A Construction of H_4 Without Miracles

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Introduction

The reflection group H_4 is the symmetry group of two four-dimensional regular polytopes, the 600-cell and its dual, the 120-cell. It has a non-crystallographic root system $\mathcal{H}_4$ consisting of 120 elements, the vertices of the 600-cell. See Coxeter [**C**, §8.5].

In this note, we present an explicit construction of $\mathcal{H}_4$ that requires very little in the way of tedious checking, nor much in the way of miracles. It is analogous to constructing a crystallographic root system as the set of short vectors in a suitable lattice. The difference here is that no lattice is available, so as a substitute we use a finitely generated group that is not discrete. Our construction also has the benefit of demonstrating in an obvious way the fact (perhaps not widely known) that $\mathcal{H}_4$ includes a copy of the root system $\mathcal{D}_4$, and hence that there is a corresponding inclusion of the Weyl group D_4 as a subgroup of H_4.

Before proceeding, let us briefly describe what is probably the "standard" construction, as found in Exercise VI.4.12 of Bourbaki [**B**] and Section 2.13 of Humphreys [**H**]. In fact this construction is due to Witt [**W**], although explicit coordinates for the 120-cell and 600-cell go back at least to Schläfli in the 1850's and Schoute in the 1900's [**C**, §8.9]. One shows that every finite subgroup of $\mathbf{H}^*$ (the multiplicative group of the quaternions) that includes -1 is a root system, and then one miraculously produces a suitable 120-element subgroup that meets the requirements, the so-called *icosian group* [**CS**]. To verify that the icosians do form a group (or simply a root system) is rather tedious.

The icosian group can be rendered less mysterious by noting that the alternating group of degree five has a three-dimensional representation as rotational symmetries of the icosahedron. Lifting this from $SO(3)$ to Spin(3) yields a 120-element group, a double cover of

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the alternating group. However, Spin(3) is isomorphic to the unit-norm subgroup of $\mathbf{H}^*$, so the 120-element group embeds in $\mathbf{H}^*$. Admittedly, this may make the standard construction seem *more* miraculous, not less, and in any case leaves aside the unpleasant task of identifying explicit coordinates.

A New Construction

Let A be a subring of $\mathbf{R}$ (the real field), and L an A-submodule of the Euclidean space $\mathbf{R}^n$ with inner product $\langle\ ,\ \rangle$.

PROPOSITION. *If Φ is any finite subset of $L_2 = \{\alpha \in L : \langle\alpha,\alpha\rangle = 2\}$ that is maximal with respect to the property that $\langle\alpha,\beta\rangle \in A$ for all $\alpha,\beta \in \Phi$, then Φ is a root system.*

Proof. Let $\alpha,\beta \in \Phi$. The reflection of α through the hyperplane orthogonal to β is $\gamma = \alpha - \langle\alpha,\beta\rangle\beta$. Hence $\gamma \in L_2$, since L is an A-module and reflections preserve length. Furthermore, the inner product of γ with any other member of Φ is clearly in A, since α and β have this property. Therefore $\gamma \in \Phi$ by maximality. Hence every reflection through a hyperplane orthogonal to a member of Φ permutes Φ, so Φ is a root system. $\square$

Now, to construct $\mathcal{H}_4$, let $a = 2\cos(\pi/5) = (1+\sqrt{5})/2$ denote the golden ratio, $A = \mathbf{Z}[a]$, and $\varepsilon_1,\ldots,\varepsilon_4$ an orthonormal basis of $\mathbf{R}^4$. Let

$$\begin{aligned} L &= \{a_1\varepsilon_1 + \cdots + a_4\varepsilon_4 : a_i \pm a_j \in A \text{ for all } i,j\} \\ &= \{\alpha \in \mathbf{R}^4 : \langle\alpha,\beta\rangle \in A \text{ for all } \beta \in \mathcal{D}_4\}, \end{aligned}$$

where $\mathcal{D}_4 = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\}$ denotes the usual realization of the root system for the Weyl group D_4.

There are only finitely many members of L_2. Indeed, every member of L has the form

$$\alpha = \tfrac{1}{2}(m_1 + an_1)\varepsilon_1 + \cdots + \tfrac{1}{2}(m_4 + an_4)\varepsilon_4,$$

with $m_i, n_i \in \mathbf{Z}$, $m_1 = \cdots = m_4 \bmod 2$ and $n_1 = \cdots = n_4 \bmod 2$, and the Diophantine equation $\langle\alpha,\alpha\rangle = 2$ involves a positive definite quadratic form in the variables m_i, n_i. In fact, it is easy to check that the members of L_2 consist of all signed permutations of

$$\begin{aligned} \alpha_1 &= \varepsilon_1 + \varepsilon_2, \\ \alpha_2 &= \tfrac{1}{2}(1-a)(\varepsilon_1+\varepsilon_2+\varepsilon_3) + \tfrac{1}{2}(1+a)\varepsilon_4, \\ \alpha_3 &= \tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3) + (a - \tfrac{1}{2})\varepsilon_4, \\ \alpha_4 &= \tfrac{1}{2}a(\varepsilon_1+\varepsilon_2+\varepsilon_3) + (\tfrac{1}{2}a - 1)\varepsilon_4, \end{aligned}$$

for a total of $24 + 64 + 64 + 64 = 216$ vectors.

We cannot fit all of L_2 into a single root system of the sort described by the proposition, since it is not true that $\langle\alpha,\beta\rangle \in A$ for all $\alpha,\beta \in L_2$. For if $\alpha = a_1\varepsilon_1 + \cdots + a_4\varepsilon_4$ is any member of L_2, then the result of negating the coordinate ε_1 yields $\alpha' = \alpha - 2a_1\varepsilon_1 \in L_2$. Hence $\langle\alpha,\alpha'\rangle = 2 - 2a_1^2$, and therefore $\langle\alpha,\alpha'\rangle \in A$ if and only if $a_1^2 \in A/2$. However, one can easily check that the coordinates of $\alpha_2, \alpha_3, \alpha_4$ do not have this property, so any subset of L_2 whose pairwise inner products belong to A can contain at most half of the signed permutations of α_2, α_3, and α_4, leaving a maximum size of $24 + 32 + 32 + 32 = 120$.

On the other hand, the inner product of every $\alpha \in \mathcal{D}_4$ with every $\beta \in L_2$ belongs to A, by construction of L. Hence every subset Φ of L_2 that is maximal with respect to A-valued inner products must necessarily include all of $\mathcal{D}_4$, and must also form a root system, by the proposition. It follows that the root system Φ must be a union of D_4-orbits, with D_4 acting as an index-two subgroup of the group of all signed permutations of the coordinates. Therefore, by checking that the pairwise inner products of $\alpha_2, \alpha_3, \alpha_4$ belong to A, we may immediately conclude that

$$\mathcal{H}_4 := \mathcal{D}_4 \;\cup\; D_4\alpha_2 \;\cup\; D_4\alpha_3 \;\cup\; D_4\alpha_4$$

is a root system of order 120. A set of simple roots is given by $\varepsilon_2 - \varepsilon_1$, $\varepsilon_3 - \varepsilon_2$, $\varepsilon_4 - \varepsilon_3$, and $\frac{1}{2}(a+1)\varepsilon_1 - \frac{1}{2}(a-1)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, as can be verified by checking that the matrix of inner products is consistent with the geometry implied by the Coxeter diagram of H_4.

References

[**B**] N. Bourbaki, "Groupes et Algèbres de Lie, Chp. IV–VI," Masson, Paris, 1981.

[**CS**] J. H. Conway and N. J. A. Sloane, "Sphere packings, lattices and groups," 2nd ed., Springer-Verlag, New York, 1993.

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[**W**] E. Witt, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, *Abh. Math. Sem. Univ. Hamburg* **14** (1941), 289–322.