A Weyl Group Generating Function That Ought To Be Better Known

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November 1996 edition

Abstract. Let W be a finite Weyl group with root system Φ, simple roots $\alpha_1, \dots, \alpha_n$, exponents $e_1, \dots, e_n$, and index of connection f. Let $b_1, \dots, b_n$ denote the simple root coordinates for the sum of all positive roots, and for $w \in W$ let $\ell(w)$ denote the length and $D(w) = \{i : w^{-1}\alpha_i < 0\}$ the descent set of w. By analyzing the structure of the corresponding affine Weyl group, we prove that $\sum_{w \in W} q^{\sigma(w)-\ell(w)} = f \cdot \prod_{i=1}^{n} (1-q^{b_i})/(1-q^{e_i})$, where $\sigma(w) = \sum_{i \in D(w)} b_i$.

1. Introduction

In this paper, we derive an interesting and unusual generating function for finite Weyl groups by analyzing the Poincaré series of the corresponding affine groups (Theorem 1.1). An equivalent result can be found in a 1965 paper of Iwahori and Matsumoto [**IM**, §1.10] (see Remark 2.3 below), but it seems to have become a "forgotten" part of the literature on reflection groups. Nevertheless, in our opinion the identity is sufficiently interesting that it deserves to be resurrected from obscurity. Along the way, we will also have the opportunity to explain a "mystery" in the exercises of Bourbaki [**B**].

Let Φ be a (reduced) crystallographic root system of rank n in a real Euclidean space V with inner product $(\ ,\ )$. (Standard references are [**B**] and [**H**].) We let $\alpha_1, \dots, \alpha_n$ denote a choice of simple roots, and Φ^+ the resulting set of positive roots. We write $\alpha > 0$ or $\alpha < 0$ according to whether $\alpha \in \Phi^+$ or $-\alpha \in \Phi^+$.

Let W denote the Weyl group generated by the reflections $s_1, \dots, s_n$ corresponding to $\alpha_1, \dots, \alpha_n$. For each $w \in W$, we define $\ell(w)$ to be the minimum length among all expressions of the form $w = s_{i_1} \cdots s_{i_l}$. It is well-known that $\ell(w) = |\Phi(w)|$, where

$$\Phi(w) = \left\{\alpha \in \Phi^+ : w^{-1}\alpha < 0\right\}.$$

The descent set of w is defined to be $D(w) = \{i : w^{-1}\alpha_i < 0\}$.

Research of the first author supported by a grant from the NSF.

In the following, we use the notation $e_1, \ldots, e_n$ for the exponents of W, f for the index of the root lattice in the weight lattice, ρ for half the sum of all $\alpha \in \Phi^+$, and $b_1, \ldots, b_n$ for the coordinates of 2ρ relative to the simple roots, so that $2\rho = b_1\alpha_1 + \cdots + b_n\alpha_n$. Tables of these quantities can be found in [**B**].

THEOREM 1.1. *We have*

$$\sum_{w \in W} q^{\sigma(w)-\ell(w)} = f \cdot \prod_{i=1}^{n} \frac{1-q^{b_i}}{1-q^{e_i}},$$

where $\sigma(w) = \sum_{i \in D(w)} b_i$.

In the limit $q \to 1$, we obtain the following.

COROLLARY 1.2. $|W| = f \cdot (b_1 \cdots b_n)/(e_1 \cdots e_n)$.

REMARK 1.3. In case Φ is irreducible, there is a well-known formula for $|W|$ that is quite similar to Corollary 1.2; namely,

$$|W| = f \cdot n! \cdot c_1 \cdots c_n,$$

where $c_1, \ldots, c_n$ are the coordinates of the highest root $\bar{\alpha}$ relative to the simple roots (e.g., [$\mathbf{H}$, §4.9]). Thus we have

$$|W|/f = n! \cdot c_1 \cdots c_n = (b_1 \cdots b_n)/(e_1 \cdots e_n).$$

We thank James Humphreys for pointing out that the latter of these two equalities is Exercise VI.4.6(a) of Bourbaki [**B**]. It is interesting to note that the wording of the exercise ("Vérifier dans chaque cas que...") suggests a solution that relies on the classification of root systems, and provides no indication of the original source. In fact, it seems that the most likely source is a paper of Bott [**Bt**], where the identity appears as Proposition 13.1.

REMARK 1.4. An extension of Corollary 1.2 that is valid for arbitrary (i.e., not necessarily crystallographic) reflection groups has been given recently by Heckman and Opdam (Example 3.22 of [**HO**]). Normalizing the roots so that $(\alpha, \alpha) = 2$, they prove

$$|W| = (b_1 \cdots b_n) \cdot \det(\alpha_i, \alpha_j) \cdot \prod_{\mathrm{ht}(\alpha)>1} \frac{\mathrm{ht}(\alpha)-1}{\mathrm{ht}(\alpha)},$$

where $\mathrm{ht}(\alpha)$ denotes the sum of the simple root coordinates of α (i.e., the height of α), and as before, $b_1\alpha_1 + \cdots + b_n\alpha_n$ denotes the sum of the positive roots. The fact that this agrees with Corollary 1.2 in the crystallographic case is a consequence of the fact that for Weyl groups, the partition formed by the exponents is conjugate to the partition formed by the number of roots of each height (e.g., [**H**, §3.20]). We thank Ian Macdonald for bringing this generalization to our attention.

REMARK 1.5. Even in the symmetric group case ($W = S_n$, $\Phi = A_{n-1}$), Theorem 1.1 seems to have gone unnoticed in the combinatorics literature. To describe this particular case in more detail, let $\varepsilon_1, \ldots, \varepsilon_n$ denote an orthonormal basis of $\mathbf{R}^n$, and choose simple roots $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$. For $w \in S_n$, we have $D(w) = \{i : w_i > w_{i+1}\}$ and $\ell(w) = |\{(i,j) : i < j,\ w_i > w_j\}|$, where w_i denotes the coefficient of ε_i in $w(\varepsilon_1 + 2\varepsilon_2 + \cdots + n\varepsilon_n)$. It is easy to check that $b_i = i(n-i)$, so in this case Theorem 1.1 takes the form

$$\sum_{w \in S_n} q^{\sigma(w) - \ell(w)} = n \cdot \prod_{i=1}^{n-1} \frac{1 - q^{i(n-i)}}{1 - q^i}.$$

2. Proof of the Main Result

We will assume henceforth that Φ is irreducible. This provides several advantages, including the existence of a highest root $\bar{\alpha} \in \Phi^+$. At the same time, this poses no loss of generality, since Theorem 1.1 can be easily reduced to this case.

The proof relies on the theory of affine Weyl groups, so we first briefly review the main features of this theory, as found in [**B**] or [**H**]. The affine Weyl group $\widetilde{W}$ associated with Φ is the semidirect product of W and the co-root lattice $\mathbf{Z}\Phi^\vee$, the latter acting on V via translations. It is known that $\widetilde{W}$ is a Coxeter group relative to the generating set $\{s_0, s_1, \ldots, s_n\}$, where s_0 denotes reflection through the (affine) hyperplane $\langle \bar{\alpha}, 1 \rangle$. (We use the notation $\langle \alpha, k \rangle$ for the hyperplane $\{\gamma \in V : (\gamma, \alpha) = k\}$.) We will use $\ell(\cdot)$ to denote the length function on $\widetilde{W}$ relative to this generating set; since W is a parabolic subgroup of $\widetilde{W}$, it follows that this agrees with and extends the length function on W.

We let $\mathcal{H}$ denote the affine hyperplane arrangement $\{\langle \alpha, k \rangle : \alpha \in \Phi^+,\ k \in \mathbf{Z}\}$, and similarly we let $\mathcal{H}_0$ denote the central arrangement $\{\langle \alpha, 0 \rangle : \alpha \in \Phi^+\}$. The actions of $\widetilde{W}$ on the connected components of $V - \bigcup \mathcal{H}$ (alcoves), and W on the connected components of $V - \bigcup \mathcal{H}_0$ (chambers) are simply transitive. Let

$$C_0 = \{\gamma \in V : (\gamma, \alpha_i) > 0 \text{ for } 1 \le i \le n\},$$
$$A_0 = \{\gamma \in C_0 : (\gamma, \bar{\alpha}) < 1\},$$

denote the fundamental chamber and alcove, respectively. Points in the closure of the fundamental chamber are said to be *dominant.*

The group $\widetilde{W}$ is embedded in a larger group $\widehat{W}$ of affine transformations that preserve the set of alcoves; namely, the semidirect product of W and the group of translations by the co-weight lattice $\Lambda^\vee = \{\lambda \in V : (\lambda, \alpha) \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$. For $w \in \widetilde{W}$, it is well-known that $\ell(w)$ is the number of hyperplanes in $\mathcal{H}$ that separate A_0 and wA_0. It will be convenient to use this as the *definition* of $\ell(w)$ for $w \in \widehat{W}$, even though in this larger group, it no longer refers to the length of a minimal expression for w relative to some generating set.

For each $w \in \widetilde{W}$, there is a unique $u \in W$ such that $u^{-1}(wA_0) \subset C_0$. It follows that

$$\widetilde{W}^0 := \{w \in \widetilde{W} : wA_0 \subset C_0\}$$

is a set of coset representatives for W in $\widetilde{W}$.

For $\lambda \in \Lambda^\vee$, let $t_\lambda \in \widehat{W}$ denote translation by λ.

Lemma 2.1. *If $w \in \widetilde{W}^0$ and $\lambda \in \Lambda^\vee$ is dominant, then $\ell(t_\lambda w) = \ell(w) + (\lambda, 2\rho)$.*

Proof. Since $wA_0 \subset C_0$, the hyperplanes in $\mathcal{H}$ that separate A_0 from wA_0 are of the form $\langle \alpha, k\rangle$, where $\alpha \in \Phi^+$ and $k > 0$. Similar remarks apply to $t_\lambda wA_0 = \lambda + wA_0$, the only difference being that there are (λ, α) additional hyperplanes perpendicular to α that separate $t_\lambda wA_0$ from A_0. Thus we have $\ell(t_\lambda w) - \ell(w) = \sum_{\alpha \in \Phi^+}(\lambda, \alpha)$. $\square$

Now consider the (open) parallelepiped

$$P = \{\gamma \in V : 0 < (\gamma, \alpha_i) < 1 \text{ for } 1 \le i \le n\}.$$

The bounding walls of P are members of $\mathcal{H}$, and thus $P - \bigcup \mathcal{H}$ is a union of alcoves. For each alcove A, there is a unique translation of A to an alcove in P, so

$$\widehat{W}_P := \{w \in \widehat{W} : wA_0 \subset P\}$$

is a set of coset representatives for $\Lambda^\vee$ in $\widehat{W}$. In particular, since $\widehat{W}$ is the semidirect product of W and $\Lambda^\vee$, it follows that $|W| = |\widehat{W}_P|$. In fact, for each $w \in W$ let us define w_P to be the unique member of $\widehat{W}_P$ such that wA_0 is a translation of w_PA_0. Thus $w \mapsto w_P$ is a bijection $W \to \widehat{W}_P$.

Lemma 2.2. *For $w \in W$, we have $\ell(w_P) = \sigma(w) - \ell(w)$.*

Proof. For every $\gamma \in A_0$ and $\alpha \in \Phi^+$, we have $0 < (\gamma, \alpha) < 1$. Therefore

$$\begin{aligned} 0 < (w\gamma, \alpha) < 1 &\quad \text{if } w^{-1}\alpha > 0, \\ -1 < (w\gamma, \alpha) < 0 &\quad \text{if } w^{-1}\alpha < 0. \end{aligned}$$

In particular, the unique translation that sends wA_0 to an alcove in P is of the form t_λ, where $(\lambda, \alpha_i) = 0$ or 1 according to whether $w^{-1}\alpha_i$ is a positive or negative root. In other words, $\lambda = \sum_{i \in D(w)} \omega_i$, where $\omega_1, \ldots, \omega_n$ denote the fundamental co-weights (i.e., the basis dual to the simple roots).

Furthermore, it follows that for each $\alpha \in \Phi^+$, the number of hyperplanes in $\mathcal{H}$ that are perpendicular to α and separate A_0 from $w_P A_0 = \lambda + wA_0$ is (λ, α) or $(\lambda, \alpha) - 1$ according to whether $w^{-1}\alpha$ is positive or negative. Thus we obtain

$$\ell(w_P) = \sum_{\alpha \in \Phi^+} (\lambda, \alpha) - |\Phi(w)| = (\lambda, 2\rho) - \ell(w) = \sum_{i \in D(w)} (\omega_i, 2\rho) - \ell(w),$$

proving the claim. $\square$

Proof of Theorem 1.1. Let $\widetilde{W}_P = \widetilde{W} \cap \widehat{W}_P$. For every $X \subset \widehat{W}$, let $X(q) = \sum_{w \in X} q^{\ell(w)}$ denote the corresponding length generating function. By Lemma 2.2, we have

$$\sum_{w \in W} q^{\sigma(w) - \ell(w)} = \widehat{W}_P(q) = f \cdot \widetilde{W}_P(q), \tag{2.1}$$

since the stabilizer of A_0 in $\widehat{W}$ has order f. On the other hand, for each alcove A in C_0, there is a unique dominant $\lambda \in \Lambda^\vee$ such that $A \subset \lambda + P$. Conversely, every translation of an alcove in P by a dominant co-weight is in C_0, so by Lemma 2.1 we have

$$\widetilde{W}^0(q) = \sum_{w \in \widetilde{W}_P} \sum_{\lambda \in \Lambda^\vee_+} q^{\ell(t_\lambda w)} = \sum_{w \in \widetilde{W}_P} q^{\ell(w)} \sum_{\lambda \in \Lambda^\vee_+} q^{(\lambda, 2\rho)} = \widetilde{W}_P(q) \cdot \prod_{i=1}^{n} \frac{1}{1 - q^{b_i}}, \tag{2.2}$$

where $\Lambda^\vee_+$ denotes the set of dominant co-weights. Since $w \in \widetilde{W}$ is a member of $\widetilde{W}^0$ if and only if A_0 and wA_0 are on the same side of the hyperplanes $\langle \alpha_i, 0 \rangle$, it follows that

$$\widetilde{W}^0 = \{w \in \widetilde{W} : \ell(s_i w) > \ell(w) \text{ for } 1 \le i \le n\},$$

and hence $\widetilde{W}^0$ is the set of *minimal* coset representatives for W in $\widetilde{W}$ (e.g., [**H**, §5.12]). However in that case, and more generally for any pair consisting of a Coxeter group and a parabolic subgroup (*ibid.*), we have

$$\widetilde{W}^0(q) = \widetilde{W}(q)/W(q). \tag{2.3}$$

Combining (2.1)–(2.3), we obtain

$$\sum_{w \in W} q^{\sigma(w) - \ell(w)} = f \cdot \frac{\widetilde{W}(q)}{W(q)} \cdot \prod_{i=1}^{n} (1 - q^{b_i}). \tag{2.4}$$

To complete the proof, recall that $\widetilde{W}(q) = W(q) / \prod_i (1 - q^{e_i})$, by a theorem of Bott [**Bt**]. $\square$

Remark 2.3. As we indicated previously, Theorem 1.1 is equivalent to a result of Iwahori and Matsumoto [**IM**]. To be more precise, our (2.4) is their Proposition 1.30. Of course from there it is a trivial matter to use Bott's theorem to provide an explicit formula. Following [**IM**], it should also be noted that the series $\widetilde{W}_P(t)$ is implicit in [**Bt**, p. 277] (where it is denoted $Q(t)$), although it appears without any explicit description along the lines of (2.1). Let us also note that (2.2) occurs as Lemma 5.4 of [**BHO**] as well.

Remark 2.4. Let Ω denote the stabilizer of A_0 in $\widehat{W}$, a subgroup of order f. Since $\widehat{W}$ is the semidirect product of W and $\Lambda^\vee$, there is a natural group isomorphism $\widehat{W}/\Lambda^\vee \to W$. Since the intersection of Ω and $\Lambda^\vee$ is trivial, Ω is isomorphic to its image $\bar{\Omega}$ in W. For example, in the case $W = S_n$, $\bar{\Omega}$ is generated by the n-cycle x such that $x\varepsilon_1 = \varepsilon_2$, $x\varepsilon_2 = \varepsilon_3, \ldots, x\varepsilon_n = \varepsilon_1$, using the notation of Remark 1.5.

By Lemma 2.2, it follows that $\bar{\Omega} = \{w \in W : \sigma(w) = \ell(w)\}$, and moreover that $\sigma - \ell$ is right-invariant with respect to $\bar{\Omega}$; i.e.,

$$\sigma(wx) - \ell(wx) = \sigma(w) - \ell(w)$$

for all $w \in W$ and $x \in \bar{\Omega}$. Hence as an alternative formulation of Theorem 1.1, we have

$$\sum_{w \in W/\bar{\Omega}} q^{\sigma(w)-\ell(w)} = \sum_{w \in \widetilde{W}_P} q^{\ell(w)} = \prod_{i=1}^{n} \frac{1-q^{b_i}}{1-q^{e_i}}.$$

Remark 2.5. In case there is more than one orbit of roots in Φ, it is possible to give a multivariate refinement of Theorem 1.1 as follows. Assuming Φ is irreducible, there are exactly two such orbits, distinguishable by length ("short" and "long"). Define integers b'_i and b''_i so that $b'_1\alpha_1 + \cdots + b'_n\alpha_n$ and $b''_1\alpha_1 + \cdots + b''_n\alpha_n$ are the sums of the short and long positive roots (respectively), and set

$$\sigma'(w) = \sum_{i \in D(w)} b'_i, \qquad \sigma''(w) = \sum_{i \in D(w)} b''_i$$

for all $w \in W$. Similarly, define $\ell'(w)$ and $\ell''(w)$ to be the number of occurrences of reflections corresponding respectively to short and long roots in some (equivalently, every) reduced expression for $w \in \widetilde{W}$. With only minor adjustments, the same argument used to prove (2.4) also proves

$$\sum_{w \in W} q_1^{\sigma'(w)-\ell'(w)} q_2^{\sigma''(w)-\ell''(w)} = f \cdot \frac{\widetilde{W}(q_1,q_2)}{W(q_1,q_2)} \cdot \prod_{i=1}^{n} (1-q_1^{b'_i})(1-q_2^{b''_i}),$$

where $X(q_1, q_2) := \sum_{w \in X} q_1^{\ell'(w)} q_2^{\ell''(w)}$ for any $X \subset \widetilde{W}$. Furthermore, there is an explicit product formula for $\widetilde{W}(q_1, q_2)/W(q_1, q_2)$ due to Macdonald [**M1**, Thm. 3.3]. See also [**M2**] for a new proof that is classification-free.

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