Admissible W-Graphs and Commuting Cartan Matrices

John R. Stembridge

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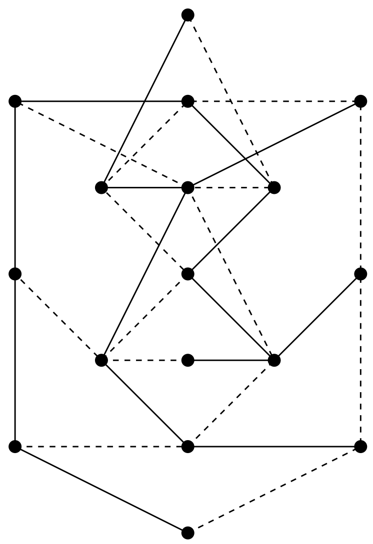
Contents



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0. Introduction

A W-graph is a weighted directed graph that encodes certain actions of a Coxeter group W or its Iwahori-Hecke algebra $\mathcal{H}(W)$. It is admissible if it is bipartite and has nonnegative integer edge weights that satisfy a simple symmetry condition. Of particular interest are the admissible W-graphs and $W \times W$-graphs that encode the one-sided and two-sided actions of the standard generators on the Kazhdan-Lusztig basis of $\mathcal{H}(W)$, as well as the strongly connected components of these graphs—the latter are the so-called Kazhdan-Lusztig cells.

In a previous paper [**S**], we posed the problem of classifying the admissible W-cells, and remarked on the possibility that there may only be finitely many such cells for each finite Coxeter group W. One of the complications related to this problem is the fact that there is no obvious way to reduce the classification of admissible cells for a direct product $W_1 \times W_2$ to those for its factors W_1 and W_2. For example, while there is a natural notion of outer tensor product for cells, it is not true that every admissible $W_1 \times W_2$-cell is a tensor product. Indeed, many two-sided Kazhdan-Lusztig cells cannot be expressed as the tensor product of two one-sided cells.

The dihedral groups $I_2(p)$ (i.e., the finite Coxeter groups of rank 2) provide an interesting illustration of these issues. As shown in [**S**], the nontrivial admissible $I_2(p)$-cells are the simply-laced Dynkin diagrams whose Coxeter number divides p. As we explain in this paper, classifying the admissible cells for a direct product $I_2(p) \times I_2(q)$ turns out to be equivalent to classifying pairs of simply-laced (but not necessarily irreducible) Cartan matrices that commute, have Coxeter numbers dividing p and q, and satisfy a simple parity condition.

The main result of this paper is a classification of these "admissible" pairs of Cartan matrices without any conditions on the Coxeter numbers (see Theorem 3.1). What we find is that they occur in 6 infinite families, along with 11 exceptional pairs whose sizes range from 12 to 32 vertices. A key ingredient of the proof is the Perron-Frobenius Theorem on the eigenvectors of nonnegative matrices. In particular, we make extensive use of the eigenvectors of Cartan matrices to identify or eliminate potential admissible pairs (see Lemma 4.5 and Remark 4.6).

After proving the classification theorem in Sections 4–5, we review the W-graph concept in Section 6, and describe precisely how the classification of admissible pairs of Cartan matrices enables the classification of admissible $I_2(p) \times I_2(q)$-cells (see Theorem 6.5). In Section 7, we discuss the relationship between admissible cells for a general direct product $W_1 \times W_2$ and admissible cells for the factors W_1 and W_2. In particular, we identify various sufficient conditions that enable one to bound the number of admissible $W_1 \times W_2$-cells (e.g., see Remark 7.5 and Corollary 7.7).

The final section is an appendix providing explicit coordinates for the dominant eigenvectors of simply-laced Cartan matrices.

1. Admissible Bigraphs

Given a simple graph Γ on the vertex set $[n] = \{1, \ldots, n\}$, let $A(\Gamma)$ denote the corresponding adjacency matrix. Thus $A(\Gamma)$ is a symmetric $0,1$-matrix in which a 1 in position (i, j) indicates the presence of an edge connecting vertices i and j.

We will be concerned here primarily with *ADE graphs*; i.e., disjoint (finite) unions of copies of the Dynkin diagrams A_n ($n \geqslant 1$), D_n ($n \geqslant 4$), E_6, E_7, and E_8. It is well known that the adjacency matrices of such graphs may be intrinsically characterized as the symmetric $0,1$-matrices with zero diagonals and eigenvalues < 2; or equivalently, as the matrices A such that $2 - A$ is a symmetric positive definite Cartan matrix.

DEFINITION 1.1. A *bigraph* is an ordered pair of simple graphs (Γ, Δ) that share a common vertex set. If Γ and Δ are both ADE graphs, then we say that (Γ, Δ) is an *ADE bigraph*. The pair (Γ, Δ) is *admissible* if

(i) the adjacency matrices $A(\Gamma)$ and $A(\Delta)$ commute, and
(ii) it has a 2×2 *vertex partition*; i.e., the vertices may be partitioned into four blocks $V(i, j)$ ($i, j \in \{0, 1\}$) so that every vertex of $V(i, j)$ is adjacent via edges of Γ only to vertices in $V(1 - i, j)$, and via edges of Δ only to vertices in $V(i, 1 - j)$.

The *dual* of the bigraph (Γ, Δ), denoted $(\Gamma, \Delta)^*$, is defined to be (Δ, Γ).

REMARK 1.2. It is convenient to regard a bigraph (Γ, Δ) as a single graph with colored edges: red for the edges of Γ, blue for the edges of Δ. In these terms, having adjacency matrices that commute may be reformulated in graph-theoretic terms as the property that for every pair of vertices u, v, the number of red-blue paths from u to v (i.e. two-step paths consisting of a red edge followed by a blue edge) equals the number of blue-red paths from u to v. Similarly, one may reformulate (ii) as the property that every closed path has an even number of red edges and an even number of blue edges.

Note that disjoint unions of admissible bigraphs are again admissible, so for classification purposes, it suffices to consider only those admissible bigraphs that are connected.

Assuming that (Γ, Δ) is connected, it is not hard to see that the 2×2-partition of its vertices is essentially unique. Indeed, one may arbitrarily assign a vertex to (say) the block $V(0, 0)$. Once the block containing a given vertex v is known, the conditions in (ii) determine the blocks containing each neighbor of v.

As noted in the introduction, our motivation for considering admissible bigraphs is related to the theory of W-graphs. More precisely, we shall see in Theorem 6.5 that for each pair of dihedral groups W_1 and W_2, every nontrivial admissible $W_1 \times W_2$-graph that is a cell (i.e., strongly connected) has the structure of a connected admissible ADE bigraph. Conversely, all such bigraphs arise in this way for some pair of dihedral groups.

EXAMPLE 1.3 (Tensor Product). Let Δ_1 and Δ_2 be simple bipartite graphs with vertex sets V_1 and V_2, respectively. Identifying graphs with their edge sets, we define the *tensor*

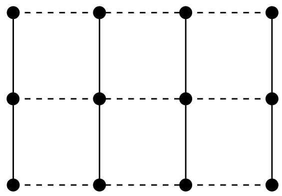

FIGURE 1.1: $A_3 \otimes A_4$.

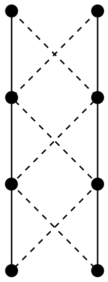

FIGURE 1.2: $A_4 \times A_4$.

product $\Delta_1 \otimes \Delta_2$ to be the bigraph (Γ_1, Γ_2) on the vertex set $V_1 \times V_2$ given by

$$\Gamma_1 = \big\{\{(u_1, u), (v_1, u)\} : u \in V_2,\ \{u_1, v_1\} \in \Delta_1\big\},$$
$$\Gamma_2 = \big\{\{(u, u_2), (u, v_2)\} : u \in V_1,\ \{u_2, v_2\} \in \Delta_2\big\}.$$

It is easy to see that the adjacency matrices of Γ_1 and Γ_2 are Kronecker products of the form $A(\Delta_1) \otimes I$ and $I \otimes A(\Delta_2)$, and hence they commute. Moreover, any pair of 2-colorings of Δ_1 and Δ_2 induces a 2×2-partition of $V_1 \times V_2$, and thus tensor products are admissible bigraphs. Recalling that ADE graphs are acyclic (hence bipartite), it follows that tensor products of connected ADE graphs yield connected admissible ADE bigraphs.

As a special case, note that $A_m \otimes A_n$ is a grid graph (see Figure 1.1[1]).

EXAMPLE 1.4 (Twists). Let Δ be a simple bipartite graph with vertex set V. Again identifying graphs with their edge sets, we define a Δ-*twist*, denoted $\Delta \times \Delta$, to be the bigraph (Γ_1, Γ_2) on the vertex set $V \times \{0, 1\}$ given by

$$\Gamma_1 = \big\{\{(u, i), (v, i)\} : i \in \{0, 1\},\ \{u, v\} \in \Delta\big\},$$
$$\Gamma_2 = \big\{\{(u, i), (v, 1 - i)\} : i \in \{0, 1\},\ \{u, v\} \in \Delta\big\}.$$

The adjacency matrices of Γ_1 and Γ_2 have the form

$$A(\Gamma_1) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad A(\Gamma_2) = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix},$$

and hence they commute. Moreover, if $V_0 \cup V_1$ is a partition of V induced by a 2-coloring of Δ, then the partition of $\Delta \times \Delta$ whose (i, j)-block consists of all vertices of the form (v, j) with $v \in V_{i+j}$ (subscript mod 2) is a 2×2-partition. It follows that if Δ is a connected ADE graph with at least two vertices, then $\Delta \times \Delta$ is a connected admissible ADE bigraph. Note that Γ_1 and Γ_2 both consist of two disjoint copies of Δ, although this would fail if we did not require Δ to be bipartite. (Indeed, Γ_2 is bipartite regardless of Δ.)

An A_4-twist is illustrated in Figure 1.2.

[1]In monochromatic versions of this article, we use solid and dotted lines to represent red and blue edges in the figures, respectively.

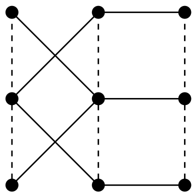

FIGURE 2.1: $A_5 * D_4$.

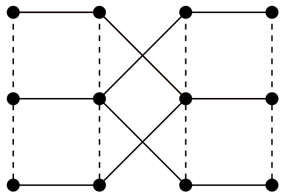

FIGURE 2.2: $E_6 * E_6$.

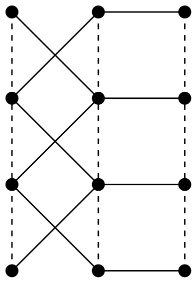

FIGURE 2.3: $D_6 * D_6$.

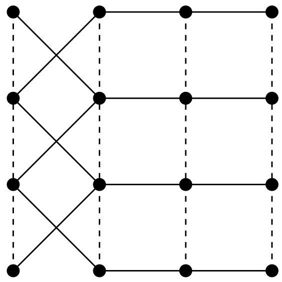

FIGURE 2.4: $E_8 * E_8$.

2. Bindings

Let (Γ, Δ) be a bigraph, and let $\Gamma_1, \ldots, \Gamma_\ell$ denote the connected components of Γ.

If $\ell = 2$ and every edge of Δ connects a vertex of Γ_1 to a vertex of Γ_2 (and there is at least one such edge), then we say that (Γ, Δ) is a *binding* of Γ_1 and Γ_2. For example, if Ψ is a connected bipartite graph with at least two vertices, then the twisted bigraph $\Psi \times \Psi$ provides a binding of two copies of Ψ. Similarly, the tensor product of Ψ with A_2 is also a binding of two copies of Ψ; we use the notation

$$(\Psi \equiv \Psi) \ := \ (\Psi \otimes A_2)$$

and refer to this as a *parallel* binding.

If Γ has only one component and Δ has at least one edge, we regard (Γ, Δ) as a *self-binding* of $\Gamma = \Gamma_1$. (Note that such bigraphs are not admissible.)

In general, (Γ, Δ) is completely determined by the bindings involving the connected components of Γ. To make this precise, it is convenient to introduce a *component graph* $\mathcal{C} = \mathcal{C}(\Gamma, \Delta)$ on the vertex set $[\ell]$ that has an edge $\{i, j\}$ (a loop, if $i = j$) if there is an edge of Δ that connects vertices of Γ_i and Γ_j. In these terms, one may recover (Γ, Δ) from the components Γ_i, the graph $\mathcal{C}$, and for each edge or loop $\{i, j\}$, the binding of Γ_i and Γ_j.

In this way, the notation $A_3 \times A_3 \equiv A_3$ specifies a bigraph (Γ, Δ) in which Γ consists of three copies of A_3, the component graph is a path of length 2, and there is one parallel and one twisted binding. As we have illustrated in Figure 2.1, the resulting graph Δ has two components, one isomorphic to A_5, and one isomorphic to D_4. Thus the dual bigraph $(A_3 \times A_3 \equiv A_3)^*$ provides a binding of A_5 and D_4. More generally, one may define a binding of A_{2n-1} and D_{n+1} $(n \geqslant 3)$ by setting

$$A_{2n-1} * D_{n+1} \ := \ (A_3 \times A_3 \equiv \cdots \equiv A_3)^* \qquad (n \text{ copies of } A_3). \tag{2.1}$$

This is sensible even when $n = 2$ if we identify D_3 with A_3 and $A_3 * D_3$ with $A_3 \times A_3$.

One may similarly construct bindings of two copies of E_6, D_6, or E_8 by dualizing strings of parallel and twisted bindings of A_3 or A_4 as follows:

$$\begin{aligned} E_6 * E_6 &:= (A_3 \equiv A_3 \times A_3 \equiv A_3)^*, \\ D_6 * D_6 &:= (A_4 \times A_4 \equiv A_4)^*, \\ E_8 * E_8 &:= (A_4 \times A_4 \equiv A_4 \equiv A_4)^*. \end{aligned}$$

These are illustrated in Figures 2.2–4.

Remark 2.1. A potential ambiguity in the above constructions arises from the possibility that specifying the isomorphism classes of each binding in (Γ, Δ) might not uniquely determine the isomorphism class of (Γ, Δ). However, given two copies of any graph Ψ, it is easy to see that the automorphism group of either copy of Ψ acts transitively on the set of parallel bindings between the two copies. Thus if the component graph of (Γ, Δ) is acyclic, and at most one binding is not parallel, then there is no ambiguity—specifying the isomorphism class of each binding suffices to specify the isomorphism class of (Γ, Δ).

Proposition 2.2. *An ADE bigraph is admissible if and only if all of its bindings are admissible bigraphs.*

Postponing the proof temporarily, note that each of the connected components in the above bindings (both red and blue) are *ADE* graphs, and we know that parallel and twisted bindings are admissible. Hence,

Corollary 2.3. *Each of the bindings $A_{2n-1} * D_{n+1}$, $E_6 * E_6$, $D_6 * D_6$, and $E_8 * E_8$ defined above are admissible ADE bigraphs.*

Lemma 2.4. *The adjacency matrices of a bigraph commute if and only if the adjacency matrices of its bindings (including self-bindings) commute.*

Proof. As noted in Remark 1.2, having $A(\Gamma)$ and $A(\Delta)$ commute is equivalent to a statement about independence of red-blue and blue-red path counts between pairs of vertices u, v in (Γ, Δ). However, if there are any such paths from u to v, then the intermediate vertices must each be adjacent via a red edge to either u or v, and hence all such paths are confined to a single binding. □

Lemma 2.5. *Let (Γ, Δ) be a bigraph with commuting adjacency matrices.*

(a) *If (Γ, Δ) is a binding of components Γ_1 and Γ_2, then every vertex of Γ_1 is adjacent along an edge of Δ to a vertex of Γ_2.*
(b) *If Δ is acyclic, then the component graph of (Γ, Δ) is acyclic.*

Proof. (a) Let $\{u, v\}$ be an edge of Δ with u in Γ_1 and v in Γ_2, and u' a vertex adjacent to u in Γ_1. In this situation, the triple (u', u, v) is a red-blue path in (Γ, Δ). Given that the adjacency matrices commute, there must also be a blue-red path of the form (u', v', v);

i.e., $\{u', v'\} \in \Delta$ and v' is a neighbor of v in Γ_2. Iterating, we reach a similar conclusion for the neighbors of u' in Γ_1, and so on.

(b) From (a), it follows that every path in the component graph $\mathcal{C}$ induces paths in Δ that visit the same components of Γ in the same order. Thus by traversing a cycle in $\mathcal{C}$ sufficiently many times, the induced paths in Δ must eventually repeat vertices. That is, the existence of a cycle in $\mathcal{C}$ forces the existence of a cycle in Δ. $\square$

Proof of Proposition 2.2. Let (Γ, Δ) be an ADE bigraph. If (Γ, Δ) is admissible, then the adjacency matrices of its bindings commute (Lemma 2.4), and the existence of a 2×2-partition of its vertices immediately provides a 2×2-partition for every sub-bigraph. Thus each binding is admissible.

For the converse, given that each binding of (Γ, Δ) is admissible, it follows that the adjacency matrices of Γ and Δ commute (Lemma 2.4 again), so it suffices to find a 2×2-partition of the vertices. Arguing by contradiction, we may suppose that (Γ, Δ) has a closed path with an odd number of red edges or an odd number of blue edges (recall Remark 1.2). Since Δ is an ADE graph, it is acyclic, and thus the component graph $\mathcal{C}$ of (Γ, Δ) is acyclic (Lemma 2.5(b)). It follows that the number of blue edges in any closed path is necessarily even (and nonzero, since Γ is an ADE graph and hence bipartite), so the number of red edges must be odd. Among all such closed paths, choose one with a minimum number of blue edges.

A second consequence of the fact that $\mathcal{C}$ is acyclic is that the sequence of components of Γ visited while traversing the path must "double back" at some point; i.e., the path has a segment of the form $(u_0, u_1, \ldots, u_l)$, where the first and last edges in this segment are blue, all others are red, and u_0, u_l belong to the same component of Γ, say Γ_1. If the distance from u_0 to u_l in Γ_1 is not $l \bmod 2$, we contradict the hypothesis that the binding between Γ_1 and the component containing $u_1, \ldots, u_{l-1}$, say Γ_2, has a 2×2 vertex partition. Thus there must be a red path from u_0 to u_l whose length is $l \bmod 2$. By replacing the segment $u_0, u_1, \ldots, u_l$ with this path, we obtain a closed path with an odd number of red edges and fewer blue edges, a contradiction. $\square$

3. The Classification

We are now in a position to describe the classification of admissible ADE bigraphs in terms of the bindings $A_{2n-1} * D_{n+1}$, $E_6 * E_6$, $D_6 * D_6$, and $E_8 * E_8$ of Section 2.

Theorem 3.1. *The connected, admissible ADE bigraphs are*

(a) *tensor products* $\Gamma \otimes \Delta$ *(where* Γ *and* Δ *are connected ADE graphs),*
(b) *twists* $\Psi \times \Psi$ *(where* Ψ *is a connected ADE graph of rank* $\geqslant 4$*),*
(c) $(AD^{m-1})_n := (A_{2n-1} * D_{n+1} \equiv \cdots \equiv D_{n+1})$ *(m factors;* $m, n \geqslant 2$*),*
(d) $(A^{m-1}D)_n := (A_{2n-1} \equiv \cdots \equiv A_{2n-1} * D_{n+1})$ *(m factors;* $m, n \geqslant 2$*),*
(e) $A_{2n-1} \equiv A_{2n-1} * D_{n+1} \equiv D_{n+1}$ $(n \geqslant 2)$,
(f) $E_6 * E_6 \equiv \cdots \equiv E_6$ *(n factors;* $n \geqslant 2$*),* $E_6 \equiv E_6 * E_6 \equiv E_6$,

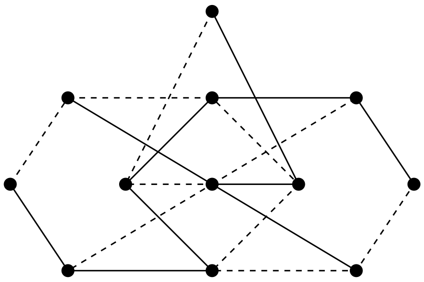

FIGURE 3.1: $D_5 \boxtimes A_7$.

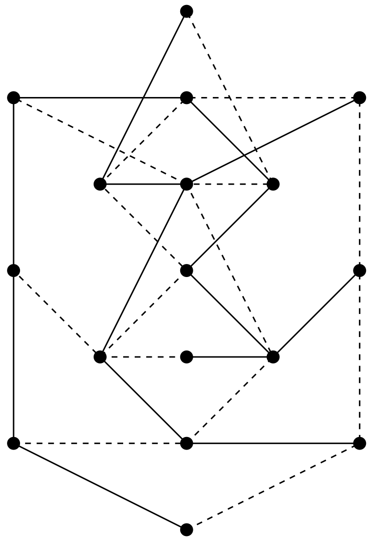

FIGURE 3.2: $E_7 \boxtimes D_{10}$.

(g) $A_4 \times A_4 \equiv A_4$, $\quad A_4 \times A_4 \equiv A_4 \equiv A_4$,
(h) $D_6 * D_6$, $\quad D_6 * D_6 \equiv D_6$, $\quad D_6 * D_6 \equiv D_6 \equiv D_6$,
(i) $E_8 * E_8$, $\quad E_8 * E_8 \equiv E_8$, $\quad E_8 * E_8 \equiv E_8 \equiv E_8$, *and*
(j) *the exceptional bigraphs* $D_5 \boxtimes A_7$ *and* $E_7 \boxtimes D_{10}$ *(see Figures 3.1 and 3.2).*

Note that the bigraphs listed in (c) and (d) coincide when $m = 2$ or $n = 2$. Indeed,

$$\begin{aligned}(AD^1)_n = (A^1 D)_n = (AD)_n = A_{2n-1} * D_{n+1},\\ (AD^{m-1})_2 = (A_3 \times A_3 \equiv \cdots \equiv A_3) \cong (A^{m-1}D)_2,\end{aligned} \tag{3.1}$$

and $(AD)_2$ (the case $m = n = 2$) is an A_3-twist.

Regarding dual relationships, it is clear that the dual of $\Gamma \otimes \Delta$ is $\Delta \otimes \Gamma$, and that twists are self-dual. Also, it is apparent from Figures 3.1 and 3.2 that $D_5 \boxtimes A_7$ and $E_7 \boxtimes D_{10}$ are self-dual. For the remaining admissible ADE bigraphs, we claim the following.

THEOREM 3.2. *We have the dual isomorphisms*

(a) $(AD^{m-1})^*_n \cong (AD^{n-1})_m$, $\quad (A^{m-1}D)^*_n \cong (A^{n-1}D)_m \quad (m, n \geqslant 2)$,
(b) $(A_{2n-1} \equiv A_{2n-1} * D_{n+1} \equiv D_{n+1})^* \cong (E_6 * E_6 \equiv \cdots \equiv E_6)$ $\quad$ *(n factors; $n \geqslant 2$)*,
(c) $(D_6 * D_6)^* \cong (A_4 \times A_4 \equiv A_4)$, $\quad (E_8 * E_8)^* \cong (A_4 \times A_4 \equiv A_4 \equiv A_4)$,
(d) $(E_8 * E_8 \equiv E_8)^* \cong (D_6 * D_6 \equiv D_6 \equiv D_6)$, *and*

the bigraphs $D_6 * D_6 \equiv D_6$, $E_6 \equiv E_6 * E_6 \equiv E_6$, *and* $E_8 * E_8 \equiv E_8 \equiv E_8$ *are self-dual.*

Note that (c) is true by definition; we include it for completeness.

While the proofs of Theorems 3.1 and 3.2 will be presented in the next two sections, some of the basic issues should be mentioned in advance.

First, to prove that the bigraphs listed in Theorem 3.1 are admissible and of ADE type, it suffices by Proposition 2.2 to show that in each case, the blue edges form ADE graphs

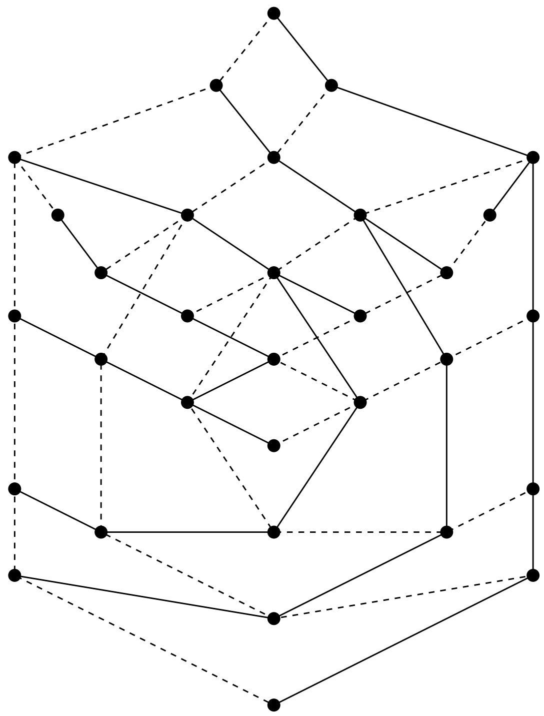

FIGURE 3.3: $E_8 * E_8 \equiv E_8 \equiv E_8$.

(the red edges are ADE by construction), and that all bindings are admissible. The latter follows in most cases from the reasoning in Section 2 (see especially Corollary 2.3). The two cases requiring further work are $D_5 \boxtimes A_7$ and $E_7 \boxtimes D_{10}$. It is easy to check that these bigraphs provide self-dual bindings of D_5 and A_7 (in the former case) and E_7 and D_{10} (in the latter case) and have 2×2 vertex partitions. However, we know no shortcut to verifying that they have commuting adjacency matrices—we leave this to the reader.

Second, we should also mention that it might seem to be tedious in some cases to verify that the blue edges form ADE graphs. For example, while it is easy to see that the bigraph in Figure 3.3 is self-dual and consists of eight copies of E_8 (four red, four blue), the reader may be reluctant to check that this bigraph is isomorphic to $E_8 * E_8 \equiv E_8 \equiv E_8$. However, during the course of proving that the list provided in Theorem 3.1 is exhaustive, it will become transparent that these bigraphs are indeed ADE.

	h	color type
A_{2n}	$2n+1$	$\{n, n\}$
A_{2n+1}	$2n+2$	$\{n+1, n\}$
D_{2n}	$4n-2$	$\{n+1, n-1\}$
D_{2n+1}	$4n$	$\{n+1, n\}$
E_6	12	$\{3, 3\}$
E_7	18	$\{4, 3\}$
E_8	30	$\{4, 4\}$

TABLE 4.1: Coxeter numbers and color types of ADE graphs.

4. Double Bindings

Proposition 2.2 provides a clear strategy for classifying the admissible ADE bigraphs: first classify all admissible ADE bindings, and then determine all of the ways they may be glued together along some (necessarily acyclic) component graph into an ADE bigraph.

The first of these steps may be further refined by the observation that even if (Γ, Δ) has at most two Γ-components, it may have more than two Δ-components. In such cases, the dual (Δ, Γ) may be decomposed into bindings of smaller bigraphs, and the duals of these bindings might be further decomposable, and so on.

Of course if Γ and Δ both have at most two connected components, then no such decompositions are possible. A particular instance of this occurs when (Γ, Δ) and (Δ, Γ) are both bindings; in such cases, we say that (Γ, Δ) is a *double binding*.

THEOREM 4.1. *The admissible ADE double bindings are twists $\Psi \times \Psi$ and the exceptional bindings $D_5 \boxtimes A_7$ and $E_7 \boxtimes D_{10}$.*

As a preliminary step, define the *color type* of a connected bipartite graph Ψ to be the multiset $\{a, b\}$, where a and b denote the number of vertices of each color in the (essentially) unique 2-coloring of Ψ. Table 4.1 lists the color types of the ADE graphs.

Now consider an admissible double binding (Γ, Δ). It is convenient to let Γ_0 and Γ_1 denote the connected components of Γ, and Δ_0 and Δ_1 the components of Δ. The virtue of this convention is that the (essentially) unique 2×2 vertex partition of (Γ, Δ) may be obtained by taking $V(i, j)$ to be the set of vertices that are endpoints of edges of both Δ_i and Γ_j. An immediate consequence of this observation is the following.

LEMMA 4.2. *If (Γ, Δ) is an admissible double binding, and the two components of Γ have color types $\{a, b\}$ and $\{c, d\}$, then the two components of Δ must have color types $\{a, c\}$ and $\{b, d\}$, or $\{a, d\}$ and $\{b, c\}$.*

Recall that if Ψ is a connected graph, then the Perron-Frobenius Theorem implies that the adjacency matrix $A(\Psi)$ has a dominant eigenvalue with multiplicity 1, the corresponding eigenvector $\mathbf{v}$ has positive coordinates, and $\mathbf{v}$ is the unique eigenvector of $A(\Psi)$ with nonnegative coordinates (up to scalar multiples).

LEMMA 4.3. *In a connected admissible bigraph* (Γ, Δ), *the dominant eigenvalues of the components of* Γ *are all equal, and hence also dually, the dominant eigenvalues of the components of* Δ *are all equal.*

Proof. By Lemma 2.4, we may assume that Γ has only two components, say Γ_0 and Γ_1. Letting λ_i and $\mathbf{v}_i$ denote the dominant eigenvalue and eigenvector for Γ_i, we have

$$A(\Gamma)A(\Delta)\mathbf{v}_0 = A(\Delta)A(\Gamma)\mathbf{v}_0 = \lambda_0 A(\Delta)\mathbf{v}_0.$$

That is, $A(\Delta)\mathbf{v}_0$ is in the λ_0-eigenspace of $A(\Gamma)$. Since $\mathbf{v}_0$ has positive coordinates and $A(\Delta)$ is nonzero (by connectivity), it follows that $A(\Delta)\mathbf{v}_0$ is nonzero and has nonnegative coordinates. However, $A(\Delta)$ maps vectors supported on one component of Γ to the other, so $A(\Delta)\mathbf{v}_0$ must be a multiple of the dominant eigenvector $\mathbf{v}_1$ and hence $\lambda_0 = \lambda_1$. $\square$

Recall that the Coxeter number h of an irreducible crystallographic root system may be defined as one more than the height of the highest root. It is known that the dominant eigenvalue of the associated Dynkin diagram is $2\cos(\pi/h)$ (see Exercise V.6.4 of [**B**]).

The Coxeter numbers of the ADE graphs are listed in Table 4.1.

COROLLARY 4.4. *In a connected admissible* ADE *bigraph* (Γ, Δ), *there exist positive integers* p, q *such that every component of* Γ *has Coxeter number* p *and every component of* Δ *has Coxeter number* q.

Let $\mathbf{v}(\Psi)$ denote the dominant eigenvector of the connected graph Ψ.

LEMMA 4.5. *If* (Γ, Δ) *is an admissible double binding, then for any two connected components* Γ_0 *of* Γ *and* Δ_0 *of* Δ, *the projections of* $\mathbf{v}(\Gamma_0)$ *and* $\mathbf{v}(\Delta_0)$ *onto the space spanned by the vertices common to* Γ_0 *and* Δ_0 *are proportional.*

Proof. Let $\mathbf{v}_0 = \mathbf{v}(\Gamma_0)$ and $\mathbf{v}_1 = \mathbf{v}(\Gamma_1)$. As vectors supported on Γ, we know that $\mathbf{v}_0$ and $\mathbf{v}_1$ have nonnegative coordinates, and from the proof of Lemma 4.3, we know that $A(\Delta)\mathbf{v}_i$ is a (necessarily positive) multiple of $\mathbf{v}_{1-i}$. By the Perron-Frobenius Theorem (or an easy 2×2 matrix calculation), it follows that some vector in the strictly positive span of $\mathbf{v}_0$ and $\mathbf{v}_1$ is an eigenvector of Δ. In other words, there is a vector $\mathbf{v}$ with strictly positive coordinates that is simultaneously an eigenvector for both Γ and Δ. It follows that the projection of $\mathbf{v}$ onto any connected component of either Γ or Δ is necessarily a (positive) multiple of the dominant eigenvector for that component, and thus the projection of $\mathbf{v}$ onto the subspace spanned by the vertices common to Γ_0 and Δ_0 is a multiple of the projections of the dominant eigenvectors for those components. $\square$

REMARK 4.6. To appreciate the power of the above condition, consider the problem of attempting to reconstruct an admissible double binding (Γ, Δ) from knowledge of the isomorphism classes of the components $\Gamma_0, \Gamma_1, \Delta_0, \Delta_1$. One may compute in advance the dominant eigenvectors of these components, thereby obtaining the common projections $\mathbf{u}_{ij}$ of $\mathbf{v}(\Gamma_j)$ and $\mathbf{v}(\Delta_i)$ guaranteed by Lemma 4.5. In case the (positive) coordinates within each projection $\mathbf{u}_{ij}$ are distinct, these coordinates in effect provide canonical numberings of the vertices, and specify the bigraph uniquely up to isomorphism. In any case, the only ambiguities in reconstructing the bigraph amount to permutations of the vertices that preserve the coordinates of the projections $\mathbf{u}_{ij}$.

LEMMA 4.7. *If* (Γ, Δ) *is an admissible ADE double binding such that the two components of* Γ *are not isomorphic, then* (Γ, Δ) *is isomorphic to* $D_5 \boxtimes A_7$ *or* $E_7 \boxtimes D_{10}$.

Proof. The two components of Γ must have the same Coxeter number (Corollary 4.4), so as a pair they must be isomorphic to one of $\{D_{n+1}, A_{2n-1}\}$, $\{E_6, D_7\}$, $\{E_6, A_{11}\}$, $\{E_7, D_{10}\}$, $\{E_7, A_{17}\}$, $\{E_8, D_{16}\}$, or $\{E_8, A_{29}\}$. Let $\{\{a,b\},\{c,d\}\}$ denote the color types of one of these pairs, arranged so that $a \geqslant c$, $a \geqslant b$, and $c \geqslant d$. Since there is a connected *ADE* graph with color type $\{p,q\}$ if and only if $|p-q| \leqslant 2$ (see Table 4.1), it follows from Lemma 4.2 that $a - c \leqslant 2$. This leaves the possibilities

$$\begin{array}{rlrl} \{A_5, D_4\} & \{\{3,2\},\{3,1\}\}, & \{A_7, D_5\} & \{\{4,3\},\{3,2\}\}, \\ \{A_9, D_6\} & \{\{5,4\},\{4,2\}\}, & \{A_{11}, D_7\} & \{\{6,5\},\{4,3\}\}, \\ \{A_{13}, D_8\} & \{\{7,6\},\{5,3\}\}, & \{D_7, E_6\} & \{\{4,3\},\{3,3\}\}, \\ \{D_{10}, E_7\} & \{\{6,4\},\{4,3\}\}. & & \end{array}$$

Since none of these color type pairs has the form $\{\{a,b\},\{a,b\}\}$ or $\{\{a,a\},\{b,b\}\}$, the two components of Δ cannot be isomorphic. Moreover, since the dual bigraph (Δ, Γ) is also an admissible *ADE* double binding, and the seven 4-tuples $\{a,b,c,d\}$ above are distinct from each other, it follows that the two components of Δ must be isomorphic as a pair to the two components of Γ.

If we index the components of Γ and Δ so that $\Gamma_0 \cong \Delta_0$ and $\Gamma_1 \cong \Delta_1$, then the 2×2 matrix whose (i,j)-entry is the number of vertices common to Γ_j and Δ_i is symmetric, and the entries of this matrix are a, b, c, d, in some order. In particular, at least two of these entries must be equal, so we may eliminate the pairs $\{A_{11}, D_7\}$ and $\{A_{13}, D_8\}$ from further consideration; the matrices of vertex cardinalities for the remaining pairs are

$$\begin{array}{c} A_5 \\ D_4 \end{array}\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}, \quad \begin{array}{c} A_7 \\ D_5 \end{array}\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}, \quad \begin{array}{c} A_9 \\ D_6 \end{array}\begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix}, \quad \begin{array}{c} D_7 \\ E_6 \end{array}\begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}, \quad \begin{array}{c} D_{10} \\ E_7 \end{array}\begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}.$$

Lemma 4.5 implies that there must be color classes of Γ_0 and Δ_1 such that the coordinates of $\mathbf{v}(\Gamma_0)$ and $\mathbf{v}(\Delta_1)$ within these color classes are proportional. From the eigenvector data provided in Table 8.1, one sees that this condition fails for the pairs $\{A_5, D_4\}$, $\{A_9, D_6\}$, and $\{D_7, E_6\}$. For example, the coordinates of the 4-vertex color classes of the dominant eigenvectors for A_9 and D_6 are proportional to $(1,1,\gamma,\gamma)$ and $(1,\gamma,\gamma,1+\gamma)$ (respectively), where $\gamma=(1+\sqrt{5})/2$.

The only remaining possibilities are $\{A_7, D_5\}$ and $\{D_{10}, E_7\}$.

In the case $\{\Gamma_0,\Gamma_1\}=\{A_7,D_5\}$, consider labeling the vertices of Γ as follows:

$$a - 1 - b - 2 - b - 1 - a \qquad \begin{array}{c} \phantom{1' - {}} 1' \phantom{{} - 2' - \alpha} \\ \phantom{1' - {}} | \phantom{{} - 2' - \alpha} \\ 1' - \beta - 2' - \alpha. \end{array}$$

This labeling has been chosen so that two vertices have the same label if and only if they are in the same color class of the same component, and they have equal coordinates in the corresponding dominant eigenvector. Moreover, as can be seen from data in Table 8.1, the eigenvector coordinates for the color classes with vertices labeled $1,2,1$ and $1',2',1'$ are proportional. By Lemma 4.5 (see also the discussion in Remark 4.6), it follows that if the edges of Δ are superimposed on the above vertices with the labels intact, there must be an isomorphism between Γ and Δ that interchanges all vertices labeled 1 with vertices labeled $1'$, similarly 2 with $2'$, and preserves all other vertex labels.

Consequently, in Δ, the vertex labeled α must be adjacent to the vertex labeled 2, the vertex labeled $2'$ must be adjacent to both of the vertices labeled b, and so on. The only ambiguity preventing the complete reconstruction of Δ is that the vertices labeled $1'$ must each be adjacent to vertices labeled a and b. Up to isomorphism, there are two ways to achieve this: each $1'$ vertex must be adjacent to vertices a and b that are at a distance 2 apart in Γ, or they are at distance 4. However, in the former case, there are two red-blue paths from a 1-vertex to a $1'$-vertex, but only one blue-red path (via β), contradicting the hypothesis that the adjacency matrices of Γ and Δ commute. This resolves the ambiguities in Δ, and it is easy to check that the resulting bigraph is $D_5 \boxtimes A_7$ as in Figure 3.1.

In the case $\{\Gamma_0,\Gamma_1\}=\{D_{10},E_7\}$, the analogous eigenvector-induced labeling of Γ is

$$\begin{array}{c} \phantom{b - {}} b \phantom{{} - d - 3 - c - 2 - b - 1 - a} \\ \phantom{b - {}} | \phantom{{} - d - 3 - c - 2 - b - 1 - a} \\ b - 4 - d - 3 - c - 2 - b - 1 - a \end{array} \qquad \begin{array}{c} \phantom{\gamma - 3' - \beta - {}} 2' \phantom{{} - \alpha - 1'.} \\ \phantom{\gamma - 3' - \beta - {}} | \phantom{{} - \alpha - 1'.} \\ \gamma - 3' - \beta - 4' - \alpha - 1'. \end{array}$$

Here, the coordinates of the dominant eigenvectors at the vertices labeled $1,2,3,4$ and $1',2',3',4'$ are correspondingly proportional, so there must be an isomorphism between Γ and Δ that interchanges i and i' $(1 \leqslant i \leqslant 4)$ and preserves all other vertex labels. The only repeated vertex label is b, so the only ambiguity preventing the reconstruction of Δ from this data is knowing which of the three vertices labeled b is adjacent in Δ to the vertices labeled $1'$ and $2'$; the other two b vertices must be adjacent only to $4'$. If the latter two vertices are the two that are also adjacent to 4 (in Γ), then there are three red-blue paths from 4 to $4'$ (one via d), but only two blue-red paths (via α and β). Hence, one of the two b vertices adjacent to 4 in Γ must be the vertex adjacent to $1'$ and $2'$, and up to isomorphism it does not matter which one. Thus Δ may be reconstructed, and it is not hard to check that the result is the bigraph $E_7 \boxtimes D_{10}$ displayed in Figure 3.2. $\square$

The following lemma completes the proof of Theorem 4.1.

LEMMA 4.8. *If* (Γ, Δ) *is an admissible ADE double binding such that the two components of* Γ *are isomorphic, then* (Γ, Δ) *is a twist* $\Psi \times \Psi$ *for some ADE graph* Ψ*.*

Proof. Since the dual (Δ, Γ) is also a double binding, and we are given that the components Γ_0 and Γ_1 of Γ are isomorphic, it follows that the components Δ_0 and Δ_1 of Δ must also be isomorphic; otherwise, we contradict Lemma 4.7. Moreover, since Δ_0 and Δ_1 are isomorphic, it follows that if $\{a, b\}$ is the color type of Γ_0 and Γ_1, then $\{a, b\}$ must also be the color type of Δ_0 and Δ_1 (Lemma 4.2).

Thus, the possibilities for $\{\Gamma_0, \Delta_0\}$ and their color types $\{a, b\}$ are

$$\begin{array}{llllll} \{A_{2n+1}, D_{2n+1}\} & \{n+1, n\}, & \{A_6, E_6\} & \{3,3\}, & \{A_7, E_7\} & \{4,3\}, \\ \{D_7, E_7\} & \{4,3\}, & \{A_8, E_8\} & \{4,4\}, & & \end{array}$$

or else Γ_0 and Δ_0 are isomorphic. By Lemma 4.5, the projections of $\mathbf{v}(\Gamma_0)$ onto the two color classes of Γ_0 must be proportional to the analogous projections of $\mathbf{v}(\Delta_0)$. Consulting the eigenvector data in the appendix, one sees that this constraint eliminates all but the possibility that $\Gamma_0, \Gamma_1, \Delta_0, \Delta_1$ are all isomorphic to some connected ADE graph Ψ. (For example, the coordinates of $\mathbf{v}(D_{2n+1})$ within the n-vertex color class are distinct, whereas for $\mathbf{v}(A_{2n+1})$, there is always at least one pair of equal coordinates in the n-vertex color class for $n \geqslant 2$.) To complete the proof, it therefore suffices to show that for each choice of Ψ, there is at most one possibility for (Γ, Δ) up to isomorphism (namely, a Ψ-twist).

Case 1: $\Psi = A_{2n+1}$. The sets of vertices of A_{2n+1} with equal dominant eigenvector coordinates are precisely the vertex orbits of the automorphism group. Thus if we use the labels $-n, \dots, -1, 0, 1, \dots, n$ and $-n', \dots, -1', 0', 1', \dots, n'$ for the vertices of Γ_0 and Γ_1 in path order (respectively), then Lemma 4.5 implies that vertex i must be adjacent in Δ to vertices $(i-1)'$ or $-(i-1)'$ (if $i > -n$) and $(i+1)'$ or $-(i+1)'$ (if $i < n$). Up to isomorphism, we claim that one may assume that i is Δ-adjacent to $(i-1)'$ and $(i+1)'$, and thus the isomorphism class of (Γ, Δ) is uniquely determined.

Arguing by induction with respect to $|i|$, note that the claim is trivial when $i = 0$, since 0 must be adjacent to $1'$ and $-1'$. Also, 1 and -1 must both be adjacent to $0'$, so for $i = 1$, the only ambiguity is whether 1 is adjacent to $2'$ and -1 to $-2'$, or vice-versa. We can force the former by an isomorphic relabeling of Γ_1 if necessary. Now suppose $i \geqslant 2$. By the induction hypothesis, $i-1$ is Δ-adjacent to i' and $(i-2)'$, so there are two blue-red paths from $i-1$ to $(i-1)'$. Hence there must also be two red-blue paths, which forces i to be adjacent to $(i-1)'$. Again by the induction hypothesis, there is a blue-red path from $i-1$ to $(i+1)'$ (via i'), so there must also be at least one red-blue path, which forces i to be adjacent to $(i+1)'$ and completes the induction.

Case 2: $\Psi = A_{2n}$. The graph A_{2n} has a color-reversing automorphism, so all of the ways of reassembling the four color classes of Γ_0 and Γ_1 into vertex sets for Δ_0 and Δ_1

are equivalent. Once this choice is fixed, the fact that the projection of the dominant eigenvector for A_{2n} onto either of its color classes has distinct coordinates (Corollary 8.2) leaves only one way to construct Δ that is consistent with Lemma 4.5.

Case 3: $\Psi = D_n$. Number the vertices of $\Gamma_0 \cong D_n$ from 1 to n as in (8.1), and similarly number the vertices of Γ_1 from $1'$ to n'. By Corollary 8.3, one sees that $\mathbf{v}(D_n)$ has equal coordinates at vertices 1 and 2, and the only other possible collision is that vertices 1, 2 and k have matching coordinates if $\cos((k-2)\pi/(2n-2)) = 1/2$; i.e., $k = 2(n+2)/3$. Note also that if k exists (i.e., $n = 1 \bmod 3$), it is necessarily even, and therefore belongs to the same color class as vertices 1 and 2.

Since vertices 1 and 2 have the same Γ-neighbors (both adjacent to 3), it follows that if k does not exist, there is only one way to construct Δ that is consistent with Lemma 4.5. Also, in the case $n = 4$, all three vertices (1, 2 and $k = 4$) have the same Γ-neighbors, so again Δ is uniquely determined. Finally, if $n = 1 \bmod 3$ and $k \geqslant 6$, then the only ambiguity in constructing Δ is knowing which of the three vertices $1, 2, k$ is Δ-adjacent to both $(k-1)'$ and $(k+1)'$, and similarly, which one of $1', 2', k'$ is Δ-adjacent to both $k-1$ and $k+1$. If (say) vertex 1 is Δ-adjacent to $(k-1)'$ and $(k+1)'$, then there are two blue-red paths from 1 to k', whereas 1 has only one Γ-neighbor, so there cannot be two red-blue paths from 1 to k'. Thus vertex k must be Δ-adjacent to $(k \pm 1)'$ and similarly k' must be Δ-adjacent to $k \pm 1$, whence Δ is uniquely determined.

Case 4: $\Psi = E_n$. The dominant eigenvectors for E_7 and E_8 have distinct coordinates, so there is only one way to construct (Γ, Δ) that is consistent with Lemma 4.5. For $\mathbf{v}(E_6)$, the sets of vertices with equal coordinates are precisely the vertex orbits of the automorphism group. If we number the vertices of $\Gamma_0 \cong E_6$ from 1 to 6 as in (8.2), and similarly number the vertices of Γ_1 from $1'$ to $6'$, the only ambiguities in reconstructing Δ via Lemma 4.5 are that vertices 3 and 5 must be adjacent in Δ to vertices $1'$ and $6'$ (in some order), and similarly, 1 and 6 must be Δ-adjacent to $3'$ and $5'$ (in some order). Replacing Γ_1 with an isomorphic relabeling if necessary, we may assume that $\{3, 1'\} \in \Delta$ and $\{5, 6'\} \in \Delta$. Since 3 and 5 are also Δ-adjacent to $4'$, it follows that there are two blue-red paths from 3 to $3'$ and from 5 to $5'$. In order to have equal numbers of red-blue paths, it is therefore necessary that $\{1, 3'\}, \{6, 5'\} \in \Delta$, whence Δ is uniquely determined. $\square$

5. Proof of the Classification

By Proposition 2.2 and a simple induction argument, it follows that the class of connected admissible ADE bigraphs is the smallest bigraph class $\mathcal{G}$ that

(i) contains $A_1 \otimes A_1$, $A_1 \otimes A_2$, and the admissible ADE double bindings,
(ii) contains all ADE bigraphs that can be obtained by gluing together bindings in $\mathcal{G}$ along some connected component graph, and
(iii) contains the dual of every bigraph (Γ, Δ) in $\mathcal{G}$ such that Δ has $\leqslant 2$ components.

Thus it suffices to show that the list of bigraphs provided in the statement of Theorem 3.1 satisfies (i)–(iii), and that each listed bigraph is indeed admissible and of ADE type.

Certainly Theorem 4.1 implies (i).

Proof of (ii). Consider a connected ADE bigraph (Γ, Δ) whose bindings are all listed in Theorem 3.1. Note that the sublist of bindings consists of

(1) the parallel and twisted bindings $\Psi \equiv \Psi$ and $\Psi \times \Psi$,
(2) $A_{2n-1} * D_{n+1}$, $E_6 * E_6$, $D_6 * D_6$, $E_8 * E_8$, and
(3) the exceptional bindings $D_5 \boxtimes A_7$ and $E_7 \boxtimes D_{10}$.

The fact that these bigraphs are admissible has been observed previously over the course of Sections 1–3. Note also that since ADE graphs are acyclic, we know that the component graph $\mathcal{C}$ of (Γ, Δ) must be a tree (Lemma 2.5(b)).

If every binding is parallel, then (Γ, Δ) is necessarily a tensor product of ADE graphs, so we may assume that at least one binding is not parallel. In particular, note that since $A_2 \times A_2$ is isomorphic to $A_2 \equiv A_2$, one should regard A_2-twists as parallel bindings.

If $\mathcal{C}$ has a node of degree $\geqslant 3$, then every vertex of the corresponding component Γ_0 of Γ must have Δ-degree $\geqslant 3$ (Lemma 2.5(a)). Moreover, in each of the non-parallel bindings listed above, every red component has a vertex whose blue degree is $\geqslant 2$. Independent of where the non-parallel bindings occur, it follows (again via Lemma 2.5(a)) that some component of Δ contains at least two vertices of Γ_0. However, this contradicts the fact that (connected) ADE graphs have at most one vertex of degree $\geqslant 3$.

Thus $\mathcal{C}$ must be a path with at least one non-parallel binding. We claim that there must be exactly one non-parallel binding. Toward a contradiction, suppose that there are at least two such bindings. Passing to a sub-bigraph if necessary, we may assume that there are exactly two, and that they occur at the ends of $\mathcal{C}$. Since Δ is acyclic, a maximal Δ-path through distinct vertices necessarily terminates at vertices with Δ-degree 1. These end vertices must occur in the components of Γ at the ends of $\mathcal{C}$, since the vertices in all other components necessarily have degree $\geqslant 2$ (Lemma 2.5(a) again). However, in all of the non-parallel bindings listed above, every blue component has at least two edges. Hence, the vertices one step away from the endpoints of a maximal Δ-path must have degree $\geqslant 3$ in Δ. Since ADE graphs have at most one vertex of degree $\geqslant 3$, this forces all maximal paths to have length 2, and hence all components of Δ are isomorphic to A_3 or D_4. This is a contradiction, since every component of Δ contains at least one blue component from every binding, and therefore has at least four edges.

Thus $\mathcal{C}$ is a path with exactly one non-parallel binding. If a red component of this binding has a vertex with blue degree 3 or two vertices with blue degree 2, then this component is necessarily one of the end nodes of $\mathcal{C}$. Otherwise, some component of Δ would have a vertex of degree 4 or two vertices of degree 3, and thus could not be an ADE graph. This eliminates $D_5 \boxtimes A_7$, $E_7 \boxtimes D_{10}$ and all twists except $A_3 \times A_3$ and $A_4 \times A_4$ from further consideration, since *both* red components of these bindings have this property.

Now suppose that the non-parallel binding is one of $A_4 \times A_4$, $D_6 * D_6$, or $E_8 * E_8$. Each of these bindings has blue components isomorphic to A_4. In such cases, one of the red components of the binding must be an end node of $\mathcal{C}$. Otherwise, if there were parallel bindings involving both red components, then the two non-leaf vertices of a blue A_4-component would both have degree 3 in Δ, and hence Δ could not be an ADE graph. In addition, the path $\mathcal{C}$ must have length $\leqslant 3$; otherwise, a Δ-component containing a blue copy of A_4 from the non-parallel binding would have a vertex of degree 3 and two branches of length $\geqslant 3$, an impossibility for ADE graphs. This leaves only the bigraphs

$$\begin{array}{lll} A_4 \times A_4, & A_4 \times A_4 \equiv A_4, & A_4 \times A_4 \equiv A_4 \equiv A_4, \\ D_6 * D_6, & D_6 * D_6 \equiv D_6, & D_6 * D_6 \equiv D_6 \equiv D_6, \\ E_8 * E_8, & E_8 * E_8 \equiv E_8, & E_8 * E_8 \equiv E_8 \equiv E_8, \end{array} \tag{5.1}$$

each of which is listed in Theorem 3.1.

The remaining possibility is that the non-parallel binding is $A_{2n-1} * D_{n+1}$ (this includes $A_3 \times A_3$ as the special case $n = 2$) or $E_6 * E_6$. Each of these bindings has blue components isomorphic to A_3. In such cases, if the non-parallel binding is at distance at least two from one endpoint of $\mathcal{C}$ and at least one from the other, then a component of Δ containing a blue copy of A_3 from the non-parallel binding would have a vertex of degree 3 and either three branches of length $\geqslant 2$ or two branches of length $\geqslant 3$, depending on how the copy of A_3 is situated. However, no ADE graph has either feature, so the non-parallel binding is either at an end of $\mathcal{C}$ or at distance one from both ends. This leaves the possibilities

$$\begin{gathered} (AD^{m-1})_n, \quad (A^{m-1}D)_n, \quad A_{2n-1} \equiv A_{2n-1} * D_{n+1} \equiv D_{n+1} \quad (m, n \geqslant 2), \\ E_6 * E_6 \equiv \cdots \equiv E_6, \quad E_6 \equiv E_6 * E_6 \equiv E_6. \end{gathered} \tag{5.2}$$

Each of these bigraphs is listed (and in the first two cases, defined) in parts (c) through (f) of Theorem 3.1. $\square$

Proof of (iii). Let (Γ, Δ) be one of the bigraphs listed in Theorem 3.1, and assume that Δ has $\leqslant 2$ connected components. Since tensor products and twists are dual-closed, and the bigraphs $D_5 \boxtimes A_7$ and $E_7 \boxtimes D_{10}$ are self-dual, we may assume that (Γ, Δ) is one of the bigraphs listed in (5.1) or (5.2). Each of these bigraphs has at least two Δ-components, so we need only consider those with exactly two. In such cases, the one non-parallel binding in (Γ, Δ) must also have exactly two blue components. Examining (5.1) and (5.2), we conclude that the non-parallel binding must be an A_4-twist or an A_3-twist. In the former case, (Γ, Δ) is either $A_4 \times A_4 \equiv A_4$ or $A_4 \times A_4 \equiv A_4 \equiv A_4$ (or a self-dual A_4-twist), and these are dual to $D_6 * D_6$ and $E_8 * E_8$ by definition. In the latter case, (Γ, Δ) is isomorphic to either $A_3 \equiv A_3 \times A_3 \equiv A_3$ or $(AD^{m-1})_2 \cong (A^{m-1}D)_2$, and these are dual to $E_6 * E_6$ and $A_{2m-1} * D_{m+1}$ by definition (recall (2.1) and (3.1)). $\square$

To confirm that the bigraphs listed in Theorem 3.1 are admissible and of ADE type, it remains only to show that the blue portions of each bigraph (Γ, Δ) listed in (5.1) and (5.2) are ADE (by Proposition 2.2). In the first group, all blue components within the non-parallel binding are isomorphic to A_4, and it is easy to check that the Δ-component containing any of these copies of A_4 will be isomorphic to either D_6 or E_8 if one or two parallel bindings are present.

In the second group, all blue components in the non-parallel binding are copies of A_3. If both red components of this binding also have parallel bindings attached, then the Δ-component containing a given blue copy of A_3 will be isomorphic to E_6. Otherwise, one of these red components, say Γ_0, is an endpoint of the component graph. If the component graph has m nodes, then the Δ-component containing a given blue copy of A_3 will either be A_{2m-1} or D_{m+1}, depending on whether this copy of A_3 contains only one vertex from Γ_0 (the former case) or two vertices (the latter case).

Thus all listed bigraphs are of ADE type and the proof of Theorem 3.1 is complete. $\square$

In the course of proving that the blue components of the bigraphs listed in (5.1) and (5.2) are ADE, we also identified the isomorphism classes of these components. For example, the blue components of the bigraphs listed in (5.1) are all isomorphic to A_4, D_6, or E_8, according to whether the component graph has either two, three, or four nodes.

For the remaining bigraphs listed in Theorem 3.1, it is a trivial matter to determine the isomorphism classes of the blue components, and a comparison of these isomorphism classes reveals the following unexpected result.

Theorem 5.1. *If* (Γ, Δ) *and* (Γ', Δ') *are connected, admissible* ADE *bigraphs, then they are isomorphic if and only if* $\Gamma \cong \Gamma'$ *and* $\Delta \cong \Delta'$.

Proof of Theorem 3.2. The binding $A_{2n-1} * D_{n+1}$ has $n-1$ blue copies of A_3 that each have only one vertex in the red copy of D_{n+1}, and one blue A_3 that has two vertices in the red copy of D_{n+1}. Thus by the above analysis, it follows that $(AD^{m-1})_n$ has $n-1$ blue copies of D_{m+1} and 1 blue copy of A_{2m-1}. The dual of $(AD^{m-1})_n$ must therefore be isomorphic to $(AD^{n-1})_m$, by Theorem 5.1. Similarly, the bigraph $E_8 * E_8 \equiv E_8 \equiv E_8$ has four red and four blue components, all isomorphic to E_8. Since it is the only connected admissible bigraph with this property, it must be self-dual. The remaining cases are similar and left to the reader. $\square$

6. Bigraphs and Cells

Let W be a Coxeter group with distinguished generators $s_1, \dots, s_n$, and let $T_1, \dots, T_n$ denote the corresponding generators of the associated Iwahori-Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ over the ground ring $\mathbb{Z}[q^{\pm 1/2}]$. Recall that a defining presentation of $\mathcal{H}$ consists of the quadratic relations $(T_i - q)(T_i + 1) = 0$ and the braid relations inherited from W.

Definition 6.1. A *W-graph* is a triple $\Gamma = (V, m, \tau)$ such that

(i) V is a vertex set (in this paper, always assumed to be finite),
(ii) $m : V \times V \to \mathbb{Z}[q^{\pm 1/2}]$ is a matrix of edge weights,

(iii) $\tau : V \to 2^{[n]}$ is an assignment of subsets of $[n]$ to the vertices, and

(iv) the free $\mathbb{Z}[q^{\pm 1/2}]$-module generated by V has an $\mathcal{H}$-module structure such that

$$T_i(u) = \begin{cases} qu & \text{if } i \notin \tau(u), \\ -u + q^{1/2} \sum_{v : i \notin \tau(v)} m(u,v)v & \text{if } i \in \tau(u) \end{cases} \tag{6.1}$$

for all vertices $u \in V$ and $1 \leqslant i \leqslant n$.

In addition, if $m(u,v) = 0$ whenever $\tau(u) \subseteq \tau(v)$, then we say that Γ is *reduced*.

Remark 6.2. (a) It is easy to check that any operator having the form of (6.1) satisfies the quadratic relation $(T_i - q)(T_i + 1) = 0$; the content of the above definition is that the braid relations should hold as well.

(b) The matrices representing the action of $\mathcal{H}$ in (6.1) are transposed from those used in the original W-graph definition due to Kazhdan and Lusztig [**KL**]. However, one should also note that this has no effect on what constitutes a W-graph.

(c) If $\tau(u) \subseteq \tau(v)$, then the value of $m(u,v)$ has no effect on the operators in (6.1). Thus there is no loss of generality in restricting our attention to reduced W-graphs.

(d) One should interpret the value of $m(u,v)$ as indicating the weight of an edge directed from u to v (or the absence of such an edge, if $m(u,v) = 0$). Since many of the W-graphs under consideration here will be reduced and have symmetric, 0,1-edge weights, one may thus view such W-graphs as vertex-labeled simple graphs.

(e) A W-graph is called a *cell* if it is strongly connected; i.e., there is a directed path from u to v and v to u for every pair of vertices u, v. The vertex set of any directed graph may be partitioned uniquely into strongly connected components (i.e., maximal vertex subsets that induce strongly connected subgraphs), and it is easy to show that in the case of a W-graph Γ, these components are themselves (cellular) W-graphs. Moreover, the $\mathcal{H}$-modules that they induce are subquotients of the $\mathcal{H}$-module induced by Γ. Thus one may view W-cells as the combinatorially irreducible W-graphs.

(f) If Γ is a reduced W-graph, then any vertex v with $\tau(v) = \varnothing$ or $\tau(v) = [n]$ necessarily has outdegree 0 or indegree 0, and therefore the singleton $\{v\}$ forms a cell of Γ. We say that such cells are *trivial*.

The main motivation for studying W-graphs concerns the Kazhdan-Lusztig basis of $\mathcal{H}$ and involves applications to the representation theory of complex semisimple Lie groups. This also extends to a more general setting that involves Kazhdan-Lusztig-Vogan polynomials and representations of real semisimple Lie groups (e.g., see [**LV**] and [**M**]).

Recall from [**KL**] that $\mathcal{H}$ has a distinguished basis $\{C_w : w \in W\}$ (the Kazhdan-Lusztig basis), and that the (transposed) matrices representing the action of the generators T_i on this basis has the form of a W-graph $\Gamma_L(W) = (W, m, \tau_L)$ with

$$\tau_L(v) := \{i : \ell(s_i v) < \ell(v)\} \qquad (v \in W),$$

where $\ell(\cdot)$ denotes the length function on W, and the edge weights are

$$m(u,v) := \mu(u,v) + \mu(v,u) \tag{6.2}$$

where $\mu(u,v)$ is defined to be the coefficient of $q^{(\ell(v)-\ell(u)-1)/2}$ in the Kazhdan-Lusztig polynomial $P_{u,v}(q)$. To convert Γ_W to reduced form, one should modify (6.2) so that

$$m(u,v) := \begin{cases} \mu(u,v) + \mu(v,u) & \text{if } \tau_L(u) \not\subseteq \tau_L(v), \\ 0 & \text{if } \tau_L(u) \subseteq \tau_L(v). \end{cases} \tag{6.3}$$

Note also that since $P_{v,v}(q) = 1$, and $P_{u,v}(q) \neq 0$ only if $u \leqslant v$ in the Bruhat order, we have either $\mu(u,v) = 0$ or $\mu(v,u) = 0$ for all $u, v \in W$ (often both).

More generally, letting $[n]' := \{1', \ldots, n'\}$ index the generators of a second copy of W, the same edge weight matrix as in (6.2) may be used to encode the left and right action of the generators on $\mathcal{H}$ as a $W \times W$-graph $\Gamma_{LR}(W) = (W, m, \tau_{LR})$, where

$$\tau_{LR}(v) := \{i : \ell(s_i v) < \ell(v)\} \cup \{i' : \ell(v s_i) < \ell(v)\}.$$

Again, in order to convert this $W \times W$-graph to reduced form, one should modify (6.2) so that $m(u,v) = 0$ whenever $\tau_{LR}(u) \subseteq \tau_{LR}(v)$.

Bearing in mind that (6.3) destroys the edge-symmetry in (6.2), we define a W-graph $\Gamma = (V, m, \tau)$ to be *edge-symmetric* if $m(u,v) = m(v,u)$ unless $\tau(u) \subseteq \tau(v)$ or $\tau(v) \subseteq \tau(u)$. In [**S**], we defined a W-graph $\Gamma = (V, m, \tau)$ to be *admissible* if it

(i) is bipartite,
(ii) has nonnegative edge weights, and
(iii) is reduced and edge-symmetric (in the above sense).

Our primary motivation for this definition is that it captures three essential combinatorial features of the Kazhdan-Lusztig W-graphs. Indeed, one knows that the Kazhdan-Lusztig polynomial $P_{u,v}(q)$ only involves integral powers of q, so $\mu(u,v)$ and $m(u,v)$ are nonzero only if $\ell(u) - \ell(v)$ is odd (hence (i) holds), and for finite or crystallographic (and conjecturally all) Coxeter groups, one knows that $P_{u,v}(q)$ has nonnegative integer coefficients.

Evidence accumulated so far (see [**S**]) raises the following.

QUESTION 6.3. *If W is finite, are there finitely many admissible W-cells?*

In case W is a dihedral Coxeter group $I_2(p)$, we know that the admissible W-cells may be described in terms of ADE graphs.

THEOREM 6.4 [**S**]. *The nontrivial admissible $I_2(p)$-cells (V, m, τ) are the connected ADE graphs having Coxeter number dividing p, with τ being a proper vertex 2-coloring using the labels $\{1\}$ and $\{2\}$, and the edges all being symmetric and of unit weight.*

For example (the case $p = 6$), there are seven nontrivial admissible G_2-cells up to isomorphism, corresponding to the two 2-colorings of the ADE graphs A_1, A_5, and D_4, and the unique 2-coloring of A_2. The nontrivial cells that occur in the Kazhdan-Lusztig graph $\Gamma_L(I_2(p))$ are the 2-colorings of A_{p-1}.

If $\Gamma = (U, m_1, \tau_1)$ is a W_1-graph, and $\Delta = (V, m_2, \tau_2)$ is a W_2-graph, with the generators of W_1 and W_2 being indexed by disjoint sets, then the *outer product* $\Gamma \otimes \Delta$ is defined to be the $W_1 \times W_2$-graph with vertex set $U \times V$ and

$$\tau(uv) := \tau_1(u) \cup \tau_2(v),$$
$$m(uv, u'v') := m_1(u, u')\delta_{v,v'} + m_2(v, v')\delta_{u,u'}$$

for all $u, u' \in U$ and $v, v' \in V$. It is not hard to see that if Γ and Δ are admissible, then so is $\Gamma \otimes \Delta$. In addition, if Γ and Δ are simple graphs (i.e., symmetric unit edge weights), then $\Gamma \otimes \Delta$ is a tensor product in the sense of Example 1.3 (so our notation is consistent).

As noted in the introduction, there exist admissible $W_1 \times W_2$-cells that are not outer products. Thus if Question 6.3 is to be resolved affirmatively, it is not sufficient to focus only on the irreducible Coxeter groups. As an illustration of this, we claim that the admissible cells for the direct product of two dihedral groups, say $I_2(p) \times I_2(q)$, are admissible ADE bigraphs. To state this more precisely, let us index the generators so that $\{s_1, s_2\}$ generates $I_2(p)$ and $\{s_3, s_4\}$ generates $I_2(q)$.

Theorem 6.5. *The admissible $I_2(p) \times I_2(q)$-cells are*

(a) *outer products of admissible $I_2(p)$-cells with trivial $I_2(q)$-cells (or vice-versa), and*
(b) *connected admissible ADE bigraphs with respective Coxeter numbers dividing p and q, symmetric unit edge weights (both red and blue), and vertex labels assigned so that $V(i, j) := \{v : \tau(v) = \{i + 1, j + 3\}\}$ $(i, j \in \{0, 1\})$ is a 2×2-partition.*

Thus for example, the bigraphs $A_{2n-1} * D_{n+1}$ and $E_8 * E_8 \equiv E_8 \equiv E_8$ yield admissible cells for $I_2(2n) \times I_2(4)$ and $I_2(30) \times I_2(30)$, respectively. Also, the bigraph corresponding to the nontrivial $I_2(p) \times I_2(p)$-cell in $\Gamma_{LR}(I_2(p))$ is the twisted product $A_{p-1} \times A_{p-1}$.

Proof. Let $\Gamma = (V, m, \tau)$ be an admissible $I_2(p) \times I_2(q)$-cell. By the Compatibility Rule (see Proposition 4.1 of [**S**]), we know that if there is an edge directed from u to v, then for every $i \in \tau(u) \setminus \tau(v)$ and $j \in \tau(v) \setminus \tau(u)$, the corresponding generators s_i and s_j may not commute. Moreover, $\tau(u) \setminus \tau(v)$ is necessarily nonempty (otherwise, Γ would not be reduced), so exactly one of the following holds:

(i) $\tau(v)$ is a proper subset of $\tau(u)$,
(ii) $\tau(u) \setminus \tau(v) = \{1\}$ and $\tau(v) \setminus \tau(u) = \{2\}$ (or vice-versa), or
(iii) $\tau(u) \setminus \tau(v) = \{3\}$ and $\tau(v) \setminus \tau(u) = \{4\}$ (or vice-versa).

Now consider the possibility that $\tau(u)$ omits both 3 and 4. If so, then the same must be true for $\tau(v)$, and hence also for the τ-label of every vertex, since Γ is strongly connected. It follows that the restriction of Γ to $I_2(q)$ is (after reduction) a disjoint union of copies of a trivial $I_2(q)$-cell Δ, while the restriction of Γ to $I_2(p)$ involves no edge deletions, and thus consists of a single $I_2(p)$-cell. Furthermore, Γ is the outer product of this admissible $I_2(p)$-cell with Δ.

On the other hand, if $\tau(v)$ includes both 3 and 4, then the same must be true for $\tau(u)$, and hence also for the τ-label of every vertex. Similar reasoning shows that again, Γ is isomorphic to the outer product of an admissible $I_2(p)$-cell and a trivial $I_2(q)$-cell.

Symmetric considerations apply if there is a vertex of Γ whose τ-label omits or includes both 1 and 2, so the remaining possibility is that the τ-label of every vertex includes exactly one member of $\{1, 2\}$ and one member of $\{3, 4\}$. In particular, there are no edges of type (i) and all edge weights are therefore symmetric. It follows that we can impose a connected bigraph structure on Γ by declaring the edges of types (ii) and (iii) to be red and blue, respectively. Furthermore, it is easy to see that the Compatibility Rule implies that one may construct a 2×2 vertex partition for Γ by setting

$$V(i, j) = \{v \in V : \tau(v) = \{i + 1, j + 3\}\} \quad (i, j \in \{0, 1\}). \tag{6.4}$$

Restriction of Γ to $I_2(p)$ or $I_2(q)$ amounts to deleting the blue or red edges (respectively), and thus yields a disjoint union of admissible $I_2(p)$-cells or $I_2(q)$-cells. By Theorem 6.4, we conclude that Γ is an ADE bigraph whose red and blue Coxeter numbers divide p and q (respectively), and that all edges have unit weight. To see that the adjacency matrices for the red and blue edges commute, note that these matrices are encoded by the actions of the commuting elements $q^{-1/2}(T_1+T_2)-(q^{1/2}-q^{-1/2})$ and $q^{-1/2}(T_3+T_4)-(q^{1/2}-q^{-1/2})$ in the $\mathcal{H}$-module corresponding to Γ (see (6.1)). Thus Γ is admissible as a bigraph.

Conversely, suppose that $\Gamma = (\Gamma_1, \Gamma_2)$ is a connected admissible ADE bigraph with 2×2 vertex partition $V(i, j)$, and that the Coxeter numbers of the components of Γ_1 and Γ_2 divide p and q, respectively. If the edges are taken to have unit weight and τ-labels are assigned so as to be consistent with (6.4), then Theorem 6.4 implies that one obtains admissible $I_2(p)$-graphs and $I_2(q)$-graphs from suitable restrictions of Γ. To conclude that Γ is a (necessarily admissible) $I_2(p) \times I_2(q)$-cell, it remains to establish that in the setting of (6.1), the operators T_1 and T_2 commute with T_3 and T_4.

A simple calculation (or application of the $r = 2$ case of Lemma 4.6 in [**S**]) reveals that for any bigraph with a τ-labeling induced by a 2×2-partition as in (6.4), the coefficient of v in $q^{-1}(T_1T_3 - T_3T_1)(u)$ vanishes unless $\tau(u) = \{1, 3\}$ and $\tau(v) = \{2, 4\}$, in which case it is the difference between the number of blue-red paths and red-blue paths from u to v. Thus given that the adjacency matrices of Γ_1 and Γ_2 commute, it follows that T_1 and T_3 commute, and the remaining commutativity relations follow by symmetry. □

7. Remarks on Admissible Two-Sided Cells

Let W_1 and W_2 be two finite Coxeter groups. As noted in the previous section, there exist admissible $W_1 \times W_2$-cells that are not outer products, and it is a delicate question to resolve whether or not there are finitely many admissible $W_1 \times W_2$-cells, given that there are finitely many for W_1 and W_2.

To explore this in more detail, let us consider an admissible $W_1 \times W_2$-cell $\Gamma = (V, m, \tau)$. For each vertex v, we let $\tau_1(v)$ and $\tau_2(v)$ denote those members of $\tau(v)$ that index generators of W_1 and W_2, respectively. By the Compatibility Rule in [**S**] (cited previously in the proof of Theorem 6.5) and the fact that admissible cells are reduced, it follows that an edge directed from u to v in Γ has one of three possible types:

Type 1: $\tau_1(u) \not\subseteq \tau_1(v)$ and $\tau_2(u) = \tau_2(v)$.
Type 2: $\tau_1(u) = \tau_1(v)$ and $\tau_2(u) \not\subseteq \tau_2(v)$.
Type 12: $\tau_1(u) \supsetneq \tau_1(v)$ and $\tau_2(u) \supsetneq \tau_2(v)$.

To view Γ as a (reduced) W_1-graph, one should delete all of the type 2 edges, and use the vertex labeling provided by τ_1. Subsequently, one may partition the vertices of Γ into admissible W_1-cells; similar remarks apply to type 1 edges, τ_2-labels, and W_2-cells.

Proposition 7.1. *In an admissible $W_1 \times W_2$-cell, the value of τ_2 is constant within a W_1-cell. In particular, the endpoints of every edge of type 12 belong to distinct W_1-cells and distinct W_2-cells.*

Proof. Note that τ_2 is monotone decreasing along any edges of type 1 or type 12, and hence must be constant along a closed path composed of such edges. □

Lemma 7.2. *For every pair of vertices u, v in an admissible $W_1 \times W_2$-cell Γ, there is a directed path from u to v in which all edges of type 1 and 12 precede all edges of type 2.*

Proof. Since Γ is strongly connected, there must be a directed path from u to v. Among all shortest paths from u to v, choose one that minimizes the number of type 2 edges. Now suppose that this path has a two-step segment $v_1 \to v_2 \to v_3$ in which a type 2 edge precedes a type 1 or type 12 edge. The first edge is part of the W_2-restriction of Γ, and the second edge is part of the W_1-restriction, so there must be commuting generators $T_i \in \mathcal{H}(W_2)$ and $T_j \in \mathcal{H}(W_1)$ such that the coefficients of v_2 in $(1+T_i)(v_1)$ and v_3 in $(1+T_j)(v_2)$ are nonzero. Recalling that admissible edge weights are positive, it follows that the coefficient of v_3 in $(1+T_i)(1+T_j)(v_1)$ is also nonzero, so there must be a two-step path $v_1 \to v_2' \to v_3$ such that the first edge is part of the W_1-restriction of Γ and the second edge is part of the W_2-restriction. The first edge could be of type 1 or 12, but the second edge must be of type 2, otherwise we contradict the minimality of type 2 edges. In this way, by repeated switches we can reduce the total number of $2, 1$ and $2, 12$ inversions until all type 2 edges appear at the end of the path. □

If the strongly connected components of a directed graph are contracted to points, one obtains an acyclic directed graph. Passing to the transitive closure, one obtains a poset. In this way, one may regard the W_1-cells in a $W_1 \times W_2$-graph as being partially ordered.

Lemma 7.3. *If Γ_0 is minimal (respectively, maximal) in the partial ordering of W_1-cells in an admissible $W_1 \times W_2$-cell Γ, then for every vertex v in Γ, there is a vertex u in Γ_0 and a directed path from u to v (respectively, v to u) through edges of type 2.*

Proof. Let v be an arbitrary vertex in Γ, and fix a vertex u_0 in Γ_0. By Lemma 7.2, there is a directed path from u_0 to v in which all edges of type 2 occur last. Edges of the other types are present in the W_1-restriction of Γ, so if Γ_0 is minimal, no such edge can exit Γ_0. Thus if u is the last vertex of Γ_0 along this path, then the remainder of the path from u to v traverses only type 2 edges. Similarly, if Γ_0 is maximal, then there cannot be any edges of type 1 or 12 directed into Γ_0 except for edges that are internal to Γ_0. Thus by following a directed path from v to u_0 along a path in which the edges of type 2 occur first (the existence of which follows from an obvious variation of Lemma 7.2), the last type 2 edge must terminate at some vertex u in Γ_0. $\square$

Let $\tau(\Gamma) := \{\tau(v) : v \in V\}$ denote the τ-support of $\Gamma = (V, m, \tau)$.

Proposition 7.4. *Let Γ be an admissible $W_1 \times W_2$-cell.*

(a) *Every maximal or minimal W_1-cell Γ_0 in Γ has full τ_1-support; i.e., $\tau_1(\Gamma_0) = \tau_1(\Gamma)$.*
(b) *Every minimal W_2-cell in Γ meets every maximal W_1-cell.*

Proof. (a) Apply Lemma 7.3, noting that τ_1 is constant along type 2 edges.

(b) Choose a vertex v in some minimal W_2-cell. Since a type 2 edge cannot exit a minimal W_2-cell, it follows that the set of vertices reachable from v by a directed path of type 2 edges must belong to the same W_2-cell. By Lemma 7.3, this set includes vertices from every maximal W_1-cell. $\square$

Remark 7.5. (a) Considering part (a) of the above proposition, it is natural to arrange the admissible $W_1 \times W_2$-cells Γ into families according to their τ_1-support and τ_2-support. These supports must match those of some admissible W_1-cell and W_2-cell, respectively.

(b) If the partial ordering of (say) the W_2-cells in an admissible $W_1 \times W_2$-cell Γ is an antichain, then every W_2-cell is both minimal and maximal. In that case, part (b) of the above proposition implies that every maximal or minimal W_1-cell meets *all* of the W_2-cells, and hence Γ has at most $N_1 N_2$ vertices, where N_1 denotes the minimum size of any maximal or minimal W_1-cell, and N_2 denotes the maximum size of any W_2-cell in Γ.

(c) A condition sufficient to force the W_2-cells in Γ to form an antichain is that every admissible W_2-graph with two related cells $\Gamma_1 < \Gamma_2$ whose τ_2-support matches $\tau_2(\Gamma)$ must have a third cell Γ_3 whose τ_2-support is not a subset of $\tau_2(\Gamma)$ and satisfies $\Gamma_1 < \Gamma_3 < \Gamma_2$.

(d) In an admissible W_2-graph with an edge $u \to v$ between distinct W_2-cells, edge-symmetry forces $\tau_2(u)$ to properly contain $\tau_2(v)$. Thus, a sufficient (but not necessary) condition for (c) is that $\tau_2(\Gamma)$ has no proper inclusions.

The following result eliminates the possibility of unbounded edge weights.

Proposition 7.6. *There are only finitely many admissible $W_1 \times W_2$-cells on N vertices that contain a given maximal W_1-cell and minimal W_2-cell.*

Proof. Let $\Gamma = (V, m, \tau)$ be an N-vertex admissible $W_1 \times W_2$-cell containing a maximal W_1-cell Γ_1 and a minimal W_2-cell Γ_2. Of course there are only finitely many (unweighted) graphs on N vertices, and finitely many ways to assign index sets to their vertices, so we need only to establish a bound on the weight of each edge $u \to v$ in Γ.

By Lemma 7.3, we can find paths of length $< N$ through type 2 edges from v to a vertex v_0 in Γ_1, and through type 1 edges from some vertex u_0 in Γ_2 to u. It follows that there exist monomials t_1 and t_2 of degree $< N$ in the elements $X_i := q^{-1/2}(1+T_i)$ of $\mathcal{H}(W_1)$ and $\mathcal{H}(W_2)$ (respectively) such that the coefficients of u in $t_1(u_0)$ and v_0 in $t_2(v)$ are positive integers. Also, there is an index j such that the coefficient of v in $X_j(u)$ is $m(u,v)$. Thus, the coefficient of $q^0 v_0$ in $t_2 X_j t_1(u_0)$ is at least $m(u,v)$.

We may assume that T_j and X_j belong to (say) $\mathcal{H}(W_2)$.

In that case, since $\mathcal{H}(W_1)$ and $\mathcal{H}(W_2)$ commute, we know that $m(u,v)$ is bounded by the coefficient of $q^0 v_0$ in $t_1 t_2 X_j(u_0)$. However, u_0 is in the minimal W_2-cell Γ_2, and thus $\mathcal{H}(W_2)u_0$ is confined to the span of Γ_2. Similarly, v_0 is in the maximal W_1-cell Γ_1, so the coefficient of v_0 in every element of $\mathcal{H}(W_1)u$ is 0 unless u is a vertex of Γ_1. Since there is an absolute bound (depending only on N and Γ_2) on the coefficients in $t_2 X_j(u_0)$, and there is a similar bound on the coefficient of $q^0 v_0$ in $t_1(u)$ for all u in Γ_1, it follows that $m(u,v)$ is also bounded. □

Corollary 7.7. *If there are finitely many admissible W_2-cells with τ_2-support $\mathcal{T}$, then there are finitely many admissible $W_1 \times W_2$-cells with τ_2-support $\mathcal{T}$ that contain a given maximal or minimal W_1-cell and whose W_2-cells form an antichain.*

Proof. As noted in Remark 7.5(b), the number of vertices in such a cell has an upper bound of $N_1 N_2$, where N_1 denotes the size of the given W_1-cell, and N_2 is the maximum size of an admissible W_2-cell with τ_2-support $\mathcal{T}$. Apply Proposition 7.6. □

Note that the dihedral groups $I_2(p)$ and the Weyl group S_4 both have the property that the τ-supports of their admissible cells have no proper inclusions (Theorem 6.4 above and Remark 4.5 of [**S**], respectively), and thus satisfy the condition discussed in Remark 7.5. Since there are finitely many such cells, we obtain

Corollary 7.8. *If there are finitely many admissible cells for W, then the same is true for $W \times I_2(p)$ and $W \times S_4$.*

8. Appendix: Dominant Eigenvectors for ADE graphs

Recall that $\mathbf{v}(\Psi)$ denotes the dominant eigenvector of a (connected) graph Ψ.

Proposition 8.1. *If Ψ is a connected ADE graph and $i_1, \ldots, i_p$ is a path through distinct vertices of Ψ such that i_1 has degree 1 and $i_2, \ldots, i_{p-1}$ have degree 2, then $\mathbf{v}(\Psi)$ may be scaled so that the coordinates in positions $i_1, i_2, \ldots, i_p$ are $\sin\theta, \sin 2\theta, \ldots, \sin p\theta$, where $\theta = \pi/h$ and h is the Coxeter number.*

Proof. This follows from the trigonometric identity

$$\sin(k-1)\theta + \sin(k+1)\theta = 2\cos\theta \sin k\theta$$

and the previously noted fact that $2\cos\theta$ is the dominant eigenvalue of Ψ. □

Corollary 8.2. *If the vertices of A_n are numbered $1 - 2 - \cdots - n$, then*

$$\mathbf{v}(A_n) = \Big(\sin\Big(\frac{\pi}{n+1}\Big),\ \sin\Big(\frac{2\pi}{n+1}\Big),\ \ldots,\ \sin\Big(\frac{n\pi}{n+1}\Big)\Big).$$

Corollary 8.3. *If the vertices of D_n are numbered*

$$\begin{array}{c} \quad 2 \\ \quad | \\ 1 - 3 - 4 - \cdots - n, \end{array} \tag{8.1}$$

then $\mathbf{v}(D_n) = (1/2,\, 1/2,\, \cos\theta,\, \cos 2\theta,\, \ldots,\, \cos(n-2)\theta)$, *where* $\theta = \pi/(2n-2)$.

Proof. Partition D_n into three paths terminating at vertex 3. Now apply Proposition 8.1 along each path, rescaling the coordinates so that each path terminates with the value $\cos\theta$. (Note also that $\sin k\theta = \cos(n-k-1)\theta$.) □

In a similar way, one obtains

Corollary 8.4. *If the vertices of E_n $(n = 6, 7, 8)$ are numbered*

$$\begin{array}{c} \qquad 2 \\ \qquad | \\ 1 - 3 - 4 - 5 - \cdots - n, \end{array} \tag{8.2}$$

and $\theta = \pi/h$ (where $h = 12, 18$ and 30, respectively), then

$$\mathbf{v}(E_n) = \Big(\frac{\sin\theta}{\sin 3\theta},\ \frac{\sin\theta}{\sin 2\theta},\ \frac{\sin 2\theta}{\sin 3\theta},\ 1,\ \frac{\sin(n-2)\theta}{\sin(n-3)\theta},\ \ldots,\ \frac{\sin\theta}{\sin(n-3)\theta}\Big).$$

To facilitate applications of Lemma 4.5, in Table 8.1 we have provided projections of the dominant eigenvectors for ADE graphs onto their respective color classes. The projections are rescaled so that the smallest coordinate within each color class is 1, and then the coordinates are sorted into increasing order, one line per color class. The vertex number (following the above conventions) is also appended to each coordinate, in parentheses.

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Department of Mathematics, University of Michigan, Ann Arbor MI 48109–1043 USA
E-mail address: jrs@umich.edu

A_4	1.000(1)	1.618(3)				
	1.000(4)	1.618(2)				
A_5	1.000(5)	1.000(1)	2.000(3)			
	1.000(4)	1.000(2)				
A_6	1.000(1)	1.802(5)	2.247(3)			
	1.000(6)	1.802(2)	2.247(4)			
A_7	1.000(1)	1.000(7)	2.414(3)	2.414(5)		
	1.000(2)	1.000(6)	1.414(4)			
A_8	1.000(1)	1.879(7)	2.532(3)	2.879(5)		
	1.000(8)	1.879(2)	2.532(6)	2.879(4)		
A_9	1.000(9)	1.000(1)	2.618(7)	2.618(3)	3.236(5)	
	1.000(8)	1.000(2)	1.618(6)	1.618(4)		
A_{10}	1.000(1)	1.919(9)	2.682(3)	3.229(7)	3.513(5)	
	1.000(10)	1.919(2)	2.682(8)	3.229(4)	3.513(6)	
D_4	1.000(1)	1.000(2)	1.000(4)			
	1.000(3)					
D_5	1.000(1)	1.000(2)	1.414(4)			
	1.000(5)	2.414(3)				
D_6	1.000(6)	1.618(1)	1.618(2)	2.618(4)		
	1.000(5)	1.618(3)				
D_7	1.000(1)	1.000(2)	1.000(6)	1.732(4)		
	1.000(7)	2.732(5)	3.732(3)			
D_8	1.000(8)	2.247(1)	2.247(2)	2.802(6)	4.049(4)	
	1.000(7)	1.802(5)	2.247(3)			
D_9	1.000(8)	1.307(1)	1.307(2)	1.848(6)	2.414(4)	
	1.000(9)	2.848(7)	4.262(5)	5.027(3)		
D_{10}	1.000(10)	2.879(1)	2.879(2)	2.879(8)	4.411(6)	5.411(4)
	1.000(9)	1.879(7)	2.532(5)	2.879(3)		
E_6	1.000(1)	1.000(6)	2.732(4)			
	1.000(2)	1.366(3)	1.366(5)			
E_7	1.000(7)	1.879(2)	2.532(3)	2.879(5)		
	1.000(1)	1.532(6)	2.879(4)			
E_8	1.000(8)	1.618(1)	2.956(6)	4.783(4)		
	1.000(7)	1.209(2)	1.618(3)	1.956(5)		

TABLE 8.1: Color-class projections of dominant eigenvectors.