#### Remark 5.4

The value of  $\theta$  is normally restricted to a specified interval, usually,  $-\pi < \theta \leq \pi$  or  $0 \leq \theta < 2\pi$ .

#### Example 5.10

- Find the cylindrical coordinates of the point  $P$  with Cartesian coordinates  $(6, 6, 8)$ .
- Find the Cartesian coordinates of the point  $P$  with cylindrical coordinates  $\left(8, \frac{2\pi}{3}, -3\right)$ .

#### SOLUTION

- We have that  $r = \sqrt{6^2 + 6^2} = 6\sqrt{2}$  and  $\theta = \tan^{-1} \left( \frac{6}{6} \right) = \frac{\pi}{4}$ . This value of  $\theta$  holds for both  $(-\pi, \pi]$  and  $[0, 2\pi)$ . Hence the cylindrical coordinates of  $(6, 6, 8)$  are  $\left(6\sqrt{2}, \frac{\pi}{4}, 8\right)$ .

(b) We have that  $x = 8 \cos \left( \frac{2\pi}{3} \right) = -4$  and  $y = 8 \sin \left( \frac{2\pi}{3} \right) = 8 \left( \frac{\sqrt{3}}{2} \right) = 4\sqrt{3}$ . Hence the Cartesian coordinates of the point  $P$  are  $(-4, 4\sqrt{3}, -3)$ .

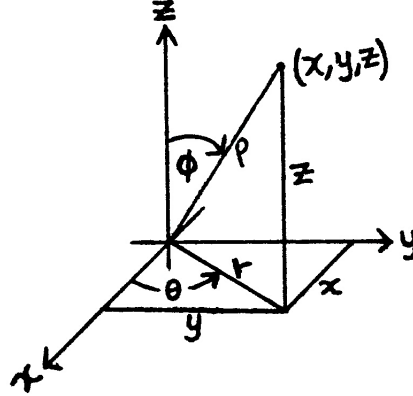


Figure 5.9: The spherical coordinate system

Let us consider Figure(5.9) above. We have  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1} \left( \frac{y}{x} \right)$ . Also,  $z = \rho \cos \phi$  and  $r = \rho \sin \phi$  where  $0 \leq \phi \leq \pi$ .

Let  $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . Then we have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{k} &= |\mathbf{v}| |\mathbf{k}| \cos \phi = |\mathbf{v}| \cos \phi \\ \cos \phi &= \frac{\mathbf{v} \cdot \mathbf{k}}{|\mathbf{v}|} \\ \phi &= \cos^{-1} \left( \frac{\mathbf{v} \cdot \mathbf{k}}{|\mathbf{v}|} \right) \end{aligned}$$

Given the Cartesian coordinates  $(x, y, z)$  of a point  $P$ , the spherical coordinates are  $(\rho, \theta, \phi)$ .

**Definition 5.6 (Spherical coordinates)**

The spherical coordinates  $(\rho, \theta, \phi)$  of a point  $P$  with Cartesian(rectangular) coordinates  $(x, y, z)$  are defined by

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

where  $\rho \geq 0$ ,  $0 \leq \theta < 2\pi$  or  $-\pi < \theta \leq \pi$  and  $0 \leq \phi \leq \pi$ .

**Example 5.11**

Express the surface  $x^2 + y^2 - z^2 = 1$  in spherical coordinates.

SOLUTION

$$x^2 + y^2 - z^2 = 1$$

$$x^2 + y^2 + z^2 - 2z^2 = 1$$

$$\rho^2 - 2\rho^2 \cos^2 \phi = 1$$

$$\rho^2(1 - 2\cos^2 \phi) = 1$$

$$-\rho^2 \cos 2\phi = 1$$

Exercise Five

1. Simplify the following expressions:

(a)  $(\mathbf{a} + \mathbf{d}) \cdot (\mathbf{b} - \mathbf{c}) + (\mathbf{a} - \mathbf{d}) \cdot (\mathbf{b} + \mathbf{c})$

(b)  $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) + \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) + \mathbf{c} \cdot (\mathbf{a} - \mathbf{b})$

(c)  $\mathbf{a} \cdot (\lambda \mathbf{b} - \mu \mathbf{c}) + \mathbf{b} \cdot (\lambda \mathbf{c} - \mu \mathbf{a}) + \mathbf{c} \cdot (\lambda \mathbf{a} - \mu \mathbf{b})$

2. A vector  $\mathbf{c}$  is expressed in the form

$$\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are linearly independent. By taking scalar products of this equation with  $\mathbf{a}$  and  $\mathbf{b}$ , express  $\lambda$  and  $\mu$  in terms of scalar products of the three vectors.

3. Two vectors are given by

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} - \mathbf{k},$$

$$\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}.$$

Find the unit vectors corresponding to  $\mathbf{a}$  and  $\mathbf{b}$ , and the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

4. The vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are given by

$$\mathbf{a} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k},$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k},$$

$$\mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Calculate  $\mathbf{a} \times \mathbf{b}$  and  $\mathbf{a} \times \mathbf{c}$ . Show that  $\mathbf{b}$  and  $\mathbf{c}$  are orthogonal, but that  $\mathbf{a} \times \mathbf{b}$  and  $\mathbf{a} \times \mathbf{c}$  are not orthogonal. Find a vector  $\mathbf{d}$  such that  $\mathbf{d} \times \mathbf{b}$  and  $\mathbf{d} \times \mathbf{c}$  are orthogonal.

5. If vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  satisfy  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ , prove that  $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$ .

6. Vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are defined by

$$\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

$$\mathbf{b} = \mathbf{i} - \mathbf{j} - 3\mathbf{k},$$

$$\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Evaluate  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ ,  $(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$ ,  $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b}$ ,  $\mathbf{a} \times (\mathbf{c} \times \mathbf{b})$  and  $[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \times \mathbf{c}$ .

7. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$  are vectors, show that

$$(a) \quad \mathbf{a} \times [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] = (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \times \mathbf{d}),$$

$$(b) \quad (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

$$(c) \quad |\mathbf{a} \times \mathbf{b}|^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

8. Prove that  $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$  if and only if  $(\mathbf{u} \times \mathbf{w}) \times \mathbf{v} = \mathbf{0}$ .

9. Find the area of the parallelogram with sides the vectors  $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ .

10. Find the volume of the parallelepiped determined by the vertices  $(0, 1, 0)$ ,  $(1, 1, 1)$ ,  $(0, 2, 0)$  and  $(3, 1, 2)$ .

11. (a) Prove that the area of the triangle in the plane with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  is the absolute value of

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

(b) Find the area of the triangle with vertices  $(1, 2)$ ,  $(0, 1)$ ,  $(-1, 1)$ .

12. Convert the following points from Cartesian to cylindrical and spherical coordinates:

$$(a) \quad (-2\sqrt{3}, -2, 3) \quad (b) \quad (-1, 0, 1) \quad (c) \quad (-\sqrt{2}, 1, 0)$$

13. Convert the following points from cylindrical to Cartesian and spherical coordinates:

$$(b) \quad \left(1, \frac{\pi}{4}, 1\right) \quad (b) \quad \left(-2, -\frac{\pi}{2}, 1\right) \quad (c) \quad \left(3, \frac{\pi}{6}, -4\right)$$

14. Convert the following points from spherical to Cartesian and cylindrical coordinates:

$$(c) \quad \left(1, \frac{\pi}{2}, \pi\right) \quad (b) \quad \left(2, -\frac{\pi}{2}, \frac{\pi}{6}\right) \quad (c) \quad \left(-1, \pi, \frac{\pi}{6}\right)$$