Calculus II (MATH 223)

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<u>LECTURE NOTES</u>

Contents

Chapter 1

Introduction

1.1 Review of Calculus I

1.1.1 Maximum & Minimum Values

Definition 1. A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in the domain D of f. Similarly, f has an absolute minimum (global minimum) at c if $f(c) \leq f(x)$ for all x in D. The maximum and minimum vlaues of f are called the extreme values of f.

Figure 1.1 illustrates these concepts.

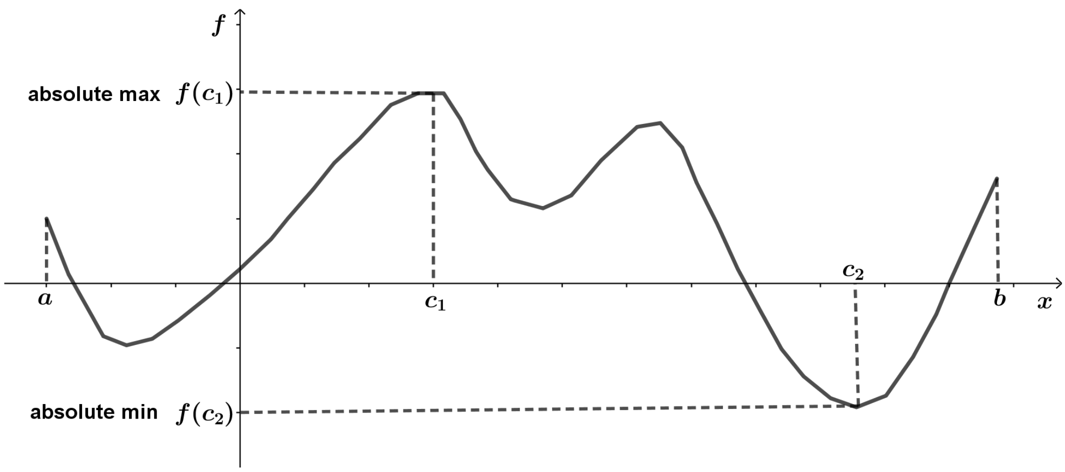

Figure 1.1: Schematic plot showing the absolute maximum and minimum positions.

Definition 2. A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ for all x in some open interval containing c. Similarly, f has a local minimum at c if $f(c) \leq f(x)$ for all x near c.

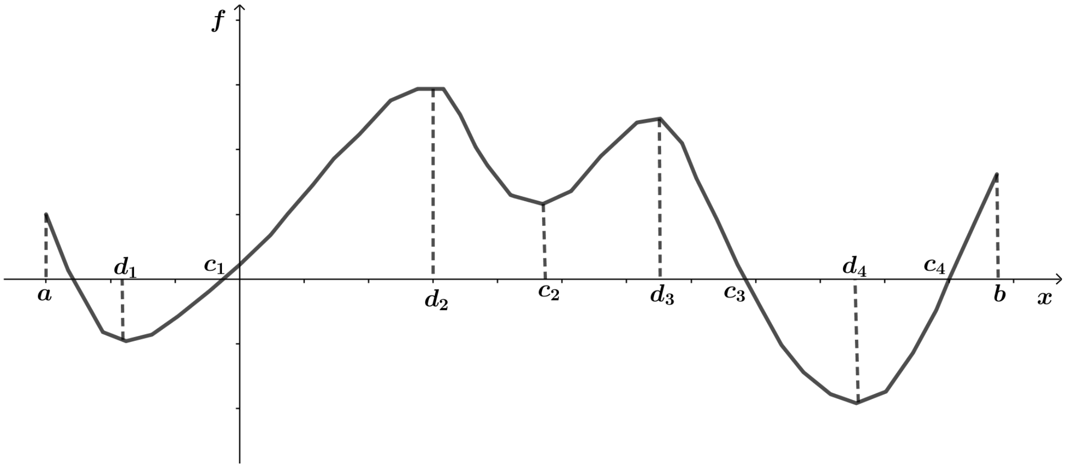

Figure 1.2: Schematic plot showing intervals of increase and decrease.

Example 1. See Figure 1.2. On the interval (a, c_1), $f(d_1)$ is a local minimum value; similarly $f(c_2)$ is a local minimum in the interval (d_2, d_3). On (c_1, c_2), $f(d_2)$ is a local maximum.

Theorem 1 (**The Extreme Value Theorem**)**.** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ for some number c in $[a, b]$ and an absolute minimum value $f(d)$ for some number d in $[a, b]$.

NOTE: An extreme value can be taken on more than once, as illustrated in Figure 1.3.

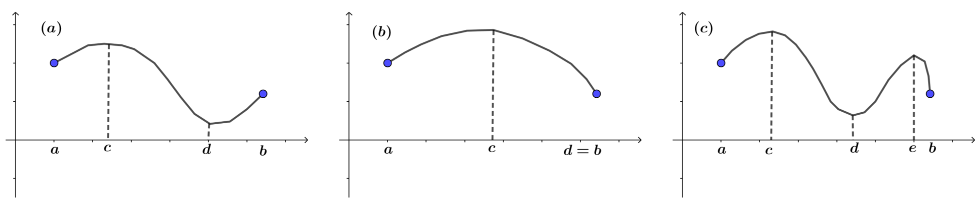

Figure 1.3: A schematic showing extreme values.

Theorem 2 (**Fermat's Theorem**)**.** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

NOTE: When $f'(c) = 0$, f does not necessarily have a maximum or minimum at c. For example, the function $f(x) = x^3$ has $f'(x) = 3x^2$. Thus $f'(0) = 0$, but f has no maximum or minimum value (see Figure 1.4).

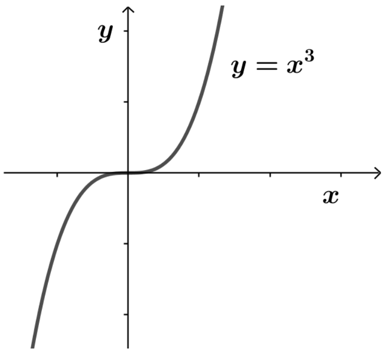

Figure 1.4: Graph of $y = x^3$.

Definition 3 (**Critical Number**)**.** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 3 (**Rolle's Theorem**)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exist at least one number c in (a, b) such that $f'(c) = 0$.

An illustration is given in Figure 1.5.

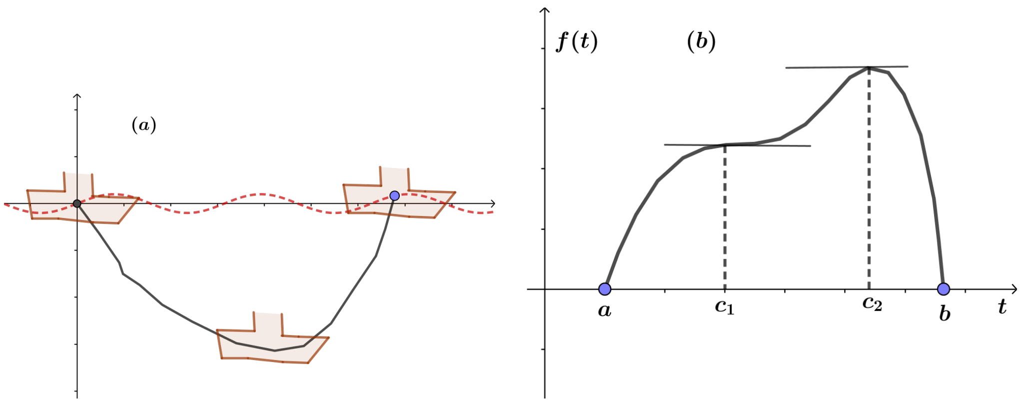

Figure 1.5: Schematic illustrations of Rolle's Theorem.

Proof. Figure 1.6 gives an illustration of the two cases involved in the proof of the theorem.

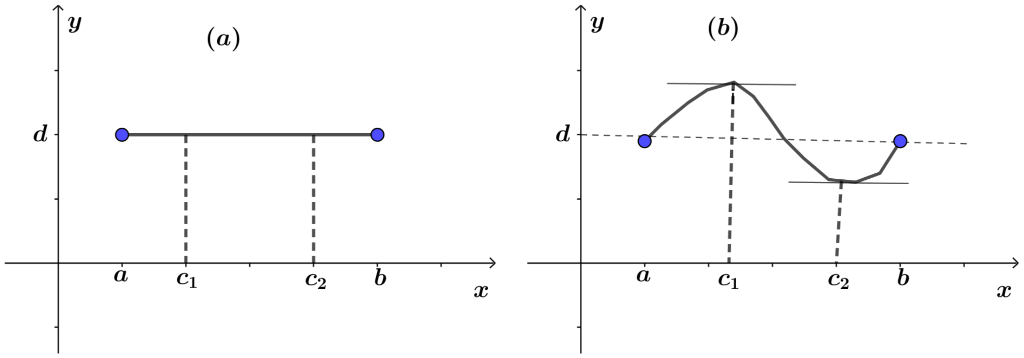

Figure 1.6: Schematic illustrations of Rolle's Theorem.

Let $f(a) = f(b) = d$.

CASE 1:

$f(x) = d$ for x in $[a, b]$. This implies that $f'(x) = 0$ for all x in $[a, b]$. Thus $f'(x) = 0$ for some c in (a, b).

CASE 2:

$f(x) \neq d$ for at least one x in $[a, b]$. So there must be a number x in (a, b) where $f(x) > d$ or $f(x) < d$. Supoose first that $f(x) > d$. Since f is continuous on $[a, b]$, then by the Extreme Value Theorem, f has an absolute maximum value at some number c in $[a, b]$. The number c cannot be an end point since $f(a) = f(b) = d$ and we have assumed that $f(x) > d$ for some x in (a, b). Thus, c must lie in (a, b). Since f is differentiable on (a, b), $f'(x)$ exists and so by Fermat's Theorem $f'(c) = 0$. The case $f(x) < d$ can be proved in a similar manner. □

Example 2. Verify that the function satisfies the hypotheses of Rolle's Theorem (RT) on the given interval, and find all values of c that satisfy the conclusion of the theorem

(1) $f(x) = x^2 - 4x + 3; \quad [1, 3]$

(2) $f(x) = x^3 - x; \quad [-1, 1]$

(3) $f(x) = x^3 - 9x; \quad [-3, 3]$

Solution. (1) $f(x) = x^2 - 4x + 3; \quad [1, 3]$.

$$f(1) = 1 - 4 + 3 = 0$$
$$f(3) = 9 - 12 + 3 = 0$$

The polynomial $f(x)$ is continuous and differentiable on $[1, 3]$. Therefore the hypotheses of Rolle's Theorem are satisfied.

$$f'(x) = 2x - 4$$

$$f'(c) = 0 \Longrightarrow 2c - 4 = 0$$
$$\Longrightarrow c = 2.$$

Sketch:

$$f(x) = x^2 - x - 3x + 3 = x(x-1) - 3(x-1)$$
$$= (x-1)(x-3)$$

So the zeros are $x = 1$ and $x = 3$. See Figure 1.7a.

(2) $f(x) = x^3 - x; \quad [-1, 1]$
$f(x)$ is a polynomial so it's continuous and differentiable on $[-1, 1]$.

$$f(-1) = -1 + 1 = 0$$
$$f(1) = 1 - 1 = 0$$
$$\Longrightarrow f(-1) = f(1)$$

Thus, Rolle's Theorem is satisfied.

$$f'(x) = 3x^2 - 1$$
$$f'(c) = 0 \Longrightarrow 3c^2 - 1 = 0$$
$$\Longrightarrow c^2 = \frac{1}{3}$$
$$\Longrightarrow c = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}. \tag{1.1}$$

A graph of $f(x) = x(x^2 - 1) = x(x-1)(x+1)$ is shown in Figure 1.7b.

Example 3 (**Real life example**)**.** During a test dive of a prototype of a submarine, the depth in meters of the submarine at time t in minutes is given by
$$h(t) = t^3(t-7)^4$$
where $0 \le t \le 7$.

(a) Use Rolle's Theorem to show that there is some instant of time $t = c$ between 0 and 7 when $h'(c) = 0$.

(b) Find the number c and interpret your results.

Solution.
$$h(t) = t^3(t-7)^4; \quad 0 \le t \le 7.$$

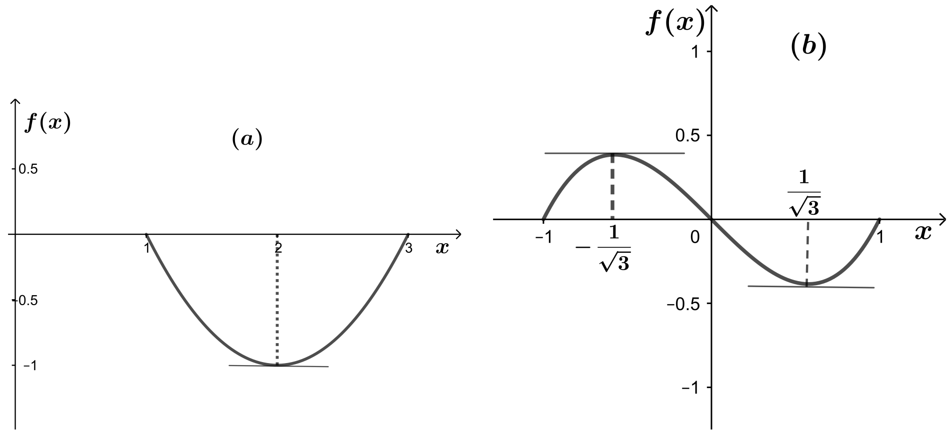

Figure 1.7: Plot of (a) $f(x) = x^2 - 4x + 3$ and (b) $f(x) = x^3 - x$.

(a) The polynomial function $h(t)$ is continuous and differentiable on $\mathbb{R}$ or $(-\infty, \infty)$.

$$h(0) = 0; \quad h(7) = 0$$

$$\Longrightarrow h(0) = h(7).$$

So the hypotheses of Rolle's Theorem is satisfied.

(b)

$$\begin{aligned} h'(t) &= 3t^2(t-7)^4 + 4(t-7)^3t^3 \\ &= t^2(t-7)^3\left[3(t-7)+4t\right] \\ &= t^2(t-7)^3(7t-21) \\ &= 7t^2(t-7)^3(t-3) \end{aligned}$$

We find a number c in $[0, 7]$ such that $h'(t) = 0$.

$$h'(c) = 0 \Longrightarrow 7t^2(t-7)^3(t-3) = 0$$

$$\Longrightarrow t = 0,\ 3,\ 7.$$

The number $c = 3$ is in $(0, 7)$.
The submarine is on the surface at $t = 0$ and returns after 7 minutes, $h(7) = 0$. The submarine reaches a maximum depth of $h(3) = 6912$ at $t = 3$ minutes. See Figure 1.8.

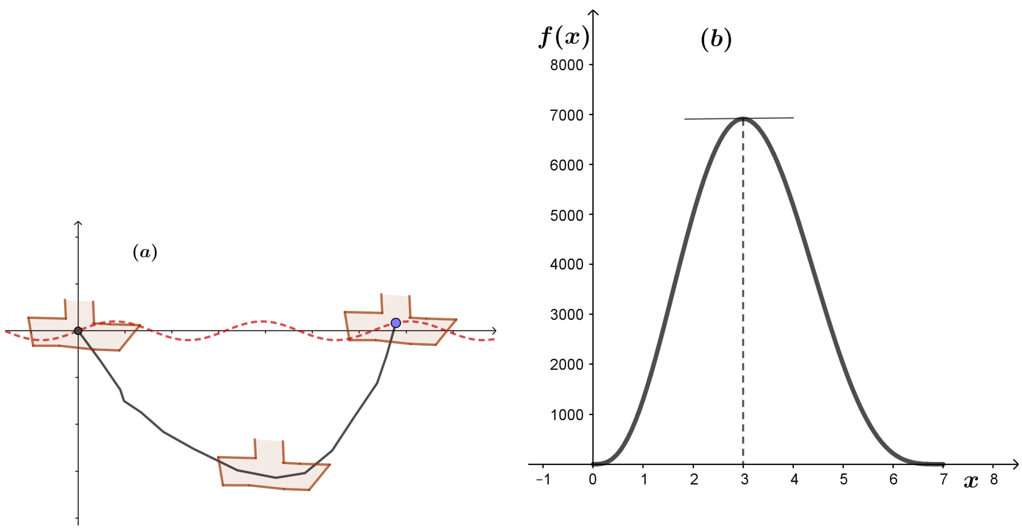

Figure 1.8: (a) Schematic of submarine dive (b) Plot of the submarine function $h(t) = t^3(t-7)^4$.

Chapter 2

The Mean Value Theorem and its Applications

2.1 The Mean Value Theorem (MVT)

Theorem 4. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a};$$

or $f(b) - f(a) = f'(c)(b - a)$.

Interpretation:
Case (a): From Figure 2.1, the slope of the secant line is

$$S = \frac{f(b) - f(a)}{b - a}$$

and the slope of the tangent line, T, to the function f at c is $f'(c)$. MVT tells us that there is at least one point $(c, f(c))$ in the interval (a, b) such that T is parallel to S.
Case (b): Note that if $f(a) = f(b)$ then we recover Rolle's Theorem.

Proof. The equation of a line with slope m and passing through (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

Now

$$m = S = \frac{f(b) - f(a)}{b - a}.$$

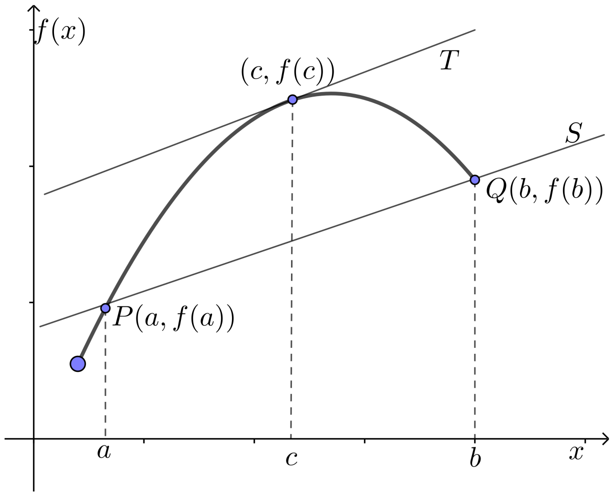

Figure 2.1: Schematic illustration of the Mean Value Theorem.

Since $(b, f(b))$ is on S, the equation of the secant becomes

$$y - f(b) = \frac{f(b) - f(a)}{b - a}(x - b)$$

$$\Longrightarrow y = f(b) + \frac{f(b) - f(a)}{b - a}(x - b)$$

Let $D(x)$ be the function for the distance between $f(x)$ and y such that

$$D(x) = f(x) - \left[f(b) + \frac{f(b) - f(a)}{b - a}(x - b)\right]$$

Note that $D(a) = 0 = D(b)$. Also, $D(x)$ is continuous on $[a, b]$ and differentiable on (a, b), therefore we can apply Rolle's Theorem that there is a c in (a, b) such that $D'(c) = 0$. Now,

$$D'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

So, $D'(x) = 0$ implies that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

Example 4. Show that the function satisfies the hypotheses of the Mean Value Theorem on the given interval, and find all values of c that satisfy the conclusion of the theorem.

(a) $f(x) = x^2 + 1; \quad [0, 2]$

(b) $f(x) = x^3; \quad [-1, 1]$

Solution. (a) $f(x) = x^2 + 1; \quad [0, 2]$
$f(x)$ is a polynomial and so it is continuous on $[0, 2]$ and differentiable on $(0, 2)$. So the hypotheses of the MVT are satisfied.

$$c \in [0, 2] : \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\implies f'(x) = 2x = \frac{f(2) - f(0)}{2 - 0}$$

$$\implies 2x = \frac{5 - 1}{2} = 2$$

$$\implies x = 1$$

$$\therefore c = 1 \in [0, 2].$$

See illustration of this in Figure $2.2a$.
(b) $f(x) = x^3; \quad [-1, 1]$
$f(x)$ is a polynomial so MVT is satisfied.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\implies f'(x) = 3x^2 = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$\implies 3x^2 = \frac{1 - (-1)}{2} = 1$$

$$\implies x^2 = \frac{1}{3}$$

$$\implies x = \pm\frac{1}{\sqrt{3}}.$$

This is illustrated in Figure $2.2b$.

Exercise. Use the graph of f to estimate the values of c that satisfy the conclusion of the MVT for the given interval shown in Figure 2.3.

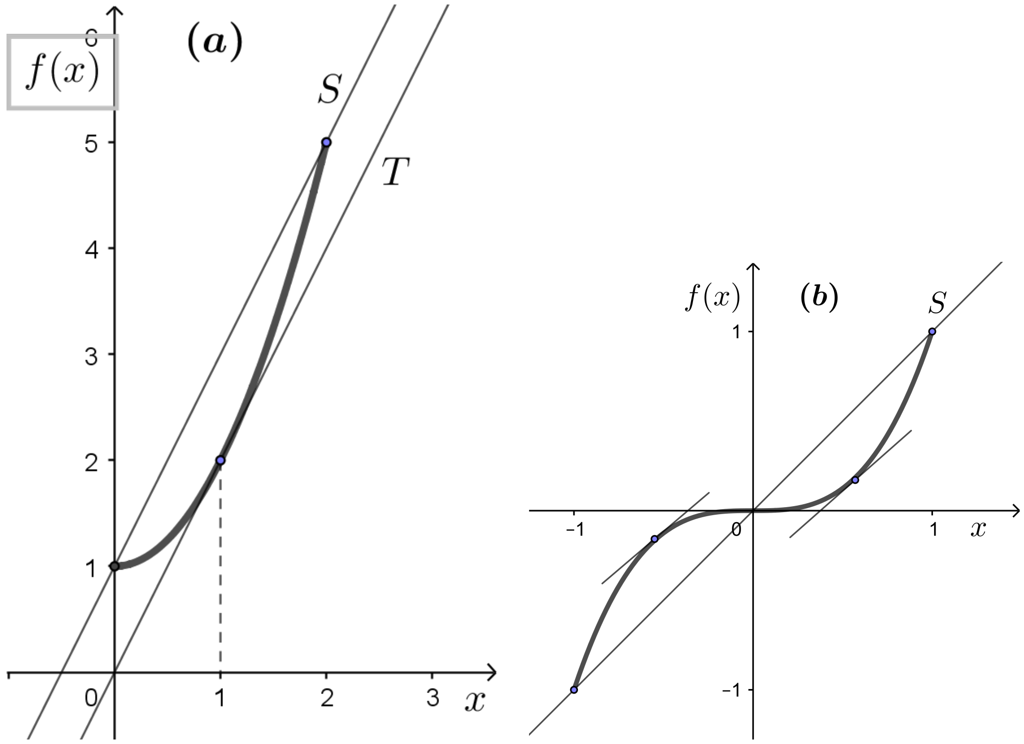

Figure 2.2: Plot of (a) $f(x) = x^2 + 1$ and (b) $f(x) = x^3$

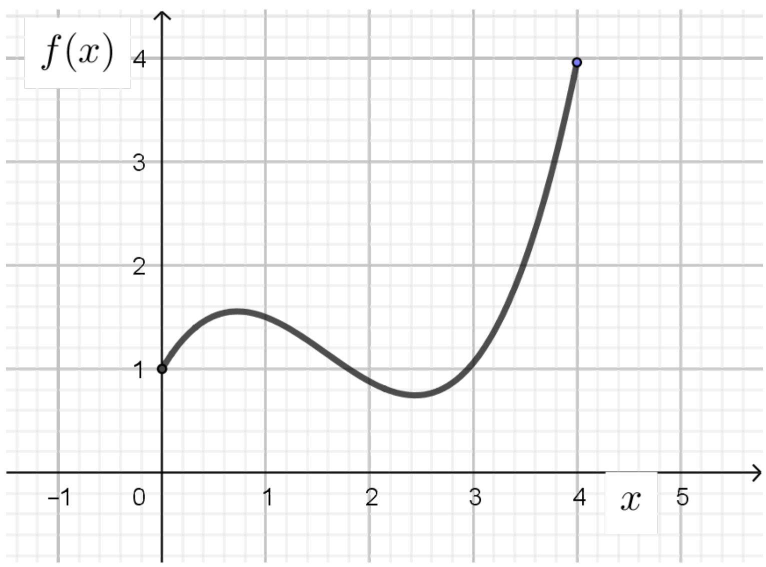

Figure 2.3:

Example 5 (Real Life Example). The position of a maglev (magnetic levitation) moving along a straight, elevated monorail track is given by

$$S = f(t) = 4t^2, \quad 0 \leq t \leq 30,$$

where S is measured in meters and t in seconds.

(a) Find the average velocity of the maglev during the first 4 seconds of the run.

(b) Find the number c in $(0, 4)$ that satisfy the conclusion of the MVT.

(c) Interpret your results.

Solution. $S = f(t) = 4t^2, \quad 0 \leq t \leq 30$

(a) $Average\,velocity = \dfrac{f(b) - f(a)}{b - a}$.
First 4 seconds implies the interval $[0, 4]$, such that $a = 0$ and $b = 4$. Thus

$$Average\,velocity = \frac{f(4) - f(0)}{4 - 0}$$
$$\Longrightarrow \frac{64 - 0}{4} = 16\,m/s$$

(b) Find c in $(0, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Now, $f'(t) = 8t$. Thus $8c = 16 \Longrightarrow c = 2$.

(c) Since $f'(t)$ measures the instantaneous velocity of the maglev at any time t, the MVT tells us that at $t = 2\,\mathrm{s}$ (between $t = 0$ and $t = 4$), the maglev must attain an intantaneous velocity equal to the average velocity over $[0, 4]$.

Example 6. Let

$$f(x) = \begin{cases} x^2, & \text{if} \quad x < 1 \\ 2 - x, & \text{if} \quad x \geq 1 \end{cases} \tag{2.1}$$

Considering Figure 2.4(a), does f satisfy the hypotheses of the MVT on $[0, 2]$? Explain.

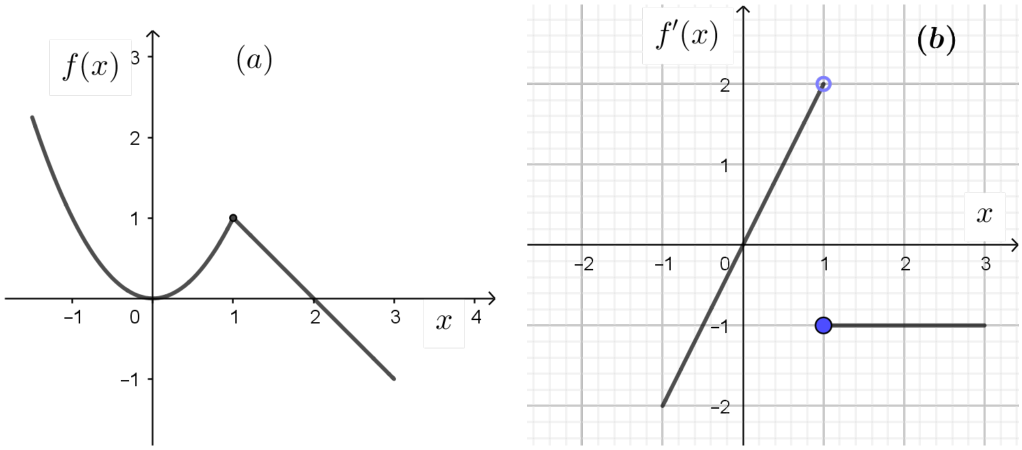

Figure 2.4: Plot of (a) a function, and (b) it's derivative.

Solution.

$$f'(x) = \begin{cases} 2x, & x < 1 \\ -1, & x \geq 1 \end{cases} \tag{2.2}$$

Figure 2.4(b) shows the graph of $f'(x)$. We note f does not satisfy the hypothesis of the MVT since $f'(x)$ does not exist at $x = 1$ which lies in $[0, 2]$.

Example 7. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. The inequality gives a restriction on the rate of growth of f, which then imposes a restriction on the possible values of f. Use the MVT to determine how large $f(4)$ can possibly be.

Solution. Since we are given $f'(x) \leq 5$, the function is differentiable and therefore continuous on $[0, 4]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\implies f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\implies f'(c) = \frac{f(4) + 3}{4}$$

Since $f'(x) \leq 5$, we have

$$\frac{f(4) + 3}{4} \leq 5$$

$$\implies f(4) \leq 17.$$

So the answer is 17.

Example 8. At $2:00$ P.M. a car's speedometer reads 30 mi/h. At $2:10$ P.M. it reads 50 mi/h. Show that at some time between $2:00$ and $2:10$ the acceleration is exactly 120 mi/h.

Solution. At $2:00$ P.M. let $t = 0$ and speed be 30 mi/h. This implies that at $2:10$ P.M., $t = 10$ min, and speed is 50 mi/h.
By assumption, the car's speed, say $f(t)$, is continuous and differentiable everywhere. Note that acceleration is the derivative of speed. Using the MVT

$$f'(t) = \frac{f(b) - f(a)}{b - a} = \frac{f(10) - f(0)}{10 - 0} = \frac{50 - 30\,\text{mi/h}}{10\,\text{min}}$$
$$= \frac{20}{10/60} = 120\,\text{mi/h}^2.$$

2.2 Consequences of the MVT

In this section, we examine some consequences of the Mean Value Theorem. We will especially consider

- Functions with zero derivatives,
- Functions with equal derivatives, and
- Increasing and decreasing functions.

2.2.1 Functions with Zero Derivatives

Theorem 5 (Zero Derivatives). If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof. We show that for any pair of numbers in (a, b), f has the same value. Let x_1 and x_2 be arbitrary numbers in (a, b) such that $x_1 < x_2$. Since $f(x)$ is differentiable in (a, b), it is differentiable in (x_1, x_2) and continuous in $[x_1, x_2]$. Thus f satisfies the hypotheses of MVT on $[x_1, x_2]$. This implies that there is a c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$\implies f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

But $f'(x) = 0$ for all x in (a, b).

$$\implies f'(c) = 0$$

$$\implies f(x_2) - f(x_1) = 0$$

$$\implies f(x_2) = f(x_1).$$

□

2.2.2 Functions with Equal Derivatives

Theorem 6 (**Equal Derivatives**)**.** If $f'(x) = g'(x)$ for all x in an interval (a, b), then f and g differ by a constant on (a, b); that is, there exist a constant c such that

$$f(x) = g(x) + c$$

for all x in (a, b).

Proof. Let $h(x) = f(x) - g(x)$. This implies that

$$h'(x) = f'(x) - g'(x) = 0$$

for every x in (a, b). By Theorem 5, $h(x)$ is constant. Thus

$$f(x) - g(x) = c$$

$$\implies f(x) = g(x) + c.$$

□

2.2.3 Increasing and Decreasing Functions

Definition 4. A function f is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
It is called decreasing on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ on I.

Figure 2.5 gives an illustration of increasing and decreasing functions. **Question:** How can we use the derivative of a function to determine if it's increasing or decreasing? The following theorem provides the answer.

Theorem 7 (**Increasing/Decreasing Test or I/D Test**)**.** (a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

The theorem is illustrated in Figure 2.6.

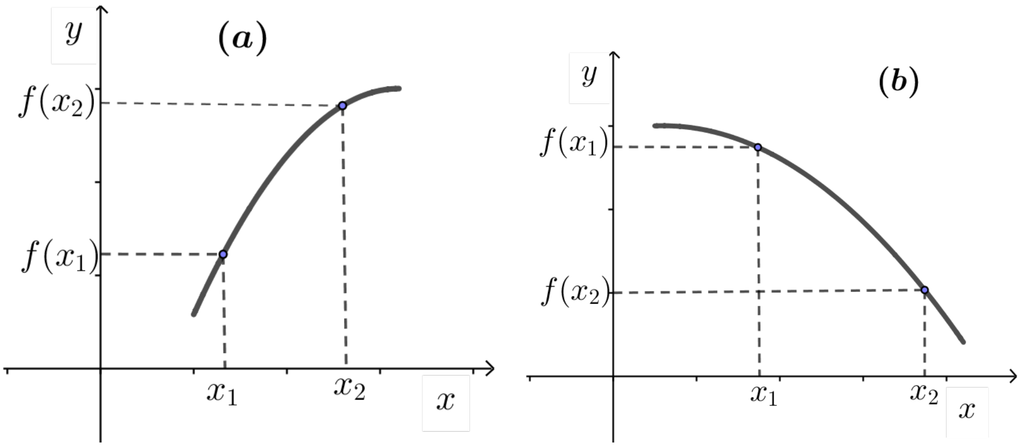

Figure 2.5: (a) Increasing function (b) Decreasing function.

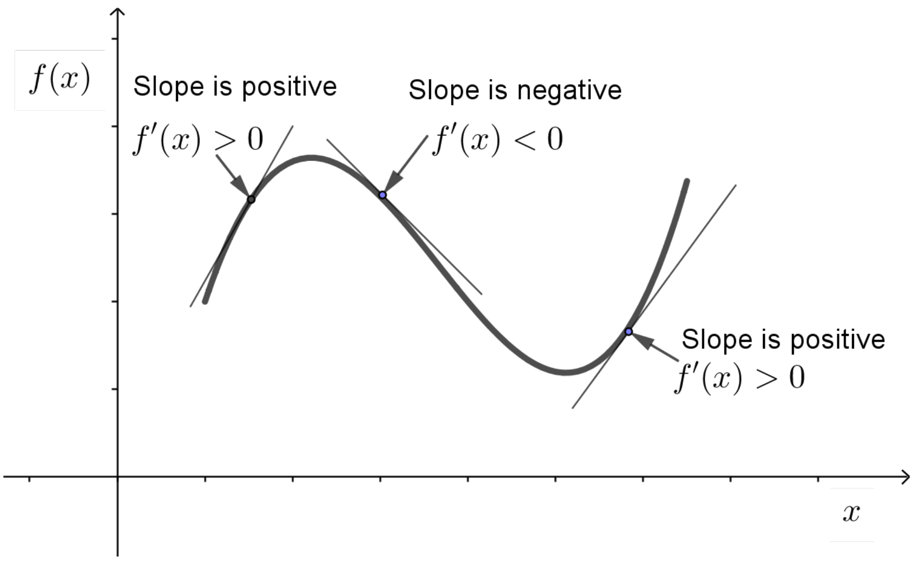

Figure 2.6: Positive and negative slopes of a function.

Proof. (a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. Given that $f'(x) > 0$, we know f is differentiable in $[x_1, x_2]$. So by the MVT there is a number c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Longrightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ since $f'(x) > 0$ for all x in the interval. Also $x_2 - x_1 > 0$, therefore

$$f(x_2) - f(x_1) > 0$$

$$\implies f(x_2) > f(x_1)$$

Therefore f is increasing.

(b) Try proving this.

□

Finding Intervals of Increase and Decrease

Knowing the intervals where a function is increasing or decreasing helps in sketching a graph of the function. The procedure is outlined below:

1) Find all values x for which $f'(x) = 0$ or $f'(x)$ does not exist (the critical values). Use the critical values to partition the domain of f into open intervals.

2) In each interval, select a test value c and determine the sign of $f'(c)$.

   (a) If $f'(c) > 0$, then f is increasing on that interval.

   (b) If $f'(c) < 0$, then f is decreasing on that interval.

   (c) If $f'(c) = 0$, then f is constant on that interval. That is, it has no local maximum or minimum at c.

Example 9. Find the intervals on which f is increasing or decreasing:

(1) $f(x) = -x^3 + 3x^2 + 1$.

(2) $f(x) = \dfrac{x^2}{x-1}$.

(3) $f(x) = x^3 - 3x^2 + 2$.

Solution. (1) $f(x) = -x^3 + 3x^2 + 1$

$$f'(x) = -3x^2 + 6x = -3x(x-2)$$

This implies $f'(x)$ is continuous everywhere and has zeros at $x = 0$ and $x = 2$. Using the zeros, we now partition the domain $(-\infty, \infty)$ into open intervals as shown in Table 2.1. Figure 2.7 illustrates the increasing and decreasing parts of the function.

Interval	$-3x$	$x-2$	$f'(x)$	Description
$(-\infty, 0)$	$+$	$-$	$-$	decreasing
$(0, 2)$	$-$	$-$	$+$	increasing
$(2, \infty)$	$-$	$+$	$-$	decreasing

Table 2.1: The intervals of increase and decrease for $f(x) = -x^3 + 3x^2 + 1$.

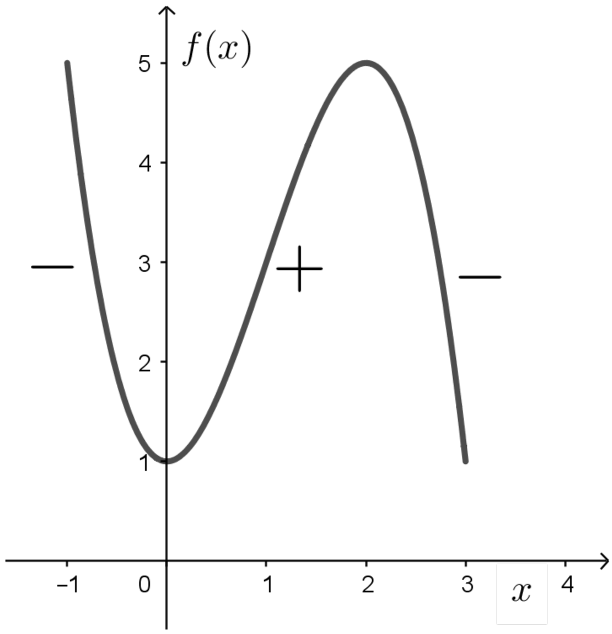

Figure 2.7: Increasing (+) and decreasing (−) parts of the function.

(2) $f(x) = \dfrac{x^2}{x-1}$.

$$f'(x) = \frac{(x-1)(2x) - x^2 \cdot 1}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - x^2}{(x-1)^2}$$

$$\Longrightarrow f'(x) = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

We note that $f'(x)$ does not exist at $x = 1$. Now,

$$f'(x) = 0 \Longrightarrow x = 0, 2.$$

So the critical values are $0, 1$ and 2. Table 2.2 shows the intervals of increase and decrease. Figure 2.8a shows the graph of $f(x) = \dfrac{x^2}{x-1}$,

Interval	x	$x-2$	$f'(x)$	Description
$(-\infty, 0)$	$-$	$-$	$+$	increasing
$(0, 1)$	$+$	$-$	$-$	decreasing
$(1, 2)$	$+$	$-$	$-$	decreasing
$(2, \infty)$	$+$	$+$	$+$	increasing

Table 2.2: The intervals of increase and decrease for $f(x) = \dfrac{x^2}{x-1}$.

confirming the intervals where the function is increasing and decreasing.

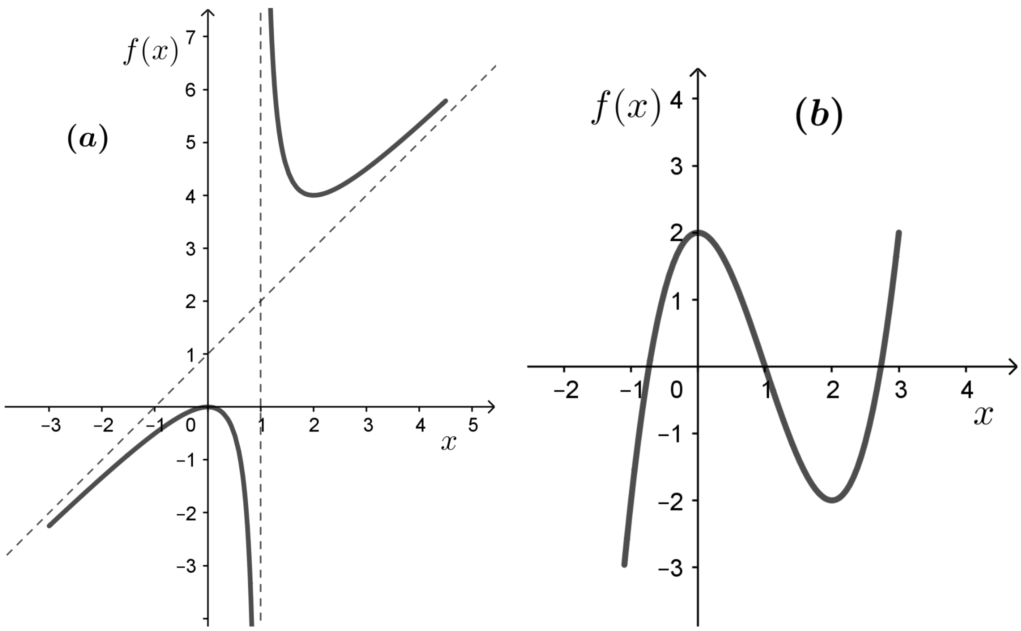

Figure 2.8: Plots of (a) $f(x) = \frac{x^2}{x-1}$ and (b) $f(x) = x^3 - 3x^2 + 2$.

(3) $f(x) = x^3 - 3x^2 + 2$. Try solving this before looking at the solution.

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = 3x(x-2)$$

$$f'(x) = 0 \Longrightarrow x = 0, 2$$

The intervals of increase and decrease are displayed in Table 2.3 The graph of $f(x) = x^3 - 3x^2 + 2$ is displayed in Figure 2.8b, showing the intervals where the function is increasing and decreasing.

Interval	$3x$	$x-2$	$f'(x)$	Description
$(-\infty, 0)$	$-$	$-$	$+$	increasing
$(0, 2)$	$+$	$-$	$-$	decreasing
$(2, \infty)$	$+$	$+$	$+$	increasing

Table 2.3: The intervals of increase and decrease for $f(x) = x^3 - 3x^2 + 2$.

2.3 Applications of the MVT: Using the MVT to Establish an Inequality

Example 10. Use the Mean Value Theorem to prove the following inequalities

(a) $\cos x \geq 1 - x$ for $x \geq 0$

(b) $\tan x > x$ for $0 < x < \dfrac{\pi}{2}$

Solution. (a) $\cos x \geq 1 - x$ for $x \geq 0$. Let $f(x) = \cos x$ for $[0, x]$. f is continuous and differentiable on the given interval, so by MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $a = 0$ and $b = x$.

$$\implies \frac{f(x) - f(0)}{x - 0} = -\sin c$$

$$\implies \frac{\cos x - 1}{x} = -\sin c$$

But $\sin c \leq 1 \implies -\sin c \geq -1$

$$\implies \frac{\cos x - 1}{x} \geq -1$$

$$\implies \cos x - 1 \geq -x$$

$$\therefore \cos x \geq 1 - x$$

(b) $\tan x > x$ for $0 < x < \dfrac{\pi}{2}$. Let $f(x) = \tan x$ on $[0, x]$ where $0 < x < \frac{\pi}{2}$. By the MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $a = 0$ and $b = x$.

$$\Longrightarrow f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\tan x}{x}$$

$$\Longrightarrow \sec^2 c = \frac{\tan x}{x}, \quad x > 0$$

Note that

$$\sec x = \frac{1}{\cos x}.$$

$$\text{Thus, } \cos x < 1 \Longrightarrow \frac{1}{\cos x} > 1$$

$$\Longrightarrow \sec x > 1 \Longrightarrow \sec^2 c > 1$$

$$\Longrightarrow \frac{\tan x}{x} > 1$$

$$\therefore \tan x > x, \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

Example 11. Suppose that $1 \leq f'(x) \leq 4$ for all x in $[2, 5]$. Show that

$$3 \leq f(5) - f(2) \leq 12$$

Solution. $1 \leq f'(x) \leq 4$ for all x in $[2, 5]$. Since $f'(x)$ is finite, f is differentiable and continuous on the given interval. By the MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad [2, 5]$$

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{f(5) - f(2)}{3}$$

But $1 \leq f'(c) \leq 4$. Thus

$$1 \leq \frac{f(5) - f(2)}{3} \leq 4$$

$$\therefore 3 \leq f(5) - f(2) \leq 12$$

Example 12. (a) Show that $e^x \geq 1 + x$ for $x \geq 0$.

(b) Deduce that

$$e^x \geq 1 + x + \frac{1}{2}x^2 \quad \text{for} \quad x \geq 0$$

(c) Use mathematical induction to prove that for $x \geq 0$ and any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Solution. (a) Let $f(x) = e^x - (1 + x)$ and show that $f(x) \geq 0$. Now $f(0) = e^0 - (1+0) = 1-1 = 0$, and $f'(x) = e^x - 1$. Note that if $x > 0$, $e^x > 1 \Longrightarrow f'(x) > 0$. Since $f'(x) > 0$ for $x > 0$, it implies that $f(x)$ is increasing for $x > 0$.

$$\Longrightarrow f(x) > f(0) \quad \text{for} \quad x > 0$$

$$\Longrightarrow f(x) > 0$$

$$\Longrightarrow e^x - (1 + x) > 0$$

$$\Longrightarrow e^x > 1 + x.$$

Thus,

$$e^x \geq 1 + x.$$

(b)

$$e^x \geq 1 + x + \frac{1}{2}x^2 \quad \text{for} \quad x \geq 0$$

Let $g(x) = e^x - \left(1 + x + \frac{1}{2}x^2\right)$, and show that $g(x) \geq 0$.

For $x = 0$:

$$g(0) = e^0 - (1 + 0 + 0) = 0 \Longrightarrow g(x) = 0$$

$$\Longrightarrow e^x = 1 + x + \frac{1}{2}x^2 \quad \text{when} \quad x = 0$$

For $x > 0$:

$$g'(x) = e^x - (0 + 1 + x)$$

$$\Longrightarrow g'(x) = e^x - (1 + x) = f(x)$$

from solution to (a). Since $f(x) > 0$ for $x > 0$, $g'(x) > 0$ for $x > 0$. This implies that $g(x)$ is increasing, and so

$$g(x) > g(0) \quad \text{for} \quad x > 0$$

$$\Longrightarrow e^x - \left(1 + x + \frac{1}{2}x^2\right) > 0$$

$$\Longrightarrow e^x > 1 + x + \frac{1}{2}x^2$$

Thus,

$$e^x \geq 1 + x + \frac{1}{2}x^2 \quad \text{for} \quad x \geq 0$$

(c) Let S_n be a statement about the positive integer n. Let

$$S_n(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)$$

$S_1(x)$ is true from the solution to problem (a).
$S_2(x)$ is true from the solution to problem (b).
Assume $S_k(x)$ is true. Then

$$S_k(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right) \geq 0$$

By mathematical induction, we need to show that $S_{k+1}(x)$ is true. Now

$$S_{k+1}(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k+1}}{(k+1)!}\right)$$

$$S_{k+1}(0) = 0 \quad \text{for} \quad x = 0$$

$$S'_{k+1}(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right) = S_k(x) \geq 0$$

Thus, $S_{k+1}(x)$ is increasing, and so

$$S_{k+1}(x) > S_{k+1}(0) = 0$$

$$\Longrightarrow S_{k+1}(x) > 0$$

Hence,

$$S_{k+1}(x) \geq 0.$$

Thus, $S_{k+1}(x)$ is true. Therefore, by mathematical induction, $S_n(x)$ is true. That is

$$S_n = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \geq 0.$$

Chapter 3

Inverse Functions

In this chapter, we examine inverse functions and some of their properties. We will investigate the reflective property of an inverse function, examine the existence of an inverse function, and find the derivative of an inverse function.

3.1 Introduction

Consider the arrow diagrams in Figure 3.1.

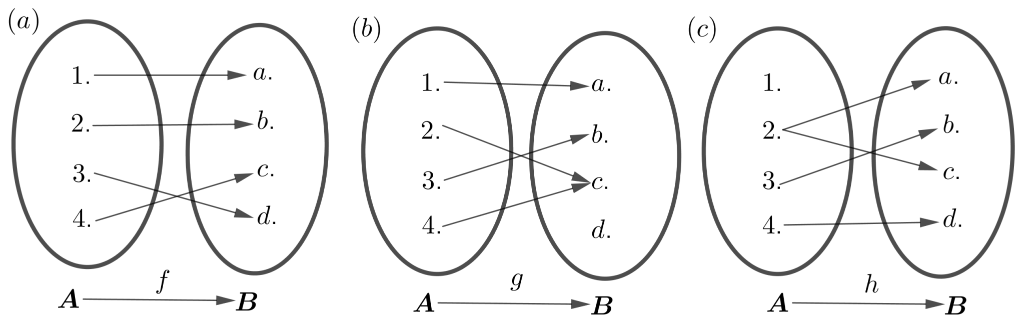

Figure 3.1: Arrow diagrams. (a) one-to-one function (b) A function, but not one-to-one (c) Not a function.

(1) $f : A \to B$ is a function, and it never takes on the same value twice. That is, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. It is a one-to-one function (defined below).

(2) $g : A \to B$ is a function. However, note that

$$g(2) = g(4) = c$$

Thus, g takes on the same value twice. So g is not a one-to-one function.

(3) $h : A \to B$ is not a function since $h(2)$ takes on both a and c.

Definition 5. A function f is called a one-to-one function if it never takes on the same value twice; that is

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

or if

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once. Figure 3.2 illustrates the use of the horizontal line test.

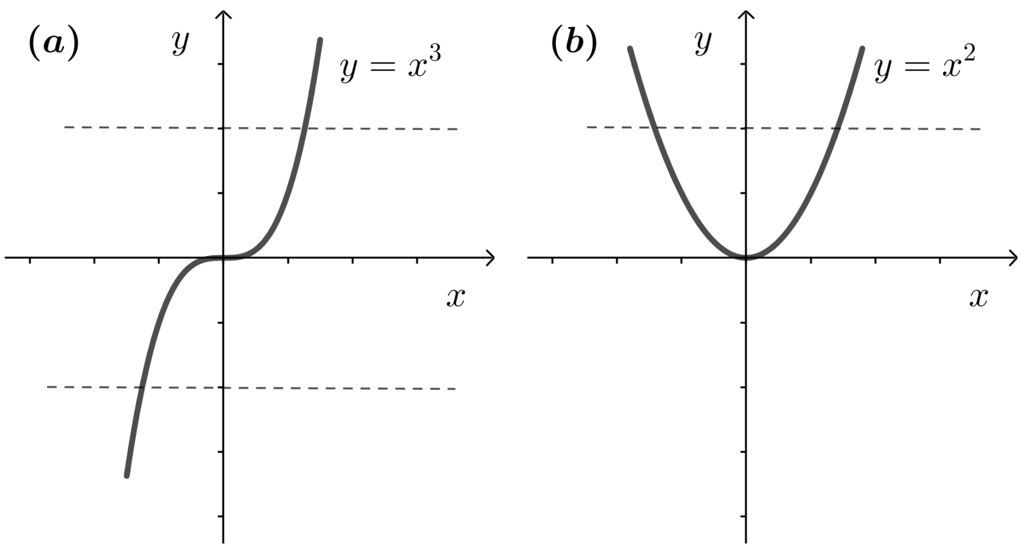

Figure 3.2: The horizontal line test shows that (a) $y = x^3$ is a one-to-one function (b) $y = x^2$ is not a one-to-one function.

Definition 6. Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y$$

for any y in B. Equivalently, we may reverse x and y such that

$$f^{-1}(x) = y \Longleftrightarrow f(y) = x$$

This illustrated in Figure 3.3

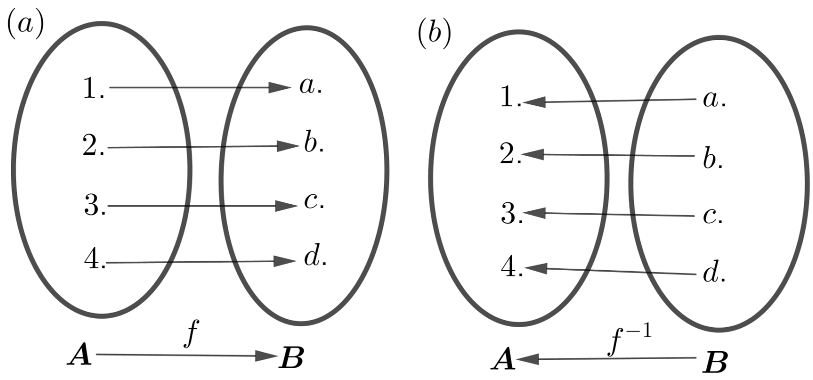

Figure 3.3: Arrow diagram of a function and its inverse.

Remark. Domain of f^{-1} = range of f, and
Range of f^{-1} = domain of f.

Remark. Warning! $f^{-1}(x) \neq \dfrac{1}{f(x)}$. For reciprocals we write

$$\frac{1}{f(x)} = [f(x)]^{-1}$$

Example 13. If $f(2) = 7$ and $f(-10) = -17$, find $f^{-1}(7)$ and $f^{-1}(-17)$.

Solution.

$$f^{-1}(7) = 2, \quad f^{-1}(-17) = -10$$

Cancellation Equations

Let $g(x)$ be the inverse of f. That is, $g(x) = f^{-1}(x)$. Then

$$g[f(x)] = f^{-1}[f(x)] = x \quad \text{for all} \quad x \in A$$

$$f[g(x)] = f\left[f^{-1}(x)\right] = x \quad \text{for all} \quad x \in B$$

Example 14. Show that $f(x) = x^{1/3}$ and $g(x) = x^3$ are inverses of each other.

Solution.

$$f\left[g(x)\right] = f(x^3) = \left(x^3\right)^{1/3} = x$$

$$g\left[f(x)\right] = g(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

This implies that f and g are inverses of each other. That is

$$f^{-1}(x) = g(x) \quad \text{and} \quad g^{-1}(x) = f(x).$$

The graphs of $f(x)$ and $g(x)$ are displayed in Figure 3.4. Note that the graph of x^3 is a reflection of the graph of $x^{1/3}$ about the line $y = x$.

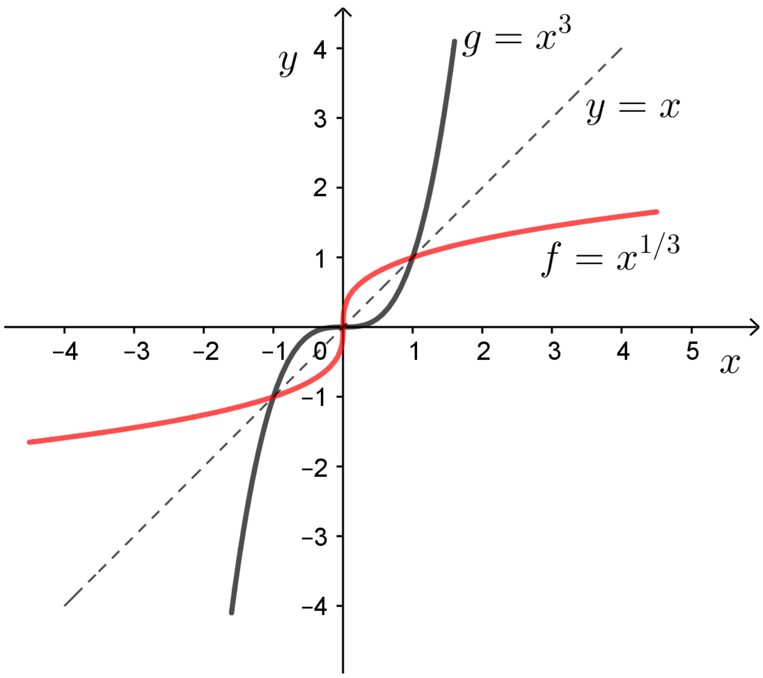

Figure 3.4: Plots of $g = x^3$ and $f = x^{1/3}$.

3.2 Reflective Property of Inverse Functions

If (a, b) is on the graph of $f(x)$, then $b = f(a)$, and we have

$$f^{-1}(b) = f^{-1}\left[f(a)\right] = a$$

As shown in Figure 3.5, the graph of f^{-1} is the reflection of the graph of f about the line $y = x$ and vice versa.

Example 15. Examples of the reflective property of inverse functions are illustrated in Figure 3.6.

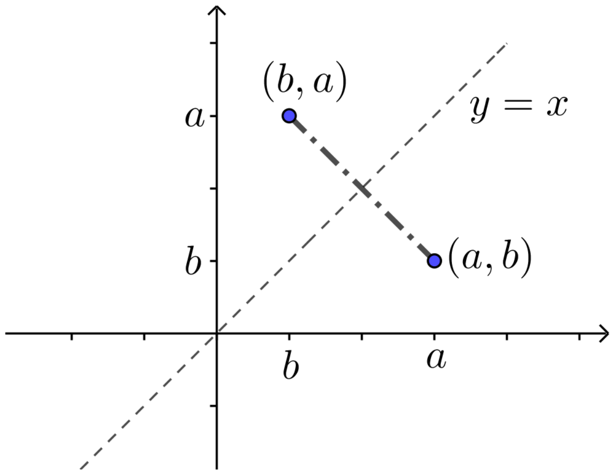

Figure 3.5: Schematic of the reflective property of inverse functions.

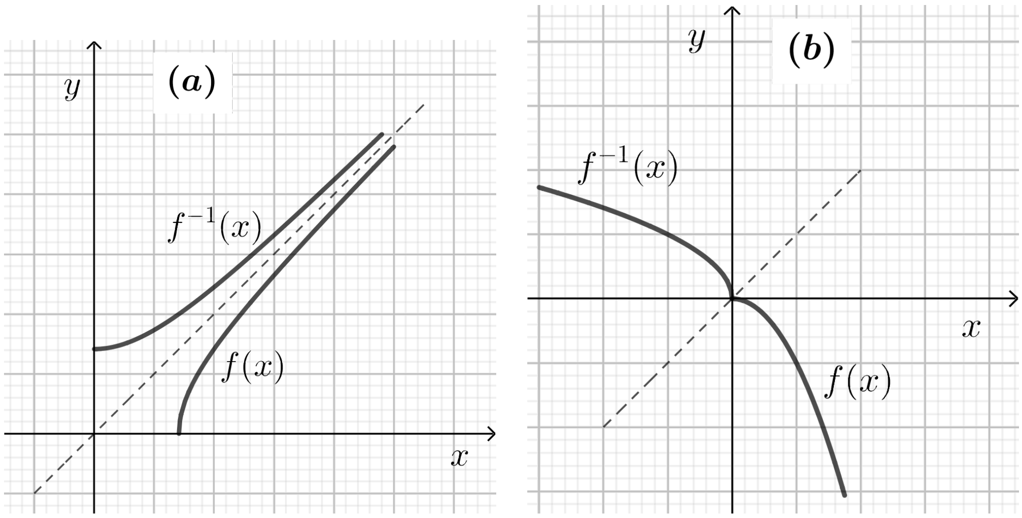

Figure 3.6: Illustration of the reflective property of some inverse functions.

3.3 Existence of an Inverse Function

Theorem 8. A function has an inverse if and only if it is one-to-one.

Proof. Suppose $f : A \to B$ has an inverse function $y = g(x)$. Then

$$f(y) = f[g(x)] = x$$

If $f(a) = f(b)$, then

$$g[f(a)] = g[f(b)] \Longrightarrow a = b$$

Therefore, f is one-to-one.
Now suppose f is one-to-one. For $x \in B$, there exist $y \in A$ such that $f(y) = x$. There is only one such y since f is one-to-one. Let

$$y = g(x)$$

$$\Longrightarrow g(f(y)) = g(x) = y, \quad \text{(we used } f(y) = x\text{)}$$

for any y, and

$$f[g(x)] = f(y) = x$$

for any x. So g is an inverse of f and vice versa. □

3.3.1 Finding Inverse Functions

Below is a procedure for finding inverse functions:

(1) Write $y = f(x)$

(2) Solve for x in terms of y (if possible)

(3) Interchange x and y to obtain $y = f^{-1}(x)$.

Example 16. Find the inverse of the following functions

(a) $y = \dfrac{1}{\sqrt{2x-3}}$

(b) $y = \dfrac{1+3x}{5-2x}$

(c) $y = \ln(x+3)$

Solution. (a) $y = \dfrac{1}{\sqrt{2x-3}}$

$$\Longrightarrow y^2 = \frac{1}{2x-3}$$

$$\Longrightarrow y^2(2x-3) = 1 \Longrightarrow 2y^2x - 3y^2 = 1$$

$$\Longrightarrow x = \frac{1+3y^2}{2y^2}$$

$$\therefore f^{-1}(x) = \frac{1+3x^2}{2x^2}$$

(b) $y = \dfrac{1+3x}{5-2x}$

$$\begin{aligned}
&\Longrightarrow 5y - 2yx = 1 + 3x \\
&\Longrightarrow 3x + 2yx = 5y - 1 \\
&\Longrightarrow x(3 + 2y) = 5y - 1 \\
&\Longrightarrow x = \frac{5y-1}{3+2y} \\
&\therefore f^{-1}(x) = \frac{5x-1}{3+2x}
\end{aligned}$$

(c) $y = \ln(x+3)$

$$\begin{aligned}
&\Longrightarrow x + 3 = e^y \\
&\Longrightarrow x = e^y - 3 \\
&\therefore f^{-1}(x) = e^x - 3
\end{aligned}$$

3.4 Continuity and Differentiability of Inverse Functions

Theorem 9. Let f be one-to-one, so that it has an inverse f^{-1}.

(a) If f is continuous on its domain, then f^{-1} is continuous on its domain.

(b) If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Theorem 10. Let f be differentiable on its domain and have an inverse function $g = f^{-1}$. Then the derivative of g is given by

$$g'(x) = \frac{1}{f'[g(x)]},$$

provided that $f'[g(x)] \neq 0$.

Proof. Since f is differentiable, g is also differentiable by Theorem 9. Since g is the inverse function;

$$f[g(x)] = x$$

Differentiating (and using the Chain Rule), we get

$$f'[g(x)]g'(x) = 1$$

$$\therefore g'(x) = \frac{1}{f'[g(x)]}.$$

□

Example 17. Let g denote the inverse of the function f. Show that the point (a, b) lies on the graph of f. Find $g'(b)$.

(a) $f(x) = 2x + 1; \quad (2, 5)$

(b) $f(x) = x^5 + 2x^3 + x - 1; \quad (0, -1)$

Solution. (a) $f(x) = 2x + 1; \quad (2, 5)$

$$\Longrightarrow f(2) = 2(2) + 1 = 5$$

Therefore $(2, 5)$ lies on the graph of f. Also $f^{-1}(5) = g(5) = 2$. Now

$$g'(b) = \frac{1}{f'[g(b)]} \Longrightarrow g'(5) = \frac{1}{f'[g(5)]}$$

$$\Longrightarrow g'(5) = \frac{1}{f'(2)}$$

Now, $f'(x) = 2 \Longrightarrow f'(2) = 2$

$$\therefore g'(5) = \frac{1}{2}$$

(b) $f(x) = x^5 + 2x^3 + x - 1; \quad (0, -1)$.

$$f(0) = 0 + 0 + 0 - 1 = -1$$

Thus, $(0, -1)$ lies on f. And $f^{-1}(-1) = g(-1) = 0$.

$$g'(-1) = \frac{1}{f'[g(-1)]} = \frac{1}{f'(0)}$$

$f'(x) = 5x^4 + 6x + 1$

$$\Longrightarrow f'(0) = 0 + 0 + 1 = 1$$

$$\therefore g'(-1) = 1$$

Exercise. (1) Suppose $f(x) = x^2$ for $x \in [0, \infty)$, and let g be the inverse of f.

(a) Compute $g'(x)$ using $g'(x) = \dfrac{1}{f'[g(x)]}$

(b) Find $g'(x)$ by first computing $g(x)$.

(2) Suppose that g is the inverse of a function f. If $f(2) = 4$ and $f'(2) = 3$, find $g'(4)$.

(3) Find $f^{-1}(a)$ for the function f and the real number a.

(a) $f(x) = x^3 + x - 1; \quad a = -1$

(b) $f(x) = 2x^5 + 3x^3 + 2; \quad a = 2.$

(4) The graph of f is given in Figure 3.7

(a) Why is f one-to-one?

(b) State the domain and range of f^{-1}.

(c) Estimate the value of $f^{-1}(1)$.

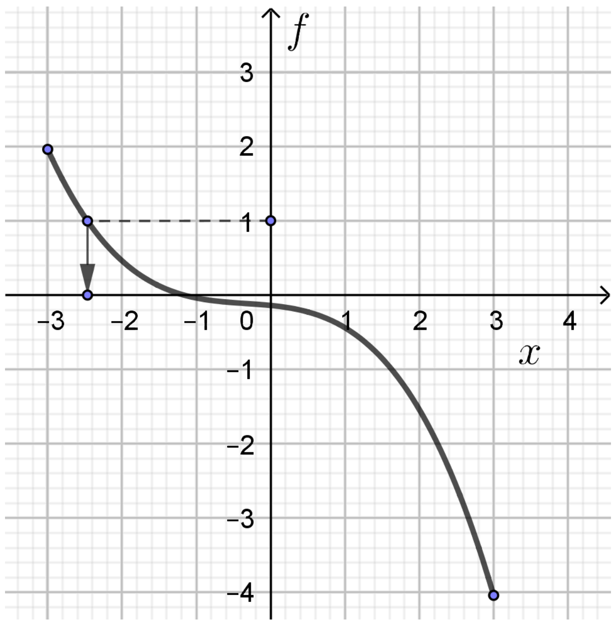

Figure 3.7: Plot of a function.

Solution to Selected Exercises

Solution. (1) $f(x) = x^2$ for $x \in [0, \infty)$.

(a) $g'(x) = \dfrac{1}{f'[g(x)]}$

$$y = x^2 \Longrightarrow x = \sqrt{y}, \quad x \geq 0$$

$$\therefore f^{-1}(x) = g(x) = \sqrt{x}$$
$$f'(x) = 2x \Longrightarrow f'[g(x)] = 2\sqrt{x}$$
$$\Longrightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

(b) From $g(x) = \sqrt{x}$

$$\Longrightarrow g'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

(2)
$$g'(4) = \frac{1}{f'[g(4)]}$$

$f(2) = 4 \Longrightarrow f^{-1}(4) = g(4) = 2$. From the first equation, we get

$$g'(4) = \frac{1}{f'(2)} = \frac{1}{3}$$

(3) $f(x) = x^3 + x - 1; \quad a = -1.$

(a) Let $f^{-1}(-1) = x$

$$\Longrightarrow f(x) = -1$$
$$\Longrightarrow x^3 + x - 1 = -1$$
$$\Longrightarrow x^3 + x = 0 \Longrightarrow x(x^2 + 1) = 0$$

But $(x^2 + 1) > 0$, thus $x = 0$.

$$\therefore f^{-1}(-1) = 0$$

(b) $f(x) = 2x^5 + 3x^3 + 2; \quad a = 2$. Let $f^{-1}(2) = x$

$$\Longrightarrow f(x) = 2$$
$$2x^5 + 3x^3 + 2 = 2$$
$$\Longrightarrow 2x^5 + 3x^3 = 0 \Longrightarrow x^3(2x^2 + 3) = 0$$
$$\Longrightarrow x = 0$$
$$\therefore f^{-1}(2) = 0.$$

(4) (a) Using the horizontal line test; any horizontal line through the graph of f intersects it at only one point.

(b) Domain of $f^{-1} = [-4, 2]$. (The range of $f^{-1} = [-3, 3]$).

(c) Let $f^{-1}(1) = x$. Thus $f(x) = 1$. From the graph, we have

$$f^{-1}(1) \approx -2.5$$

Chapter 4

Logarithmic and Exponential Functions

In this chapter, we will first study *general exponential* and *general logarithmic* functions, and move on to consider special cases of these two functions: the *natural logarithmic function* and the *exponential function*, and learn about their graphs. We will delve into the proofs of some basic properties of these two functions, and the behaviour of the functions at infinity and close to the origin. The natural logarithmic and the exponential functions may be represented as *limits*, this chapter addresses this aspect of the functions. The chapter concludes by looking at logarithmic inequalities and limits, the *scales of infinity*, that is, comparison of the magnitude of two functions, and finally, how to *differentiate logarithmic functions*.

4.1 General Logarithmic Functions

The logarithmic (or log) function with base a is denoted by $\log_a$, such that

$$\boxed{\log_a x = y \Longleftrightarrow a^y = x; \quad a > 0, a \neq 1.}$$

The log function is the inverse of the exponential function, for example $f(x) = a^x$. Using the **cancellation equations** with $f(x) = a^x$, $f^{-1}(x) = \log_a x$,

$$\log_a a^x = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0.$$

4.1.1 Laws of Logarithms

If x and y are positive numbers, then

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^r = r \log_a x$
   where r is any real number.

Proof. 1. $\log_a xy = \log_a x + \log_a y$. Let $m = \log_a x$ and $n = \log_a y$

$$\implies x = a^m, \quad y = a^n$$

$$\implies xy = a^n \cdot a^m = a^{m+n}$$

The last line would be proved later from the properties of the general exponential functions.

$$\implies \log_a xy = \log_a a^{m+n} = (m+n) \log_a a$$

Using the cancellation property, we get

$$\therefore \log_a xy = m + n = \log_a x + \log_a y$$

2. $\log_a \frac{x}{y} = \log_a x - \log_a y$.
   Using the definition of m and n above, we have

$$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}$$

$$\implies \log_a \frac{x}{y} = \log_a a^{m-n} = (m-n) \log_a a$$

$$\therefore \log_a \frac{x}{g} = \log_a x - \log_a y$$

3. $\log_a x^r = r \log_a x$. Let $m = \log_a x$

$$\implies x = a^m$$

Raising both sides to the power r gives

$$x^r = (a^m)^r$$

$$\implies \log_a x^r = \log_a a^{mr} = mr$$

$$\therefore \log_a x^r = r \log_a x$$

□

4.2 The Natural Logarithm

The natural logarithmic function is a special case of the logarithmic function; it is the log function to the base e, where the number $e \approx 2.71828$. By replacing a with e in the general logarithm, we get the natural logarithm:

$$\log_e x = \ln x$$

Properties

1. Here are a first set of properties for the natural log function:

$$\ln x = y \Longleftrightarrow e^y = x$$

$$\ln e^x = x, \quad x \in \mathbb{R}$$

$$e^{\ln x} = x$$

   Note that if $x = 1$, we get

$$\ln e = 1$$

2. Below is another important property of the natural log function. For $a > 0 (a \neq 1)$,

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

   Proof. Let $m = \log_a x$, then

$$x = a^m \Longrightarrow \ln x = \ln a^m$$

$$\Longrightarrow \ln x = m \ln a$$

$$\Longrightarrow m = \frac{\ln x}{\ln a}$$

$$\therefore \log_a x = \frac{\ln x}{\ln a}$$

   □

   The relation above is helpful for using calculators to compute the logarithm with any base. Another definition of the natural log is given by using integrals, as shown below.

Definition 7. The **natural logarithm** function is defined by

$$\ln x = \int_1^x \frac{1}{t} \mathrm{d}t \quad \text{for all} \quad x > 0.$$

If $x > 0$, $\ln x$ can be interpreted as the area of the region under the graph of $y = \dfrac{1}{t}$ on the interval $[1, x]$. Figure 4.1 gives illustrations of the above definition.

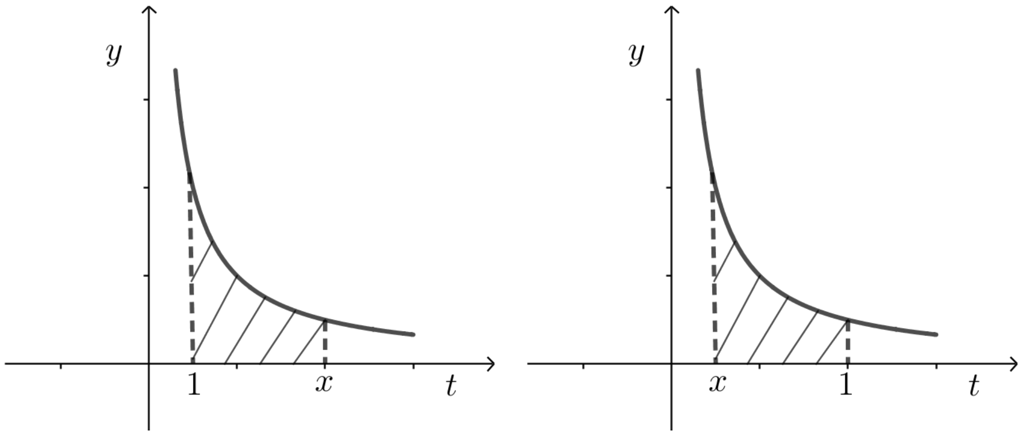

Figure 4.1: Schematic plot showing

Graph of $\ln x$

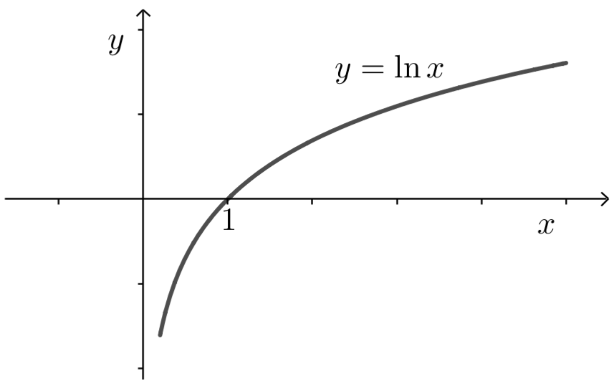

Figure 4.2: Graph of $\ln x$

4.2.1 Laws of Natural Logarithms

Theorem 11. Let x and y be positive numbers and let r be a rational number. Then

(a) $\ln 1 = 0$

(b) $\ln xy = \ln x + \ln y$

(c) $\ln \frac{x}{y} = \ln x - \ln y$

(d) $\ln x^r = r \ln x$

Proof. The proof in (a) is straightforward. For the proof of the expressions in (b)-(d), you may use the approach used in the proof of the general logarithmic functions; the proofs shown below rely on the derivative of the logarithmic function which will be treated shortly.

(a) $\ln x = \displaystyle\int_1^x \frac{1}{t} \mathrm{d}t, \Longrightarrow \ln 1 = 0.$

(b) Let $F(x) = \ln ax$ for $a > 0$. Then $F'(x) = \dfrac{1}{ax}(a) = \dfrac{1}{x}$. But

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

This implies that $\ln x$ and $F(x) = \ln ax$ differ by a constant, say c. That is

$$\ln ax = \ln x + c$$

If $x = 1 \Longrightarrow c = \ln a$. Note that $x = 1$ is the intial value of $\ln x$ from definition. Thus

$$\ln ax = \ln x + \ln a$$

Since a can be any positive number, we can replace a by y to get

$$\ln xy = \ln x + \ln y.$$

(c) Put $x = \dfrac{1}{y}$ in (b) to get

$$\ln 1 = \ln \frac{1}{y} + \ln y$$

$$\Longrightarrow \ln \frac{1}{y} = -\ln y$$

$$\therefore \ln \left(\frac{x}{y}\right) = \ln \left(x \cdot \frac{1}{y}\right) = \ln x + \ln \frac{1}{y} = \ln x - \ln y.$$

(d) Let $F = \ln x^r$, and $G = r\ln x$. Then

$$F'(x) = \frac{r}{x}, \quad G'(x) = \frac{r}{x}$$

$$\Longrightarrow \ln x^r = r\ln x + c$$

For $x = 1$, we get $c = 0$. Thus

$$\ln x^r = r\ln x.$$

□

4.2.2 Derivatives of Logarithmic Functions

Prove the following identities

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a} \tag{4.1}$$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}} \tag{4.2}$$

Proof.

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Let $a = e^{\ln a}$. Then

$$a^x = e^{(\ln a)x}$$

$$\frac{d}{dx}(a^x) = e^{(\ln a)x} \cdot \ln a = e^{\ln a^x} \cdot \ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a$$

□

Proof.

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Let $y = \log_a x$. Then $a^y = x$.

$$\Longrightarrow \frac{d}{dx}(a^y) = 1$$

$$\Longrightarrow \frac{d}{dy}(a^y)\frac{dy}{dx} = 1$$

Using equation (4.1), we get

$$\implies a^y (\ln a) \frac{dy}{dx} = 1$$

But $a^y = x$. Thus

$$x \ln a \frac{dy}{dx} = 1$$
$$\therefore \frac{dy}{dx} = \frac{1}{x \ln a}.$$

□

Remark. Replacing a with e in equation (4.2), we get

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

A more general form of this relation would be shown later.

Example 18. (1) Expand the following expressions using the laws of logarithms

(a) $\ln \frac{2\sqrt{3}}{5}$ (b) $\ln \frac{x^{1/3}y^{2/3}}{z^{1/2}}$

Solution. (a)

$$\ln \frac{2\sqrt{3}}{5} = \ln 2\sqrt{3} - \ln 5$$

(b)

$$\begin{aligned} \ln \frac{x^{1/3}y^{2/3}}{z^{1/2}} &= \ln \left(x^{1/3}y^{2/3}\right) - \ln z^{1/2} \\ &= \ln x^{1/3} + \ln y^{2/3} - \ln z^{1/2} \\ &= \frac{1}{3} \ln x + \frac{2}{3} \ln y - \frac{1}{2} \ln z \end{aligned}$$

Example 19. Use the laws of logarithms to write the expression as the logarithm of a single quantity.

(a) $\ln 4 + \ln 6 - \ln 12$ (b) $3 \ln 2 - \frac{1}{2} \ln (x + 1)$

Solution. (a) $\ln 4 + \ln 6 - \ln 12 = \ln (4 \times 6) - \ln 12 = \ln \frac{24}{12} = \ln 2$

(b) $3 \ln 2 - \frac{1}{2} \ln (x + 1) = \ln 2^3 - \ln (x + 1)^{1/2} = \ln \left(\frac{8}{\sqrt{x + 1}} \right)$

Theorem 12. Let u be a differentiable function of x. Then

(a) $\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0$

(b) $\frac{d}{dx} \ln |u| = \frac{1}{u} \frac{du}{dx}, \quad u \neq 0$

Note: (b) is similar to the derivative $\frac{d}{dx} \ln[g(x)] = \frac{g'(x)}{g(x)}$.

Proof. (a) Already discussed in a remark above.

(b) Try proving it using using the chain rule.

□

Example 20. Find the derivatives of the following funtions

(a) $f(x) = \ln(2x+3)$ (b) $g(x) = \frac{\ln x}{x+1}$ (c) $y = \ln|\cos x|$

Solution. (a)

$$f(x) = \ln(2x+3) \Longrightarrow f'(x) = \frac{1}{2x+3} \frac{d}{dx}(2x+3)$$

$$= \frac{1}{2x+3}(2) = \frac{2}{2x+3}$$

(b) $g(x) = \frac{\ln x}{x+1}$

$$g'(x) = \frac{(x+1)\frac{1}{x} - \ln x}{(x+1)^2}$$

$$= \frac{\frac{x+1}{x} - \ln x}{(x+1)^2} = \frac{(x+1) - x \ln x}{x(x+1)^2}$$

(c) $y = \ln|\cos x|$

$$\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

4.2.3 Logarithmic Differentiation

Used to simplify complicated expressions for differentiation.

Example 21.

(a) $y = (2x+1)^2(3x^2-4)^3$ (c) $y = x^{\sqrt{x}}$

(b) $y = \sqrt[3]{\dfrac{x-1}{x^2+1}}$

Solution. (a) $y = (2x+1)^2(3x^2-4)^3$

$$\Longrightarrow \ln y = 2\ln(2x+1) + 3\ln(3x^2-4)$$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2}{2x+1}(2) + \frac{3}{3x^2-4}(6x)$$

$$= \frac{4}{2x+1} + \frac{18x}{3x^2-4}$$

$$\therefore \frac{dy}{dx} = \left(\frac{4}{2x+1} + \frac{18x}{3x^2-4}\right) \cdot \left[(2x+1)^2(3x^2-4)^3\right]$$

(b) $y = \sqrt[3]{\dfrac{x-1}{x^2+1}}$

$$\Longrightarrow y = \left(\frac{x-1}{x^2+1}\right)^{1/3}$$

$$\Longrightarrow \ln y = \frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x^2+1)$$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{3(x-1)} - \frac{1}{3(x^2+1)}(2x)$$

$$= \frac{1}{3(x-1)} - \frac{2x}{3(x^2+1)}$$

(c) $y = x^{\sqrt{x}}$
Sorry, it's an exercise! :)

4.2.4 The Natural Logarithm as a Limit

For any number $y > 0$

$$\ln y = \lim_{x\to 0} \frac{y^x - 1}{x} \quad \text{and} \quad \ln a = \lim_{x\to 0} \frac{a^x - 1}{x}$$

Remark. We note the following important relations:

(a) $f'(a) = \lim\limits_{x\to 0} \dfrac{f(x+a) - f(a)}{x-a}$

$$\Longrightarrow f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} \tag{4.3}$$

(b) If $f(x) = z = y^x$, then by logarithmic differentiation

$$\ln z = x \ln y$$

$$\frac{1}{z}\frac{dz}{dx} = \ln y \Longrightarrow \frac{dz}{dx} = z \ln y$$

$$\Longrightarrow f'(x) = y^x \ln y$$

$$f'(0) = \ln y \tag{4.4}$$

Thus, the derivative of $f(x) = y^x$ at $x = 0$ is $\ln y$. Applying (4.4) in (4.3) gives

$$\ln y = \lim_{x \to 0} \frac{y^x - 1}{x}$$

$$\ln a = \lim_{x \to 0} \frac{a^x - 1}{x}$$

4.3 General Exponential Functions

An exponential function is a function of the form

$$f(x) = a^x, \quad a > 0$$

where a is a constant. The graph of a^x with $a = 2$ is shown in Figure 4.3

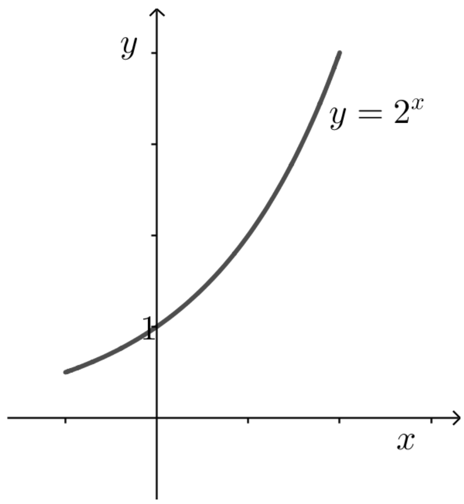

Figure 4.3: Graph of $y = 2^x$

4.3.1 Laws of Exponents

Let a and b be positive numbers. If x and y are real numbers, then

(a) $a^{x+y} = a^x \cdot a^y$

(b) $(a^x)^y = a^{xy}$

(c) $(ab)^x = a^x b^x$

(d) $\dfrac{a^x}{a^y} = a^{x-y}$

(e) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Proof. (a)

$$\begin{aligned} a^x a^y &= a^{x+y} \\ \log_a(a^x \cdot a^y) &= \log_a a^x + \log_a a^y \\ &= x + y = \log_a \left(a^{x+y}\right) \end{aligned}$$

Since the logarithm function is one-to-one

$$a^x \cdot a^y = a^{x+y}$$

(b) $(a^x)^y = a^{xy}$. Let $r = a^x$

$$\Longrightarrow \log_a (a^x)^y = \log_a r^y = y \log_a r$$

$$= y \log_a a^x = xy \log_a a = xy$$

$$\Longrightarrow \log_a (a^x)^y = \log_a a^{xy}$$

Since the logarithm is a one-to-one function, we get

$$(a^x)^y = a^{xy}$$

□

Remark. To prove the relation in (c), we first establish the following identity

$$\boxed{\log_a xy = \log_a x + \log_a y}$$

We have already proved the above relation under "general logarithms". An alternative proof is shown below, you may skip it if you want.

Proof. Let $y = \log_a x$. Then

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

Also let $g = \log_a bx$. Then

$$a^g = bx \Longrightarrow \frac{d}{dx}(a^g) = b \Longrightarrow \frac{d}{dg}(a^g)\frac{dg}{dx} = b$$

$$\Longrightarrow a^g \ln a \cdot \frac{dg}{dx} = b$$

$$\Longrightarrow \frac{dg}{dx} = \frac{b}{bx \ln a} = \frac{1}{x \ln a}$$

Since $dy/dx = dg/dx$, y and g differ by a constant, say C. Thus

$$\log_a x = \log_a bx + C$$

$$x = 1 \Longrightarrow \log_a 1 = \log_a b + C \Longrightarrow C = -\log_a b$$

$$\Longrightarrow \log_a bx = \log_a x + \log_a b$$

Since b is arbitrary, we get

$$\boxed{\log_a xy = \log_a x + \log_a y}$$

□

Proof. We're now ready to prove the relation in (c); $(ab)^x = a^x b^x$. Taking the lagarithm of both sides, we get

$$\Longrightarrow \log_a a^x b^x = \log_a a^x + \log_a b^x$$

$$= x \log_a a + x \log_a b$$

$$= x (\log_a a + \log_a b)$$

$$= x \log_a (ab)$$

$$\Longrightarrow \log_a a^x b^x = \log_a (ab)^x$$

$$\therefore a^x b^x = (ab)^x.$$

□

4.3.2 Derivative of Exponential functions

1). Let $f(x) = a^x$ by definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Note that: $f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$. Therefore

$$\boxed{f'(x) = f'(0) a^x}$$

Remember:

$$\ln y = \lim_{x\to 0}\frac{y^x - 1}{x} \Longrightarrow \ln a = \lim_{x\to 0}\frac{a^x - 1}{x} = \lim_{h\to 0}\frac{a^h - 1}{h}$$

where $f^{'}(0) = \ln a$. Therefore,

$$\boxed{f^{'}(x) = a^x \ln a}$$

Alternatively:

$$f(x) = a^x = e^{\ln a^x} = e^{x\ln a}$$

$$\Longrightarrow f^{'}(x) = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a) = e^{x\ln a}\cdot(\ln a)$$

$$\therefore \boxed{f^{'}(x) = a^x\cdot(\ln a)}$$

Theorem 13. Show that

$$\frac{d}{dx}a^u = (\ln a)a^u\cdot\frac{du}{dx}$$

where a is a positive number $(a \neq 1)$ and u is the differentiable function of x.

Proof. Let $y = a^u$, then

$$\frac{d}{dx}(a^u) = \frac{d}{du}(a^u)\frac{du}{dx}$$

$$\boxed{\frac{d}{dx}(a^u) = a^u\ln a\frac{du}{dx}}$$

□

Question: Show that $\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}$ where u is the differentiable, $a > 0(a \neq 1)$

4.3.3 Natural Exponential functions

The natural exponential function denoted by $\exp(x)$ or e^x, is the function satisfying

1. $\ln(e^x) = x, \quad \forall x \in [-\infty, \infty]$
2. $e^{\ln x} = x, \quad \forall x \in [0, \infty]$

Equivalently, $e^x = y \Leftrightarrow \ln y = x$. It is the inverse of the natural log function: **Diagram below**

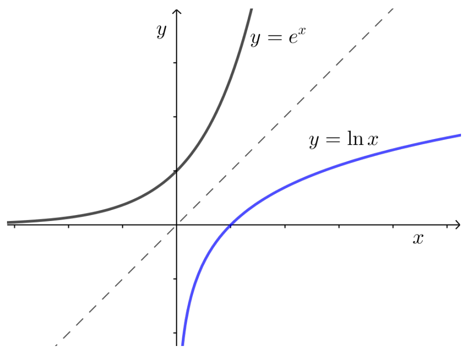

Figure 4.4: Graphs of $y = \ln x$ and $y = e^x$

The number e

The number e is the number such that:

$$\ln e = \int_1^e \frac{1}{t} dt = 1$$

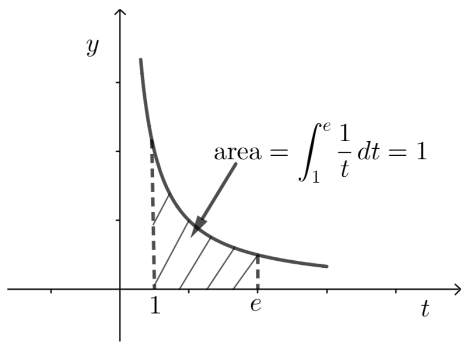

Figure 4.5: Graph of $y = \dfrac{1}{t}$

$$e \approx 2.718281828459$$

Theorem 14. a) $\ln e^x = x \ \forall \ x \in [-\infty, \infty]. \implies \ln e = 1$

b) $e^{\ln x} = x \ \forall \ x \in [0, \infty]$

Example 22. 1) Solve $\ln(x + 3) = 6$

$$\implies x + 3 = e^6 \implies x + 3 \approx 403.43$$
$$\therefore x \approx 400.43$$

2) Solve $e^{1-2x} = 5$

$$\Longrightarrow 1 - 2x = \ln 5 \Longrightarrow 2x = 1 - \ln 5$$

$$\Longrightarrow 2x = 1 - 1.609 = -0.609$$

$$\therefore x \approx -0.305$$

The number e as a Limit

Let $f(x) = \ln x$ and use the definition of derivative to compute f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

Let $x = 1$, we get.

$$\Longrightarrow f'(1) = \lim_{h \to 0} \frac{\ln(1+h)}{h} = \lim_{h \to 0} \frac{1}{h} \ln(1+h) = \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}}$$

But $f'(x) = \dfrac{1}{x} \Rightarrow f'(1) = 1$

$$\Longrightarrow \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}} = 1,$$

since ln is a continuous function.

$$\ln \left[\lim_{h \to 0} (1+h)^{\frac{1}{h}} \right] = 1$$

$$\therefore \boxed{\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e}$$

Note that a similar expression for e^x as a limit may be obtained if the substitution $x = 1$ is not made in the above derivation.

Let n $= \dfrac{1}{h} \Rightarrow n \to \infty$ as $h \to 0$

$$\therefore \boxed{\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e}$$

Exercise. 1) Show that $e^x = \lim_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n$

2) Prove that $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

Theorem 15 (Laws of exponents)**.** Let x and y be real numbers and r be a rational number. Then we have the following:

1) $e^x e^y = e^{x+y}$

2) $\dfrac{e^x}{e^y} = e^{x-y}$

3) $(e^x)^r = e^{xr}$

Proof. 1)

$$e^x \cdot e^y = e^{x \times y}$$

$$\Longrightarrow \ln(e^x \cdot e^y) = \ln e^x + \ln e^y = x + y$$

$$\Longrightarrow \ln(e^x e^y) = \ln e^{x+y}.$$

Since ln is one to one: $e^x e^y = e^{x+y}$

2) Let $y = -y$ in **1)**. Therefore we have:

$$e^x \cdot e^{-y} = e^{x-y} \Longrightarrow \frac{e^x}{e^y} = e^{x-y}$$

□

4.3.4 Derivatives of Exponential Functions

Theorem 16. Let u be a differentiable function of x. Then we have the following:

(a) $\dfrac{d}{dx}e^x = e^x$

(b) $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$

Proof. (a) $\dfrac{d}{dx}e^x = e^x$ Let y = e^x

$$\ln y = x \Longrightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = y$$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

(b) $\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$ Let y $= e^u$

$$\ln y = u(x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = y\frac{du}{dx}$$

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

□

Exercise. (1) Differentiate the following:

a) $f(x) = \frac{e^{2x}}{1+e^{-x}}$

b) $g(x) = e^{\cos(x)}$

c) $f(x) = x^2 \ln(e^{2x} + 1)$

d) $g(t) = e^{t \ln t}$ **e).** $f(x) = 2x^2$

(2) Find the Limit of the following:

a) $\lim_{t\to\infty} \left(\frac{3t^2+1}{2t^2-1}\right) e^{-0.1t}$

b) $\lim_{x\to\infty} \left(\frac{2e^x+1}{3e^x+2}\right)$

c) $\lim_{x\to\infty} \sin \frac{1}{x}$

(3) Show that

$$\lim_{x\to 0} \frac{\log(3+x) - \log 3}{x} = \frac{1}{3\ln 10}$$

4.4 Logarithmic and Exponential Integration

(1) Let u be a differentiable function and $u \neq 0$. Then

$$\int \frac{1}{u}du = \ln |u| + c.$$

NB:

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \Rightarrow \ln |x| = \int \frac{1}{x}dx$$

(2) Let u be a differentiable function of x. Then:

$$\int e^u du = e^u + c$$

(3) $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + c, a > 0, a \neq 1$. We use

$$y = a^x \Rightarrow \ln y = x \ln a$$

$$\Rightarrow \frac{y'}{y} = \ln a \Rightarrow y' = a^x \ln a$$

$$\Rightarrow \frac{d}{dx}(a^x) = a^x \ln a$$

$$a^x + c_1 = (\ln a) \int a^x dx$$

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + c}$$

Examples

Integrate the following: **a).** $\displaystyle\int \frac{1}{2x+1} dx$ **b).** $\displaystyle\int \frac{\sqrt{\ln x}}{x} dx$ **c).** $\displaystyle\int e^{5x} dx$

d). $\displaystyle\int \frac{e^{\frac{2}{x}}}{x^2} dx$

e). $\displaystyle\int_0^1 \frac{e^x}{1+e^x} dx$ **f).** $\int_0^3 2^x dx$

Solution. a) $\displaystyle I = \int \frac{1}{2x+1} dx$, Let $u = 2x + 1 \Rightarrow du = 2dx$

$$I = \int \frac{1}{u}\left(\frac{1}{2}\right) du = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + c$$

$$\therefore \boxed{I = \frac{1}{2}\ln|2x+1| + c}$$

b) $\displaystyle I = \int \frac{\sqrt{\ln x}}{x} dx$, Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$I = \int \sqrt{u} du = \int u^{\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} + c$$

$$\therefore \boxed{I = \frac{2}{3}(\ln x)^{\frac{3}{2}} + c}$$

c) $I = e^{5x}dx$, Let $u = 5x \Rightarrow du = 5dx$
$I = \int e^u \cdot \frac{du}{5} = \frac{1}{5}\int e^u du \Rightarrow \frac{1}{5}e^u + c$

$$\therefore \boxed{\frac{1}{5}e^{5x} + c}$$

d). $I = \int \frac{e^{\frac{2}{x}}}{x^2}dx$, Let $u = \frac{2}{x} \Rightarrow du = -\frac{2}{x^2}dx \Rightarrow -\frac{1}{2}du = \frac{dx}{x^2}$
$I = \int e^u \left(-\frac{1}{2}\right) du = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c$

$$\therefore \boxed{I = -\frac{1}{2}e^{\frac{2}{x}+c}}$$

e). $I = \int_0^1 \frac{e^x}{1+e^x}dx$ Let $u = 1 + e^x \Rightarrow du = e^x dx$
$I = \int_0^1 \frac{e^x}{u} \cdot \left(\frac{du}{e^x}\right) = \int_0^1 \frac{1}{u}du \Rightarrow \ln|u| \Rightarrow \ln|1+e^x| \Rightarrow \ln|1+e^1| - \ln|1+e^0|$

$$\therefore \boxed{I = \ln\left(\frac{1+e}{2}\right) \approx 0.6201}$$

f). $I = \int_0^3 2^x dx = \frac{2^x}{\ln 2}\Big|_0^3 = \frac{2^3}{\ln 2} - \frac{1}{\ln 2} = \frac{7}{\ln 2}$

$$\therefore \boxed{I = \frac{7}{\ln 2}}$$

4.5 Logarithmic Inequalities

a) For a number $a > 1$, $y = \log_a x$ is an increasing function of x and it is a decreasing function for $0 < a < 1$.

b) If $y = \log_a x$, then $x > 0$. For example $y = \log_3(x-1) \Rightarrow x - 1 > 0 \Rightarrow x > 1$.

c) $a > 1$, $\log_a x > \log_a y \Rightarrow x > y$.

d) $0 < a < 1$, $\log_a x > \log_a y \Rightarrow x < y$.

Example 23. **1.** Find the values of x that satisfy the ff inequalities:
a) $\log_3(2x+3) > \log_3(3x)$ **b)** $\log_2(x+1) > \log_4(x^2)$

2. Find the solution set of $2\log_{49}(2x+1) - 1 \leqslant 0$

Solution. 1a)

$$\log_3(2x+3) > \log_3(3x)$$

$$2x+3 > 3x \Rightarrow x < 3$$

Also,

$$2x+3 > 0 \Rightarrow x > -\frac{3}{2}$$

And

$$3x > 0 \Rightarrow x > 0$$

Solution set: $\boxed{0 < x < 3}$. See Figure 4.6.

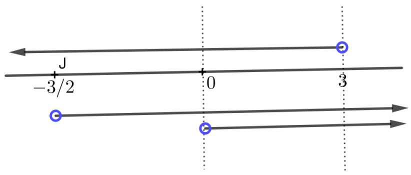

Figure 4.6:

(1b) $\log_2(x+1) > \log_4 x^2$
NB: $y = \log_2(x+1) \Rightarrow 2^y = x+1 \Rightarrow 4^y = (x+1)^2$
$y = \log_4(x+1)^2 \therefore \log_4(x+1)^2 > \log_4 x^2 \Rightarrow (x+1)^2 > x^2 \Rightarrow x^2+2x+1 > x^2$
$\Rightarrow 2x+1 > 0 \Rightarrow \boxed{x > -\frac{1}{2}}$

2) $2\log_{49}(2x+1) - 1 \leqslant 0$

$$\log_{49}(2x+1) \leqslant \frac{1}{2} \Rightarrow \log_{49}(2x+1) \leqslant \frac{1}{2}\log_{49} 49$$

$$\Rightarrow \log_{49}(2x+1) \leqslant \log_{49} 7 \Rightarrow 2x+1 \leqslant 7 \Rightarrow x \leqslant 3$$

Also: $2x+1 > 0 \Rightarrow x > -\frac{1}{2} \therefore \boxed{-\frac{1}{2} < x \leqslant 3}$

Example 24. 1. Use the definition of the natural log function to prove that, for $x > 0$, $x(1+x) > (1+x)\ln(1+x) > x$. Hence show that $\ln(1+x) > x - x^2 + x^3 - x^4 + \cdots$ and that as

$$x \to 0, \quad \frac{\ln(1+x)}{x} \to 1$$

Solution. NB: $\ln(1+x) = \int_1^{1+x} \frac{1}{t} dt$, for $x > 0 \Rightarrow x + 1 > 1$
Let $1 < t < 1 + x \Rightarrow \frac{1}{1+x} < \frac{1}{t} < 1$.
Integrating with respect to t gives

$$\int_1^{1+x} \frac{1}{1+x} dt < \int_1^{1+x} \frac{1}{t} dt < \int_1^{1+x} dt$$

$\Rightarrow \frac{x}{1+x} < \ln(1+x) < x$**.......(1)**

$$x < (1+x)\ln(1+x) < x(1+x) \qquad \therefore x(1+x) > (1+x)\ln(1+x) > x$$

From **(1):**,

$$\ln(1+x) > \frac{x}{1+x} \Rightarrow \ln(1+x) > x(1 - x + x^2 - x^3 + \cdots)$$

$$\Rightarrow \ln(1+x) > x - x^2 + x^3 - x^4 + \cdots$$

Also from **(1):**

$$\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1 \Rightarrow \lim_{x\to 0} \frac{1}{1+x} < \lim_{x\to 0} \frac{\ln(1+x)}{x} < 1$$

$$\Rightarrow 1 < \lim_{x\to 0} \frac{\ln(1+x)}{x} < 1.$$

$\therefore$ **By the squeeze theorem:** $\lim_{x\to 0} \frac{\ln(1+x)}{x} \to 1 \Rightarrow \ln \lim_{x\to 0}(1+x)^{\frac{1}{x}} \Rightarrow \ln e = 1$.

Exercise. 1. Show that for all $x > 0$, $x - \frac{1}{2}x^2 < \ln(1+x)$, and for all $x > 1$, we have $\frac{x-1}{x} < \ln x < x - 1$

Chapter 5

Indeterminate Forms and L'Hopital's Rule

A function $f(x)$ is said to be continuous at a point a, if $\lim_{x \to a} f(x) = f(a)$. However, some limits cannot be evaluated by a simple substitution. For example, $\lim_{x \to 0} \frac{2x}{x} = \lim_{x \to 0} 2 = 2$ and $\lim_{x \to 0} \frac{\sin x}{x} = 1$. In each example, a direct substitution results in the indeterminate form $\frac{0}{0}$. Simarlarly, using direct substitution for $\lim_{x \to \infty} \frac{10x}{x^2} = \lim_{x \to \infty} \frac{10}{x} = 0$ results in the indeterminate form $\frac{\infty}{\infty}$.

Definition 8. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of the type $\frac{0}{0}$.

Definition 9. If $\lim_{x \to a} f(x) = +\infty$ and $\lim_{x \to a} g(x) = +\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type $\frac{\infty}{\infty}$.

5.1 L'Hopital's Rule

Theorem 17. (L'Hopital's Rule) Suppose f and g are differentiable on an open interval I that contains a, with the possible exception of a itself, and $g^{'}(x) \neq 0$ for all x in I. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an interminate form of the type $\frac{0}{0}$ or

$\dfrac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{5.1}$$

provided the limit on right-hand side exists or is infinite.

Remark. (1) $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]} \neq \lim_{x\to a}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$
[Don't use the Quotient Rule!!]

(2) Make sure the limit does have one of the indeterminate forms $\left(\dfrac{0}{0} \text{ or } \dfrac{\infty}{\infty}\right)$ before applying **L'Hopital's Rule!**

Example 25. 1) Use L'Hopital's rule to evaluate the following:

(a) $\displaystyle\lim_{x\to 1}\frac{x^3+x^2-2x}{x-1}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sqrt{9-3x}-3}{x}$

2) [Repeated application of L'Hopital's rule]:

(a) $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{x^2}$ (c) $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{2x}}$

(b) $\displaystyle\lim_{x\to 2}\frac{x^3-3x^2+4}{x^4-4x^3+7x^2-12x+2}$

3. Evaluate the following:

(a) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$ (b) $\displaystyle\lim_{x\to 1^+}\frac{\sin\pi x}{\sqrt{x-1}}$ (c) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\frac{1+\tan x}{\sec x}$

5.2 Indeterminate Forms $\infty-\infty$ and $0-\infty$

Definition 10. If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then the limit $\lim_{x\to a}[f(x)-g(x)]$ is said to be an indeterminate fom of the type $\infty-\infty$.

Definition 11. If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=+\infty$, then $\lim_{x\to a} f(x)g(x)$ is said to be indeterminate form of the type $0-\infty$.

Remark. Indeterminate forms of the $\infty-\infty$ and $0-\infty$ can be expressed as one of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic manipulation.

Example 26. Evaluate the following:

(a) $\lim_{x\to\infty} x^2 \sin\frac{1}{4x^2}$

(b) $\lim_{x\to\infty}(x-\sqrt{x^2-3x})$

(c) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

(d) $\lim_{x\to 0^+} x\ln x$

5.3 Indeterminate Forms of $0^0, \infty^0, 1^\infty$

The limit $\lim_{x\to a}[f(x)]^{g(x)}$ is said to be an indeterminate form of the type:

I. 0^0 if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$

II. ∞^0 if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$

III. 1^∞ if $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=+\infty$

Remark. The above indeterminate forms can usually be converted to the type $0\cdot\infty$ by taking logarithm or using $[f(x)]^{g(x)}=e^{g(x)\ln f(x)}$.
Procedure:

(i) Evaluate $\lim_{x\to a} g(x)\ln f(x)=L$

(ii) Then $\lim_{x\to a}[f(x)]^{g(x)}=e^L$

Example 27. (1) Evaluate:

(a) $\lim_{x\to 0^+} X^x$ (b) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ (c) $\lim_{x\to 0^+}\left(\frac{1}{x}\right)$

(2) Evaluate the following:

(a) $\lim_{h\to 0}(1+2h)^{\frac{1}{h}}$

(b) $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

(c) $\lim_{x\to 0^+}(\sin x)\sqrt{\frac{1-x}{x}}$

(d) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}$

5.4 Scales of Infinity(Order of Magnitude)

5.4.1 Motivation

Consider the solution to the following limits.

(1) $\displaystyle\lim_{x\to\infty}\frac{3x^3+5}{4x^2+x}$

$$\Longrightarrow \lim_{x\to\infty}\frac{3x+\frac{5}{x^2}}{4+\frac{1}{x}}=\infty$$

The limit going to infinity implies that the numerator, $3x^3+5$, is growing much faster than the denominator, $4x^2+x$.

(2) $\displaystyle\lim_{x\to\infty}\frac{3x+5}{4x^2+3x+1}$

$$\Longrightarrow \lim_{x\to\infty}\frac{3/x+5/x^2}{4+3/x+1/x^2}=0$$

This implies that the denominator, in this case, is growing faster than the numerator. Equivalently, we can say that the numerator is growing more slowly than the denomintor.

(3) $\displaystyle\lim_{x\to\infty}\frac{3x^3+5}{4x^3+x}$

$$\Longrightarrow \lim_{x\to\infty}\frac{3+\frac{5}{x^3}}{4+\frac{1}{x^2}}=\frac{3}{4}$$

Unlike the previous two examples, the numerator and denominator have similar rates of growth, because the limit of their ratio is a constant.

In the examples, the limits are determined by the order of magnitude of the polynomials in the numerator and denominator. In other words, the limits could have been obtained if we considered only the term with highest order of magnitude in the numerator and the denominator.

Motivational Questions

The main question to address is this: Apart from polynomials, is it possible to determine the limit $f(x)/g(x)$ of the two function $f(x)$ and $g(x)$ as x approaches infinity or zero?
Why is this important? An answer to the question could tell us whether $f(x)$ is growing faster or slower than $g(x)$. Think of the economies of two countries, say Ghana and Nigeria, that are both growing very fast. Which economy is growing more rapidly? Conversely, if both economies are decaying, can we tell which one is decaying more slowly?

Consider the functions $f(x)=x^2$ and $g(x)=x$. Notice that both functions grow without bound as x becomes infinitely large (see Figure 5.1). But f grows rapidly than g. In general, how can you tell which function is growing

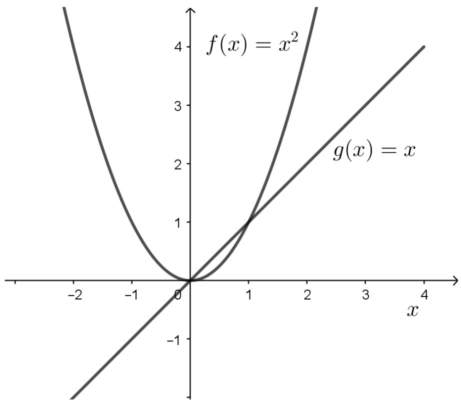

Figure 5.1: Graph of $\ln x$

(decaying) more rapidly (slowly) than the other? The following definitions and theorems will help in answering the question above.

Definition 12. (Order of Magnitude) A function $f(x)$ is said to be of the k-th order of magnitude if $\frac{f(x)}{x^k} \to L \neq 0$ as $x \to \infty$, where L is a constant.

Definition 13. (Relative Rates of Growth) Let $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^*(\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\})$, and $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$.

a) We say f approaches ∞ on a higher order of magnitude than g if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = \infty$ or we can say that $f(x) \to \infty$ faster than $g(x)$, or $g(x) \to \infty$ slower than f.

b) We can say f approach ∞ on a lower order of magnitude than $g(x)$ if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = 0$ or we can say that $f(x) \to \infty$ slower than $g(x)$ or $g(x) \to \infty$ faster than $f(x)$.

c) We say f and g approaches ∞ on the same order of magnitude if $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = C$, $C \in \mathbb{R}, C \neq 0$.

Theorem: As $x \to \infty$, the magnitude of the exponential, power and logarithmic functions approach ∞. Moreover, the order of magnitude with which they approach ∞ is given in the following order, with exponential functions approaching the fastest, and logarithmic functions approaching the slowest.

(1) Exponentials of the form a^x, where $a > 0, a \neq 1$. If $a > b$ then a^x approaches ∞ faster than b^x as $x \to \infty$.

(2) Power functions, x^n, $n\epsilon\mathbb{N}$. If $n > m$, then x^n approaches ∞ faster than x^m.

(3) Log functions, $\log_a(x)$ for $a > 0$ and $a \neq 1$. All log functions approach ∞ at the same rate, because $\dfrac{\log_a x}{\log_b x} = \dfrac{\ln(b)}{\ln(a)}$.
If any of the above functions is multiplied by a non-zero constant , it does not change the relative order of magnitude at which the function approaches ∞. 5

Example

1. $\lim_{x\to\infty} \dfrac{e^x}{\ln x} = \infty$, since e^x grows faster than $\ln x$ as $x \to \infty$.

2. $\lim_{x\to\infty} \dfrac{\ln x}{e^x} = 0$.

NB: For a linear combination (sums and differences) of these functions, only the dominant term matters.

Definition:Dominant term
Suppose f can be written as a linear combination of the functions $\{f_1, f_2, ..., f_n\}$ and $\lim_{x\to a} |f(x)| = \infty$, where $a\epsilon\mathbb{R}^*$. The dominant term of f is the function f_i which approaches ∞ on the highest order of magnitude. Denote this by $\tilde{f}$.

Theorem: Dominant term
Consider $f(x)$ and $g(x)$, with $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$. If $\tilde{f}$ and $\tilde{g}$ are the dominant terms of f and g respectively, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{\tilde{f}(x)}{\tilde{g}(x)}$. eg:
$\lim_{x\to\infty} \dfrac{e^x - 12x^2 + x}{x^4 + 1} = \lim_{x\to\infty} \dfrac{e^x}{x^4} = \infty$.

Theorem: Ranking growth rates as $x \to \infty$
Let $f << g$ mean that g grows faster than f as $x \to \infty$ with positive real numbers p, q, r and s and $b > 1$, then $\ln^q x << x^p << x^p \ln^r x << x^{p+s} << b^x << x^x$.

5.4.2 Relative Rates of Decay

Definition: Relative rates of Decay
Let $f, g : \mathbb{R} \to \mathbb{R}, a\epsilon\mathbb{R}^*$, and $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = 0$.

a. We say f approaches 0 on a higher order of magnitude than g if $\lim\limits_{x\to a} |\frac{f(x)}{g(x)}| = 0$.
b. We say f approaches 0 on a lower order of magnitude than g if $\lim\limits_{x\to a} |\frac{f(x)}{g(x)}| = \infty$.
c. We say f and g approach 0 on the same order of magnitude if $\lim\limits_{x\to a} |\frac{f(x)}{g(x)}| = C, C \in \mathbb{R}$, $c \neq 0$.

NB: Because e^x grows faster than x^2 as $x \to \infty$, it also implies that e^{-x} decays faster than x^{-2} as $x \to \infty$. So looking at the reciprocals of the exponential and power functions helps in determining their relative rates of decay.

Example 28. 1. $\lim\limits_{x\to\infty} \dfrac{e^{-x}+1}{x^{-2}} = \lim\limits_{x\to\infty} \dfrac{e^{-x}}{x^{-2}} + \dfrac{1}{x^{-2}} = \lim\limits_{x\to\infty} \left(\dfrac{x^2}{e^x} + x^2\right) = \infty$

2 Evaluate $\lim\limits_{x\to\infty} \dfrac{e^x - 12x^2 + x}{x^4+1}$.

3. Show that the exponential function e^x increases more rapidly as $x \to +\infty$ than any fixed power of x. That is $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0, n > 0$ **.....(1)**
or $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = +\infty, n > 0$**.....(2)**

Solution

$\ln\left(\dfrac{e^x}{x^n}\right) = \ln e^x - \ln x^n = x - n\ln x = \ln x\left(\dfrac{x}{\ln x} - n\right)$

$\therefore \lim\limits_{x\to\infty} \ln\left(\dfrac{e^x}{x^n}\right) = \lim\limits_{x\to\infty} \ln x \cdot \lim\limits_{x\to\infty}\left(\dfrac{x}{\ln x} - n\right) = \infty \cdot \infty = \infty$

$\therefore \lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \infty$

2. Show that the function $\ln x$ increases more slowly than any positive integral power of x. That is: $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^n} = 0, x > 0, n > 1$ (n is an integer).

Solution

NB: If $t \geqslant 1, \dfrac{1}{t} \leqslant \frac{1}{t^{\frac{1}{2}}} \Rightarrow \ln x = \int_1^x \dfrac{1}{t}dt \leqslant \int_1^x \dfrac{1}{t^{\frac{1}{2}}}dt = \int_1^x t^{-1/2}dt = 2t^{\frac{1}{2}}\Big|_1^x = 2x^{\frac{1}{2}} - 2$

$\therefore \ln x \leqslant 2x^{\frac{1}{2}} - 2 \Rightarrow \dfrac{\ln x}{x} \leqslant \dfrac{2}{x^{\frac{1}{2}}} - \dfrac{2}{x}$

$x > 1 : 0 < \dfrac{\ln x}{x^n} \leqslant \dfrac{\ln x}{x} \leqslant \dfrac{2}{x^{\frac{1}{2}}} - \dfrac{2}{x}$. As $x \to \infty$, $0 < \lim\limits_{x\to\infty} \dfrac{\ln x}{x^n} \leqslant 0$.

$\therefore$ **By squeeze(sandwich) theorem**: $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^n} = 0$. This implies: $x^n > \ln x$.

Examples

1. Prove the following: **a).** $\frac{d}{dx}(\ln x)^k = \frac{k}{x(\ln x)^{1-k}}$ **b).** $\frac{d}{dx}(\ln \ln x)^k = \frac{k}{x \ln x (\ln \ln x)^{1-k}}$

2. Find: **a)** $\lim_{x\to\infty} \frac{\ln \ln x}{\ln x} = \lim_{x\to\infty} \frac{\ln^2 x}{\ln x}$ **b)** $\lim_{x\to\infty} \frac{\ln^3 x}{\ln x}$

3.Determine which function has a higher order of magnitude: **a).** $f(x) = \sqrt[3]{x}, g(x) = \sqrt{x}$

b). $f(x) = \sqrt{\ln x}, g(x) = \sqrt[3]{\ln x}$ **c).** $f(x) = \sqrt{x}, g(x) = \ln^3 x$

4. Arrange the functions according to their order of magnitude: $g = \frac{x}{\sqrt{\ln x}}$, $f = \frac{x\sqrt{\ln x}}{\ln \ln x}$, $h = \frac{x \ln \ln x}{\sqrt{\ln x}}$, $q = \frac{x \ln \ln \ln x}{\sqrt{\ln \ln x}}$

5. Find: **a).** $\lim_{x\to -4} \frac{1}{(x+4)^4}$ **b).** $\lim_{x\to 3} \frac{3-x}{(x-3)^4}$ **c).** $\lim_{x\to 3} \frac{4x^2-9x-9}{x-3}$

6. Show that for large values of x, $\sqrt{x} > e^{\sqrt{\ln x}} > \ln^3 x$

7. $\lim_{x\to 0} \frac{2x^3 - x^2 + x}{x^3 + 2x}$

8. $\lim_{x\to 0} \frac{2x^3 - x^2 + x + 1}{x^3 + 2x + 2}$

Chapter 6

Hyperbolic Functions

The analysis of certain problems such as the shape of a hanging cable and pursuit curve of a missile involves combinations of exponential functions such as e^{-cx} and e^{cx}, where c is a constant. Because they occur frequently in maths, these combinations have been given the special name of hyperbolic functions. They are related to the hyperbola, just as trigonometric functions are related to the circle. Before defining hyperbolic functions, we first take a look at odd and even functions.

6.1 Odd and Even Functions

Definition 14. A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all x in its domain.

For example, $f(x) = x^{2k}$ is an even function for some $k \epsilon \mathbb{Z}$ and $f(x) = x^{2k+1}$ is an odd function for the same $k \epsilon \mathbb{Z}$.

Symmetry

(i) Even functions are symmetric about the coordinate(vertical) axis. eg: $f(x) = x^2 + 2$

(ii) Odd functions are symmetric about the origin. eg: $f(x) = x^3$. See Figure 6.1.

Theorem 18. Let $f(x)$ be any given function. Then the functions $\frac{1}{2}\left(f(x) + f(-x)\right)$ and $\frac{1}{2}(f(x) - f(-x))$ are even and odd functions respectively. That is, every function can be written as the sum of an even and an odd function.

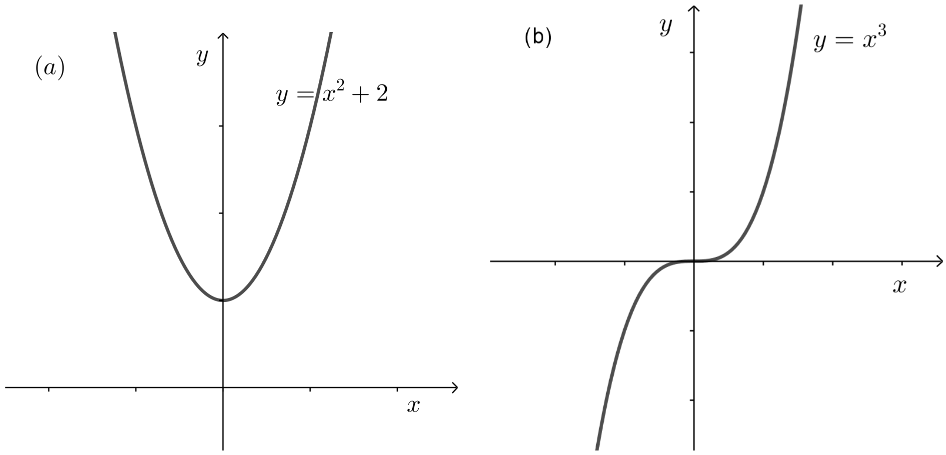

Figure 6.1: Graph of (a) $y = x^2 + 2$ (even), and (b) $y = x^3$ (odd).

Proof.

$$\text{Let} \quad f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right) \quad \text{and} \quad f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

$$f_e(-x) = \frac{1}{2}\left(f(-x) + f(x)\right) = \frac{1}{2}\left(f(x) + f(-x)\right) = f_e(x)$$

$$f_o(-x) = \frac{1}{2}\left(f(-x) - f(x)\right) = -\frac{1}{2}\left(f(x) - f(-x)\right) = -f_o(x)$$

$$\text{Now,} \quad f_e(x) + f_o(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$

$$= \frac{1}{2}\left[f(x) + f(-x) + f(x) - f(-x)\right] = \frac{1}{2}[2f(x)] = f(x)$$

□

6.1.1 Basic Properties of Even and Odd Functions

a. The sum or difference of two even (odd) functions is even (odd), and any constant multiple of an even (odd) function is even (odd).

b. The product of two even functions is even, and the product of two odd functions is even. The product of an even and odd function is odd.

c. The quotient of two even or two odd functions is an even function, but the quotient of an even and an odd function is odd.

d. The derivative of an even function is odd and the derivative of an odd function is even.

e. For any $a\epsilon\mathbb{R}$, we have $\int_{-a}^{a} f_o(x) = 0$ and $\int_{-a}^{a} f_e(x) = 2\int_0^a f_e(x)$

6.1.2 Hyperbolic functions: Definitions

Consider the function $f(x) = e^x$. Then $e^x = \dfrac{1}{2}(e^x + e^{-x}) + \dfrac{1}{2}(e^x - e^{-x})$ and $e^{-x} = \dfrac{1}{2}(e^x + e^{-x}) - \dfrac{1}{2}(e^x - e^{-x})$. Note that $\dfrac{1}{2}(e^x + e^{-x})$ is even and $\frac{1}{2}(e^x - e^{-x})$ is odd.

Definition 15. (DEFINITIONS OF THE HYPERBOLIC FUNCTIONS)

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\tanh x = \frac{\sinh x}{\cosh} \qquad \coth x = \frac{1}{\tanh x}$$

Applications

Hyperbolic functions have several applications in science and engineering. A well known application is the use of the hyperbolic cosine to describe the shape of a hanging cable. If a heavy flexible wire (or cable; such as a telephone or power line) is suspended between two points at the same height, it takes the shape of a curve with equation $y = A + B\cosh(x/B)$ (see Figure 6.2a).

A second application occurs in the description of the velocity of ocean waves. The velocity of a water wave with wave-length L moving across the ocean (or a water body) with average depth d, is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity (see Figure 6.2b). Besides the two physical applications mentioned above, hyperbolic functions also have several mathematical applications. Most especially, they are often used to simplify complicated expressions under the techniques of integration. We would see some of these uses later in the course.

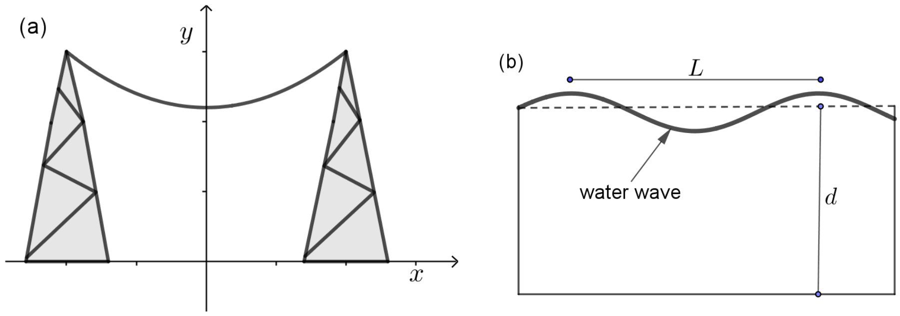

Figure 6.2: (a) A hanging cable with equation of the form $y = A + B\cosh(x/B)$. (b) Schematic of a water wave.

Hyperbolic cosine (cosh) and Hyperbolic sine (sinh)

Definition 16. The even function $\cosh : \mathbb{R} \to [1, \infty)$ is defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\frac{e^{2x}+1}{e^x}\right) = \frac{1}{2}\left(\frac{1+e^{-2x}}{e^{-x}}\right),\ x\epsilon\mathbb{R}$$

The odd function $\sinh : \mathbb{R} \to (-\infty, \infty)$ is defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(\frac{e^{2x}-1}{e^x}\right) = \frac{1}{2}\left(\frac{1-e^{-2x}}{e^{-x}}\right),\ x\epsilon\mathbb{R}$$

6.1.3 Graphs

(1) Use the additions of $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$ to graph the cosh and sinh functions.

(2) You may also use the approach of curve sketching in calculus I to sketch the graphs of hyperbolic functions. See Figure 6.3.

Example 29. Graph of $\tanh x$. Let $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}$

$$\implies f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{e^{2x}-1}{e^{2x}+1}$$

$f(0) = \dfrac{1-1}{2} = 0, f(x) = 0 \Rightarrow \dfrac{e^{2x}-1}{e^{2x}+1} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$
$\therefore (0,0)$ is an intercept on both axes.
As $x \to \infty, f(x) = \dfrac{1-e^{-2x}}{1+e^{-2x}} = 1$.
As $x \to -\infty, f(x) = \dfrac{e^{2x}-1}{e^{2x}+1} = -1$.
So $f(x) = \pm 1$ are horizontal asymptotes.

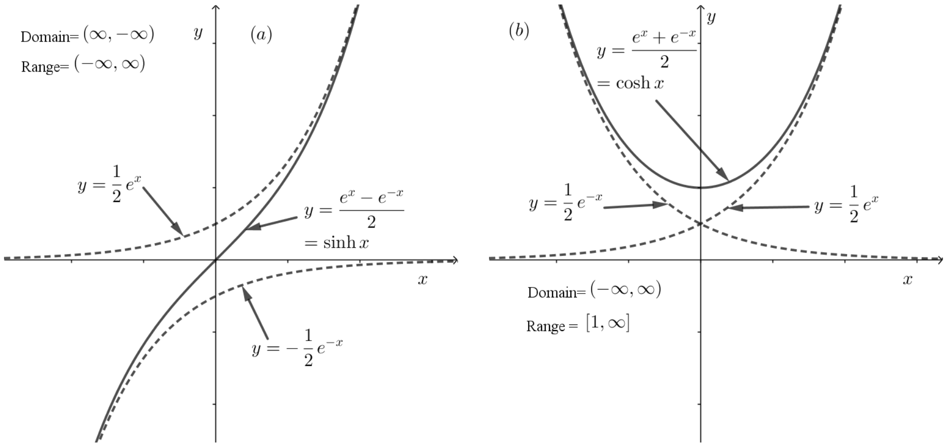

Figure 6.3: Sketch of (a) $y = \sinh x$ and (b) $y = \cosh x$.

We can show that $f^{'}(x) = \text{sech}^2\, x = \left(\dfrac{2e^x}{e^{2x}+1}\right) > 0 \Rightarrow$ **No turning points** Also, $f^{''}(x) = 0 \Rightarrow x = 0 \therefore (0, 0)$ is a point of inflection. See the graph in Figure 6.4a. The graphs of other hyperbolic functions are displayed in Figure 6.4b-d.

6.2 Hyperbolic Identities and Osborn's Rule

(a) $\sinh(-x) = -\sinh x$

(b) $\cosh(-x) = \cosh x$

(c) $\cosh^2 x - \sinh^2 x = 1$

(d) $\text{sech}^2\, x = 1 - \tanh^2 x$

(e) $1 + \text{csch}^2\, x = \coth^2 x$

Proof. (a) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\Rightarrow \sinh(-x) = \frac{e^{-x} - e^{+x}}{2} = -\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= -\sinh x$$

(b) Try it.

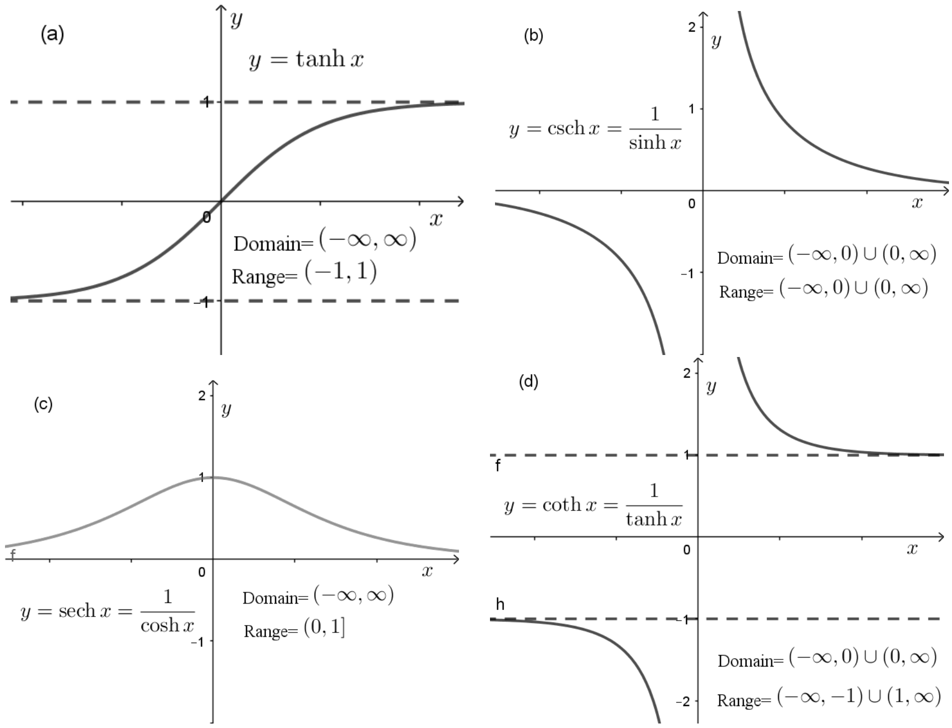

Figure 6.4: Graph of (a) $y = \tanh x$ (b) $y = \operatorname{csch} x$ (c) $y = \operatorname{sech} x$ (d) $y = \coth x$.

(c) $$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

(d) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

From (c), we have

$$\cosh^2 - \sinh^2 = 1 \Rightarrow \frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2} = \frac{1}{\cosh^2 x}$$
$$\Longrightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x$$

Alternatively,

$$\operatorname{sech}^2 x + \tanh^2 x = \frac{1}{\cosh^2 x} + \frac{\sinh^2 x}{\cosh^2 x}$$
$$= \frac{1 + \sinh^2 x}{\cosh^2 x} = \frac{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}{\left(\frac{e^x + e^{-x}}{2}\right)^2}$$

$$= \frac{\frac{4+e^{2x}-2+e^{-2x}}{4}}{\frac{e^{2x}+2+e^{-2x}}{4}}$$

$$\therefore \operatorname{sech}^2 x + \tanh^2 x = \frac{e^{2x}+2+e^{-2x}}{e^{2x}+2+e^{-2x}} = 1$$

(e) $1 + \operatorname{csch}^2 x = \coth^2 x$
From **c.**: $\cosh^2 x - \sinh^2 x = 1$
$\dfrac{\cosh^2 x}{\sinh^2 x} - \dfrac{\sinh^2 x}{\sinh^2 x} = \dfrac{1}{\sinh^2 x} \Rightarrow \coth^2 x - 1 = \operatorname{csch}^2 x$

$$\therefore 1 + \operatorname{csch}^2 x = \coth^2 x$$

□

Exercise. Prove the following:

1. $\sinh(2x) = 2\sinh(x)\cosh(x)$
2. $\cosh(2x) = \cosh^2 x + \sinh^2 x$
3. $\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(x)\sinh(x)$
3. $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

6.2.1 Relationship to the Exponential function

It can be shown that:

(i) $e^x = \cosh(x) + \sinh(x)$

(ii) $e^{-x} = \cosh(x) - \sinh(x)$

These are very similar to trigonometric expressions using Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x) \tag{6.1}$$
$$e^{-ix} = \cos(x) - i\sin(x) \tag{6.2}$$

(6.1) + (6.2) gives

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \cosh(ix) \tag{6.3}$$

(6.1) - (6.2) results in

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = -i\left(\frac{e^{ix} - e^{-ix}}{2}\right) = -i\sinh(ix) \tag{6.4}$$

Thus, (6.3) implies that

$$\boxed{\cos(iy) = \cosh(-y) = \cosh(y)}$$

and (6.4) means that

$$\boxed{\sin(iy) = -i\sinh(-y) = i\sinh(y)}$$

So

$$\boxed{\cosh(y) = \cos(iy) \quad \text{and} \quad \sinh(y) = -i\sin(iy)}$$

Thus, trigonometric formulae can be converted to formulas for hyperbolic functions using Osborn's rule:

6.2.2 Osborn's Rule

cos can be converted to cosh and sin into sinh, except when there is a product of two sines, then a sign change must be effected. That is

$$\cos \to \cosh \quad \text{and} \quad \sin \to i\sinh$$

Example 30. 1) $\cos(2x) = 1 - 2\sin^2 x \Rightarrow \cosh(2x) = 1 + 2\sinh^2 x$.

2) $\sin(2x) = 2\sin(x)\cos(x) \Rightarrow \sinh(2x) = 2\sinh(x)\cosh(x)$

6.3 Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we notice that all of them are one-to-one functions on $(-\infty, \infty)$ except $\cosh x$ and $\operatorname{sech} x$. However, if we restrict the domain of $\cosh x$ and $\operatorname{sech} x$ to $[0, \infty)$. They become one-to -one with inverses $\cosh^{-1} x$ and $\operatorname{sech}^{-1} x$.

Definitions	**Domain**
$y = \sinh^{-1} x$	$\to (-\infty, \infty)$
$y = \cosh^{-1} x$	$\to [1, \infty)$
$y = \tanh^{-1} x$	$\to (-1, 1)$
$y = \operatorname{csch}^{-1} x$	$\to (-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x$	$\to (0, 1]$
$y = \coth^{-1} x$	$\to (-\infty, -1) \cup (1, \infty)$

Graphs

You may sketch the graph of the inverse hyperbolic functions by reflecting the graph of the corresponding hyperbolic function about the line $y = x$ (see the section on Inverse Functions). The graphs of some inverse hyperbolic functions are displayed in Figure 6.5.

Exercise. Sketch the graph of $y = \operatorname{sech}^{-1} x$.

6.3.1 Representation of Inverse Hyperbolic Functions as Logarithmic Functions

For some applications, it is more convenient to represent hyperbolic functions as logarithmic functions. The following exercises illustrates some of these representations and how to proof them.

Example 31. Prove that:

(1) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1});\qquad (-\infty, \infty)$

(2) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1});\qquad [1, \infty)$

(3) $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right);\qquad (-1, 1)$

Proof. (1) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. Let

$$y = \sinh^{-1} x \Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\Longrightarrow x = \frac{e^{2y} - 1}{2e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

Let $t = e^y$, then $t^2 - 2xt - 1 = 0$ is a quadratic in $t = e^y$.

$$\Rightarrow t = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$\Longrightarrow e^y = x \pm \sqrt{x^2 + 1}$$

. The root $x + \sqrt{x^2 + 1}$ is admissible but $x - \sqrt{x^2 + 1}$ is not, since $x - \sqrt{x^2 + 1} < 0$ because $x < \sqrt{x^2 + 1}$; however $e^y > 0$.

$$\therefore e^y = x + \sqrt{x^2 + 1}$$

$$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$$

$$\therefore \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

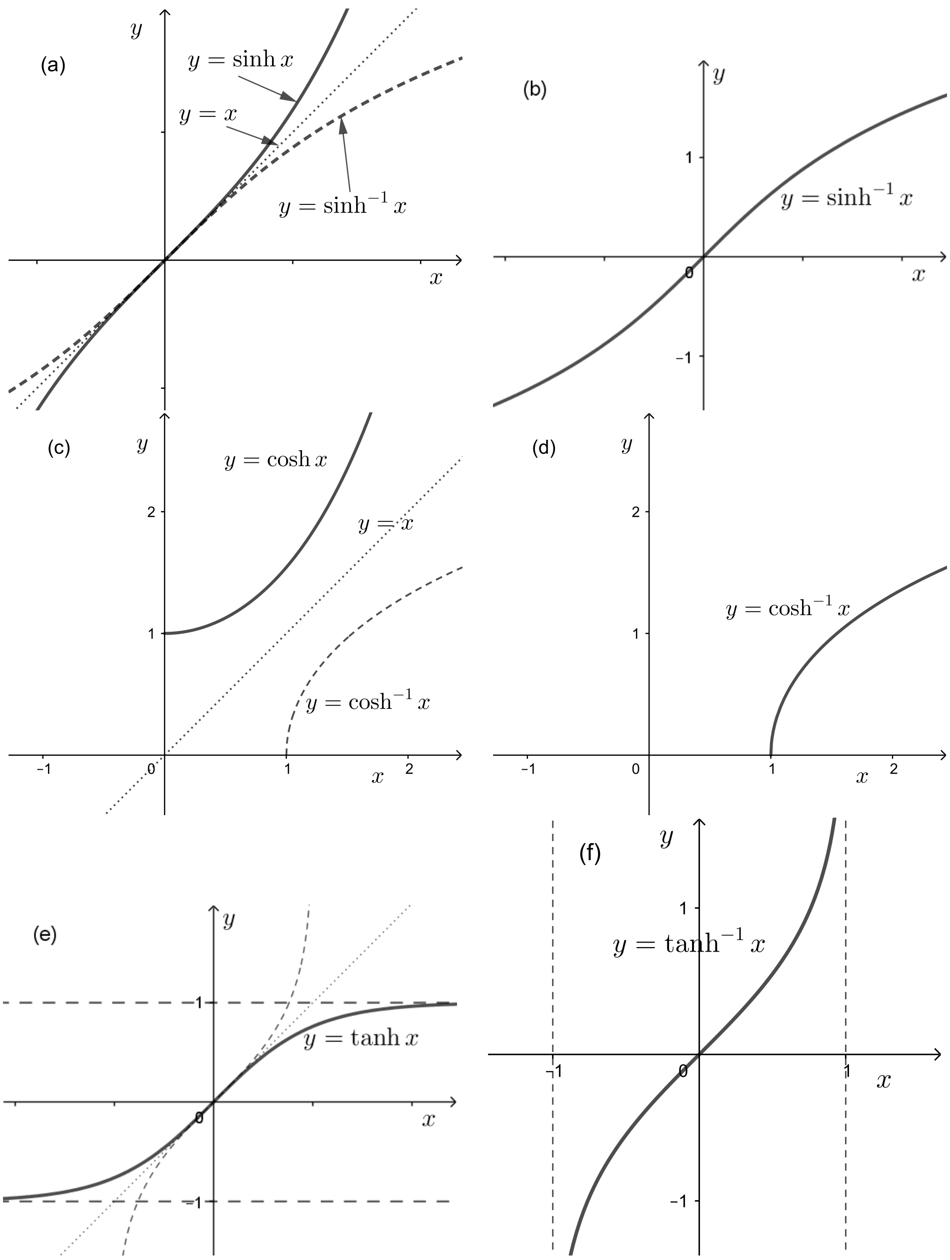

Figure 6.5: Graphs of hyperbolic functions (left panels) and their inverse functions (right panels). The dashed curves (left panels) indicate the reflection of the hyperbolic function (i.e. the inverse function) about the line $y = x$.

(2) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad [1, \infty)$
Let $y = \cosh^{-1} x \Rightarrow x = \cosh y$

$$\Longrightarrow x = \frac{e^y + e^{-y}}{2} = \frac{e^{2y}+1}{2e^y}$$
$$\Rightarrow e^{2y} - 2xe^y + 1 = 0$$
$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$
$$\therefore e^y = x \pm \sqrt{x^2-1}$$

Note that

$$(x-1) < x+1 \Rightarrow (x-1)(x-1) < (x-1)(x+1) \Rightarrow (x-1)^2 < (x^2-1)$$
$$x - 1 < \sqrt{x^2-1} \Rightarrow x - \sqrt{x^2-1} < 1$$
$$\Rightarrow y = \ln(x - \sqrt{x^2-1})$$

$\Rightarrow y = \cosh^{-1} x < 0$, since $x - \sqrt{x^2-1} < 1$.
But $\cosh^{-1} x > 0$ for all $x \geqslant 1$. So we discard $x - \sqrt{x^2-1}$ and retain $e^y = x + \sqrt{x^2-1}$.

$$\therefore y = \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

An alternative approach to discarding $x - \sqrt{x^2-1}$ is given as follows:

$$e^y = x - \sqrt{x^2-1} = (x - \sqrt{x^2-1})\frac{(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})}$$

$$\Rightarrow e^y = \frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \frac{1}{x+\sqrt{x^2-1}} = (x+\sqrt{x^2-1})^{-1}$$
$$\Longrightarrow y = \cosh^{-1} x = -\ln(x + \sqrt{x^2-1}) < 0$$

. But by definition, $\cosh^{-1} x > 0 \, \forall x \geqslant 1$, so we discard $x - \sqrt{x^2-1}$.

(3) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad (-1,1)$

Let $y = \tanh^{-1} x$

$$\Rightarrow x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^y}$$
$$\Rightarrow x = \frac{e^{2y}-1}{e^{2y}+1} \Rightarrow xe^{2y} + x = e^{2y} - 1$$
$$\Rightarrow e^{2y}(x-1) = -(x+1)$$
$$\Rightarrow e^{2y} = \frac{-(x+1)}{(x-1)} = \left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

□

Exercise. (a) Show that $\lim_{x\to\infty} \tanh(\ln x) = 1$.

(b) Simplify: $\dfrac{\cosh(\ln x) + \sinh(\ln x)}{\cosh(\ln x) - \sinh(\ln x)}$ [**Ans:** x^2]

6.4 Derivatives of hyperbolic functions

The derivatives of hyperbolic functions are easily computed because they are defined in terms of e^x and e^{-x}.

Example 32. Show that

(a) $\dfrac{d}{dx}(\sinh x) = \cosh x$

(b) $\dfrac{d}{dx}(\cosh x) = \sinh x$

(c) $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(d) $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Solution. (a) $y = \sinh x = \dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}e^x - \dfrac{1}{2}e^{-x}$

$$\frac{dy}{dx} = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x - e^{-x}}{2} = \cosh x$$

(c) $y = \tanh x = \dfrac{\sinh x}{\cosh x}$

$$\Rightarrow \frac{dy}{dx} = \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\therefore \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

(d) $y = \coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$

$$\begin{aligned}\Longrightarrow \frac{dy}{dx} &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} \\ &= -\operatorname{csch}^2 x\end{aligned}$$

6.5 Derivatives of Inverse Hyperbolic Functions

Procedure:

(a) Write the function as a log function and differentiate directly, or

(b) Use implicit differentiation.

Example 33.

$$\begin{aligned} y &= \sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x)\right] \\ &= \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{x}{\sqrt{x^2+1}}\right] \\ &= \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

Alternatively:

$$y = \sinh^{-1} x \Rightarrow \sinh y = x$$

Differentiate:

$$\begin{aligned}\cosh y \cdot \frac{dy}{dx} &= 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\cosh y}\end{aligned}$$

But $\cosh^2 y - \sinh^2 y = 1$

$$\Rightarrow \cosh^2 y = 1 + \sinh^2 y$$

$$\Rightarrow \cosh y = \sqrt{1+\sinh^2 y}$$
$$= \sqrt{1+x^2}$$
$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

Exercise. Show that

(a) $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$

(b) $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$ [**Note:** $\operatorname{sech}^2 y = 1-\tanh^2 y$]

(c) $\dfrac{d}{dx}\coth^{-1} x = \dfrac{1}{x^2-1}$

Chapter 7

Infinite Sequences

7.1 Definitions & Notation

Definition 17. A sequences is a list of numbers written in a definite order:

$$a_1, a_2, \ldots, a_n, \ldots$$

where a_1 = first term, a_2 = second term,... $a_n = n^{th}$ term.

A sequence $\{a_n\}$ may also be defined as a function $f(n) = a_n$ whose domain is the set of positive integers.

Notation

The sequence $a_1, a_2, \ldots$ may be denoted by $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$.
Some sequences have a simple definition with a formula for the n^{th} term, others are not so straightforward to define.

Example 34. 1. Simple definitions:

(i) $\left\{\dfrac{2n}{n+2}\right\}_1^{\infty}, \quad a_n = \dfrac{2n}{n+2}.$

(ii) $\left\{\dfrac{(-1)^{n+1}(n+1)}{2^n}\right\}_0^{\infty}, \quad a_n = \dfrac{(-1)^{n+1}(n+1)}{2^n}$

The sequences above may be written as shown below, displaying the first few terms:

$$\left\{\frac{2n}{n+2}\right\}_1^{\infty} = \left\{\frac{2}{3}, \frac{4}{4}, \frac{6}{5}, \frac{8}{6}, \cdots\right\}$$

$$\left\{\frac{(-1)^{n+1}(n+1)}{2^n}\right\}_0^\infty = \left\{-1, \frac{2}{2}, -\frac{3}{4}, \frac{4}{8}, \cdots\right\}$$

The terms of this sequence show alternating signs of positive and negative terms. Such a sequence is called an **alternating sequence**.

2. Not so simple definitions:
The **Fibonacci** sequence $\{f_n\}$ is defined by

$$f_1 = 1, \quad f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

The first few terms are given by $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Graphs

The first few terms of an infinite sequence may be pictured by plotting the points (a, a_n), as displayed in Figure 7.1, for some of our previous examples.

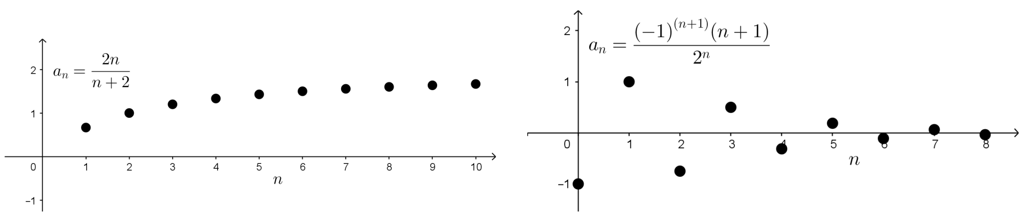

Figure 7.1:

Exercise. (1) List the first 5 terms of each of the sequence $\{a_n\}$.

(a) $a_n = \dfrac{n+1}{3n-1}$

(b) $a_n = \dfrac{3(-1)^n}{n!}$

(c) $a_n = \dfrac{(-1)^{(n+1)}2^n}{n+1}$

(d) $a_n = \sin\dfrac{n\pi}{2}$

(e) $a_1 = 3, \quad a_{n+1} = 2a_n - 1$

(f) $a_2 = 2, \quad a_{n+1} = 3a_n - 1$

(2) Find a formula for the general term of the sequence, assuming that the pattern of the first few terms continues.

(a) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$ (c) $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$

(b) $\left\{-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \ldots\right\}$ (d) $\{5, 1, 5, 1, 5, 1, \ldots\}$

7.1.1 Convergence & Divergence of a Sequence

We observe from Figure 7.1a that the sequence a_n approaches the number 2 as n gets larger and larger. In order words, we can make the difference, $2 - a_n$, as small as we like by taking n sufficiently large. This written as

$$\lim_{n\to\infty} \frac{2n}{n+2} = 2$$

Similarly, we notice that the sequence in Figure 7.1b approaches 0 as n get larger. Thus, the limit as $n \to \infty$ of that sequence is 0. A working or intuitive (or less precise) definition of the limit of a sequence is given below.

Definition 18. A sequence $\{a_n\}$ has the **limit** L, and written as

$$\lim_{n\to\infty} a_n = L,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is **convergent**. Otherwise the sequence **diverges** or is **divergent**.

We now give a more precise definition of the limit of a sequence.

Definition 19. (Precise defintion) A sequence $\{a_n\}$ **converges** and has **limit** L, written as

$$\lim_{n\to\infty} a_n = L,$$

if for every $\epsilon > 0$ there exists a positive integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

An illustration of this precise definition is displayed in Figure 7.2, showing that for any ϵ, the sequence $\{a_n\}$ must always lie between the horizontal lines $L - \epsilon$ and $L + \epsilon$ for some $n > N$. Usually, the smaller the value of ϵ, the larger the N.

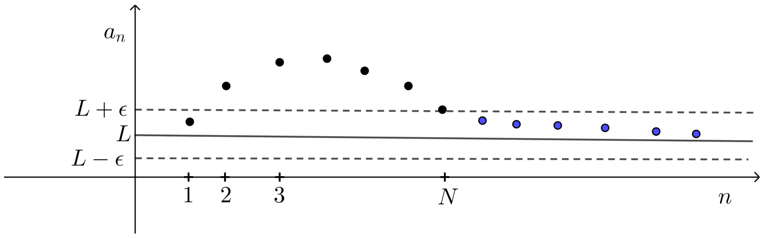

Figure 7.2:

7.1.2 Some Useful Theorems:

We notice that the precise definition of the limit of a sequence is similar to that of the limit of a function at infinity. The main difference between a function $y = f(x)$ with domain $(0, \infty)$ and that of a sequence $\{a_n\}$ defined as $f(n) = a_n$ is that n is an integer. Theorem 19 below summarizes this idea.

Theorem 19. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Example 35. Since we know that $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$, it follows by theorem 19 that

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad (r > 0)$$

Example 36. Find $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. Notice that as $n \to \infty$ both the numerator and denominator approach ∞. We cannot apply L'Hopital's rule directly since doesn't appy to sequences but to functions of real variables. But we can apply it to the related function $f(x) = \ln x/x$. Thus

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

By theorem 19, then implies that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\displaystyle\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$

4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

6. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

Theorem 20. (**Squeeze theorem**) If $a_n \leq b_n \leq c_n$ for $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$

The squeeze (or sandwich) theorem is illustrated in Figure 7.3.

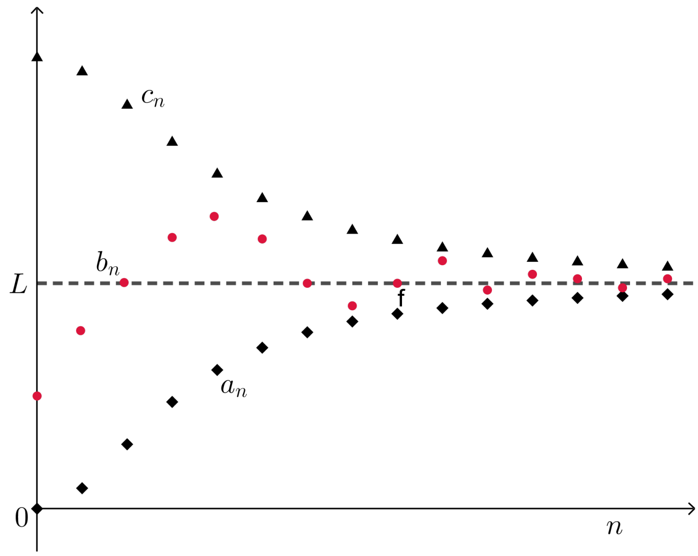

Figure 7.3:

Example 37. Find $\lim_{n\to\infty} \frac{n!}{n^n}$.

Solution. Let $a_n = n!/n^n$. The first few terms of $\{a_n\}$ are

$$a_1 = \frac{1!}{1} = 1, \quad a_2 = \frac{2!}{2} = \frac{2\cdot 1}{2\cdot 2}, \quad a_3 = \frac{3!}{3} = \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}$$

and the nth term is given by

$$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

Note that the expression in parentheses is at most 1. That is

$$\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right) \leq 1$$

$$\Longrightarrow a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right) \leq \frac{1}{n}.$$

Thus, $a_n \leq 1/n$, and because $a_n > 0$, we obtain the inequality

$$0 < a_n \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty} 1/n = 0$, the Squeeze Theorem gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Some sequences cannot be written as functions that we can take the limit of. This is especially true of the alternating sequence that we saw earlier. The next theorem will help in finding the limit of such sequences.

Theorem 21. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. We note that

$$-|a_n| \leq a_n \leq |a_n|.$$

Also, we have

$$\lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0.$$

Thus, we have $\lim\limits_{n\to\infty} (-|a_n|) = 0 = \lim\limits_{n\to\infty} |a_n|$, and so by the Squeeze Theorem, we have

$$\lim_{n\to\infty} a_n = 0.$$

□

Example 38. Find $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$, if it exists.

Solution. We have

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, by Theorem 21, we have

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

The following theorem says that, given a convergent sequence $\{a_n\}$, if we apply a continuous function to the terms of the sequence, $f(a_n)$, we again obtain a convergent sequence.

Theorem 22. If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example 39. Find $\lim\limits_{n\to\infty} \cos(\pi/n)$.

Solution. Let $a_n = \pi/n$. Then $\cos(\pi/n) = f(a_n)$ where $f(x) = \cos x$. Now $\lim\limits_{n\to\infty} \pi/n = 0$, and since the cosine function is continuous at 0, we have

$$\lim_{n\to\infty} \cos(\pi/n) = \cos\left(\lim_{n\to\infty} \pi/n\right) = \cos(0) = 1.$$

Example 40. Find $\lim\limits_{n\to\infty} e^{\sin(\pi/n)}$.

Solution. Let $a_n = \sin(\pi/n)$, then $e^{\sin(\pi/n)} = f(a_n)$ where $f(x) = e^x$. Now,

$$\lim_{n\to\infty} \sin(\pi/n) = 0$$

and since f is continuous at 0, we have

$$\lim_{n\to\infty} e^{\sin(\pi/n)} = e^{\lim_{n\to\infty} \sin(\pi/n)} = e^0 = 1$$

Example 41. Show that the sequence $\{r^n\}_0^\infty$ converges if $-1 < r \leq 1$ and diverges for all other values of r. Also show that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if}\, r = 1 \end{cases} \tag{7.1}$$

Solution. The proof/solution is given below as a series of different cases. Note that, from Calculus I, we have

$$\lim_{n\to\infty} a^x = \infty \quad \text{if} \quad a > 1, \quad \text{and} \quad \lim_{n\to\infty} a^x = 0 \quad \text{if} \quad 0 < a < 1.$$

Case 1: For $r > 1$

In this case, since $\lim_{x\to\infty} r^x = \infty$ for $r > 1$, we apply Theorem 19 to get $\lim\limits_{n\to\infty} r^n = \infty$. So the sequence diverges for $r > 1$.

Case 2: For $r = 1$

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1.$$

So the sequence converges for $r = 1$ and the limit is equal to 1.

Case 3: For $0 < r < 1$
From Calculus I, $\lim_{x\to\infty} r^x = 0$ if $0 < r < 1$, so by Theorem 19, we get $\lim\limits_{n\to\infty} r^n = 0$. So the sequence converges if $0 < r < 1$ and the limit is equal to 0.

Case 4: For $r = 0$

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$$

So the sequence converges for $r = 0$, and the limit is equal to 0.

Case 5: For $-1 < r < 0$
Note that if $-1 < r < 0$ then $0 < |r| < 1$. Thus, we have

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = \left(\lim_{n\to\infty} |r|\right)^n = 0^n = 0.$$

Hence, by Theorem 21, we have $\lim\limits_{n\to\infty} r^n = 0$. So the sequence converges and the limit is equal to 0.

Case 6: For $r = -1$
In this case we get the alternating sequence

$$\{r^n\}_0^\infty = \{(-1)^n\}_0^\infty = 1, -1, 1, -1, \cdots$$

Thus, the limit $\lim\limits_{n\to\infty} (-1)^n$ does not exist. So the sequence diverges for $r = -1$. Recall that Theorem 21 only applies if $|a_n| = 0$ as $n \to \infty$. We cannot use it if $\lim_{n\to\infty} |a_n| \neq 0$. For instance $\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} |-1|^n = \lim_{n\to\infty} 1^n = 1$ **does not** imply that the sequence $(-1)^n$ converges to 1.

Case 7: For $r < -1$
In this case we obtain an alternating sequence for every value of r as in Case 6. For instance, if $r = -3$ we get the alternating sequence

$$\{(-3)^n\}_0^\infty = \{1, -3, 9, -27, 81, \cdots\}$$

whose values keeps getting larger and larger as n increases. Thus, the sequence diverges.

Example 42. Determine if the sequence converges or diverges, and find the limit if it converges.

(a) $\left\{\dfrac{2n^2+3}{3n-7n^2}\right\}_{n=2}^\infty$

(b) $\left\{\dfrac{e^{2n}}{n^2}\right\}_{n=1}^{\infty}$

Solution. (a) $\left\{\dfrac{2n^2+3}{3n-7n^2}\right\}_{n=2}^{\infty}$

$$\lim_{n\to\infty}\frac{2n^2+3}{3n-7n^2}=\lim_{n\to\infty}\frac{2+\frac{3}{n^2}}{\frac{3}{n}-7}=-\frac{2}{7}$$

Therefore the sequence converges, and the limit is $-2/7$.

(b) $\left\{\dfrac{e^{2n}}{n^2}\right\}_{n=1}^{\infty}$

We need to find $\lim_{n\to\infty}\dfrac{e^{2n}}{n^2}$, by applying L'Hopital's rule. But, the rule can not be applied to sequences so we define the related function

$$f(x)=\frac{e^{2x}}{x^2},\quad \text{such that}\quad f(n)=\frac{e^{2n}}{n^2}.$$

Thus, using Theorem 19, we have

$$\lim_{n\to\infty}\frac{e^{2n}}{n^2}=\lim_{x\to\infty}\frac{e^{2x}}{x^2}$$

Applying L'Hopital's rule twice, we have

$$\lim_{n\to\infty}\frac{e^{2n}}{n^2}=\lim_{x\to\infty}\frac{2e^{2x}}{2x}=\lim_{x\to\infty}\frac{2e^{2x}}{1}=\infty$$

Therefore, the sequences diverges.

7.2 Bounded Monotonic Sequences

Definition 20. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. It is called **monotonic** if it is either increasing or decreasing.

Example 43. Show that the sequence $\left\{\dfrac{n}{n+1}\right\}$ is increasing.

Solution. Let $a_n = \dfrac{n}{n+1}$. We need to show that $a_n < a_{n+1}$ for all $n \geq 1$. That is

$$\frac{n}{n+1} < \frac{n+1}{(n+1)+1}$$

$$\implies \frac{n}{n+1} < \frac{n+1}{n+2}$$

$$\implies n(n+2) < (n+1)^2$$

$$\implies n^2 + 2n < n^2 + 2n + 1$$

$$\implies 0 < 1$$

which is true for $n \geq 1$. Thus $a_n < a_{n+1}$, and so a_n is increasing.
Alternatively, let $a_n = f(n) = n/(n+1)$, and consider the function

$$f(x) = \frac{x}{x+1}$$

$$\implies f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

Thus, $f'(x) > 0$ if $x > 0$, and so f is increasing on $(0, \infty)$. Therefore, the sequence is increasing.

Definition 21. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all} \quad n \geq 1.$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all} \quad n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example 44. Determine if the following sequence is monotonic and/or bounded

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

Solution. (a) $\left\{-n^2\right\}_{n=0}^{\infty}$
The sequence is decreasing, and therefore monotonic, since

$$-n^2 > -(n+1)^2 \quad \text{for every} \quad n \geq 0.$$

Note that the terms of the sequence will either be zero or negative. Thus, the sequence is bounded above by 0.

Remark. Any positive number or zero may be chosen as the upper bound, say M, but it is standard to choose the smallest possible bound, also called the **supremum**. That is why we chose $M = 0$, since $-n^2 \leq 0$ for every n. Note that this sequence is not bounded below since we can always choose another number below any potential lower bound.
On the other hand, the sequence $\{n^2\}_0^\infty$ is increasing from 0, and therefore bounded below by 0. Here, any negative number qualifies as a lower bound, say m, but it's standard to choose the largest possible bound, also called the **infimum**. So the infimum or lower bound is $m = 0$.

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$
The sequence is an alternating sequence, with the terms alternating between 1 and -1. Thus, the sequence is neither increasing nor decreasing, and so the sequence is not monotonic. However, the sequence is bounded above by 1 and bounded below by -1.

The next theorem shows that if a sequence is both bounded and monotonic, then it must be convergent. Note that not every bound sequence is convergent. For example, the sequence $\{(-1)^n\}$, is bounded as $-1 \leq a_n \leq 1$ but it is divergent. Also, not every monotonic sequence is bounded, for example $a_n = n$ is monotonic but $a_n = n \to \infty$.

Theorem 23. (Monotonic Sequence Theorem) Every bounded, monotonic sequence is convergent.

Example 45. Show that $\left\{\dfrac{2^n}{n!}\right\}$ is convergent and find its limit.

Solution. Let $a_n = \dfrac{2^n}{n!}$. The first few terms of the sequence are

$$a_1 = 2, \quad a_2 = \frac{4}{2} = 2, \quad a_3 = \frac{8}{6} = 1.3333, \quad a_4 = \frac{16}{24} = 0.6667$$

$$a_5 = \frac{32}{120} = 0.2667, \quad a_6 = \frac{64}{720} = 0.0889, \quad \cdots, \quad a_{10} = \frac{1024}{3628800} = 0.0003$$

Thus, the sequence appears to be decreasing from $n = 2$ onwards. To prove this, we need to show that $a_n \geq a_{n+1}$. That is $\dfrac{a_{n+1}}{a_n} \leq 1$. Now

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^n \cdot 2 \cdot n!}{(n+1) \cdot n! \cdot 2^n}$$

$$\Longrightarrow \frac{a_{n+1}}{a_n} = \frac{2}{n+1}$$

So we have

$$\frac{a_{n+1}}{a_n} \leq 1 \quad \text{if} \quad n \geq 1.$$

Therefore $a_n \geq a_{n+1}$ if $n \geq 1$. Since all the terms of the sequence are positive and the sequence is decreasing, $\{a_n\}$ is bounded below by 0. Hence, by Theorem 23, the sequence is convergent.
We next find the limit of the sequence, using

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+1}$$

$$\Longrightarrow a_{n+1} = \frac{2}{n+1}a_n$$

Let $\lim_{n\to\infty} a_n = L$ then, obviously, $\lim_{n\to\infty} a_{n+1} = L$. Thus, we have

$$L = \lim_{n\to\infty} a_{a+1} = \lim_{n\to\infty} \frac{2}{n+1}a_n = \lim_{n\to\infty} \frac{2}{n+1} \cdot \lim_{n\to\infty} a_n = 0 \cdot L = 0$$

Hence $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Exercise. 1. Find

(a) $\lim_{n\to\infty} \frac{3+5n^2}{n+n^2}$ (b) $\lim_{n\to\infty} \frac{\ln n}{n}$ (c) $\lim_{n\to\infty} \frac{(-1)^n}{n}$

2. Determine whether the given sequence converges or diverges. If it converges then find the limit.

(a) $\left\{\frac{3n^2-5n+3}{1+8n-3n^2}\right\}_{n=3}^{\infty}$

(b) $\left\{\frac{(-1)^{n-2}n^2}{4+n^3}\right\}_{n=0}^{\infty}$

(c) $\left\{\frac{e^{4n}}{4-e^{2n}}\right\}_{n=1}^{\infty}$

(d) $\left\{\frac{\ln(n+2)}{\ln(1+4n)}\right\}_{n=1}^{\infty}$

(e) $\{\sin(n\pi)\}_{n=0}^{\infty}$

(f) $\left\{\frac{n+1}{\ln(5n)}\right\}_{n=2}^{\infty}$

(g) $\left\{\cos\left(\frac{3}{n+1}\right)\right\}_{n=1}^{\infty}$

(h) $\{\ln(4n+1) - \ln(2+7n)\}_{n=0}^{\infty}$

(3) Consider the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for} \quad n = 1, 2, 3, \cdots$$

(i) Use mathematical induction to show that the sequence is increasing.

(ii) Use mathematical induction to show that the sequence is bounded above by 6, that is, $a_n < 6$ for all n.

(iii) Use the Monotonic Sequence Theorem to show that the sequence converges, and that the limit is 6.

(4) Show that the sequence

$$\left\{ \frac{n^3}{n^4 + 10000} \right\}_{n=0}^{\infty}$$

is not monotonic.

Chapter 8

Infinite Series

8.1 Introduction

8.1.1 Definitions & Notation

Definition 22. An infinite series (or a series) is the sum of the terms of an infinite sequence of the form

$$a_1 + a_2 + \ldots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n.$$

Each number a_k is a **term** of the series, and a_n is the **nth term**.

Remark. Let's start by removing some confusion between the concept of a series and that of a sequence. A sequence is a list or collection of numbers written in a particular order, whereas a series is an expression of an **infinite sum** of numbers.

Since only finite sums may be added algebraically, what do we mean by an "infinite sum"? As the following example illustrates, and as we shall explain later on, the key to the definition is to consider the **sequence of partial sums** $\{S_n\}$, where S_k is **the sum of the first** k **numbers of the infinite series**. Consider the following infinite series

Example 46. $0.6 + 0.06 + 0.006 + 0.0006 + 0.00006 + \cdots$

$$\begin{aligned} S_1 &= 0.6 \\ S_2 &= 0.6 + 0.06 = 0.66 \\ S_3 &= 0.6 + 0.06 + 0.006 = 0.666 \\ S_4 &= 0.6 + 0.06 + 0.006 + 0.0006 = 0.6666 \\ S_5 &= 0.6 + 0.06 + 0.006 + 0.0006 + 0.00006 = 0.66666 \end{aligned}$$

and so on. Thus, the sequence of partial sums $\{S_n\}$ may be written as

$$0.6, 0.66, 0.666, 0.6666, 0.66666, c \ldots.$$

It follows that (we'll appreciate this conclusion later under "Geometric series")

$$S_n \to \frac{2}{3} \quad \text{as} \quad n \to \infty.$$

We see that the more numbers of the infinite series that we add, the closer the sum gets to 2/3. Thus, we write

$$\frac{2}{3} = 0.6 + 0.06 + 0.006 + 0.0006 + \cdots$$

and call $\frac{2}{3}$ the *sum* of the infinite series.

8.2 Convergence and Divergence of a Series

Consider the **partial sums** of the series: $a_1 + a_2 + \ldots + a_n + \ldots$

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

Note that the partial sums $\{s_1, s_2, s_3, \ldots, s_n\}$ now form a sequence.

Definition 23. (**Convergence & Divergence of Series**) A series $\sum a_n$ is **convergent** (or **converges**) if its sequence of partial sums $\{s_n\}$ converges; that is, if

$$\lim_{n\to\infty} s_n = s \quad \text{for some real number} \quad s.$$

The limit s is the **sum** of the series $\sum a_n$, and we write

$$s = a_1 + a_2 + \cdots + a_n + \cdots .$$

The series $\sum a_n$ is **divergent** (or **diverges**) if $\{s_n\}$ diverges. A divergent series has no sum.

Remark. The sum of an infinite series means that by adding sufficiently many terms, we get as close as we like to the number s.

Example 47. Consider the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots ,$$

(a) find s_1, s_2, s_3, s_4, s_5, and s_6

(b) find s_n

(c) show that the series converges and find its sum.

Solution. (a) The first six partial sums are given by

$$\begin{aligned}
s_1 &= \frac{1}{1\cdot 2} = \frac{1}{2}\\
s_2 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}\\
s_3 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}\\
s_4 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}\\
s_5 &= s_4 + a_5 = \frac{4}{5} + \frac{1}{5\cdot 6} = \frac{5}{6}\\
s_6 &= s_5 + a_6 = \frac{5}{6} + \frac{1}{6\cdot 7} = \frac{6}{7}
\end{aligned}$$

(b) From the first six partial sums, it is straightforward to deduce that the nth partial sum is given by the expression $s_n = \frac{n}{n+1}$. However, we shall derive this in a different way. Using partial fractions, we can show that the nth term is given by

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Thus, the nth partial sum of the series may be written as

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right). \\ &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right). \end{aligned}$$

We see that all the numbers except the first and last terms cancel, and so we get

$$s_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Remark. This series is an example of a **telescoping series**, because of the cancellation nature of the terms of the series. That is, writing the partial sums as shown above causes the terms to *telescope* to $1 - \frac{1}{n(n+1)}$. Note that the terms of different telescoping series cancel out in different ways, not just the order shown in our example above.

(c) Using the expression for s_n obtained above, we get

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Thus, the series converges and has a sum of 1. Hence, we can write

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots.$$

Example 48. Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \cdots + (-1)^{n-1} + \cdots,$$

(a) find s_1, s_2, s_3, s_4, s_5, and s_6

(b) find s_n

(c) show that the series diverges.

Solution. (a) The first six partial sums are given by

$$s_1 = 1, \quad s_2 = 0, \quad s_3 = 1, \quad s_4 = 0, \quad s_5 = 1, \quad \text{and} \quad s_6 = 0.$$

(b) The nth partial sum may be written as

$$s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{8.1}$$

(c) Since the sequence of partial sums $\{s_n\}$ oscillate between 1 and 0, it follows that $\lim_{n\to\infty} s_n$ does not exist. Hence the series diverges.

8.3 Special Series

The Geometric Series

Theorem 24. The geometric series

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

(a) converges and has the sum $s = \dfrac{a}{1-r}$ if $|r| < 1$

(b) diverges if $|r| \geq 1$.

Proof. If $r = 1$, then we have

$$a + a + a + \ldots + a = na$$

and $\lim_{n\to\infty} na = \pm\infty$, so the series will diverge.
If $r \neq 1$, we get

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} \tag{8.2}$$

$$rs_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n \tag{8.3}$$

$$s_n - rs_n = a - ar^n \qquad [(8.2) - (8.3)]$$

Thus, we get

$$\begin{aligned} s_n &= \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} + \frac{ar^n}{1-r} \\ \Longrightarrow \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a}{1-r} + \lim_{n\to\infty} \frac{ar^n}{1-r} \\ &= \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r} \end{aligned}$$

If $-1 < r < 1$, $\lim_{n\to\infty} r^n = 0$ and so

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

If $r \leq -1$ or $r > 1$, $\{r^n\}$ diverges and so $\lim_{n\to\infty} s_n$ does not exit. $\square$

In summary, the sum of the geometric series is:

$$\boxed{\begin{array}{l} \displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad |r| < 1. \\ \text{If} \quad |r| \geq 1, \text{ the series diverges.} \end{array}}$$

Example 49. Prove that the following series converges, and find its sum:

$$2 + \frac{2}{3} + \frac{2}{3^2} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

Solution. The series is geometric with $a = 2$ and $r = \dfrac{1}{3} < 1$. Thus, the series converges to the sum

$$s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3.$$

Example 50. Express the number $0.\bar{2} = 0.2222\ldots$ as a ratio of integers.

Solution.

$$0.\bar{2} = \frac{2}{10} + \frac{2}{10^2} + \cdots$$

is a geometric series with $a = 2/10$, $r = 1/10$ and the sum is

$$s = \frac{a}{1-r} = \frac{2/10}{1-1/10} = \frac{2}{9}.$$

Exercise. (1) Find the sum of the geometric series $3 + 2 + \dfrac{4}{3} + \dfrac{8}{9} + \ldots$

(2) Prove that the following series converges, and find its sum

$$0.6 + 0.06 + 0.006 + 0.0006 + 0.00006 + \cdots$$

Note: this was the series we considered at the beginning of this chapter.

The harmonic series

Prove that the the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots +$$

is divergent.

Solution.

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
&\vdots \\
s_{32} &> 1 + \frac{5}{2} \\
s_{64} &> 1 + \frac{6}{2}
\end{aligned}$$

In general $s_{2^n} > 1 + \dfrac{n}{2}$ for every positive integer n. This implies that s_n can be made as large as desired by taking n sufficiently large. That is, $\lim\limits_{n\to\infty} s_{2^n} = \infty$, which implies that $\{s_n\}$ is divergent. The harmonic series is therefore divergent.

Exercise. (1) Determine whether the series is convergent or divergent:

$$(a) \quad \sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} \qquad (b)$$

(2) Express the number as a ratio of integers

$$3.\overline{417} = 3.417417417\cdots$$

8.4 Some Useful Theorems

Theorem 25. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Proof. Let $s_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$. Then

$$\begin{aligned} s_n &= s_{n-1} + a_n \\ \Rightarrow a_n &= s_n - s_{n-1} \\ \Rightarrow \lim_{n\to\infty} a_n &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \end{aligned}$$

Let $\lim_{n\to\infty} s_n = s$. Then $\lim_{n\to\infty} s_{n-1} = s$ since $\lim_{n\to\infty} n - 1 = \infty$. Thus

$$\lim_{n\to\infty} a_n = s - s = 0$$

$\square$

Remark. The converse is not true in general; that is, if $\lim_{n\to\infty} a_n = 0$, it does not necessarity follow that the series $\sum a_n$ is convergent. An example is the harmonic series for which $\lim_{n\to\infty} a_n = 0$ even though the series is divergent. **Thus, to establish convergence of a series, it is not enough to prove that $\lim_{n\to\infty} a_n = 0$, since that may be true for divergent as well as for convergent series.**

8.4.1 The Divergence Test

(i) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

(ii) If $\lim_{n\to\infty} a_n = 0$, then further investigation is necessary to determine whether the series $\sum a_n$ is convergent or divergent.

The following examples demonstrate how to apply the divergence test.

Example 51. Determine if the following series is divergent

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{5n+1}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Solution. (1) $\displaystyle\lim_{n\to\infty} \frac{2n}{5n+1} = \lim_{n\to\infty} \frac{2}{5+\frac{1}{n}} = \frac{2}{5} \neq 0.$
Thus, by the test for divergence, the series is divergent.

(2) $\displaystyle\lim_{n\to\infty} \frac{e^n}{n^2} = \infty.$
So the series diverges by the divergence test.

(3) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. We cannot draw any conclusion, further investigation is required. Later in the text, it will be simple to show that the series is divergent.

(4) $\lim_{n\to\infty} \frac{1}{n^3} = 0$. Further investigation is required, by the test for divergence. In a short while, it will be relatively easy to show that this series is convergent.

Theorem 26. If $\sum a_n$ and $\sum b_n$ are convergent then so are

$$(a) \quad \sum ca_n = c\sum a_n$$
$$(b) \quad \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$$

Example 52. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{n(n+1)} + \frac{2}{3^n} \right)$

Solution. We can write the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \frac{1}{3^n}$. However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$ so

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 2 \cdot \frac{1}{2} = 1.$$

In example 47, we showed that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Thus

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 5 \cdot 1 = 5.$$

Therefore the sum of the series is given by

$$\sum_{n=1}^{\infty} \left(\frac{5}{n(n+1)} + \frac{2}{3^n} \right) = \sum_{n=1}^{\infty} \frac{5}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 1 + 5 = 6.$$

Theorem 27. If $\sum a_n$ is a convergent series and $\sum b_n$ is divergent, then $\sum (a_n + b_n)$ is divergent.

Example 53. Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{7}{n} \right)$$

Solution. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, since it is a geometric series with $r = 1/3$. However, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series. Therefore the given series is divergent by Theorem 27.

Exercise.

1. Show that the series $\dfrac{n^2}{5n^2 + 4}$ is divergent.

2. Prove that the following series converges, and find its sum:

$$\sum_{n=1}^{\infty} \left[\frac{7}{n(n+1)} + \frac{2}{3^{n-1}} \right]$$

3. Find the sum of the series: $\displaystyle\sum_{n=0}^{\infty}\left(4^n x^n + \frac{\cos^n x}{2^n}\right)$

The next theorem states that if corresponding terms of two series are identical after a certain term, then both series converge or both series diverge.

Theorem 28. If $\sum a_n$ and $\sum b_n$ are series such that $a_j = b_j$ for every $j > k$, where k is a positive integer, then both series converge or both series diverge.

Proof. We may write the series $\sum a_n$, by hypothesis, as

$$\sum a_n = a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} + \cdots + a_n + \cdots$$

Let S_n and T_n denote the nth partial sums of $\sum a_n$ and $\sum b_n$ respectively. The partial sums of $\sum a_n$ are given by

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_k &= a_1 + a_2 + \cdots + a_k \\ S_n &= S_k + a_{k+1} + \cdots + a_n \\ \Longrightarrow S_n - S_k &= a_{k+1} + \cdots + a_n \end{aligned}$$

Similarly, we have

$$\begin{aligned} T_n &= T_k + a_{k+1} + \cdots + a_n \\ \Longrightarrow T_n - T_k &= a_{k+1} + \cdots + a_n, \end{aligned}$$

since $a_j = b_j$ for $j > k$. Thus,

$$S_n - S_k = T_n - T_k$$

$$\Longrightarrow S_n = T_n + (S_k - T_k)$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n + (S_k - T_k)$$

and so either both of the limits exist or both do not exists. If both series converge, then their sum differ by $S_k - T_k$. □

Theorem 28 shows that if we change a finite number of terms of a series, it will have no effect on whether the series would converge or diverge. However, it does change the value or sum of a convergent series. For instance, if we replace the first k terms of $\sum a_n$ by 0, it won't affect the convergence of the series. The above theorem is re-stated differently in the next theorem.

Theorem 29. For any positive integer k, the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \quad \text{and} \quad \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \cdots$$

either both converge or both diverge.

Example 54. Show that the following series converges

$$\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+2)(n+3)} + \cdots \tag{8.4}$$

Solution. Consider the telescoping series of example 47, repeated below:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)} + \cdots$$

By deleting the first two terms, we get the series in 8.4. Since the telescoping series is convergent, by Theorem 29, the given series is also convergent.

8.5 Convergence of Positive-Term Series

In this section, we consider only **positive-term series** where the series $\sum a_n$ is such that $a_n > 0$ for every n. The convergence or divergence of an arbitrary series can often be determined from that of a related positive-term series. Thus, positive-term series a the foundation for several test for convergence of any given series.

8.5.1 The Integral Test

Consider a function f which is continuous, positive and decreasing on the interval $[1, \infty)$. The area under f is given by $A = \int_1^{\infty} f(x)\mathrm{d}x$. If this

improper integral is convergent, then A is finite. If the integral is divergent then A is infinite.
Now, if we divide the area under f into rectangular strips, each of which is taken to represent a term of the sequence $a_n = f(n)$, then since $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is the sum of these strips, the series $\sum_{n=1}^{\infty} a_n$ is convergent if $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is convergent and $\sum_{n=1}^{\infty} a_n$ is divergent if $\int_1^{\infty} f(x)\mathrm{d}x$ is divergent. Thus, we get the integral test stated below:

Theorem 30. (**Integral Test**) Suppose f is a continuous, positive decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is
convergent if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Remark. When using the integral test,

(a) it is important to verify the conditions of the test

(b) it is not necessary to start the series or the integral at $n = 1$.

(c) it is not necessary that f be always decreasing. What is important is that f is ultimately or eventually decreasing (i.e. decreasing for x larger than some number N).

Example 55. Use the integral test to prove that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent.

Solution. Since $a_n = 1/n$, we let $f(n) = 1/n$ and replace n by x to get the function $f(x) = 1/x$. Because f is positive-valued, continuous, and decreasing for $x \geq 1$, we can apply the integral test to compute

$$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} [\ln x]_1^t$$
$$= \lim_{t\to\infty} [\ln t - \ln 1] = \infty$$

Therefore the series diverges by the integral test.

Example 56. Determine whether the infinite series $\sum ne^{-n^2}$ converges.

Solution. Now $a_n = ne^{-n^2}$, so we consider the function $f(x) = xe^{-x^2}$. If $x \geq 1$, f is positive-valued and continuous. We also need to show that the function is eventually decreasing. Finding the first derivative, we have

$$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}(1 - 2x^2) < 0$$

Therefore f is decreasing on $[1, \infty]$. Thus, we can apply the integral test to compute

$$\int_1^\infty xe^{-x^2} dx = \lim_{t\to\infty} \int_1^t xe^{-x^2} dx = \lim_{t\to\infty} \left[\left(-\frac{1}{2}\right) e^{-x^2} \right]_1^t$$
$$= \left(-\frac{1}{2}\right) \lim_{t\to\infty} \left[\frac{1}{e^{t^2}} - \frac{1}{e} \right] = \frac{1}{2e}.$$

Therefore the series converges by the integral test.

Example 57. Use the Integral Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. We have $a_n = (\ln n)/n$, so we consider $f(x) = (\ln x)/x$. We note that the function f is continuous and positive on the interval $[1, \infty]$. To show that it's decreasing, we compute

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$
$$\Longrightarrow f'(x) < 0 \quad \text{for} \quad \ln x > 1 \Longrightarrow x > e$$

Thus, the function is decreasing on the interval $(e, \infty]$ or we can also use $[3, \infty]$ since $e \approx 2.7$. Using the integral test we have

$$I = \int_3^\infty \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_3^t \frac{\ln x}{x} dx$$

Let $u = \ln x \implies du = (1/x)dx$

$$\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2}u^2$$

$$\implies I = \lim_{t\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_3^t = \frac{1}{2}\lim_{t\to\infty}(\ln t)^2 - (\ln 3)^2 = \infty$$

Thus, the series diverges by the integral test.

Exercise. Test the series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

The p-series

Example 58. For what values of p is the following series, called a *p-series*, convergent?

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Solution. If $p < 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
If $p = 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$ so the series diverges by the test for divergence.
If $p > 0$, the function $f(x) = \frac{1}{x^p}$ is continuous, positive and decreasing on $[1, \infty)$. But recall that $\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x$ converges if $p > 1$ and diverges if $p \leq 1$.

We just proved the following theorem

Theorem 31. (**p-series**) The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example 59. Determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Solution. (i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is a p-series with $p = 3/2 > 1$, therefore the series converges.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

The series is a p-series with $p = 1/2 < 1$, hence the series is divergent.

Comparison Tests

The idea behind this test is to compare a given series, say $\sum_{n=1}^{\infty} a_n$, to a second series, say $\sum_{n=1}^{\infty} b_n$ (which is known to either converge or diverge).
If the terms of $\sum_{n=1}^{\infty} a_n$ are smaller than those of $\sum_{n=1}^{\infty} b_n$ (and it converges), then $\sum_{n=1}^{\infty} a_n$ also converges.
If the terms of $\sum_{n=1}^{\infty} a_n$ are larger than those of $\sum_{n=1}^{\infty} b_n$ (and it diverges) then $\sum_{n=1}^{\infty} a_n$ also diverges. Thus, we formally have:

8.5.2 The Comparison Test

Theorem 32. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is also convergent.

(ii) If $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is also divergent.

Example 60. Determine whether the series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7^n + 3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n} - 1}$

Solution. (a) Comparing the given series to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7^n}$, we have

$$\frac{1}{7^n + 3} < \frac{1}{7^n} = \left(\frac{1}{7}\right)^n \quad \text{for all} \quad n \geq 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7^n}$ is a geometric series with $r = 1/7 < 1$ therefore it is convergent. Hence, by the comparison test, the given series is convergent.

(b) Comparing the given series to the p-series $\displaystyle\sum \frac{1}{\sqrt{n}}$, we have

$$\frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}} \quad \text{for} \quad n \geq 2$$

$$\Longrightarrow \frac{5}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}} \quad \text{for} \quad n \geq 2$$

Since the series $\sum 1/\sqrt{n}$ is a divergent p-series, the given series is also divergent.

Example 61. Use the comparison test to determine whether the series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Remark. If the comparison test fails, a related test which may also be used is the limit comparison test; stated below.

8.5.3 The Limit Comparison Test

Theorem 33. (**Limit Comparison Test**) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof. This proof relies heavily on the Comparison Test, so go back and read it if you haven't yet. Let m and M be positive numbers such that $m < c < M$. If $\lim_{n\to\infty}(a_n/b_n) = c > 0$, then this implies that a_n/b_n is close to c for large n. Hence there exists an integer N such that

$$m < \frac{a_n}{b_n} < M \quad \text{whenever} \quad n > N$$

$$\implies \quad mb_n < a_n < Mb_n \quad \text{whenever} \quad n > N$$

If the series $\sum b_n$ is convergent, then the series $\sum Mb_n$ is also convergent by Theorem 26. Since $a_n < Mb_n$, the series $\sum a_n$ must converge by the Comparision Test. Also, if the series $\sum b_n$ is divergent, so is the series $\sum mb_n$. Since $mb_n < a_n$, the series $\sum a_n$ is divergent by the Comparison Test. □

Example 62. Determine whether the series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+3n}{2^n(n^2+1)}$

Solution. (a) Let

$$a_n = \frac{1}{\sqrt[3]{n^2+1}},$$

and consider $b_n = 1/\sqrt[3]{n^2}$, which is a p-series with $p = 2/3 < 1$ and therefore divergent. The term b_n is obtained by deleting the 1 in the denominator of a_n. Using the Limit Comparison Test, we compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+1}} \cdot \frac{\sqrt[3]{n^2}}{1} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+1}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+1}} = 1 > 0$$

Since $\sum b_n$ is divergent, the given series is also divergent by the Limit Comparison Test.

Remark. Note that we cannot use the basic Comparison Test in this case because

$$\frac{1}{\sqrt[3]{n^2+1}} < \frac{1}{\sqrt[3]{n^2}},$$

and since $\sum b_n$ diverges, we cannot draw any conclusion based on that test.

(b) Let

$$a_n = \sum_{n=1}^{\infty} \frac{5n^2+3n}{2^n(n^2+1)}.$$

By removing the terms of least magnitude in the numerator and denominator of a_n, we get

$$\frac{5n^2}{2^n n^2} = \frac{5}{2^n}$$

and so we choose $b_n = 1/2^n$. The series $\sum b_n$ is a convergent geometric series with $r = 1/2 < 1$. Applying the Limit Comparison Test, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^2+3n}{2^n(n^2+1)} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{5n^2+3n}{n^2+1} = 5 > 0$$

Since $\sum b_n$ is convergent, the series $\sum a_n$ is also convergent by the Limit Comparison Test.

Exercise. Test the series for convergence or divergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$

8.6 Alternating Series

An alternating series is a series whose terms are alternately positive and negative. E.g.:

$$(a)\quad 1-2+3-4+5+\ldots = \sum_{n=1}^{\infty}(-1)^{n-1}n$$

$$(b)\quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

If the terms of an alternating series decrease toward 0 in absolute value, then the series converges. Explicitly, we have

Theorem 34. (**The Alternating Series Test**) If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots \qquad (b_n > 0)$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$ then the series is convergent.

Example 63. Determine whether the alternating series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}$

Solution. (a) Let

$$a_n = \frac{2n}{4n^2-3}$$

By the Alternating Series Test, we first need to show that $a_k \geq a_{k+1}$ for every positive integer k. That is, we have to show that a_n is ultimately decreasing. Secondly, we also need to show that $\lim_{n\to\infty} a_n = 0$. One method of showing that a_n is decreasing is to define

$$f(x) = \frac{2x}{4x^2-3}$$

and prove that f is a decreasing function. Now,

$$f'(x) = \frac{(4x^2-3)(2)-(2x)(8x)}{(4x^2-3)^2} = \frac{-8x^2-6}{(4x^2-3)^2} = \frac{-(8x^2+6)}{(4x^2-3)^2} < 0$$

Thus, $f(x)$ is decreasing and therefore $f(k) \geq f(k+1)$; and hence $a_k \geq a_{k+1}$ for every $k > 0$.
Alternatively, we can also prove $a_k \geq a_{k+1}$ by showing that $a_k - a_{k+1} \geq 0$. That

$$a_k = a_{k+1} = \frac{2k}{4k^2-3} - \frac{2(k+1)}{4(k+1)^2-3}$$

$$= \frac{8k^2+8k+6}{(4k^2-3)(4k^2+8k+1)} \geq 0$$

for every positive integer k.A third method is to prove that $a_{k+1}/a_k \leq 1$. Now

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{4n^2-3} = \lim_{n\to\infty}\frac{\frac{2}{n}}{4-\frac{3}{n^2}} = 0$$

Thus, the series converges by the Alternating Series Test.

(b) Let

$$a_n = \frac{2n}{4n-3}$$

In this case, there's no need to show that $a_k \geq a_{k+1}$ since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n-3} = \lim_{n\to\infty} \frac{2}{4-\frac{3}{n}} = \frac{1}{2} \neq 0$$

implies that the second condition is not satisfied. This computation also shows that

$$\lim_{n\to\infty} (-1)^{n-1} \frac{2n}{4n-3}$$

does not exist, and so the series diverges, by the divergence test (see section 8.4.1).

Exercise. Test the series for convergence or divergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} 3n}{4n-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

8.7 Absolute Convergence, the Ratio & the Root Tests

Definition 24. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Example 64. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ is absolutely convergent. Why?

Definition 25. A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 65. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but not absolutely convergent. Why?

Theorem 35. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Note that the following inequality, which we have seen before, is true

$$-|a_n| \leq a_n \leq |a_n|.$$

$$\Longrightarrow 0 \leq a_n + |a_n| \leq 2|a_n|,$$

obtained by adding $|a_n|$ to both sides of the first inequality. Now let $b_n = a_n + |a_n|$, then

$$0 \leq b_n \leq 2|a_n|$$

If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent and therefore $\sum 2|a_n|$ is also convergent by Theorem 26. Since $b_n \leq 2|a_n|$, the series $\sum b_n$ is convergent by the Comparison Test. From $b_n = a_n + |a_n|$ we have $\sum a_n = \sum b_n - \sum |a_n|$, and so $\sum a_n$ is convergent by Theorem 26. □

Example 66. Determine whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

Solution. Note that the series is not an alternating series; the signs change irregularly. Consider

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

But the series $\sum 1/n^2$ is a convergent p-series with $p = 2$, therefore $\sum |\cos n|/n^2$ is convergent by the Comparison Test. Thus, the given series is absolutely convergent and therefore convergent by Theorem 35.

8.7.1 The Ratio Test

Theorem 36. (**Ratio Test**)

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Remark. Note that if the series has positive terms, the ratio test can still be applied without using the absolute value symbols in the theorem.

Example 67. Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2+4}{2^n}$$

Solution. Applying the Ratio Test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2+4}{2^{n+1}}\cdot\frac{2^n}{n^2+4}\right| \\ &= \lim_{n\to\infty}\frac{1}{2}\left(\frac{n^2+2n+5}{n^2+4}\right) = \frac{1}{2}(1) = \frac{1}{2} < 1.\end{aligned}$$

Therefore, by the Ratio Test, the series is absolutely convergent.

Exercise. 1. Test the series for absolute convergence

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$$

2. Test the convergence of the series

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

8.7.2 The Root Test

Theorem 37. (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Example 68. Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

Solution. Let

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\implies \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$$

Hence, the series converges by the Root Test.

Chapter 9

Power Series

9.1 Power Series

Definition 26. A power series **centered at a** is a series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_o + c_1(x-a) + c_2(x-a)^2 + \ldots \tag{9.1}$$

where the c_n's are constants called the **coefficients** of the series.

Remark. If $a = 0$ and $c_n = 1$ for all n, then we get the geometric series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x} \tag{9.2}$$

which converges for $-1 < x < 1$, diverges for $|x| \geq 1$ and has the sum $1/(1-x)$.

9.1.1 Radius and Interval of Convergence

Theorem 38. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

R is called the **radius of convergence** and the **interval of convergence** is the interval that consists of all values of x for which the series converges.

Remark. 1. In general, the Ratio Test (or the Root Test) should be used to determine the radius of convergence R.

2. Remember to always test the end points of the interval of convergence for convergence (using a different test other than the Ratio or Root Tests).

3. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is given by

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| \text{ or } R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{c_n}}$$

Alternatively, we have

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \text{ or } \frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{c_n}$$

A third method is to solve

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If the series converges, then $L < 1$ and $|x-a| < R$, where R is the radius of convergence.

(ii) If the series diverges, then $L > 1$.

We prove the first part of the third point.

Proof. Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. Suppose the limit $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \mathcal{L}$ where $0 \le \mathcal{L} \le \infty$. Applying the Ratio Test to the series, we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \cdot |x-a| = \mathcal{L}|x-a|$$

where $a_n = c_n(x-a)^n$. By the Ratio Test, the series converges if $\mathcal{L}|x-a| < 1$, that is, if $|x-a| < 1/\mathcal{L}$ and diverges if $|x-a| > 1/\mathcal{L}$. Thus, by the definition of the radius of convergence, R, we have $R = 1/\mathcal{L}$. From the equation above, we also have

$$\mathcal{L} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

$$\therefore R = \frac{1}{\mathcal{L}} = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.$$

□

Example 69. For what values of x is the series convergent?

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ (c) $\displaystyle\sum_{n=0}^{\infty}n!x^n$

Solution. Let $a_n = \dfrac{(x-3)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \left|\frac{n}{n+1}\cdot(x-3)\right| = \frac{n}{n+1}|(x-3)| = \frac{1}{1+\frac{1}{n}}|x-3|$$

$$\Longrightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}|x-3| = |x-3|$$

The series is convergent for $|x-3|<1$, which may be written as $2<x<4$.

Remark. The radius of convergence is 1, obtained from $|x-3|<1$ by using the third method stated above. Using the first approach, with $c_n = \dfrac{1}{n}$, we may also find the radius of convergence from

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{n+1}{1} = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}1+\frac{1}{n} = 1.$$

Example 70. Find the radius and interval of convergence of

(a) The **Bessel function** of order 0:

$$J_o(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n^3}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n.2^n}$

Solution. (b) Let $c_n = \dfrac{(-1)^n}{n\cdot 2^n}$. Then

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{1}{n\cdot 2^n}\cdot\frac{(n+1)\cdot 2^{n+1}}{1} = 2\lim_{n\to\infty}1+\frac{1}{n} = 2$$

So the radius of convergence is 2.

9.2 Representation of Functions as Power Series

Our aim here is to learn how to represent functions as power series. This helps in integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. The strategy is to manipulate geometric series or by differentiating or integrating the given series. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \ |x| < 1 \tag{9.3}$$

Equation (9.3) is a representation of the function $f(x) = 1/(1-x)$ as the sum of a power series (in this case the geometric series).

Example 71. Find a power series representation for the function and determine the radius of convergence:

(a) $f(x) = \dfrac{1}{1+x}$ (b) $f(x) = \dfrac{1}{1-x^3}$ (c) $f(x) = \dfrac{1}{x+2}$

Solution. (a) From the relation $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, we replace x with $-x$ to get

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\therefore \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

It is easy to show that the radius of convergence is 1.

(c)

$$\frac{1}{x+2} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\left[\frac{1}{1-(-\frac{x}{2})}\right] = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

It is straightforward to show that the radius of convergence is 1.

Differentiation and Integration of Power Series

Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, we can differentiate or integrate each term of a power series. That is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_o + c_1(x-a) + c_2(x-a)^2 + \dots \tag{9.4}$$

with radius of convergence $R > 0$ and interval of convergence $(a-R, a+R)$, then

$$(i) \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right]$$
$$= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + \dots$$

$$(ii) \qquad \int f(x)\mathrm{d}x = \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right]$$
$$= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C = C + c_o(x-a) + \dots$$

NOTE:

(a) The radii of convergence of the power series in (i) and (ii) are both R.

(b) The interval of convergence in (i) and (ii) may be different.

Example 72. Express the function as a power series. What's the radius of convergence?

$(a) \quad f(x) = \dfrac{1}{(1-x)^2}$ (differentiate equation (9.3))

$(b) \quad f(x) = \ln(5-x)$

$(c) \quad f(x) = \tan^{-1} x$

Solution. (a) Differentiating

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1} nx^{n-1}$$

This may also be written as (by replacing n by $n+1$)

$$\frac{1}{(1-x)^2} = \sum_{n=0}(n+1)x^n$$

The radius of convergence is the same as that of the original series, $R = 1$.

(b) Notice that the derivate of $\ln(5-x)$ is $-\dfrac{1}{5-x}$. This implies that

$$\ln(5-x) = -\int \frac{1}{5-x}dx$$

Now, in terms of power series, we have

$$\frac{1}{5-x} = \frac{1}{5}\left[\frac{1}{1-\frac{x}{5}}\right] = \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} = \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$$

Thus,

$$\ln(5-x) = -\int\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)} + C$$

where C is a constant of integration. To find C, we put $x = 0$ in the equation to get $C = \ln 5$. Hence, we have

$$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)} = \ln 5 - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$$

The radius of convergence of the original series is, from $\left|\dfrac{x}{5}\right| < 1 \Longrightarrow |x| < 5$, $R = 5$. Thus, the radius of convergence of the given series is also $R = 5$.

Example 73. Use a power series to approximate the definite integral to six decimal places

$$\int_0^{0.2}\frac{1}{1+x^5}\mathrm{d}x$$

9.3 Taylor Series and the Taylor Polynomial

The goal here is to determine which functions can be represented as power series and how to get these representations. We start by making use of the following theorem:

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R,$$

then its coefficients are given by $c_n = \dfrac{f^{(n)}(a)}{n!}$

Thus, we get the **Taylor series** about a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\Rightarrow f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

If $a = 0$, we get the **Maclaurin series** :

$$f(x) \quad = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Example 74. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution. If $f(x) = e^x$, then the nth derivative of f is $f^{(n)}(x) = e^x$ and

$$f^{(n)}(0) = e^0 = 1 \qquad \text{for} \quad n = 0, 1, 2, \dots$$

Hence the Taylor series about $x = 0$ (i.e. the Maclaurin series) is given by

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

9.4 nth-degree Taylor Polynomial of f at a

Question: Under what circumstances can we say that a function $f(x)$ is equal to the sum of its Taylor series? That is, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{9.5}$$

Let T_n be the partial sums of the Taylor series of f (called the **nth-degree Taylor polynomial of f**) and given by

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

For instance, the first ($n = 1$), second ($n = 2$) and third ($n = 3$) degree Taylor polynomials are given by:

$$\begin{aligned} T_1(x) &= f(a) + \frac{f'(a)}{1!}(x-a) \\ T_2(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ T_3(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3. \end{aligned}$$

Now, equation (9.5) is true (f is the sum of its Taylor series) if

$$\lim_{n\to\infty} T_n(x) = f(x) \tag{9.6}$$

However, if $R_n(x)$ is the remainder of the Taylor series then we know that

$$\begin{aligned} f(x) &= T_n(x) + R_n(x) \\ \Rightarrow \lim_{n\to\infty} f(x) &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) \\ \Rightarrow f(x) &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) \end{aligned}$$

This means that equation (9.6) is true if

$$\lim_{n\to\infty} R_n(x) = 0 \tag{9.7}$$

The following theorem therefore holds from the above argument:

Theorem 39. (Taylor Theorem) If $f(x) = T_n(x) + R_n(x)$ where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0 \tag{9.8}$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Question: How can we prove or show equation (9.7) or (9.8) for a given function? We employ the following two facts (or theorems):

Theorem 40. (Taylor Inequality) If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq R$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq R$$

Theorem 41.

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 = \lim_{n\to\infty} \frac{|x|^n}{n!}$$

for every real number x.

Examples:

1. Show that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

   and prove that it represents $\sin x$ for all x.

2. Prove that e^x is equal to the sum of its Maclaurin series.

3. Evaluate $\int e^{-x^2}\mathrm{d}x$ as an infinite series

Solution. 1. Let

$$\begin{aligned}
f(x) &= \sin x & \implies && f(0) &= 0 \\
f'(x) &= \cos x & \implies && f'(0) &= 1 \\
f''(x) &= -\sin x & \implies && f''(0) &= 0 \\
f'''(x) &= -\cos x & \implies && f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & \implies && f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & \implies && f^{(5)}(0) &= 1
\end{aligned}$$

We see that the derivatives repeat in a cycle of four, thus, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\Longrightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\therefore \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

We now need to show that $\lim_{n \to \infty} R_n(x) = 0$ in order to prove that the series actually represents $\sin x$. Since $f^{(n+1)}(x)$ is either $\pm \sin x$ or $\pm \cos x$, we have $|f^{(n+1)}(x)| \leq 1$ for all x. So letting $M = 1$ in the Taylor Inequality, we have

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}.$$

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Longrightarrow |R_n(x)| \leq 0$$

Hence, by the squeeze theorem, we have

$$\lim_{n \to \infty} |R_n(x)| = 0 \therefore \lim_{n \to \infty} R_n(x) = 0.$$

Therefore $\sin x$ is equal to the sum of its Maclaurin series.

Chapter 10

Integration

10.1 Introduction

Definition 27. A function F is an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all x in I.

Comparison Properties of the Integration:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Property 1 says that areas are positive. Property 2 implies that a bigger function has a bigger integral. Property 3 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

10.2 Fundamental Theorem of Calculus (FTC)

The Fundamental Theorem of Calculus (FTC) is the most important theorem in calculus. It states that differentiation and integration are inverse processes. In other words, if f is integrated and the result is then differentiated, we arrive back at the original function f. Also, if we take a function F, first differentiate it, and then integrate the result, we get back the original function F in the form $F(b) - F(a)$, where a and b are the lower and upper limits of the integral. In what follows, we first give an intuitive idea of the FTC before explicitly stating the theorem.

10.2.1 Intuitive idea of the FTC

Suppose f is a continuous function on $[a, b]$, and define the function g by

$$g(x) = \int_a^x f(t)dt \tag{10.1}$$

where $a \leq x \leq b$. Here, we want to show, intuitively, that the derivate of $g(x)$ is the function $f(x)$. Note that g depends on only x, the variable in the upper limit of the integral. If x is fixed number, the integral is just a number, representing the area under the graph of f if $f(x) \geq 0$. However, if x varies, then the number $\int_a^x f(t)dt$ also varies and defines the function $g(x)$. Let's try to compute $g'(x)$ from the definition of the derivative. If $f(x) \geq 0$, then $g(x+h) - g(x)$ is obtained by subtracting areas, and it is the area under the graph of f from x to $x + h$. If h is small, the area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x + h) - g(x) \approx hf(x)$$

$$\implies \frac{g(x + h) - g(x)}{h} \approx f(x)$$

Thus, intuitively, we expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

10.2.2 The Mean Value Theorem (MVT) for integrals

Definition 28. (Average Value of a Function) If f is integrable on $[a, b]$, then the **average value of** f over $[a, b]$ is the number

$$f_{av} = \frac{1}{b - a} \int_a^b f(x)dx. \tag{10.2}$$

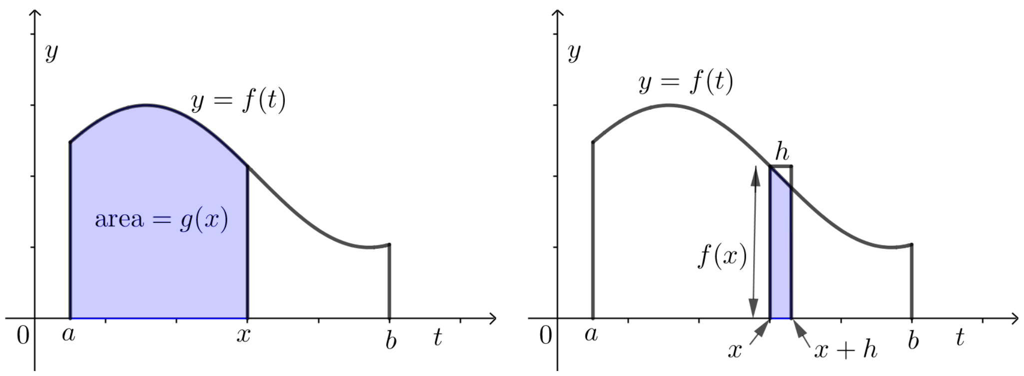

Figure 10.1:

Geometrically, the average value of f is the height of the rectangle with base on the interval $[a, b]$ and having the same area as the area of the region under the graph of f on $[a, b]$. Observe from equation (10.2) that

$$\text{area of rectangle } = (b-a)f_{av} = \int_a^b f(x)dx = \text{ area of region under } f$$

The following theorem, referred to as the Mean Value Theorem for Integrals in some text books, guarantees that f_{av} is always attained at (at least) one number in an interval $[a, b]$ if f is continuous.

Theorem 42. (The Mean Value Theorem for Integrals) If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \tag{10.3}$$

Proof. Since f is continuous on the interval $[a, b]$, the Extreme Value Theorem tells us that f attains an absolute minimum value m at some number in $[a, b]$ and an absolute maximum value M at some number in $[a, b]$. So $m \leq f(x) \leq M$ for all x in $[a, b]$. By Property 3 of integrals, we have

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

If $b > a$, then dividing by $(b-a)$ results in

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

Because the number

$$\frac{1}{b-a}\int_a^b f(x)dx$$

lies between m and M, the Intermediate Value Theorem guarantees the existence of at least one number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx,$$

as required. □

Example 75. Find the number c which is guaranteed by the MVT for Integrals for the function $f(x) = 4 - 2x$ on the interval $[0, 2]$.

Solution. The function $f(x) = 4 - 2x$ is continuous on the interval $[0, 2]$. Therefore, the MVT for Integrals states that there is a number c in $[0.2]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

where $a = 0$ and $b = 2$. Thus,

$$4 - 2c = \frac{1}{2-0}\int_0^2 (4-2x)dx = \frac{1}{2}(4) = 2$$

$$\implies 2c = 4 - 2 = 2$$

$$\therefore c = 1.$$

We now state Part 1 of the Fundamental Theorem of Calculus; abbreviated as FTC1:

Theorem 43. (The Fundamental Theorem of Calculus, Part 1) If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is an antiderivative of f, that is,

$$g'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Proof. Let x and $x+h$ be in the open interval (a, b), where $h \neq 0$. Then,

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt \right) - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

By the Mean Value Theorem for Integrals, there exists a number c between x and $x+h$ such that

$$\int_x^{x+h} f(t)dt = f(c) \cdot h$$

Therefore, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt = \frac{f(c) \cdot h}{h} = f(c).$$

Now, observe that as h approaches 0 (from either the left 0^- or the right 0^+ side), the number c, which lies between x and $x+h$, approaches x, and by continuity, $f(c)$ approaches $f(x)$. Therefore, we get

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h \to 0} f(c) = f(x),$$

as desired. □

Example 76. Find the derivative of the function $g(x) = \int_0^x \sqrt{2+t^3}dt$

Solution. Since the function $f(t) = \sqrt{2+t^3}$ is continuous, the FTC (Part 1) gives

$$g'(x) = \sqrt{2+x^3}$$

Example 77. Find $\dfrac{d}{dx} \displaystyle\int_1^{x^4} \sec t dt$

Solution. Since the upper limit of the integral is not the simple variable, x, we need to use the Chain Rule together with the FTC1. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Thus

$$\frac{d}{dx} \int_1^{x^4} \sec t dt = \frac{d}{dx} \int_1^{u} \sec t dt$$

$$= \frac{d}{du}\left(\int_1^u \sec t dt\right)\frac{du}{dx} = \sec u \frac{du}{dx}$$
$$= \sec(x^4)\cdot 4x^3$$

Exercise. Find the derivative of the function:

(a) $F(x) = \int_{-1}^{x} \frac{1}{5+t^2}dt$ (b) $G(x) = \int_x^3 \sqrt{5+t^2}dt$

Exercise. If $y = \int_0^{x^3} \sin(t^2)dt$, what is $\frac{dy}{dx}$?

Theorem 44. (The Fundamental Theorem of Calculus, Part 2 (FTC2)) If f is continuous on $[a, b]$,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

Remark.
- Part 2 of the FTC says that if a function F is first differentiated, and the result is integrated, we get back the original function F, but in the form $F(b) - F(b)$.
- Part 1 and Part 2 of the FTC state that differentiation and integration are inverse processes.
- The proof of FTC2 is an exercise for you to try.

Example 78. Evaluate the following integrals:

(a) $\int_0^3 (3x^2 + 2x - 1)dx$ (c) $\int_0^{\pi} \sin x dx$

(b) $\int_0^4 3\sqrt{x}dx$

Solution. (a) Let $I = \int_0^3 (3x^2 + 2x - 1)dx$

$$I = \int_0^3 (3x^2 + 2x - 1)dx = \left[x^3 + x^2 - x\right]_0^3$$
$$= (27 + 9 - 3) - 0 = 33$$

(b) Let $I = \int_0^4 3\sqrt{x}dx$

$$I = \int_0^4 3\sqrt{x}dx = 3\left(\frac{2}{3}x^{3/2}\right)\Bigg|_0^4$$

$$= 2x^{3/2}\Big|_0^4 = 2(8-0) = 16$$

(c) Let $I = \int_0^\pi \sin x dx$

$$I = -\cos x\Big|_0^\pi = -\cos\pi + \cos 0 = 1 + 1 = 2,$$

since $\cos\pi = -1$.

Example 79. 1. Use **FTC** to find the derivative of the function.

(a) $g(u) = \int_\pi^u \frac{1}{1+t^4}dt$

(b) $h(x) = \int_2^{\frac{1}{x}} \sin^4 t dt$

(c) $g(x) = \int_{2x}^{3x} \frac{u-1}{u+1}du$ **Hint:** $\left[\int_{2x}^{3x} = \int_{2x}^{0} + \int_0^{3x}\right]$

2. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}dt = 2\sqrt{x}$$

Solution. 1. (a) $g(u) = \int_u^\pi \frac{1}{1+t^4}dt, \quad g'(u) = \frac{1}{1+u^4}.$

(b) $h(x) = \int_2^{\frac{1}{x}} \sin^4 t dt$. Let $u = \frac{1}{x} \Rightarrow \frac{du}{dx} = -\frac{1}{x^2}$

$$\Rightarrow \frac{d}{dx}\int_2^{\frac{1}{x}} \sin^4 t dt = \frac{d}{du}\left[\int_2^u \sin^4 t dt\right]\frac{du}{dx}$$

$$= (\sin^4 u)\cdot\frac{du}{dx} = \sin^4\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) = -\frac{\sin^4(\frac{1}{x})}{x^2}$$

(c) $g(x) = \int_{2x}^{3x} \frac{u-1}{u+1} du \Rightarrow g'(x) = \frac{d}{dx} \int_{2x}^{3x} \frac{u-1}{u+1} du$

$$= \frac{d}{dx} \left[\int_{2x}^{0} \frac{u-1}{u+1} du + \int_{0}^{3x} \frac{u-1}{u+1} du \right] \tag{10.4}$$

Let $I_1 = \int_{2x}^{0} \frac{u-1}{u+1} du = -\int_{0}^{2x} \frac{u-1}{u+1} du$

Let $s = 2x \Rightarrow \frac{ds}{dx} = 2$

$$\therefore, \frac{dI_1}{dx} = -\frac{d}{ds} \left(\int_0^s \frac{u-1}{u+1} du \right) \cdot \frac{ds}{dx}$$

$$= -\left(\frac{s-1}{s+1} \right) \cdot (2) = -\left(\frac{2x-1}{2x+1} \right)(2) \tag{10.5}$$

Let $I_2 = \int_0^{3x} \frac{u-1}{u+1} du$ Let $s = 3x \Rightarrow \frac{ds}{dx} = 3$

$$\therefore, \frac{dI_2}{dx} = \frac{d}{ds} \left(\int_0^s \frac{u-1}{u+1} du \right) \cdot \frac{ds}{dx}$$

$$= \left(\frac{s-1}{s+1} \right)(3) = \frac{3x-1}{3x+1} \tag{10.6}$$

Substitute (10.5) and (10.6) into (10.4) to get

$$g'(x) = -2 \left(\frac{2x-1}{2x+1} \right) + 3 \left(\frac{3x-1}{3x+1} \right)$$

(2)

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x} \Rightarrow \int_\alpha^x \frac{f(t)}{t^2} dt = 2\sqrt{x} - 6 \tag{10.7}$$

Let

$$g(x) = \int_\alpha^x \frac{f(t)}{t^2} dt = 2\sqrt{x} - 6$$

$$\Longrightarrow g'(x) = \frac{f(x)}{x^2} = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow f(x) = x^2 \cdot x^{-\frac{1}{2}} = x^{\frac{3}{2}}$$

$$\therefore \boxed{f(x) = x^{\frac{3}{2}}}$$

From (10.7), we get

$$\int_\alpha^x \frac{t^{\frac{3}{2}}}{t^2} dt = 2\sqrt{x} - 6,$$

$$\Longrightarrow \int_{\alpha}^{x} t^{-\frac{1}{2}} dt = 2\sqrt{x} - 6 \Rightarrow 2\sqrt{t}\Big|_{\alpha}^{x} = 2\sqrt{x} - 6,$$

$$\Longrightarrow 2\sqrt{x} - 2\sqrt{\alpha} = 2\sqrt{x} - 6,$$

$$\Longrightarrow 2\sqrt{\alpha} = 6 \Longrightarrow \sqrt{\alpha} = 3,$$

$$\therefore \alpha = 9.$$

Example 80. (a) Find the average value of the function $f(x) = 1 + x^2$ on $[-1, 2]$.

(b) Find a number c in $[-1, 2]$ that satisfy the **MVT** for integrals for $f(x) = 1 + x^2$.

Solution. (a) $f_{ave} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = \dfrac{1}{2-(-1)}\displaystyle\int_{-1}^{2}(1+x^2)dx$

$$= \frac{1}{3}\left[x + \frac{1}{3}x^3\right]_{-1}^{2} = \frac{1}{3}\left[\left(2 + \frac{8}{3}\right) - \left(-1 - \frac{1}{3}\right)\right]$$

$$\therefore, f_{ave} = \frac{1}{3}\left[\frac{14}{3} + \frac{4}{3}\right] = \frac{18}{9} = 2$$

(b) By **MVT** for integrals,

$$f_{ave} = f(c) \Rightarrow 1 + c^2 = 2 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

There are two values of c in this case (see Figure 10.2).

Example 81. 1. If f is continuous and $\displaystyle\int_1^3 f(x)dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Solution. 1. By **MVT** for integrals, there exists c in $[1, 3]$ such that

$$f(c) = \frac{1}{3-1}\int_1^3 f(x)dx$$

$$\Rightarrow f(c) = \frac{1}{2}(8) = 4.$$

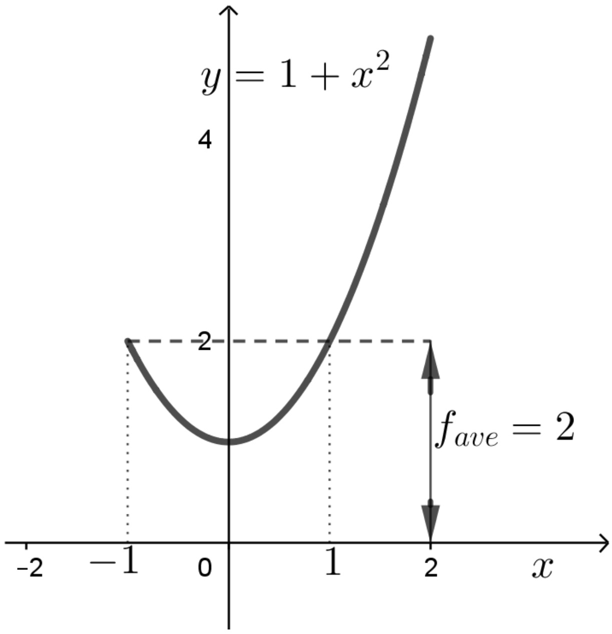

Figure 10.2:

2. $f_{ave} = \dfrac{1}{b-a}\displaystyle\int_0^b (2 + 6x - 3x^2)dx$

$$= \frac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b = \frac{1}{b}[2b + 3b^2 - b^3]$$

$$f_{ave} = 2 + 3b - b^2$$

$$f_{ave} = 3 \Rightarrow 2 + 3b - b^2 = 3 \Rightarrow b^2 - 3b + 1 = 0$$

$$b = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$\therefore, b = \dfrac{3+\sqrt{5}}{2} = 2.62$ and $\dfrac{3-\sqrt{5}}{2} = 0.38$

10.3 Riemann Sums & the Definite Integral: Integration as a Sum

The Riemann sum is a fundamental concept for the definition of the definite integral. It is based on the idea that the area under the graph of a function f can be obtained by dividing the area into smaller rectangular areas and summing them up.

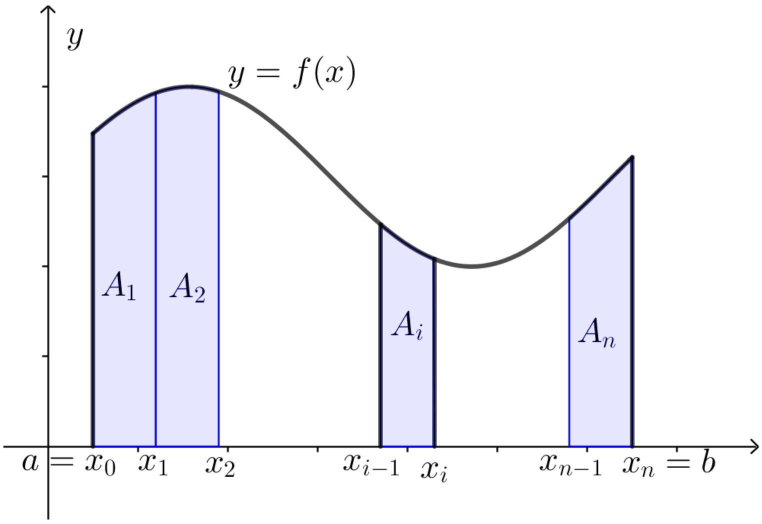

Figure 10.3:

As shown in Figure 10.3, divide the area under the graph into subintervals, or partitions:

$$[x_0, x_1], [x_1, x_2], \cdots [x_{n-1}, x_n]$$

with $x_0 = a$, $x_n = b$, and the width of each strip is $\Delta x = \dfrac{b-a}{n}$.
The right-hand endpoints of each sub-interval is:

$$x_1 = a + \Delta x_0, \quad x_2 = x_1 + \Delta x = a + 2\Delta x,$$
$$x_3 = a + 3\Delta x, \cdots, x_n = a + n\Delta x$$

That is,

$$\boxed{x_i = a + i\Delta x} \qquad i = 1, 2, \cdots, n. \tag{10.8}$$

The left-hand endpoints of each sub-interval is:

$$x_0 = a, x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \cdots, x_{n-1} = a + (n-1)\Delta x$$

That is,

$$\boxed{x_{i-1} = a + (i-1)\Delta x} \qquad i = 1, 2, \cdots n. \tag{10.9}$$

We may also use any sample point, x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$ for the height of the rectangle, $f(x_i^*)$, as illustrated in Figure 10.4. For midpoints of Riemann Sums, we have

$$x_i^* = \frac{x_{i-1} + x_i}{2} = \frac{1}{2}\left[a + (i-1)\Delta x + a + i\Delta x\right] = a + \left(i - \frac{1}{2}\right)\Delta x. \tag{10.10}$$

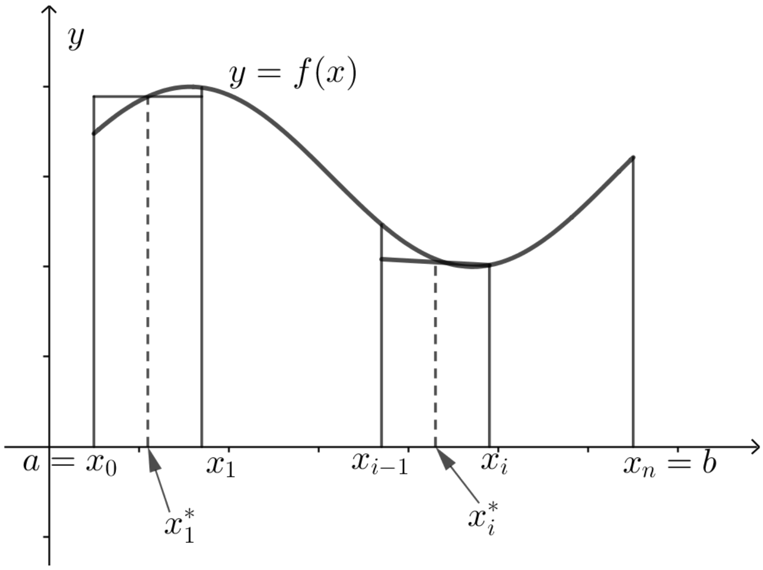

Figure 10.4:

Definition 29. The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of areas of approximating rectangles (using the right endpoints):

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

$$\Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{10.11}$$

OR (using the left endpoints)

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

$$\Rightarrow A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \tag{10.12}$$

OR (using any sample points x_i^*)

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

$$\Rightarrow A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{10.13}$$

Summation Formulas:

To compute the Riemann sums, the following summation formulas are useful:

$$\sum_{i=1}^{n} c = cn$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 82. Evaluate the Riemann sum for $f(x) = x^2$ with $a = 0$, $b = 1$ and $n = 5$, taking the sample points to be the

(a) right endpoints.

(b) left endpoints.

(c) midpoints.

Solution.

$$f(x) = x^2, \quad a = 0,\ b = 1,\ n = 5$$

$$\Longrightarrow \Delta x = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$$

(a) For the right endpoints, we have

$$x_i = a + i\Delta x = 0 + \frac{i}{5} = \frac{i}{5}; \quad i = 1, 2, \cdots, 5.$$

So the endpoints are

$$x_1 = \frac{1}{5},\ x_2 = \frac{2}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5},\ x_5 = \frac{5}{5} = 1$$

Thus, the Riemann sum using the right endpoints with $n = 5$, R_5, is given by

$$\begin{aligned}
R_5 &= \sum_{i=1}^{5} f(x_i)\Delta x \\
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&= [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]\Delta x \\
&= \left[f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1)\right] \cdot \frac{1}{5} \\
&= \left[\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + (1)^2\right] \cdot \frac{1}{5} \\
&= [2.2]\left(\frac{1}{5}\right) \\
\therefore R_5 &= 0.44
\end{aligned}$$

(b) For the left endpoints, we have

$$x_{i-1} = a + (i-1)\Delta x = 0 + \frac{i-1}{5} = \frac{i-1}{5}; \quad i = 1, 2, \cdots, 5.$$

So the endpoints are

$$x_0 = a = 0,\ x_1 = \frac{1}{5},\ x_2 = \frac{2}{5},\ x_3 = \frac{3}{5},\ x_4 = \frac{4}{5}$$

Thus, the Riemann sum using the left endpoints with $n = 5$, L_5, is given by

$$\begin{aligned}
L_5 &= \sum_{i=1}^{5} f(x_{i-1})\Delta x \\
&= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)]\Delta x \\
&= \left[f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right)\right] \cdot \frac{1}{5} \\
&= \left[(0)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right] \cdot \frac{1}{5} \\
&= [1.2]\left(\frac{1}{5}\right) \\
\therefore L_5 &= 0.24
\end{aligned}$$

(c) For the midpoints, we have

$$x_i^* = a + \left(i - \frac{1}{2}\right)\Delta x = 0 + \frac{i - 1/2}{5} = \frac{2i-1}{10}.$$

So the midpoints are

$$x_1^* = \frac{1}{10},\ x_2^* = \frac{3}{10},\ x_3^* = \frac{5}{10},\ x_4^* = \frac{7}{10},\ x_5^* = \frac{9}{10}$$

Thus, the Riemann sum using the midpoints with $n = 5$, M_5, is given by

$$\begin{aligned}
M_5 &= \sum_{i=1}^{5} f(x_i^*)\Delta x \\
&= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x \\
&= [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*)]\Delta x \\
&= \left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right)\right]\cdot\frac{1}{5} \\
&= \left[\left(\frac{1}{10}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(\frac{5}{10}\right)^2 + \left(\frac{7}{10}\right)^2 + \left(\frac{9}{10}\right)^2\right]\cdot\frac{1}{5} \\
&= [1.65]\left(\frac{1}{5}\right) \\
\therefore M_5 &= 0.33
\end{aligned}$$

The next example evaluates the exact value of the Riemann sum in example 82. And you should notice that the value of the Riemann sum using the midpoints is much closer to the exact value.

Example 83. Find the area under the graph of $y = x^2$ from 0 to 1 using the mid-point of subintervals to approximate the height of the rectangles.

Solution.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}.$$

The midpoints of the subintervals are given by (see equation (10.10)):

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = a + \left(i - \frac{1}{2}\right)\Delta x = \left(i - \frac{1}{2}\right)\left(\frac{1}{n}\right)$$

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x, \quad f(x)=x^2 \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(i-\frac{1}{2}\right)\frac{1}{n}\right]^2\cdot\frac{1}{n} \\
&= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\left(i-\frac{1}{2}\right)^2 \\
\implies A &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}\left(i^2-i+\frac{1}{4}\right)
\end{aligned}$$

Applying the summation formulas, we get

$$\begin{aligned}
\therefore A &= \lim_{n\to\infty}\frac{1}{n^3}\left\{\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}+\frac{1}{4}n\right\} \\
&= \lim_{n\to\infty}\left\{\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{1}{4n^2}\right\} \\
&= \frac{1}{6}(1)(2)-\frac{1}{2}(0+0)+0=\frac{2}{6}=\frac{1}{3}.
\end{aligned}$$

Exercise. Estimate the area under the graph of $y=x^2$ from 0 to 1 by taking the sample points to be midpoints and using four subintervals. Compare your answers to L_5 and $R_5 determined previously.$

Example 84. Use Riemann Sums to find the area of the region under the graph of $f(x)=4-x^2$ on the interval $[-2,1]$.

Solution. Partition the area under the graph (see Figure 10.5) into n subintervals with

$$\Delta x=\frac{b-a}{n}=\frac{1-(-2)}{n}=\frac{3}{n}.$$

Using the right endpoints [you may choose left endpoints or midpoints] to estimate the height of rectangles, we have (see equation (10.8)):

$$x_i^*=-2+i\left(\frac{3}{n}\right).$$

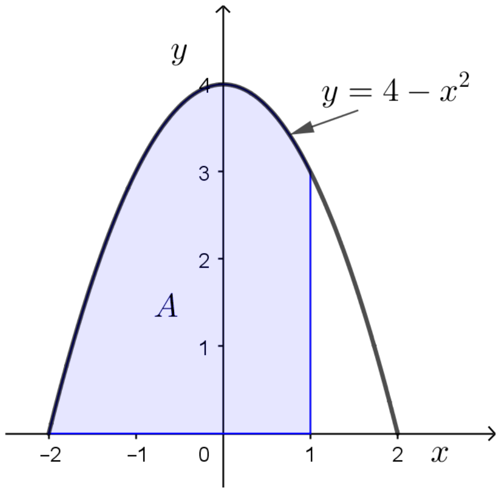

Figure 10.5:

Thus,

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \qquad f(x) = 4 - x^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2 + \frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[4 - \left(-2 + \frac{3i}{n}\right)^2\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[4 - \left(4 - \frac{12}{n}i + \frac{9}{n^2}i^2\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left(\frac{12}{n}i - \frac{9}{n^2}i^2\right) \\
&= \lim_{n\to\infty} \left[\frac{36}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \left[\frac{36}{n^2}\cdot\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[18\left(1 + \frac{1}{n}\right) - \frac{27}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] \\
&= 18 - \frac{27}{6}(2) = 18 - 9 = 9.
\end{aligned}$$

Exercise. Repeat the example above using the midpoints of subintervals to estimate the height of rectangles.

Solution.

$$\bar{x_i} = a + \left(i - \frac{1}{2}\right)\Delta x = -2 + \left(i - \frac{1}{2}\right)\frac{3}{n}$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x_i})\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left[-2 + \left(i - \frac{1}{2}\right)\frac{3}{n}\right]\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left\{4 - \left[-2 + \left(i - \frac{1}{2}\right)\frac{3}{n}\right]^2\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left\{4\left(i - \frac{1}{2}\right)\frac{3}{n} - \left(i - \frac{1}{2}\right)^2\frac{9}{n^2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{36}{n^2}\sum_{i=1}^{n}\left(i - \frac{1}{2}\right) - \frac{27}{n^3}\sum_{i=1}^{n}\left(i - \frac{1}{2}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{36}{n^2}\left[\frac{n(n+1)}{2} - \frac{1}{2}n\right] - \lim_{n\to\infty}\frac{27}{n^3}\sum_{i=1}^{n}\left[i^2 - i + \frac{1}{4}\right]$$

$$= \lim_{n\to\infty}\frac{36}{n^2}\left[\frac{n(n+1)}{2} - \frac{1}{2}n\right] - \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{1}{4}n\right]$$

$$= \lim_{n\to\infty}\frac{36}{n^2}\left[\frac{n(n+1)}{2} - \frac{1}{2}n\right] - \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{1}{4}n\right]$$

$$= \lim_{n\to\infty}36\left[\frac{1}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{2n}\right] - \lim_{n\to\infty}27\left[\frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n^2}\right) + \frac{1}{4n^2}\right]$$

$$= 36\left(\frac{1}{2}\right) - 27\left(\frac{1}{6}\right)(2) = 18 - 9 = 9$$

10.3.1 The Distance Problem

How do you find the distance travelled by an object during a certain time period if the velocity is known at all times?

Remark. (1) If the velocity is constant, then the distance is given as:

$$\boxed{\text{distance} = \text{velocity} \times \text{time}}$$

Eg: A car travelled along a straight line with a velocity of $60km/hr$ over a two hour period (see Figure 10.6a). $v = 60km/hr, 0 \leqslant t \leqslant 2$. Then the distance

$$d = 60\left[\frac{km}{hr}\right] \times 2\,\text{hr} = 120\,\text{km}.$$

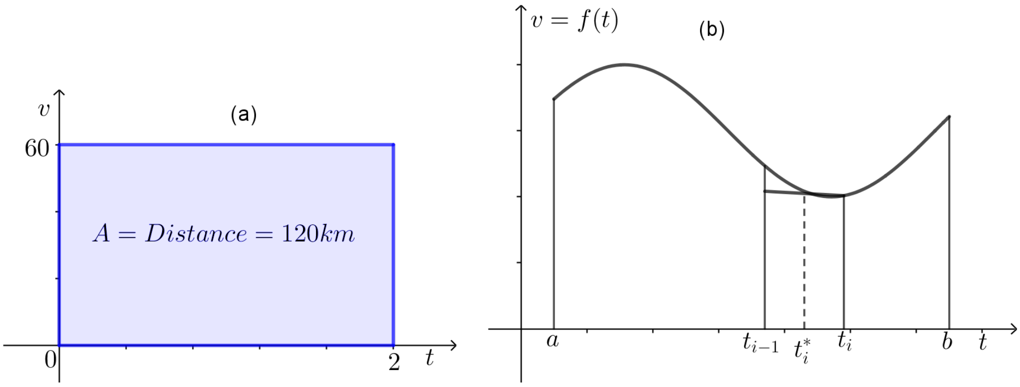

Figure 10.6:

Remark. (2) If the velocity is not constant but a function of time, t, such that $v = f(t)$ during a time interval $[a, b]$, then $d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t$ where $\Delta t = \dfrac{b-a}{n}$, $t_i^* = t_{i-1}$ for left endpoints, $t_i^* = t_i$ for right endpoints or t^* could be midpoint or any point in $[t_{i-1}, t_i]$ (see Figure 10.6b).

10.4 The Definite Integral

Definition 30. If f is a continuous function defined for $a \leqslant x \leqslant b$, we divide the integral $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

We let $x_0, x_1, x_2, \cdots, x_n$ be the endpoints of these subintervals where $a = x_0$ and $b = x_n$, and we choose sample points $x_1^*, x_2^*, \cdots, x_n^*$ in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is given as:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Remark. (a) $\displaystyle\int_a^b f(x)dx$ is a number; it does not depend on x.

(b) $f(x)$ is not necessarily a positive function. If f has both positive and negative values, then a definite integral can be interpreted as a difference of areas above the x-axis and those below the x-axis (see Figure 10.7):

$$\int_a^b f(x)dx = A_1 - A_2$$

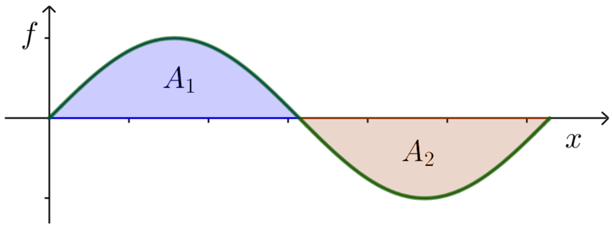

Figure 10.7:

Example 85. 1. Express $\lim_{n\to\infty} \sum_{i=1}^{n} \left[x_i^3 + x_i \sin(x_i)\right] \Delta x$ as an integral on the interval $[0, \pi]$.

2. Use Riemann summation to compute the integral:

(a) $\int_0^5 4x dx$ (b) $\int_0^4 (2x^2 + 3)dx$

Solution. (1) Comparing $\lim_{n\to\infty} \sum_{i=1}^{n} \left[x_i^3 + x_i \sin x_i\right] \Delta x$ to the left of a definite integral, we let $f(x) = x^3 + x \sin x$, $a = 0$, $b = \pi$.
Thus,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[x^3 + x \sin x\right] \Delta x = \int_0^\pi (x^3 + x \sin x)dx.$$

Solution. 2(a) $I = \int_0^5 4x dx.$

Subdivide $[0, 5]$ into n sub-intervals of equal length $\Delta x = \frac{5-0}{n} = \frac{5}{n}$.

Using right endpoints: $x_i = a + i\Delta x = 0 + i\left(\frac{5}{n}\right) = \frac{5}{n}i$

$$\therefore \int_0^5 4x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (4x_i) \cdot \frac{5}{n} = \lim_{n\to\infty} \frac{20}{n} \sum_{i=1}^{n} \frac{5}{n} i$$

$$= \lim_{n\to\infty} \frac{100}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{100}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} 50\left(1 + \frac{1}{n}\right) = 50.$$

(b)

$$I = \int_0^4 (2x^2+3)dx, \quad \Delta x = \frac{4-0}{n} = \frac{4}{n}$$

$$x_i = a + i\Delta x = \frac{4i}{n}, \quad f(x) = 2x^2 + 3$$

$$I = \int_0^4 (2x^2+3)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(\frac{4i}{n}\right)^2 + 3\right] \frac{4}{n} = \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[\frac{32}{n^2} i^2 + 3\right]$$

$$= \lim_{n\to\infty} \frac{4}{n} \left[\frac{32}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + 3n\right] = \lim_{n\to\infty} \left[\frac{128}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 12\right]$$

$$= \frac{128}{6}(2) + 12,$$

$$= \frac{128}{3} + 12 = \frac{164}{3}.$$

Example 86. Use Riemann Sums to evaluate $\int_1^2 (x+1)dx$.

Solution. $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$, $x_i = a + i\Delta x = 1 + \dfrac{i}{n}$

$$f(x) = x + 1 \Rightarrow \int_1^2 (x+1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{i}{n} + 1\right)\right] \frac{1}{n} = 2.5$$

Try the following exercises before looking up their solutions. Keep the following expressions in mind:

$$\begin{cases} \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \int_a^b f(x)dx \\ \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x = \int_a^b f(x)dx \\ \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n} = \int_a^b f(x)dx \end{cases}$$

Exercise. (1) By interpreting the limit as a definite integral, show that:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \ln 2.$

(b) $\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}.$

(2) Find the limit as $n \to \infty$ of the sum $\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2}$.

(3) Use Riemann sums to evaluate $\int_0^3 (x^3 - 6x)dx$.

Solution.(1)(a) We want to show that $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}=\ln 2$.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n(1+\frac{i}{n})}\\ \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}f(\bar{x}_i)\cdot\Delta x,\end{aligned}$$

where $\bar{x}_i = 1+\frac{i}{n} = 1+i\frac{(2-1)}{n}$,

$$a=1, b=2, \quad f(\bar{x}_i)=\frac{1}{\bar{x}_i}$$

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}.$$

Thus

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i}{n}\right)}\cdot\frac{1}{n} = \int_1^2\frac{1}{x}dx$$

$$= \ln x\Big|_1^2 = \ln(2)-\ln(1)=\ln 2.$$

Alternatively, we have

$$\bar{x}_i = \frac{i}{n} = 0+i\frac{1-0}{n},$$

$$a=0, b=1, \Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$f(\bar{x}_i) = \frac{1}{1+\bar{x}_i}.$$

So

$$\lim_{n\to\infty}\sum\frac{1}{\left(1+\frac{i}{n}\right)}\cdot\frac{1}{n} = \int_0^1\frac{1}{1+x}dx,$$

$$= \ln(1+x)\Bigg|_0^1 = \ln 2-\ln 1 = \ln 2.$$

(1)(b) $\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}} = \frac{1}{p+1}$. Let

$$S_n = \frac{1^p+2^p+\cdots+n^p}{n^{p+1}} = \left(\frac{1^p+2^p+\cdots+n^p}{n^p}\right)\cdot\frac{1}{n}$$

$$\Rightarrow \left[\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \cdots + \left(\frac{n}{n}\right)^p\right] \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^p \cdot \frac{1}{n} = \sum_{i=1}^{n} f(\bar{x_i})\Delta x,$$

where

$$\bar{x_i} = \frac{i}{n} = 0 + i \cdot \frac{1-0}{n}, a = 0, b = 1$$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}, \quad f(\bar{x_i}) = \bar{x_i}^p$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^p \cdot \frac{1}{n} = \int_0^1 x^p dx$$

$$= \frac{x^{p+1}}{p+1}\bigg|_0^1 = \frac{1}{p+1} - 0 = \frac{1}{p+1}$$

(2) $S_n = \dfrac{1}{n} + \dfrac{n}{(n+1)^2} + \cdots + \dfrac{n}{(2n-1)^2} = \dfrac{n}{n^2} + \dfrac{n}{(n+1)^2} + \cdots + \dfrac{n}{(2n-1)^2}.$

$$= \sum_{i=0}^{n-1} \frac{n}{(n+i)^2} = \sum_{i=0}^{n-1} \frac{n}{n^2\left(1+\frac{i}{n}\right)^2}.$$

Let $\bar{x_i} = 1 + \dfrac{i}{n} = 1 + \dfrac{i(2-1)}{n}, a = 1, b = 2$, and

$$\Delta x = \frac{1}{n}, f(\bar{x_i}) = \frac{1}{\bar{x_i^2}}.$$

$$\Longrightarrow \lim_{n\to\infty} S_n = \int_1^2 \frac{1}{x^2}dx = \int_1^2 x^{-2}dx = -\frac{1}{x}\bigg|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

10.5 Evaluating Definite Integrals

Theorem 45. (Evaluation Theorem:) If f is continuous on the integral $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

where F is any anti-derivative of f, that is, $F^{'} = \dfrac{dF}{dx} = f$.

Proof. Divide $[a, b]$ into n sub intervals with endpoints $x_0, x_1, x_2, \cdots, x_n$ and with width $\Delta x = \dfrac{b-a}{n}$, where $a = x_0$ and $b = x_n$. Now,

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) + \cdots \\ &\quad - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \cdots \\ &\quad + [F(x_1) - F(x_0)] \end{aligned}$$

$$\Rightarrow F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \tag{10.14}$$

Since F is continuous and differentiable on $[x_{i-1}, x_i]$, we can apply **MVT** to F. Thus, there exists a number $x_i^* \epsilon [x_{i-1}, x_i]$ such that:

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = F'(x_i^*)[x_i - x_{i-1}] = f(x_i^*)\Delta x \tag{10.15}$$

Substitute (10.15) into (10.14) to get

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Taking the limit as $n \to \infty$ on both sides, we get

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\therefore F(b) - F(a) = \int_a^b f(x)dx.$$

□

Example 87. Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \dfrac{\pi}{2}$.

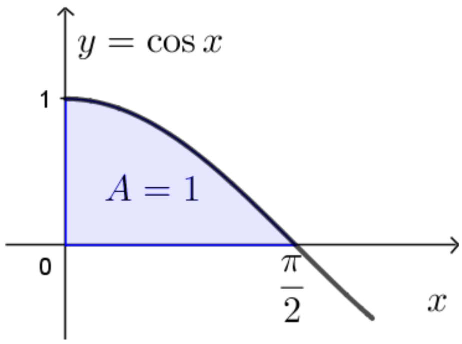

Figure 10.8:

Solution.

$$A = \int_0^b \cos(x)dx = \sin x\Big|_0^b = \sin b - \sin 0 = \sin b$$

Observe that if $b = \dfrac{\pi}{2}$, then $A = \sin\dfrac{\pi}{2} = 1$ (see Figure 10.8).

Properties of Definite Integrals

(1) $\displaystyle\int_a^b c dx = c(b-a)$

(2) $\displaystyle\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(3) $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx, c = \text{constant}$

(4) $\displaystyle\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

(5) $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(6) If $f(x) \geqslant 0$ on $[a, b]$, then $\int_a^b f(x)dx \geqslant 0$.

(7) If $f(x) \geqslant g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$

(8) If $m \leqslant f(x) \leqslant M$ on $[a, b]$, then $m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$

(9) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(10) $\int_a^a f(x)dx = 0$

Example 88. 1) Use the properties of definite integrals to evaluate:

(a) $\int_0^1 (5 + 2x^2)dx$ (b) $\int_1^2 (x^3 + 1)dx$

2) Given that $\int_{-2}^2 f(x)dx = 3$ and $\int_0^2 f(x)dx = 2$, evaluate

(a) $\int_2^0 f(x)dx$ (c) $\int_2^0 3f(x)dx - \int_0^{-2} 2f(x)dx$

(b) $\int_{-2}^0 (f(x) + 3)dx$

Solution. 1(a) $\int_0^1 (5 + 2x^2)dx = 5x + \frac{2}{3}x^3\Big|_0^1 = \left(5 + \frac{2}{3}\right) - 0 = \frac{17}{3}$

(b) $\int_1^2 (x^3 + 1)dx = \int_1^2 x^3 dx + \int_1^2 1dx$

$$= \frac{1}{4}x^4\Big|_1^2 + x\Big|_1^2$$

$$= \frac{1}{4}(16-1) + (2-1) = \frac{15}{4} + 1 = \frac{19}{4}$$

(2) We are given that $\int_{-2}^2 f(x)dx = 3$ and $\int_0^2 f(x)dx = 2$

(a) $\int_2^0 f(x)dx = -\int_0^2 f(x)dx = -2$

(b) $\int_{-2}^{0}(f(x)+3)dx = \int_{-2}^{0} f(x)dx + \int_{-2}^{0} 3dx$
Note that:

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx$$

$$\Rightarrow 3 = \int_{-2}^{0} f(x)dx + 2 \Rightarrow \int_{-2}^{0} f(x)dx = 3 - 2 = 1$$

From the equation in 2(b) above, we have

$$\Rightarrow \int_{-2}^{0}(f(x)+3)dx = 1 + 3\int_{-2}^{0} dx$$

$$= 1 + 3\Big[x\Big]_{-2}^{0} = 1 + 3(0+2) = 1 + 6 = 7$$

(c) $\int_{2}^{0} 3f(x)dx - \int_{0}^{-2} 2f(x)dx = 3\int_{2}^{0} f(x)dx - 2\int_{0}^{-2} f(x)dx$

$$= 3(-2) - 2(-1) = -6 + 2 = -4.$$

10.6 Indefinite Integrals

If $F(x)$ is an anti-derivative of $f(x)$, that is, $F'(x) = f(x)$, then $\int f(x)dx = F(x)$. The indefinite integral $\int f(x)dx$ is a function, while the definite integral $\int_a^b f(x)dx$ is a number.

Example 89. Evaluate:

(a) $\displaystyle\int x^n dx$

(b) $\displaystyle\int e^x dx$

(c) $\sin x dx$

(d) $\displaystyle\int \frac{1}{x} dx$

Solution. (a) $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(b) $\displaystyle\int e^x dx = e^x + c$ **(c)** $\int \sin x dx = -\cos x + c$

(d) $\displaystyle\int \frac{1}{x} dx = \ln|x| + c$

Exercise. Verify by differentiation that the following formula is correct:

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C,$$

where C is a constant.

10.6.1 Total change Theorem

The integral of a rate of change is the total change:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Application

(a) If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. So

$$\int_{t_1}^{t_2} v(t)dt = \int_{t_1}^{t_2} s'(t)dt = s(t_2) - s(t_1)$$

is the change of position or displacement during the time from t_1 to t_2.

(b) The total distance travelled is given by $\int_{t_1}^{t_2} |v(t)|dt$ where $|v(t)|$ is the speed (see Figure 10.9).

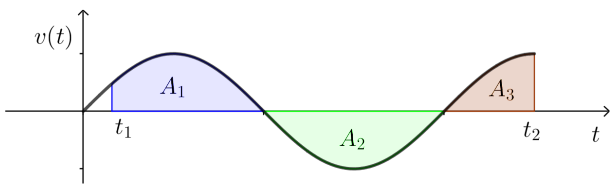

Figure 10.9:

(c) The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t) = v(t_2) - v(t_1)$$

is the change in velocity from t_1 to t_2.

10.7 Techniques of Integration

10.7.1 Method of Substitution

In general, this method works for integrals of the form $f(g(x))g'(x)dx$.

Theorem 46. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Proof. Let F be an anti-derivative of f. That is $F' = f$ or $F = \int f dx$. By chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

$$\Rightarrow \int \frac{d}{dx}F(g(x))dx = \int F'(g(x))g'(x)dx = F(g(x)) + C$$

$$\int F'(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int f(u)du$$

But $F' = f$ and $u = g(x)$

$$\therefore \int f(g(x))g'(x)dx = \int f(u)du.$$

□

Theorem 47. (For definite integrals) If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let F be an anti-derivative of f, that is $F' = f$.

$$\frac{d}{dx}F(g(x)) = F'(g(x)g'(x)) = f(g(x))g'(x))$$

$$\Rightarrow \int_a^b f(g(x))g'(x)dx = F[g(x)]_a^b$$

$$\Rightarrow \int_a^b f(g(x))g'(x)dx = F[g(b)] - F[g(a)] \tag{10.16}$$

By applying the Evaluation theorem again:

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F[g(b)] - F[g(a)] \tag{10.17}$$

From (10.16) and (10.17) we get

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

□

Example 90. 1) Evaluate the following integrals:

(a) $\int x^3 \cos(x^4 + 2)dx$

(b) $\int \frac{x}{x^2 + 1}dx$

(c) $\int e^x \sqrt{1 + e^x}dx$

(d) $\int \frac{x}{\sqrt{1 - 4x^2}}dx$

(e) $\int \tan x dx$

(f) $\int \cos^3 x dx$

(g) $\int \sec x \sqrt{\sec x + \tan x} dx$

(h) $\int \frac{1}{x(x + 1)}dx$

(i) $\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx$

(j) $\int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx$

2) Evaluate the following:

(a) $\int_0^4 \sqrt{2x + 1}dx$

(b) $\int_1^e \frac{\ln x}{x}dx$

(c) $\int_2^3 \frac{3x^2 - 1}{(x^3 - x)^2}dx$

(d) $\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x dx$

(e) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

(f) $\int_1^4 \frac{1}{x^2}\sqrt{1 + \frac{1}{x}}dx$

Solution. 1(a) $I = \int x^3 \cos(x^4+2)dx \quad$, Let $u = x^4 + 2$

$$\Rightarrow du = 4x^3dx \Rightarrow x^3dx = \frac{1}{4}du$$

$$I = \int \frac{1}{4}\cos(u)du = \frac{1}{4}\sin u + c = \frac{1}{4}\sin(x^4+2) + c$$

1(b) $I = \int \frac{x}{x^2+1}dx \quad$, Let $u = x^2 + 1$

$$\Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$$

$$I = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+1| + c$$

1(c) $I = \int e^x\sqrt{1+e^x}dx \quad$, Let $u = 1 + e^x \Rightarrow du = e^x dx$

$$\Rightarrow I = \int \sqrt{u}du = \int u^{\frac{1}{2}}du$$

$$\frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c$$

1(d) $I = \int \frac{x}{\sqrt{1-4x^2}}dx, \quad$ Let $u = 1 - 4x^2$

$$\Rightarrow du = -8xdx \Rightarrow xdx = -\frac{1}{8}du$$

$$\therefore I = \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{8}\right) du$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}du = -\frac{1}{8}(2)u^{\frac{1}{2}} + c$$

$$\Rightarrow -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + c$$

1(e) $I = \int \tan(x)dx = \int \frac{\sin x}{\cos x}dx$. Let $u = \cos x$

$$\Rightarrow du = -\sin x dx \Rightarrow \sin x dx = -du$$

$$I = \int -\frac{1}{u}du = -\ln|u| + c = -\ln|\cos x| + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$$

1(f) $I = \cos^3 x dx = \int \cos x(\cos^2 x)dx = \int \cos x(1 - \sin^2 x)dx$
Let $u = \sin x \Rightarrow du = \cos x dx$

$$\therefore, I = \int (1 - u^2)du = u - \frac{1}{3}u^3 + c = \sin x - \frac{1}{3}\sin^3 x + c$$

1(g) $I = \int \sec x\sqrt{\sec x + \tan x}dx$
Let $u = \sec x + \tan x$

$$du = (\sec x \tan x + \sec^2 x)dx = \sec x(\tan x + \sec x)dx = \sec x(u)dx$$

$$\Longrightarrow \sec x dx = \frac{du}{u}$$

$$\Longrightarrow I = \int \frac{du}{u} \cdot \sqrt{u} = \int u^{-1} \cdot u^{\frac{1}{2}} du$$

$$\therefore I = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} + c = 2(\sec x + \tan x)^{\frac{1}{2}} + c$$

1(h) $I = \int \frac{1}{x(x+1)}dx$, Let $\frac{1}{x(x+1)}$

$$= \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$$

If $x = 0, A = 1$ and if $x = -1, -B = 1 \Rightarrow B = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$I = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx = \ln|x| - \ln|x+1| + c$$

$$= \ln\left|\frac{x}{x+1}\right| + c$$

1(i) $I = \int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx$
Let $u = \sin 2x + \cos 2x \Rightarrow du = (2\cos 2x - 2\sin 2x)dx$

$$= 2(\cos 2x - \sin 2x)dx \Rightarrow \frac{1}{2}du = (\cos 2x - \sin 2x)dx$$

$$\Longrightarrow -\frac{1}{2}du = (\sin 2x - \cos 2x)dx$$

$$I = -\int \frac{\frac{1}{2}u}{u} = -\frac{1}{2}\int \frac{1}{u}du$$

$$= -\frac{1}{2}\ln|u| + c = -\frac{1}{2}\ln|\sin 2x + \cos 2x| + c.$$

1(j) $I = \int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx$

$$= \int \frac{\sin x + \cos x}{\frac{1}{e^x} + \sin x}dx = \int \frac{(\sin x + \cos x)e^x}{1 + e^x \sin x}dx$$

Let $u = 1 + e^x \sin x$, $\quad du = (e^x \sin x + e^x \cos x)dx \Rightarrow du = (\sin x + \cos x)e^x dx$

$$I = \int \frac{du}{u} = \ln|u| + c$$

$$\therefore, I = \ln|1 + e^x \sin x| + c$$

2(a) $I = \int_0^4 \sqrt{2x+1}dx$

Let $u = 2x + 1 \Rightarrow du = 2dx \Rightarrow \frac{1}{2}du = dx$

If $x = 0, u = 1; x = 4, u = 9$

$$I = \int_1^9 \sqrt{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int_1^9 u^{\frac{1}{2}}du$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_1^9$$

$$= \frac{1}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{1}{3}(\sqrt{9})^3 - 1$$

$$= \frac{1}{3}(27 - 1) = \frac{26}{3}.$$

2(b) $\int_1^e \frac{\ln x}{x}dx = \int_1^e \ln x \cdot \left(\frac{1}{x}\right)$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx \quad$ If $x = 1, u = 0; x = e, u = 1$

$$I = \int_0^1 udu = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

2(c) $I = \int_2^3 \frac{3x^2-1}{(x^3-x)^2} dx$
Let $u = x^3 - x \Rightarrow du = (3x^2-1)dx$
If $2, u = 8 - 2 = 6$ and $x = 3, u = 27 - 3 = 24$

$$I = \int_6^{24} \frac{du}{u} = \ln \left| u \right|_6^{24}$$

$$= \ln(24) - \ln(6) = \ln(4 \cdot 6) - \ln 6 = \ln 4 + \ln 6 - \ln 6 = \ln 4$$

2(d) $I = \int_0^{\frac{\pi}{2}} e^{\sin x} \cos x dx.$
Let $u = \sin x \Rightarrow du = \cos x dx$
If $x = 0, u = 0$ and $x = \frac{\pi}{2}, u = 1$

$$I = \int_0^1 e^u du = e^u \Big|_0^1 = e - 1.$$

10.7.2 Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

(a) If f is even, that is $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(b) If f is odd, that is $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0.$

Example 91. Evaluate the following integrals:

(1) $\int_{-2}^{2} (x^6 + 1)dx$

(2) $\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}$

Solution. (1) Since $f(x) = x^6 + 1$, $f(-x) = f(x)$. Thus, f is even.

$$\implies I = 2\int_0^2 (x^6+1)dx = 2\left[\frac{1}{7}x^7 + x\right]_0^2$$

$$= 2\left[\frac{128}{7} + 2\right] = 2\left(\frac{142}{7}\right)$$

$$= \frac{284}{7}.$$

(2) $I = \displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx$

$$f(-x) = -f(x) \Rightarrow f(x) \text{ is odd } \Rightarrow I = 0.$$

Exercise. Evaluate the following integrals:

(1) $I = \displaystyle\int_{-a}^{a} x\sqrt{x^2+a^2}dx$

(2) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^6} dx$

Integrals of the form

$$\int p(x)\sqrt[n]{ax+b}\,dx, \quad \int \frac{p(x)}{\sqrt[n]{ax+b}}dx, \quad \int p(x)\sqrt[n]{\frac{ax+b}{cx+d}}dx$$

Solution Approach: For the first two cases, make use of the substitutions

$$u^n = ax+b \quad \text{or} \quad u = \sqrt[n]{ax+_b}$$

and let

$$u^n = \frac{ax+b}{cx+d} \quad \text{or} \quad u = \sqrt[n]{\frac{ax+b}{cx+d}}$$

for the last case.

Example 92. Evaluate

(a) $\displaystyle\int x^2\sqrt{x-2}dx$

(b) $\displaystyle\int \frac{x}{\sqrt[3]{x+1}}dx$

(c) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}dx$

Solution. (a) $I = \displaystyle\int x^2\sqrt{x-2}dx$

Let $u = \sqrt{x-2} \Rightarrow u^2 = x-2 \Longrightarrow 2udu = dx$, and $x^2 = (u^2+2)^2$

$$\Rightarrow I = \int (u^2+2)^2 \cdot u \cdot 2udu = \int 2u^2(u^2+2)^2 du$$

$$= \int 2u^2(u^4 + 4u^2 + 4)du$$

$$I = \frac{2}{7}u^7 + \frac{8}{5}u^5 + \frac{8}{3}u^3 + c$$

$$\therefore I = \frac{2}{7}(x-2)^{\frac{7}{2}} + \frac{8}{5}(x-2)^{\frac{5}{2}} + \frac{8}{3}(x-2)^{\frac{3}{2}} + c.$$

(b) $I = \int \frac{x}{\sqrt[3]{x+1}}dx$

Let $u = (x+1)^{\frac{1}{3}} \Rightarrow u^3 = x+1 \Rightarrow 3u^2du = dx$, and $x = u^3 - 1$

$$\therefore I = \int \frac{u^3-1}{u} \cdot 3u^2du = 3\int u(u^3-1)du$$

$$= 3\int (u^4 - u)du = \frac{3}{5}u^5 - \frac{3}{2}u^2 + c$$

$$I = \frac{3}{5}(x+1)^{\frac{5}{3}} - \frac{3}{2}(x+1)^{\frac{2}{3}} + c$$

(c) $I = \int \sqrt{\frac{1+x}{1-x}}dx$

Easier approach:

$$\sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{(1+x)}{1-x} \cdot \frac{(1+x)}{(1+x)}}$$

$$= \sqrt{\frac{(1+x)^2}{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}}$$

Let $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$$\Rightarrow I = \int \frac{1+\sin\theta}{\cos\theta} \cdot \cos\theta d\theta = \int (1+\sin\theta)d\theta$$

$$= \theta - \cos\theta + c$$

$$\cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos\theta = \sqrt{1-x^2}$$

$$I = \sin^{-1}x - \sqrt{1-x^2} + c$$

Long approach:

Let $u^2 = \frac{1+x}{1-x} \Rightarrow u^2 - xu^2 = 1 + x \Rightarrow u^2 - 1 = x(u^2+1)$

$$\Rightarrow x = \frac{u^2 - 1}{u^2 + 1} \quad dx = \frac{4u}{(u^2+1)^2}du \quad \textbf{[quotient rule]}$$

$$\therefore I = \int u \cdot \frac{4u}{(u^2+1)^2}du$$

$$= \int \frac{4u^2}{(u^2+1)^2}du$$

Let $u = \tan x \Rightarrow du = \sec^2 \alpha d\alpha$

$$I = \int \frac{4\tan^2\alpha}{(\tan^\alpha + 1)^2} \cdot \sec^2 \alpha d\alpha$$

Note that $\tan^2 \alpha + 1 = \sec^2 \alpha$

$$I = \int \frac{4\tan^2\alpha}{sec^2\alpha}d\alpha$$

$$\Rightarrow I = \int \frac{4\sin^2\alpha}{\cos^2\alpha} \cdot \cos^2 \alpha d\alpha$$

$$= \int 4\sin^2 \alpha d\alpha$$

$$\int 4 \cdot \frac{1}{2}(1 - \cos 2\alpha)d\alpha = 2\int (1 - \cos 2\alpha)d\alpha$$

$$= 2\left[\alpha - \frac{1}{2}\sin 2\alpha\right] + c = 2\alpha - \sin 2\alpha + c$$

Using a right angle triangle, we have

$$\tan\alpha = u, \sin u = \frac{u}{\sqrt{1+u^2}}, \cos\alpha = \frac{1}{\sqrt{1+u^2}}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$

$$I = 2\tan^{-1} u - \frac{2u}{1+u^2} + c$$

Note: $1 + u^2 = 1 + \frac{1+x}{1-x} = \frac{2}{1-x}$

$$\Rightarrow \frac{2u}{1+u^2} = 2u \cdot \frac{(1-x)}{2}$$

$$(1-x)\cdot u = (1-x)\sqrt{\frac{1+x}{1-x}} = \sqrt{(1-x)^2\frac{(1+x)}{(1-x)}}$$
$$= \sqrt{(1-x)(1+x)} = \sqrt{1-x^2}$$
$$\therefore I = 2\tan^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right) - \sqrt{1-x^2} + c$$

10.8 Trigonometric Integrals

10.8.1 Integrating Powers of $\sin x$ and $\cos x$

Here are some guidelines for integrating $\int \sin^n x dx$ and $\int \cos^n x dx$.

(1) If n is an odd positive integer then write

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx.$$

Because $(n-1)$ is an even integer, we can employ the identity $\sin^2 x = 1 - \cos^2 x$ to get an expression that is easier to integrate.

(2) If n is an odd positive integer then write

$$\int \cos^n x dx = \int \cos^{n-1} x \cos x dx.$$

Since $(n-1)$ is even, we can use the trigonometric identity $\cos^2 x = 1 - \sin^2 x$ to get a form that can be easily integrated.

(3) If n is even, use the half-angle formula (and its variants) to simplify the integrand $\sin^n x$ or $\cos^n x$. Here are some half-angle formulae to refresh your memory:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x),$$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x), \quad \sin^2 2x = \frac{1}{2}(1-\cos 4x).$$

We will use some examples to illustrate the procedure for integrating powers trigonometric functions.

Example 93. Evaluate:

(a) $\int \cos^5 x dx$

(b) $\int \sin^4 x dx$

Solution. (a) $I = \int \cos^5 x dx = \int \cos^4 x \cos x dx = \int (1 - \sin^2 x)^2 \cos x dx$

Let $u = \sin x \Rightarrow du = \cos x dx$

$$I = \int (1-u^2)^2 du = \int (1 - 2u^2 + u^4) du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 x + c$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

(b) $I = \int \sin^4 x dx = \int [\sin^2 x]^2 dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \Rightarrow I = \frac{1}{4}\int (1 - \cos 2x)^2 dx$$

$$I = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx$$

But $\cos^2 x = \frac{1}{2}(1 + \cos 2x) \Longrightarrow \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

$$\Longrightarrow I = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right] + c$$

$$= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + c$$

$$\therefore I = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

10.8.2 Integrals of Products of Trigonometric Functions

Integrals of the form $\int \sin^m x \cos^n dx$

Here are some guidelines for integrating integrals of the form $\int \sin^m x \cos^n x\, dx$.

(1) If m is an odd integer, re-write the integral in the form

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x dx$$

and use the identity $\sin^2 x = 1 - \cos^2 x$ to express $\sin^{m-1} x$ in terms of $\cos x$. You can then make the substitution $u = \cos x$ to simplify the resulting integral.

(2) If n is an odd integer, re-write the integral in the form

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos^{n-1} x \cos x dx$$

and use the identity $\cos^2 x = 1 - \sin^2 x$ to express $\cos^{n-1} x$ in terms of $\sin x$. You can then make the substitution $u = \sin x$ to simplify the resulting integral.

(3) If both n and m are even, then use the half-angle formulas for $\cos^2 x$ and $\sin^2 x$ to reduce the exponents by one-half.

Example 94. Evaluate

(1) $\int \cos^3 x \sin^4 x dx$

Solution. (1) $\int \cos^3 x \sin^4 x dx$

$$\int \cos^3 x \sin^4 x dx = \int \cos^2 x \sin^4 x \cos x dx$$

$$= \int (1 - \sin^2 x) \sin^4 x \cos x dx$$

Let $u = \sin x$, then $du = \cos x dx$, and the integral becomes

$$\int \cos^3 x \sin^4 x dx = \int (1 - u^2) u^4 du = \int (u^4 - u^6) du$$

$$= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$$

Integrals of the form $\int \tan^m x \sec^n dx$

Here are some guidelines for integrating integrals of the form $\int \tan^m x \sec^n x dx$.

(1) If m is an odd integer, re-write the integral in the form

$$\int \tan^m x \sec^n x\, dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x\, dx$$

and use the trigonometric identity $\tan^2 x = \sec^2 x - 1$ to express $\tan^{m-1} x$ in terms of $\sec x$. You can then make the substitution $u = \sec x$ to simplify the resulting integral.

(2) If n is even, re-write the integral in the form

$$\int \tan^m x \sec^n x\, dx = \int \tan^m x \sec^{n-2} x \sec^2 x\, dx$$

and use the identity $\sec^2 x = 1 + \tan^2 x$ to express $\sec^{n-2} x$ in terms of $\tan x$. You can then make the substitution $u = \tan x$ to simplify the resulting integral.

(3) If n is odd and m is even, there is no standard procedure. You may try using integration by parts.

Example 95. Evaluate

(1) $\displaystyle\int \tan^3 x \sec^5 x dx$ (3) $\displaystyle\int \cos 5x \cos 3x dx$

(2) $\displaystyle\int \tan^2 x \sec^4 x dx$

Solution. (2) $\displaystyle\int \tan^3 x \sec^5 x dx$

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x (\sec x \tan x) dx$$

$$= \int (\sec^2 x - 1) \sec^4 x (\sec x \tan x) dx$$

Let $u = \sec x$ and $du = \sec x \tan x dx$, then we get

$$\begin{aligned}\int \tan^3 x \sec^5 x dx &= \int (u^2 - 1)u^4 du \\ &= \int (u^6 - u^4) du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\ &= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.\end{aligned}$$

(3) $\int \tan^2 x \sec^4 x dx$

$$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \sec^2 x \sec^2 x dx$$

$$= \int \tan^2 x(\tan^2 x + 1)\sec^2 x dx$$

Let $u = \tan x$, then $du = \sec^2 x dx$, and we get

$$\begin{aligned}\int \tan^2 x \sec^4 x dx &= \int u^2(u^2 + 1) du \\ &= \int (u^4 + u^2) du \\ &= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\tan^2 x + \frac{1}{3}\tan^3 x + C\end{aligned}$$

(4) Here, we apply the **Factor Formula (or Sum and Product Formula)** in trigonometry:

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Thus, we may write

$$\cos 5x \cos 3x = \frac{1}{2}(\cos 8x + \cos 2x)$$

such that

$$\int \cos 5x \cos 3x = \int \frac{1}{2}(\cos 8x + \cos 2x)\, dx$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$$

10.8.3 Integrals of Hyperbolic Functions

Remark. Go back to the chapter on hyperbolic functions and their derivatives, do a review of them, and return here. From our lectures on the definitions of hyperbolic functions and their derivatives, we obtain the integrals in Table 10.1:

$\int \cosh u du = \sinh u + C$	$\int \sinh u du = \cosh u + C$
$\int \operatorname{sech}^2 u du = \tanh u + C$	$\int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$
$\int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C$	$\int \operatorname{csch}^2 u du = -\coth u + C$

Table 10.1: Integrals of some hyperbolic functions.

Example 96. Find $\int \cosh^2 3x \sinh 3x dx$.

Solution. Let $u = 3x \Longrightarrow du = 3dx$ and $dx = \frac{1}{3}du$. Thus

$$I = \int \cosh^2 3x \sinh 3x dx = \frac{1}{3}\int \cosh^2 u \sinh u du$$

Let $t = \cosh u \Longrightarrow dt = \sinh u du$. Thus,

$$I = \frac{1}{3}\int t^2 dt = \frac{1}{9}t^3 + C$$

$$\therefore I = \frac{1}{9}\cosh^3 u + C = \frac{1}{9}\cosh^3(3x) + C.$$

Example 97. Find $\int \sinh^2 x dx$.

Solution. Note that $\cosh 2x = 1 + 2\sinh^2 x$. Thus

$$I = \int \sinh^2 x dx = \frac{1}{2}\int (\cosh 2x - 1)dx$$

$$\therefore I = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C.$$

Alternatively, we could change to exponentials:

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}(e^{2x} - 2 + e^{2x})$$

$$\Longrightarrow I = \int \frac{1}{4}(e^{2x} - 2 + e^{2x})dx = \frac{1}{8}e^{2x} - \frac{1}{2}x - \frac{1}{8}e^{-2x} + C.$$

$$= \frac{1}{4}\left(\frac{e^{2x} - e^{-2x}}{2}\right) - \frac{1}{2}x + C$$

$$\therefore I = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C,$$

which is the same answer we found earlier.

10.8.4 Trigonometric & Hyperbolic Substitutions

The following table gives substitutions that are useful for eliminating radicals from certain types of integrands like $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$. Either trigonometric or hyperbolic substitutions can be used to evaluate such integrals. In making trigonometric substitutions, we assume that the angle θ is in the range of the corresponding inverse trig function. For instance, for the substitution $x = a\sin\theta$, we have $-\pi/2 \leq \theta \leq \pi/2$. If $\sqrt{a^2 - x^2}$ occurs in a denominator, we add the restriction $|x| \neq a$, or, equivalently, $-\pi/2 < \theta < \pi/2$.

Expression in Integrand	Substitution (Trigonometric)	Substitution (Hyperbolic)
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$x = a\cosh\theta$

For hyperbolic substitution, the basic hyperbolic and inverse hyperbolic identities are relevant. For example

$$\cosh^2 x - \sinh^2 x = 1; \qquad \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$1 + \operatorname{csch}^2 x = \coth^2 x; \qquad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}); \qquad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

Example 98. Evaluate $\displaystyle\int \frac{1}{\sqrt{4 + x^2}}dx$

Solution Method 1: Trigonometric Substitution

Solution. Let $x = 2\tan\theta \implies dx = 2\sec^2\theta d\theta$. Thus

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

Therefore,

$$\int \frac{1}{\sqrt{4+x^2}}dx = \int \frac{1}{2\sec\theta}2\sec^2\theta d\theta = \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C,$$

(recall the integral of $\sec\theta$ from Calculus I). We need to put back the original x variable. Now, $\tan\theta = x/2$ implies that (using a right-angle triangle as in Figure 10.10)

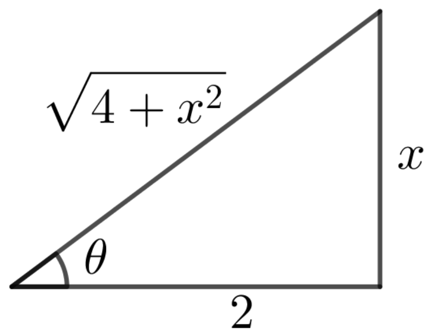

Figure 10.10:

$$\cos\theta = \frac{2}{\sqrt{4+x^2}} \implies \sec\theta = \frac{\sqrt{4+x^2}}{2}$$

Hence,

$$I = \int \frac{1}{\sqrt{4+x^2}}dx = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

This may be written in the form:

$$I = \ln\left|\sqrt{4+x^2} + x\right| - \ln 2 + C = \ln\left(\sqrt{4+x^2} + x\right) + D$$

where $D = -\ln 2 + C$ and where the absolute value sign has been removed because $\sqrt{4+x^2} + x > 0$ for every x.

Solution Method 2: Hyperbolic Substitution

Solution. Let $x = 2\sinh\theta \Longrightarrow dx = 2\cosh\theta d\theta$. Thus

$$\sqrt{4+x^2} = \sqrt{4+4\sinh^2\theta} = 2\sqrt{1+\sinh^2\theta} = 2\sqrt{\cosh^2\theta} = 2\cosh\theta$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}dx &= \int \frac{1}{2\cosh\theta}\cdot 2\cosh\theta d\theta \\ &= \int d\theta \\ \Longrightarrow \int \frac{1}{\sqrt{4+x^2}}dx &= \theta + C\end{aligned}$$

Now $x = 2\sinh\theta \Longrightarrow \theta = \sinh^{-1}\left(\frac{x}{2}\right)$. Using the identity $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$, we get

$$\begin{aligned}\theta = \ln\left[\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2+1}\right] + C &= \ln\frac{1}{2} + \ln\left(\sqrt{4+x^2}+x\right) + C \\ &= \ln\left(\sqrt{4+x^2}+x\right) - \ln 2 + C \\ \therefore \int \frac{1}{\sqrt{4+x^2}}dx &= \ln\left(\sqrt{4+x^2}+x\right) + D.\end{aligned}$$

> Using the trig substitution $x = a\tan\theta$, we can derive the formula:
>
> $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Exercise. (1) Using a hyperbolic substitution, show that

$$\int \sqrt{x^2+4}dx = \frac{x}{2}\sqrt{x^2+4} + 2\ln(x+\sqrt{x^2+4}) + C.$$

(2) Use trigonometric substitutions to show that

(a) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}dx = \sqrt{x^2-9} - \sec^{-1}\left(\frac{x}{3}\right) + C.$

(b) $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}dx = -\frac{(1-x^2)^{5/2}}{5x^5} + C.$

10.9 Integrals of Rational Functions Using Partial Fractions

Using partial fractions, rational functions can be written as a sum of simpler fractions that we already know how to integrate. If the degree of the numerator is greater than or equal to the degree of the denominator, we first perform the long division to simplify the rational function. Some times the denominator may contain irreducible quadratic factors or repeated reducible quadratic factors, in either case, you need to apply all the techniques of partial fractions to simplify the rational function before integration. In other words, if

$$f(x) = \frac{P(x)}{Q(x)}$$

we try to factor $Q(x)$ as far as possible. This may some times require "completing the square" of a quadratic expression before making a substitution to integrate the given expression.

Review of partial fraction techniques

The following examples give a review of some of the partial fraction techniques needed to simplify integrals of rational functions.

(1) $Q(x)$ is a product of linear factors, some of which are repeated.

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Multiplying both sides of the equation by the denominator on the left-hand side, we can show that

$$A = 2,\ B = 1,\ C = -3,\ D = 2.$$

We can then integrate each partial fraction on the right-hand side.

(2) $Q(x)$ contains repeated quadratic factors

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

It can be shown that

$$A = 5, \quad B = -3, \quad C = 2, \quad D = 0.$$

(3) $Q(x)$ has irreducible quadratic factors. For example, find:

$$\int \frac{2x-1}{x^2-6x+13}dx.$$

Completing the square, we can write

$$x^2-6x+13=(x-3)^2+4.$$

Thus,

$$\int \frac{2x-1}{x^2-6x+13}dx = \int \frac{2x-1}{(x-3)^2+4}dx$$

Let $u=x-3 \Longrightarrow x=u+3$ and $dx=du$. Thus

$$\begin{aligned}
\int \frac{2x-1}{x^2-6x+13}dx &= \frac{2(u+3)-1}{u^2+4}du = \int \frac{2u+5}{u^2+4}du \\
&= \int \frac{2u}{u^2+4}du + 5\int \frac{1}{u^2+4}du \\
&= \ln(u^2+4) + \frac{5}{2}\tan^{-1}\frac{u}{2} + C \\
&= \ln(x^2-6x+13) + \frac{5}{2}\tan^{-1}\frac{x-3}{2} + C.
\end{aligned}$$

Example 99. Find $\int \frac{x+5}{x^2+x-2}dx$

Solution. We can write the rational fraction as

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\Longrightarrow x+5 = A(x+2) + B(x-1)$$

If $x=1$, we get $6=3A \Longrightarrow A=2$
If $x=-2$, we get $3=-3B \Longrightarrow B=-1$. Thus, we can write

$$\frac{x+5}{x^2+x-2} = \frac{2}{x-1} - \frac{1}{x+2}.$$

Hence

$$\begin{aligned}
\int \frac{x+5}{x^2+x-2}dx &= \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx \\
&= 2\ln|x-1| - \ln|x+2| + C.
\end{aligned}$$

Example 100. Find $\displaystyle\int \frac{x^3+x}{x-1}dx$.

Solution. Using long division, it can be shown that

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

Thus,

$$\begin{aligned}\int \frac{x^3+x}{x-1}dx &= \int \left(x^2+x+2+\frac{2}{x-1}\right)dx \\ &= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C.\end{aligned}$$

10.10 The Reduction Formulae

Integration by parts can be used to derive **reduction formulas** for evaluating certain integrals. The formulas enable us to express some integrals in terms of integrals whose integrands involve lower powers. Recall the **integration by part** formulas:

$$\int udv = uv - \int vdu.$$

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu.$$

Example 101. Find a reduction formula for $\displaystyle\int \sin^n xdx$, where $n \geq 2$ is an integer.

Solution. Let

$$I = \int \sin^n xdx = \int \sin^{n-1} x \sin xdx$$

Integrating by parts, we let

$$u = \sin^{n-1} x \quad \text{and} \quad dv = \sin xdx$$

$$\Longrightarrow du = (n-1)\sin^{n-2} x\cos xdx \quad \text{and} \quad v = -\cos x$$

$$\Longrightarrow \int \sin^n xdx = uv - \int vdu$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

But $\cos^2 x = 1 - \sin^2 x$, so that

$$\int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

Moving the last term on the right to the left-hand side gives

$$\boxed{n \int \sin^n dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx}$$

or

$$\int \sin^n dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx \qquad (10.18)$$

Example 102. Use the reduction formula in (10.18) to find $\int \sin^4 dx$.

Solution. Here, $n = 4$. So we have

$$\int \sin^4 dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x dx \qquad (10.19)$$

Applying the reduction formula again to the integral $\int \sin^2 x dx$, we get

$$\int \sin^2 x dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx$$

$$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C_1$$

Substituting into equation (10.19), we obtain

$$\int \sin^4 x dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C_1 \right)$$

$$\therefore \int \sin^4 x dx = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$$

where $C = \frac{3}{4} C_1$.

10.11 Improper Integrals

10.11.1 Infinite Intervals of Integration

Improper Integrals are integrals that have infinite intervals of integration or unbounded integrands. The following definition provides a way of computing integrals over infinite intervals as integrals over finite intervals.

Definition 31. (1) If f is continuous on $[a, \infty]$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{10.20}$$

provided that the limit exists.

(2) If f is continuous on $[-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx \tag{10.21}$$

provided that the limit exists. Each improper integral in (10.20) and (10.21) is **convergent** if the limit exists and**divergent** if the limit does not exist.

(3) If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \tag{10.22}$$

where c is any real number, provided that both improper integrals on the right-hand side exist. The improper integral is **convergent** if both improper integrals on the right are convergent, and **divergent** if one or both are divergent.

Example 103. Find $\displaystyle\int_1^\infty \frac{1}{x^2}dx$

Solution.

$$I = \int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx$$

$$\therefore I = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1.$$

The improper integral is convergent in this case.

Example 104. Determine if the following integral is convergent or divergent. If it is convergent, find its value.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x}dx$

(b) $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}dx$

(c) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}dx$

(d) $\displaystyle\int_{-2}^{\infty} \sin x dx$

Solution. (a) $I = \displaystyle\int_1^{\infty} \frac{1}{x}dx$

$$I = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln(x)\Big|_1^t$$

$$\therefore I = \lim_{t\to\infty}[\ln t - \ln 1] = \infty$$

Hence, the integral is divergent.

(b) $I = \displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}dx$

$$I = \lim_{t\to-\infty}\int_t^0 \frac{1}{\sqrt{3-x}}dx = \lim_{t\to-\infty}\int_t^0 (3-x)^{-1/2}dx$$

$$= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_t^0 = \lim_{t\to-\infty}(-2\sqrt{3}+2\sqrt{3-t})$$

$$\therefore I = -2\sqrt{3}+\infty = \infty$$

Thus, the limit is infinite and hence the integral is divergent.

(c) $I = \displaystyle\int_{-\infty}^{\infty} xe^{-x^2}dx$

$$I = \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^{\infty} xe^{-x^2}dx$$

Now, let

$$I_1 = \int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty}\int_t^0 xe^{-x^2}dx$$

The integral is easily integrated by using substitution to get

$$= \lim_{t\to-\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_t^0$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}e^{-t^2}\right)$$

$$\implies I_1 = -\frac{1}{2}.$$

Now let

$$I_2 = \int_{0y}^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx$$

The integral is easily integrated by using substitution to get

$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}\right)$$

$$\implies I_2 = \frac{1}{2}$$

Thus,

$$I = I_1 + I_2 = -\frac{1}{2}+\frac{1}{2} = 0.$$

Thus, the improper integral is convergent and converges to 0.

(d) $I = \displaystyle\int_{-2}^{\infty} \sin x dx$

$$I = \lim_{t\to\infty}\int_{-2}^{t}\sin x dx = \lim_{t\to\infty}(-\cos x)|_{-2}^{t}$$

$$\therefore I = \lim_{t\to\infty}(\cos 2 - \cos t)$$

Thus, the limit does not exist and so the integral is divergent.

Exercise. (1) Determine whether the integral converges or diverges, and if it converges, find its value.

(a) $\displaystyle\int_2^{\infty}\frac{1}{(x-1)^2}dx$ (b) $\displaystyle\int_2^{\infty}\frac{1}{x-1}dx$

(2) Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

10.11.2 Improper Integrals with Discontinuous Integrands

Definition 32. (a) If f is continuous on $[a, b)$ and discontinuous at b, the

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx,$$

provided the limit exists.

(b) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx,$$

provided the limit exists.

(c) If f has a discontinuity at a number c in the open interval (a, b) but is continuous elsewhere on $[a, b]$, then

$$I = \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

provided both of the improper integrals on the right converge. If both converge, then the value of the improper integral I is the sum of the two values.

Example 105. Evaluate $I = \displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}dx$.

Solution. Note that the integrand has an infinite discontinuity at $x = 3$. Thus,

$$\begin{aligned} I = \lim_{t \to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}dx &= \lim_{t \to 3^-} \left[-2\sqrt{3-x}\right]_0^t \\ &= \lim_{t \to 3^-} -2\sqrt{3-t} + 2\sqrt{3} \\ &= 0 + 2\sqrt{3} = 2\sqrt{3}. \end{aligned}$$

Example 106. Evaluate $I = \displaystyle\int_0^3 \frac{1}{x-1}dx$ if possible.

Solution. Observe that the integrand is not continuos at $x = 1$, which occurs in the middle of the interval $[0, 3]$. So we integrate as

$$I = \int_0^1 \frac{1}{x-1}dx + \int_1^3 \frac{1}{x-1}dx$$

Now,

$$I_1 = \int_0^1 \frac{1}{x-1}dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}dx = \lim_{t\to 1^-} \ln|x-1|_0^t$$
$$= \lim_{t\to 1^-} (\ln|t-1| - \ln|-1|)$$
$$= \lim_{t\to 1^-} \ln(1-t) = -\infty$$

Thus, I_1 is divergent. Thus the whole integral I is also divergent and we don't need to evaluate $\int_1^3 \frac{1}{x-1}dx$.

Remark. WARNING!!!
If we had not noticed the discontinuity at $x = 1$ in the previous example and evaluated the integral as an ordinary integral, we would have gotten

$$\int_0^3 \frac{1}{x-1}dx = \ln|x-1|_0^3 = \ln 2 - \ln 1 = \ln 2$$

which is **wrong!** Therefore, it is important to always check whether a given integral is an ordinary integral or an improper integral, by looking at the integrand, f, on the interval $[a, b]$.

10.12 Applications of Integration

10.12.1 Area between curves

1. Recall that the area bounded by the curve $y = f(x)$ and $y = 0$ in the interval $x \in [a, b]$ (figure 10.11) is given by

$$\boxed{A = \int_a^b f(x)\mathrm{d}x.}$$

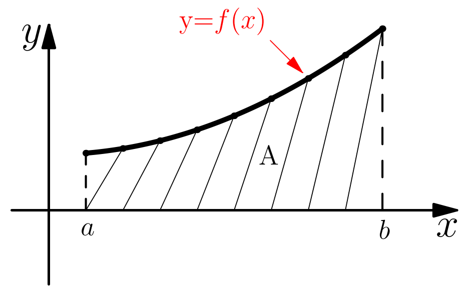

Figure 10.11: Area under a curve

2. Now consider a region S which lies between the two curves $y = f(x)$ and $y = g(x)$ in the interval $[a, b]$ such that $f(x) \geq g(x)$ for all x in $[a, b]$. To find the area between the curves:

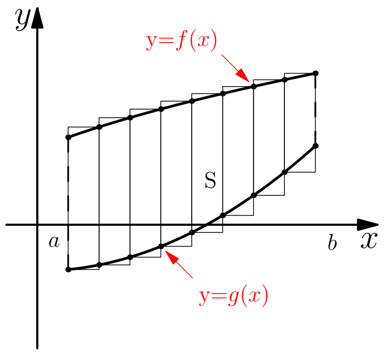

Figure 10.12: Area between two curves

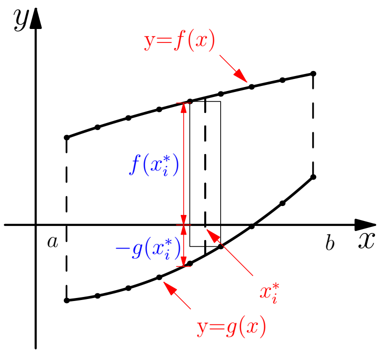

Figure 10.13: Approximating rectangle

Step 1 : Divide S into n rectangular strips of equal width Δx. The height of the i^{th} strip is $f(x_i^*) - g(x_i^*)$

Step 2 : The area of the i^{th} strip is $A_i = [f(x_i^*) - g(x_i^*)]\Delta x$. Summing all the strips we get the Riemann sum, R_n.

$$R_n = \sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x.$$

Step 3 : Find the limit as $n \to \infty$ since the approximation gets better with more strips:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x$$

Step 4 : Finally we get the area between the curves

$$\boxed{A = \int_a^b [f(x) - g(x)]\mathrm{d}x} \tag{10.23}$$

NOTE:

(a) In general, it is important to use a sketch to identify the top and bottom curves and to also find an approximating rectangle. You may also have to determine the interval $[a, b]$.

(b) If $f(x) \geq g(x)$ in some parts of the region and $g(x) \geq f(x)$ for other parts, then split the region into several parts (see figure 10.14):

$$
\begin{aligned}
A &= A_1 + A_2 \\
&= \int_a^c [f(x) - g(x)]\mathrm{d}x + \int_c^b [g(x) - f(x)]\mathrm{d}x \\
A &= \int_a^b |f(x) - g(x)|\mathrm{d}x
\end{aligned}
$$

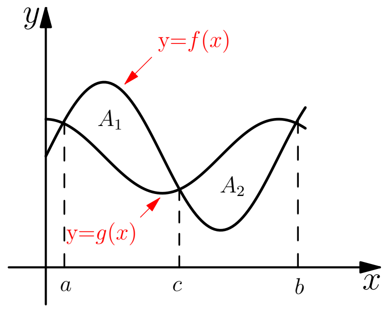

Figure 10.14: Divided region

(c) If $x = f(y)$ and $x = g(y)$ for y in the interval $[c, d]$ where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ then

$$\boxed{A = \int_c^d [f(y) - g(y)]\mathrm{d}y} \tag{10.24}$$

Example 107. Find the area bounded by the curves $y = x^2$ and $y = x$.

Solution. Let $f(x) = x, \quad g(x) = x^2$. We need to find the intersection of the two curves by equating them:

$$x^2 = x \Longrightarrow x^2 - x = 0 \quad \Longrightarrow x(x-1) = 0,$$

so they intersect at $x = 0$ and $x = 1$. Thus, the area between the curves (see Figure 10.15a) is given by

$$
\begin{aligned}
A &= \int_0^1 [f(x) - g(x)]dx \\
&= \int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 \\
&= \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6}.
\end{aligned}
$$

Alternatively, we could integrate with respect to y (see Figure 10.15b). For $y = x^2$, we solve for x to get $x = \pm\sqrt{y}$. Here, we need only the right-hand branch of the curve so we choose $f(y) = x = \sqrt{y}$. From $y = x$, we define $g(y) = y$. When $x = 0$ we have $y = 0$ and when $x = 1$ we get $y = 1$; so the interval of integration remain the same. Thus,

$$\begin{aligned} A &= \int_0^1 [f(y) - g(y)]dy \\ &= (\sqrt{y} - y)dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{2}y^{1/2}\right]_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{2}\right) - 0 = \frac{1}{6}. \end{aligned}$$

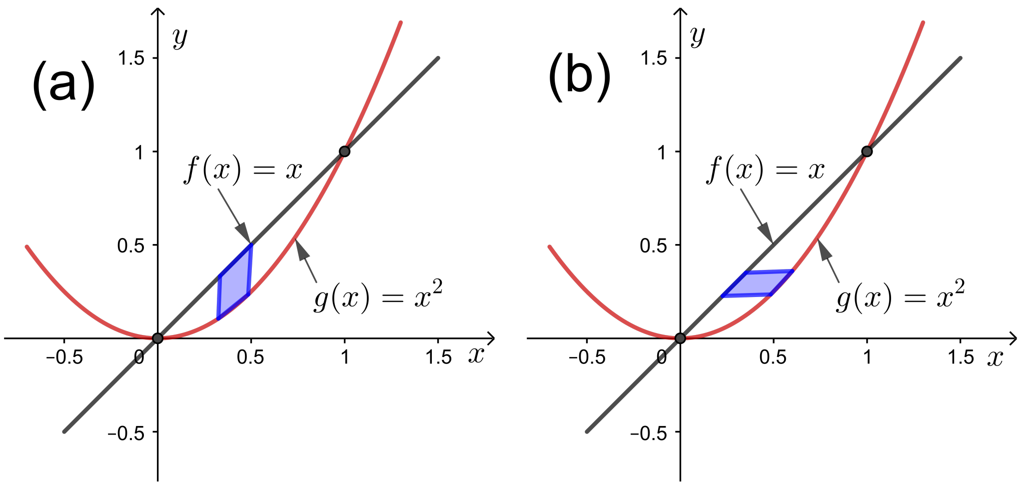

Figure 10.15:

Example 108. Find the area of the region enclosed by $f(x) = \sin x$, $g(x) = \cos x$, $x = \frac{\pi}{2}$ and the y-axis.

Solution. The graph of the required region is displayed in Figure 10.16.

The intersection point is given by

$$\sin x = \cos x \quad \Longrightarrow \tan x = 1$$
$$\therefore x = \frac{\pi}{4}$$

Thus, the required area, A, is given by the sum of the area A_1 and that of A_2 such that $A = A_1 + A_2$.

$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)dx, \qquad A_2 = \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$$

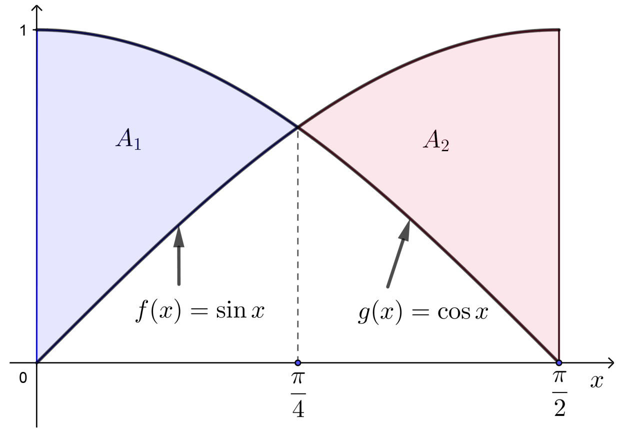

Figure 10.16:

$$\implies A = (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] + \left[(0-1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1$$

$$\therefore A = 2\sqrt{2} - 2$$

Example 109. Find the area bounded by the curves

$$y = x + 1, \quad y = 9 - x^2,$$

and the lines $x = -1$ and $x = 2$.

Solution. Here, we are given the interval $[a, b] = [-1, 2]$. By sketching the functions we see that the upper curve is $f(x) = 9 - x^2$ and the lower curve is $g(x) = x + 1$ (see Figure 10.17). So the area between the curves is given by

$$\begin{aligned} A &= \int_{-1}^{2} \left[(9 - x^2) - (x+1)\right] dx \\ &= \int_{-1}^{2} (8 - x - x^2) dx = \left[8x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} \\ &= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right) \\ &= \frac{34}{3} + \frac{49}{6} = \frac{117}{6}. \end{aligned}$$

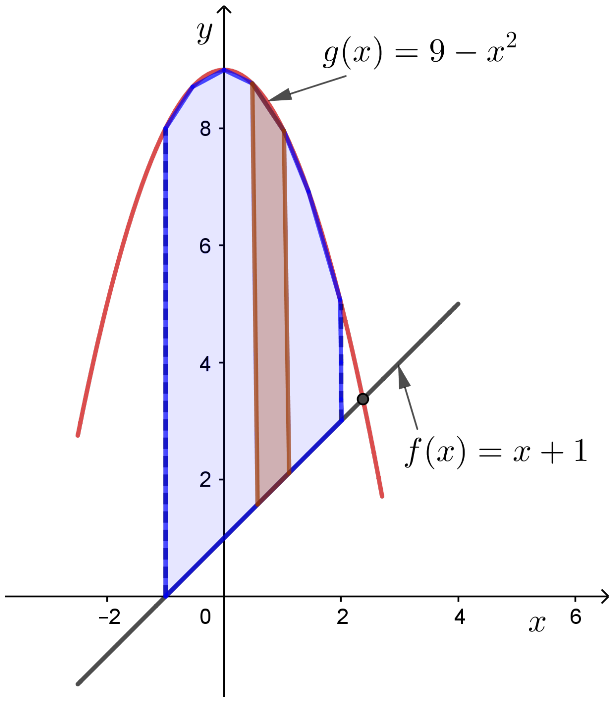

Figure 10.17: The blue shaded region is the required area, and the smaller brown region is a representative rectangle.

10.12.2 Volumes

Cylinders

The volume, V, of a cylinder of height, h, is given by

$$V = (\text{Area of cross-section}) \times \text{height} = Ah$$

Solids of revolution

Let S be a general solid which lies in the interval $[a, b]$ and let $A(x)$ be the area of a cross-section perpendicular to the $x-$axis at the point x (see Figures 10.19 and 10.20). To find the volume of S:

Step 1 : Divide S into n "slabs" of equal width Δx

Step 2 : At the point $x_i^* \in [x_{i-1}, x_i]$, the area of the slab is $A(x_i^*)$ and its height is Δx. Thus, the volume of the slab S_i is $V(S_i) \approx A(x_i^*)\Delta x$. Summing

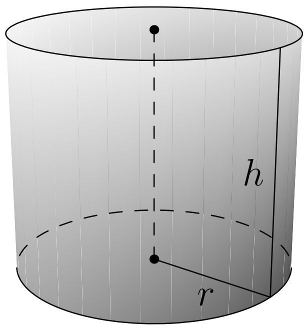

Figure 10.18: circular cylinder

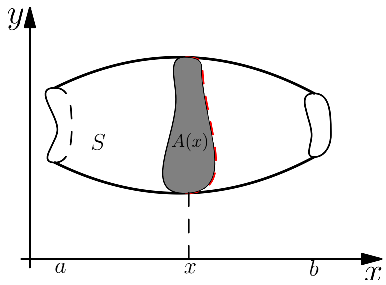

Figure 10.19:

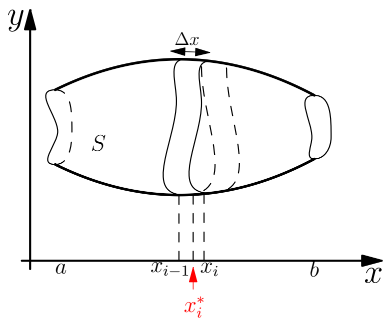

Figure 10.20:

the volume of all the slabs we get the Riemann sum:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

Step 3 Find the limit as $n \to \infty$ since the approximation gets better with more slabs:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x$$

Step 4 Finally we get the volume of S:

$$\boxed{V = \int_a^b A(x)\mathrm{d}x} \tag{10.25}$$

NOTE:

(a) The volume may also be calculated in terms of y:

$$V = \int_c^d A(y)\mathrm{d}y$$

(b) If the cross-section is a disk, then we find A as

$$A = \pi(\text{radius})^2$$

(c) If the cross-section is a washer, we find A as

$$\begin{aligned} A &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(r^2_{\text{out}} - r^2_{\text{in}}) \end{aligned}$$

Examples:
Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line

$$\begin{array}{ll} (a) & y = x^2, \quad 0 \leq x \leq 1, \quad y = 0; \quad \text{about } x\text{-axis} \\ (b) & y = e^x, \quad y = 0, \quad x = 0, \quad x = 1; \quad \text{about } x\text{-axis} \\ (c) & y = x^2, \quad 0 \leq x \leq 2, \quad y = 4, \quad x = 0; \quad \text{about } y\text{-axis} \\ (d) & y = x^2, \quad y^2 = x; \quad \text{about } x\text{-axis} \\ (e) & y = x^4, \quad y = 1; \quad \text{about } y = 2 \\ (f) & y = x^2, \quad x = y^2; \quad \text{about } x = -1 \end{array}$$

Volumes of other solids

Examples:
1. Find the volume of a pyramid whose base is a square with side L and whose height is h.
2. Find the volume of the frustrum with base radius R and top radius r.

10.12.3 Volumes by Cylindrical Shells

Consider a cylindrical shell of inner radius r_1, outer radius r_2 and height h. Let $V_1 = \pi r_1^2 h$ and $V_2 = \pi r_2^2 h$ be the volume of the inner and outer cylinders respectively. Then the volume of the shell is

$$\begin{aligned} V = V_2 - V_1 &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\frac{(r_2 + r_1)}{2}(r_2 - r_1)h. \end{aligned}$$

Let $r = \frac{r_2+r_1}{2}$ (the average radius of the cylinder)
and $\Delta r = r_2 - r_1$ (the thickness of the shell). Then

$$\boxed{V = 2\pi r h \Delta r.} \tag{10.26}$$

That is,

$$V = [\text{circumference}(2\pi r)] \times [\text{height}(h)] \times [\text{thickness}(\Delta r)].$$

In general, let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$, $y = 0$ and $x \in [a, b]$.

To obtain the volume V:

Step 1 : Divide [a,b] into n rectangular strips in $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the i^{th} strip.

Step 2 : The height of the i^{th} strip is $f(\bar{x}_i)$, radius of the shell is $\bar{x}_i$ and the thickness of the shell is Δx. Thus, the volume of the i^{th} shell is

$$V = (2\pi\bar{x}_i)f(\bar{x}_i)\Delta x.$$

Summing the volume of all shells in $[a, b]$, we get

$$V \approx \sum_{i=1}^{n}(2\pi\bar{x}_i)f(\bar{x}_i)\Delta x.$$

Step 3 : Find the limit as $n \to \infty$ since the approximation gets better with more strips:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n}(2\pi\bar{x}_i)f(\bar{x}_i)\Delta x$$

Step 4 : Finally we get the volume of the shell as:

$$\boxed{V = \int_a^b 2\pi x f(x)\mathrm{d}x} \tag{10.27}$$

NOTE:

(a) In equation (10.27),

$$\begin{aligned} 2\pi x &:= \text{Circumference} \\ f(x) &:= \text{Height} \\ dx &:= \text{Thickness} \end{aligned}$$

(b) It's important to always make a sketch to identify the radius and height of a shell.

Example 110. 1. Use both slicing and the method of cylindrical shells to obtain the volume of the solid obtained by rotating about the y-axis the region bounded by

$$y = \sqrt{x}, \quad \text{and} \quad y = x^2.$$

2. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis:

   (a) $x = 1 + y^2$, $x = 0$, $y = 1$, $y = 2$; x-axis

   (b) $x = \sqrt{y}$, $x = 0$, $y = 1$; x-axis

   (c) $y = x^2$, $y = 0$, $x = -2$, $x = -1$; y-axis

   (d) $y = x^2$, $y = 0$, $x = 1$, $x = 2$; about $x = 1$

10.12.4 Arc Length

Our aim here is to find the length of any general curve which is defined by a continuous function in a given interval. We do that by approximating the curve using a series of polygons. Let C be such a curve and $y = f(x)$ be the continuous function in the interval $a \leq x \leq b$. To obtain the length, L, of the curve, we

Step 1: divide $[a, b]$ into n subintervals with end points $x_o, x_1, \ldots x_n$ and equal width Δx.
If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots P_n$ is an approximation to C.

Step 2: The length of the arc in $[x_{i-1}, x_i]$ is approximately given by

$$\begin{aligned} L_i = |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \end{aligned}$$

Summing all polygons, we get

$$L = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Step 3: Find the limit as $n \to \infty$ since the approximation gets better with more polygons:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

By the Mean Value Theorem, there is a number $x_i^* \epsilon [x_{i-1}, x_i]$ such that

$$\begin{aligned} y_i - y_{i-1} &= f'(x_i^*)(x_i - x_{i-1}) \\ \Rightarrow \Delta y_i &= f'(x_i^*)\Delta x \end{aligned}$$

$$\begin{aligned} \Rightarrow L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \end{aligned}$$

Step 4: Finally we get the length of the curve:

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\mathrm{d}x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \tag{10.28}$$

NOTE:
If the curve has an equation of the form $x = g(y)$, $c \le y \le d$ and $g'(y)$ is continuous, then

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y} \tag{10.29}$$

Example 111. Find the Length of the arc in the given interval:

(a) $y = 1 + 6x^{3/2}, \quad 0 \le x \le 1.$

(b) $x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 9.$

(c) $y^2 = 4(x + 4)^3, \quad 0 \le x \le 2, \quad y > 0.$

The arc length function

The distance, $S(x)$, along the curve C from the initial point $[a, f(a)]$ to any point $[x, f(x)]$ on the curve is the **arc length function** and is given by

$$\boxed{S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\mathrm{d}t.} \tag{10.30}$$

Differentiating the equation gives

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}x} &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\ \implies \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\ \implies (\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2. \end{aligned}$$

Example 112. Find the arc length function for the curve $y = 2x^{3/2}$ with the starting point $P_o(1, 2)$.

10.12.5 Area of a Surface of Revolution

This is the surface obtained by rotating a curve about a line. Let's start from finding the area of the surface of a simple object and progress to the complex case of a general surface area.

1. The lateral surface area of a cone of base radius r and slant height h is given by

$$A = \pi r l$$

2. Consider a frustum of a cone with slant height l and top radius r_1 and base radius r_2. Then the area is given by

$$A = \pi r_2(l_1 + l_2) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

Using similar triangles we get

$$\begin{aligned} \frac{l_1}{r_1} &= \frac{l_1 + l}{r_2} \\ \Rightarrow r_2 l_1 &= r_1 l_1 + r_1 l \\ \Rightarrow (r_2 - r_1)l_1 &= r_1 l \end{aligned}$$

Thus,

$$A = \pi[r_1 l + r_2 l] = 2\pi\frac{(r_1 + r_2)}{2}l$$
$$\Rightarrow \boxed{A = 2\pi r l} \tag{10.31}$$

where $r = (r_1 + r_2)/2$ is the average radius.

3. In general, consider the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$ about the x-axis, where f has continuous derivatives.

To obtain the area of the surface of revolution:

Step 1: divide $[a, b]$ into n subintervals with end points $x_0, x_1, \ldots x_n$ and equal width Δx.
If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve.

Step 2: The area of the surface obtained by rotating the polygon $P_{i-1}P_i$ about the x-axis is given by (using (10.31))

$$S_i = 2\pi\frac{y_{i-1} + y_i}{2}|P_{i-1}P_i|.$$

If $x_i^* \epsilon [x_{i-1}, x_i]$, then the length of the arc is

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Thus,

$$S_i = 2\pi\frac{y_{i-1} + y_i}{2}\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Now, if Δx is small, then

$$\begin{aligned} y_i &= f(x_i) \approx f(x_i^*), \\ y_{i-1} &= f(x_i) \approx f(x_i^*), \\ \Rightarrow \frac{y_{i-1} + y_i}{2} &\approx \frac{2f(x_i^*)}{2} = f(x_i^*). \end{aligned}$$

Thus,

$$S_i \approx 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Summing over all polygons, we get

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Step 3: Finding the limit as $n \to \infty$, we get

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Step 4: Finally we get the area of the surface of revolution

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\mathrm{d}x = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \tag{10.32}$$

NOTE:
If the curve is given by $x = g(y)$, $c \le y \le d$, then

$$\boxed{S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y} \tag{10.33}$$

SUMMARY:
Equations (10.32) and (10.33) may also be written respectively as:

$$S = \int 2\pi y\mathrm{d}s \quad \text{and} \quad S = \int 2\pi x\mathrm{d}s,$$

where

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \quad \text{or} \quad \mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y$$

Example 113. 1. Find the area of the surface obtained by rotating the curve about the x-axis

$$y = \sqrt{x}, \quad 4 \le x \le 9$$

2. Find the area of the surface obtained by rotating about the y-axis (use 2 formulas)

$$y = \sqrt[3]{x}, \quad 1 \le y \le 2$$

Chapter 11

Differential Equations

A differential equation is an equation that contains an unknown function and some of its derivatives.
E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -kxy \quad \text{or} \quad y' = -kxy \tag{11.1}$$

where k is a constant.

The order of a differential equation is the order of the highest derivative that occurs in the equation. E.g.:

$$\begin{aligned} y' &= -kxy \quad &&\text{[First order]} \\ y'' + kx &= 0 &&\text{[Second order]} \\ 5y''' - ky'' + 2y &= 0 &&\text{[Third order]} \end{aligned}$$

We will consider only first order differential equations.

A function f is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. For instance, $y = f(x)$ is a solution of (11.1) if

$$f'(x) = -kxf(x)$$

for x in some interval.
Example: Show that $y = x - \frac{1}{x}$ is a solution to

$$xy' + y = 2x.$$

Solution:

$$\begin{aligned} y &= x - \frac{1}{x} \\ \Rightarrow y' &= 1 + \frac{1}{x^2} \\ \Rightarrow xy' + y &= x\left(1 + \frac{1}{x^2}\right) + x - \frac{1}{x} \\ &= x + \frac{1}{x} + x - \frac{1}{x} \\ &= 2x. \end{aligned}$$

Let's consider two types of differential equations and how to solve them.

11.1 Separable equations

These are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)f(x)$$

Solution: Separate the variables and integrate the resulting equation if possible:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= g(x)f(x) \\ \Rightarrow \int \frac{\mathrm{d}y}{f(y)} &= \int g(x)\mathrm{d}x \end{aligned}$$

Examples:
1. Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{2x}}{4y^3}$$

2. Find the solution of the differential equation that satisfies the given initial condition:

$$\begin{aligned} (a)\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 + 1, \quad y(1) = 0 \\ (b)\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y\cos x}{1 + y^2}, \quad y(0) = 1 \end{aligned}$$

Mixing problems

Mixing problems are an application of first order differential equations. If $y(t)$ is the amount of substance in a tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

Example:

A tank contains 20 Kg of salt dissolved in 5000L of water. Brine that contains 0.03 Kg of salt per liter of water enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution:

Let $y(t)$ be the amount of salt (in kilograms) after t minutes.
$y(0) = 20$, $y(30) = ?$
The rate of change of the amount of salt is:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \text{(rate in)-(rate out)} \\ \text{rate in} &= \left(0.03\frac{\text{kg}}{L}\right)\left(25\frac{L}{\text{min}}\right) = 0.75\frac{\text{kg}}{\text{min}}. \end{aligned}$$

The tank always contains 5000 L of liquid so the concentration at time t is $y(t)/5000$ [Kg/L]. Thus

$$\text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{L}\right)\left(25\frac{L}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}.$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving the equation we get

$$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is a continuous function and $y(0) = 20$ and the right hand side is never 0, we deduce that $150 - y(t)$ is always positive. Thus we get

$$\begin{aligned} y(t) &= 150 - 130e^{-t/200} \\ \Rightarrow y(30) &= 150 - 130e^{-30/200} \approx 38.1\text{kg} \end{aligned}$$

11.2 Linear Equations

A first order linear differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x)y = Q(x) \tag{11.2}$$

where P and Q are continuous functions on a given interval.
Solution to equation (11.2):

Step 1: Find the **integrating factor**

$$\mu = e^{\int P(x)\mathrm{d}x} \tag{11.3}$$

Step 2: Multiply equation (11.3) by (11.2) to get

$$\begin{aligned} e^{\int P(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)e^{\int P(x)\mathrm{d}x}y &= e^{\int P(x)\mathrm{d}x}Q(x) \\ \Rightarrow \underbrace{\mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu P(x)y}_{\frac{\mathrm{d}}{\mathrm{d}x}[\mu y]} &= \mu Q(x) \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}[\mu y] &= \mu Q(x) \end{aligned} \tag{11.4}$$

Step 3: Integrate equation (11.4) to get the solution

$$y = \frac{1}{\mu}\left[\int \mu Q(x)\mathrm{d}x + C\right] \tag{11.5}$$

where C is a constant.

Examples:
1. Which of the following differential equations is linear?

$$(a) \quad y' + e^x y = x^2y^2 \qquad\qquad (b) \quad y + \sin x = x^3y'$$

2. Solve the differential equation

$$(a) \quad y' - 2y = 2e^x \qquad\qquad (b) \quad xy' - 2y = x^2$$

3. Solve the initial-value problem (IVP):

$$y' = x + y, \quad y(0) = 2$$