Coxeter cones and their h-vectors

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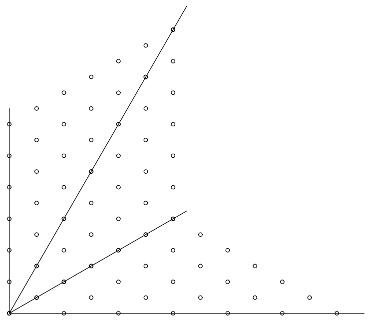

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0. Introduction

A Coxeter cone is the intersection of the nonnegative sides of the hyperplanes associated to some subset of a root system. Such cones inherit the structure of a simplicial complex from the associated Coxeter complex. They are equivalent (in fact, dual) to the "parsets" studied by Reiner in [**R1**] and [**R2**].

For example, a parset for the root system $\mathcal{A}_{n-1}$ is a poset P with n vertices, and the corresponding Coxeter cone is a conic analogue of the order polytope of P. As a simplicial complex, it is isomorphic to the order complex of the lattice of order ideals of P.

The now-disproved Neggers-Stanley conjecture (see [**B1**] and [**St2**]) asserted that the h-polynomials of type $\mathcal{A}$ Coxeter cones should have all real zeros. A weaker conjecture that remains unresolved is that these cones, or more generally all Coxeter cones, have unimodal h-vectors. (But see Remark 7.11 below for a cautionary example.)

In support of this, Reiner and Welker have constructed a simplicial polytope associated to any *graded* poset P, and show that this polytope has the same h-vector as the corresponding Coxeter cone [**RW**]. As a corollary, they deduce that these h-vectors are symmetric and unimodal. (For posets P that are not graded, the h-vector is generally not symmetric.) More recently, Brändén [**B2**] gave a direct proof that type $\mathcal{A}$ Coxeter cones have nonnegative "γ-vectors" (for the definition, see Section 1D), a result that implies unimodality of the h-vector, but is weaker than proving that all zeros of the h-polynomial are real. Brändén also gave a second proof based on the combinatorics of peak sets [**B3**]. A further consequence of his result is that the Charney-Davis conjecture holds for the Reiner-Welker polytopes whose associated complexes satisfy the flag property.

In this paper, we generalize Brändén's result to finite crystallographic root systems. More precisely, we generalize the concept of "graded poset" to root systems, and give a classification-free, constructive proof that the corresponding "graded" Coxeter cones may be partitioned into relative complexes whose h-vectors are, up to a shift, the h-vectors of Coxeter complexes of parabolic root subsystems. The shifts conspire to center these h-vectors at the same point; from this we are able to deduce that graded Coxeter cones have nonnegative γ-vectors, and hence also symmetric and unimodal h-vectors.

For future research, it would be interesting to see if the simplicial polytopes constructed by Reiner and Welker can be generalized to graded Coxeter cones. In another direction, it would also be interesting to see if a more combinatorial proof of our main result could be given, perhaps by generalizing the theory of enriched P-partitions [**St1**] and the peak combinatorics of [**B3**] (see Petersen [**P**] for an exploration of type $\mathcal{B}$ peak combinatorics). Our calculations in the Appendix below could be a small step in this direction.

The remainder of the paper is structured as follows.

In Section 1, we review standard facts about h-vectors, γ-vectors, and the Coxeter complex. In Section 2, we introduce Coxeter cones and discuss some of their basic properties. In Section 3, we introduce "labeled" Coxeter cones; these are obtained from Coxeter cones by removing the parts of the boundary on hyperplanes that separate the interior of the cone from some fixed reference point. These modified cones are relative complexes in general, and occur as intermediate objects in decompositions of larger Coxeter cones. In Section 4, we relate the h-polynomials of (labeled) Coxeter cones with the Ehrhart polynomials of certain polyhedral complexes; in some cases (such as types $\mathcal{A}$ and $\mathcal{B}$) these complexes are convex polytopes, but not in general.

The proof of the main result is spread over Sections 5–7. In the first two parts, we use Ehrhart polynomials to formulate natural combinatorial conditions that force the h-vector of a (labeled) Coxeter cone to be symmetric; in the root system $\mathcal{A}_{n-1}$, the cones that satisfy this symmetry condition correspond in the unlabeled case to graded posets, and in the labeled case to Brändén's "sign-graded" posets [**B2**]. In the last part, we prove that these symmetry conditions suffice to force the h-vectors to be unimodal.

In the Appendix, we provide explicit formulas for the flag h-polynomials of the Coxeter complexes for $\mathcal{B}_n$ and $\mathcal{D}_n$ that allow us to justify the assertion that Coxeter complexes have nonnegative γ-vectors (Theorem 1.2). Similar results for the ordinary h-polynomial have been obtained previously by Petersen [**P**] (for $\mathcal{B}_n$) and Chow [**C**] (for $\mathcal{B}_n$ and $\mathcal{D}_n$).

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1. Preliminaries

A. Ordinary and flag h-vectors.

Let Δ be a finite set of simplices that is n-colored (or balanced) in the sense that the vertices have been assigned colors from $[n] = \{1, \dots, n\}$ so that every simplex in Δ has vertices with distinct colors. Note that we are not necessarily assuming that Δ is a simplicial complex, although this case is of particular interest.

For $J \subseteq [n]$, we let $f_J(\Delta)$ denote the number of simplices in Δ whose vertices have color-set J, and set

$$h_J(\Delta) := \sum_{I \subseteq J} (-1)^{|J-I|} f_I(\Delta). \tag{1.1}$$

These subset-indexed quantities comprise the flag f-vector and flag h-vector of Δ; the corresponding generating functions

$$f(\Delta; t_1, \ldots, t_n) := \sum_{J \subseteq [n]} f_J(\Delta) \prod_{j \in J} t_j,$$
$$h(\Delta; t_1, \ldots, t_n) := \sum_{J \subseteq [n]} h_J(\Delta) \prod_{j \in J} t_j,$$

are the flag f-polynomial and flag h-polynomial. Note that (1.1) implies

$$h(\Delta; t_1, \ldots, t_n) = (1-t_1) \cdots (1-t_n) f(\Delta; t_1/(1-t_1), \ldots, t_n/(1-t_n)). \tag{1.2}$$

The ordinary f- and h-polynomials of Δ are the one-variable specializations

$$f(\Delta; t) := f(\Delta; t, \ldots, t), \quad h(\Delta; t) := h(\Delta; t, \ldots, t).$$

Note also that (1.2) specializes to

$$h(\Delta; t) = (1-t)^n f(\Delta; t/(1-t)),$$

and an easy consequence of (1.1) is that $h(\Delta; 1) = f_{[n]}(\Delta)$.

Let $\Delta_{\max}$ denote the set of simplices in Δ of dimension $n-1$, the maximum possible. In many applications (e.g., Cohen-Macaulay complexes), the ordinary or flag h-polynomial has nonnegative coefficients. A simple combinatorial way to demonstrate nonnegativity is to provide (if possible) a simplex-to-subset mapping $C : \Delta_{\max} \to 2^{[n]}$ such that

$$\Delta = \bigcup_{\sigma \in \Delta_{\max}} \{\sigma_J : C(\sigma) \subseteq J\} \qquad \text{(disjoint union)}, \tag{1.3}$$

where σ_J denotes the simplicial face of σ formed by the J-colored vertices. Indeed, one sees that each simplex with color-set J must occur as σ_J for a unique $\sigma \in \Delta_{\max}$, hence

$$f_J(\Delta) = |\{\sigma \in \Delta_{\max} : C(\sigma) \subseteq J\}|.$$

The inclusion-exclusion relation (1.1) therefore implies

$$h_J(\Delta) = |\{\sigma \in \Delta_{\max} : C(\sigma) = J\}|,$$

and hence also

$$h(\Delta; t) = \sum_{\sigma \in \Delta_{\max}} t^{|C(\sigma)|}.$$

It should be noted that a decomposition satisfying (1.3) will not be possible unless all maximal simplices (with respect to face inclusion) have dimension $n-1$.

B. The Coxeter complex.

We assume that the reader is familiar with the theory of finite reflection groups, as found (for example) in [**Hu**]; the aim of this section is to establish notational conventions and review standard facts about the Coxeter complex.

Let Φ be a finite root system in a real Euclidean space V with inner product $\langle\cdot\,,\cdot\rangle$. Let $\alpha_1,\ldots,\alpha_n \in \Phi$ denote a fixed choice of simple roots, and $s_1,\ldots,s_n \in GL(V)$ the corresponding simple reflections. These reflections generate a finite Coxeter group W, and conversely, every finite Coxeter group arises this way. One knows that Φ may be partitioned into Φ^+ (positive roots) and $-\Phi^+$ (negative roots), the former being the roots in the nonnegative span of the simple roots.

The Coxeter complex $\Sigma(\Phi)$ is a W-stable partition of V into (relatively open) convex cones of the form wC_J ($w \in W$, $J \subseteq [n]$), where

$$C_J = \{\mu \in V : \langle\mu,\alpha_j\rangle = 0 \text{ for } j \in J,\ \langle\mu,\alpha_j\rangle > 0 \text{ for } j \notin J\}.$$

By intersecting the complex with (say) the unit sphere in $\operatorname{Span}\Phi$, one obtains a collection of (topological) simplices and thus one may view $\Sigma(\Phi)$ as a simplicial complex. One may also identify $\Sigma(\Phi)$ as the normal fan of the convex hull of a generic W-orbit in V.

The maximal cells of $\Sigma(\Phi)$ are the Weyl chambers $wC_\varnothing$, and the vertices of $\Sigma(\Phi)$ are of the form $wC_{\langle j\rangle}$, where $\langle j\rangle := [n]\setminus\{j\}$. All points in the closure of $C_\varnothing$ are said to be *dominant*; one knows that the W-stabilizer of the dominant cone C_J is the parabolic subgroup $W_J = \langle s_j : j \in J\rangle$, and the orbits of C_I and C_J are disjoint for $I \neq J$. Thus by assigning color j to the vertices $wC_{\langle j\rangle}$, we obtain an n-coloring of the complex, and

$$f_J(\Sigma(\Phi)) = |W|/|W_{J^c}| \qquad (J^c = [n]\setminus J) \tag{1.4}$$

is the number of cells whose vertex color-set is J.

It is well-known that for each parabolic subgroup W_J, there is a distinguished set of coset representatives for W/W_J; namely,

$$W^J := \{w \in W : \ell(ws_j) > \ell(w) \text{ for all } j \in J\},$$

where $\ell(w)$ denotes the minimum length of a factorization of w into a product of simple reflections. In terms of the (right) descent set

$$D(w) := \{j : \ell(ws_j) < \ell(w)\},$$

one sees that $w \in W^J$ if and only if $D(w) \subseteq J^c$, hence

$$f_J(\Sigma(\Phi)) = \left|W^{J^c}\right| = |\{w \in W : D(w) \subseteq J\}|. \tag{1.5}$$

It also follows that for each cell wC_J, there is a unique chamber $w'C_\varnothing$ whose boundary contains wC_J and satisfies $D(w') \subseteq J^c$; namely, the one for which w' is the distinguished representative of the coset wW_J. The chambers-to-subsets map $wC_\varnothing \mapsto D(w)$ therefore provides a partition of $\Sigma(\Phi)$ in the form of (1.3), and hence

$$h_J(\Sigma(\Phi)) = |\{w \in W : D(w) = J\}|. \tag{1.6}$$

Alternatively, this may be seen as a direct consequence of (1.1) and (1.5).

Either way, one obtains that the h-polynomial of the Coxeter complex is

$$h(\Sigma(\Phi);t) = \sum_{w \in W} t^{|D(w)|}.$$

In the case $\Phi = \mathcal{A}_n$, this is the classical Eulerian polynomial for $W = S_{n+1}$, shifted by a factor of t from the traditional definition.

Another feature of the flag h-vector of $\Sigma(\Phi)$ (or more generally, any n-colored simplicial $(n-1)$-sphere [**BB**]) is that it satisfies the generalized Dehn-Sommerville equations

$$h_J(\Sigma(\Phi)) = h_{J^c}(\Sigma(\Phi)) \qquad (J \subseteq [n]). \tag{1.7}$$

In particular, the h-polynomial is symmetric; i.e.,

$$h(\Sigma(\Phi);t) = t^n h(\Sigma(\Phi);1/t).$$

In the context of Coxeter groups, these relations have a simple explanation. Indeed, one knows that the unique element $w_0 \in W$ of maximum length is an involution that reverses length; i.e., $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for all $w \in W$. Hence $D(w_0 w) = D(w)^c$ and one may deduce (1.7) from (1.6).

C. Unimodality and real roots.

The characterization of ordinary and flag h-vectors of various classes of complexes has been a topic of longstanding interest. Of particular interest are linear inequalities such as unimodality; i.e., the existence of some index k such that

$$h_0 \leqslant \cdots \leqslant h_{k-1} \leqslant h_k \geqslant h_{k+1} \geqslant \cdots \geqslant h_n.$$

In this direction, it is a well-known theorem of Stanley [**S2**] that the h-vectors of simplicial polytopes are unimodal (and symmetric). Specializing to Coxeter complexes, one has

THEOREM 1.1. *The coefficients of* $h(\Sigma(\Phi);t) = \sum_{w \in W} t^{|D(w)|}$ *are unimodal.*

Another type of condition on h-polynomials that has been a subject of considerable interest is the constraint that all zeroes of $h(\Delta;t)$ are real. Given that the h-vector is nonnegative, this is a much stronger condition than unimodality.

As noted in the introduction, a would-be example of this is the now-disproved Neggers-Stanley conjecture. If true, it would have implied that all zeroes of the h-polynomials of chain complexes of distributive lattices are real.

In the case of Coxeter complexes, Brenti has conjectured that the zeroes of $h(\Sigma(\Phi); t)$ are all real [**Br**]. It is easy to see that the h-polynomial is multiplicative with respect to the irreducible components of Φ, so it suffices to prove this for irreducible root systems. In the exceptional cases, this is easily checked (e.g., via (1.4)), and it is also known to be true for root systems of type $\mathcal{A}$ [**H**] and $\mathcal{B}$ [**Br**]. For type $\mathcal{D}$ (the only remaining possibility), it is still an open problem.

D. The γ-vector.

For nonnegative h-vectors that are symmetric, there is an interesting intermediate condition that is stronger than unimodality but weaker than requiring real roots. Following Brändén [**B2**] and Gal [**G**], it is clear that the set of polynomials

$$\Gamma_n = \left\{ t^i (1+t)^{n-2i} : 0 \leqslant i \leqslant n/2 \right\}$$

forms a $\mathbf{Z}$-basis for $\{h(t) \in \mathbf{Z}[t] : h(t) = t^n h(1/t)\}$, so for any complex Δ whose h-vector satisfies the Dehn-Sommerville equations $h_i = h_{n-i}$, one may define integers γ_i via

$$h(\Delta; t) = \sum_{0 \leqslant i \leqslant n/2} \gamma_i t^i (1+t)^{n-2i}. \tag{1.8}$$

The sequence $\gamma(\Delta) := (\gamma_0, \gamma_1, \dots)$ is called the γ-vector of Δ.

It is clear that the polynomials in Γ_n have symmetric unimodal coefficients with the same center of symmetry, so the same is true for the nonnegative span of Γ_n. Thus as noted by Brändén and Gal, nonnegativity of $\gamma(\Delta)$ implies unimodality of the h-vector.[1]

If the coefficients of $h(t)$ are nonnegative and symmetric, then the zeroes of $h(t)$ outside of $\{0, -1\}$ must occur in pairs $a, 1/a$. Thus if all zeroes of $h(t)$ are real, then $a < 0$ and $h(t)$ is a product of factors of the form

$$(t-a)(t-1/a) = (1+t)^2 - (2+a+1/a)t \in \mathbf{R}^+\Gamma_2,$$

along with a positive scalar and factors of $t \in \Gamma_2$ and $1+t \in \Gamma_1$. Noting the multiplicative compatibility $\Gamma_m \cdot \Gamma_n \subseteq \Gamma_{m+n}$, it follows as noted by Brändén and Gal that any such h-polynomial has a nonnegative γ-vector.

[1] The use of nonnegative expansions in the form of (1.8) to prove symmetric unimodality is an idea that can be traced back (at least) to work of Gasharov [**Ga**].

Among Coxeter complexes, the fact that all zeroes of $h(\Sigma(\Phi); t)$ are known to be real for all irreducible root systems except $\Phi = \mathcal{D}_n$ immediately implies the nonnegativity of the γ-vector in these cases. Moreover, Chow [**C**] has recently given a combinatorial proof that the γ-vector of $\Sigma(\mathcal{D}_n)$ is nonnegative; in the Appendix we also give explicit formulas for the flag h-polynomials of $\Sigma(\mathcal{B}_n)$ and $\Sigma(\mathcal{D}_n)$ that specialize to nonnegative formulas for their γ-vectors (Corollaries A.2 and A.5). Given the multiplicative compatibility of h-polynomials and the cones $\mathbf{R}^+\Gamma_n$ with respect to irreducible components, we thus have

Theorem 1.2. *The γ-vectors of finite Coxeter complexes are nonnegative.*

Note that this result implies Theorem 1.1. Also, it is a special case of Gal's conjecture that flag Gorenstein* simplicial complexes (a class that includes complete fans arising from simplicial hyperplane arrangements) have nonnegative γ-vectors [**G**].

2. Coxeter cones

We define a *Coxeter cone* to be a closed convex cone of the form

$$\Delta(\Psi) := \{\mu \in V : \langle \mu, \beta \rangle \geqslant 0 \text{ for all } \beta \in \Psi\},$$

where Ψ is any subset of the root system Φ. The bounding walls of this cone are root hyperplanes, so it is a closed union of cells of the Coxeter complex, and thus forms a simplicial complex. By abuse of notation, we will identify $\Delta(\Psi)$ with this complex.

We will always assume that Ψ and $\Delta(\Psi)$ are *nondegenerate* in the sense that $\Delta(\Psi)$ is not confined to any of the root hyperplanes, and thus contains one or more Weyl chambers. It will be convenient to let

$$X(\Psi) := \{w \in W : wC_\varnothing \subset \Delta(\Psi)\}$$

denote the subset of W indexing the chambers in $\Delta(\Psi)$.

Although it is not always convenient, there is no loss of generality in assuming that Ψ has no redundancies; i.e., that there exists no proper $\Psi' \subset \Psi$ such that $\Delta(\Psi') = \Delta(\Psi)$. This is equivalent to having Ψ be the set of extreme rays of the cone it generates.

Remark 2.1. An alternative to eliminating redundancy is to maximize it; i.e., assume that Ψ contains all roots in its positive linear span. This yields the category of "parsets" introduced by Reiner in [**R2**], although the definition there is flawed in that it does not suffice in general to check that $\alpha, \beta \in \Psi$ and $\gamma = a\alpha + b\beta \in \Phi$ $(a, b > 0)$ implies $\gamma \in \Psi$. In the terminology of [**St3**], one needs to check whether Ψ contains the apex of every irreducible circuit with a base contained in Ψ; such circuits may have rank > 2.

Example 2.2. In the case $\Phi = \mathcal{A}_{n-1} = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n\}$, one may identify each $\Psi \subseteq \Phi$ with a directed graph D on the vertex set $[n]$ by associating the root $\varepsilon_i - \varepsilon_j$

with the arc $i \to j$. In this way, Ψ is nondegenerate if and only if D is acyclic. It has no redundancies if and only if D is the covering relation of a partially ordered set P; in that case, the Coxeter cone $\Delta(\Psi)$ corresponds to the set of order-preserving maps $P \to \mathbf{R}$, and $X(\Psi) \subseteq S_n$ is the set of linear extensions of P. As a simplicial complex, $\Delta(\Psi)$ is isomorphic to the chain complex of the lattice of order ideals of P.

EXAMPLE 2.3. In the case $\Phi = \mathcal{B}_n$, nondegenerate Coxeter cones correspond to the "signed posets" introduced by Reiner [**R3**]. An alternative way to study these objects used by Fischer [**F**] can be based on a bijection between subsets of Φ and (loopless) digraphs on the vertex set $\pm[n] := \{\pm 1, \ldots, \pm n\}$ in which $i \mapsto -i$ is an orientation-reversing automorphism. More explicitly, using the standard realization

$$\mathcal{B}_n = \{\pm\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\},$$

each root corresponds to one or two arcs:

$$\begin{aligned}
\varepsilon_i \quad &\Leftrightarrow \quad i \to -i, \\
-\varepsilon_i \quad &\Leftrightarrow \quad -i \to i, \\
\varepsilon_i - \varepsilon_j \quad &\Leftrightarrow \quad i \to j,\ -j \to -i, \\
\varepsilon_i + \varepsilon_j \quad &\Leftrightarrow \quad i \to -j,\ j \to -i, \\
-\varepsilon_i - \varepsilon_j \quad &\Leftrightarrow \quad -i \to j,\ -j \to i.
\end{aligned}$$

As in $\mathcal{A}_{n-1}$, a subset $\Psi \subseteq \Phi$ is nondegenerate if and only if the corresponding digraph is acyclic. Also, the subsets with the fewest redundancies correspond to the covering relations of partial orderings P of $\pm[n]$ in which $i \mapsto -i$ is an order-reversing involution. In these terms, the Coxeter cone $\Delta(\Psi)$ corresponds to the set of order-preserving maps $f : P \to \mathbf{R}$ that commute with the involution (i.e., $f(-i) = -f(i)$), and $X(\Psi)$ corresponds to the set of linear extensions of P with the property that the involution acts as reversal.

A subtle point is that covering relations in this category of posets-with-involution may have redundancies. Indeed, if the arcs $i \to -i$ and $j \to -j$ are covering relations, then the arcs $i \to -j$ and $j \to -i$ could also be present as covering relations, but would nevertheless be redundant. Geometrically, this redundancy corresponds to the fact that $\varepsilon_i + \varepsilon_j$ is in the positive linear span of ε_i and ε_j.

PROPOSITION 2.4. *The extreme rays of the cone generated by Ψ are the members of*

$$\Psi' := \{w\alpha_i : w \in X(\Psi),\ ws_i \notin X(\Psi)\}.$$

In particular, $\Psi' \subseteq \Psi$, and equality occurs if and only if Ψ has no redundancies.

Proof. If the root β is an extreme ray, then $\beta \in \Psi$ and there must be interior points of $\Delta(\Psi)$ that are separated from external points by a single root hyperplane; namely, the hyperplane orthogonal to β. Consequently, there must be chambers internal and external to $\Delta(\Psi)$ with this property, say $wC_\varnothing \subset \Delta(\Psi)$ and $w'C_\varnothing \not\subset \Delta(\Psi)$. Two such chambers are separated by a single root hyperplane if and only if $w' = ws_i$ for some i, in which case the hyperplane is orthogonal to $w\alpha_i$ and hence $\beta = \pm w\alpha_i$. Since $wC_\varnothing$ is on the same side of this hyperplane as $w\alpha_i$, we conclude that $\beta = w\alpha_i$ and $\beta \in \Psi'$.

Conversely, if $w \in X(\Psi)$ and $ws_i \notin X(\Psi)$, then the chambers $wC_\varnothing$ are $ws_iC_\varnothing$ are on the same side of every root hyperplane except for the one orthogonal to $w\alpha_i$, so $w\alpha_i$ must belong to Ψ and we cannot omit $w\alpha_i$ from Ψ without enlarging the corresponding Coxeter cone. That is, every member of Ψ' is an extreme ray. $\square$

Remark 2.5. If we translate the above result into the poset context of Example 2.2, we obtain the observation that $i < j$ is a covering relation in a poset P if and only if there is a linear extension of P such that (1) i and j occur consecutively (in that order), and (2) reversing i and j does not produce a linear extension.

Noting that $\Delta(\Psi)$ and $\Delta(w\Psi)$ are isomorphic (as simplicial complexes) for all $w \in W$, there is no loss of generality in assuming that $C_\varnothing \subset \Delta(\Psi)$ and $1 \in X(\Psi)$. Equivalently, one may assume $\Psi \subseteq \Phi^+$. In such cases, we say that Ψ and $\Delta(\Psi)$ are in *standard form*.

Proposition 2.6. *If Ψ is in standard form, then $X(\Psi)$ is an order ideal of the right weak order; i.e., if $w \in X(\Psi)$ and $\ell(ws_i) < \ell(w)$, then $ws_i \in X(\Psi)$.*

Proof. One knows that if $\ell(ws_i) < \ell(w)$, then $w\alpha_i$ is a negative root (e.g., see [**Hu**]). Given that Ψ is in standard form, it follows that $w\alpha_i \notin \Psi$ and hence $w \in X(\Psi)$ implies $ws_i \in X(\Psi)$, otherwise we contradict Proposition 2.4. $\square$

Remark 2.7. By a theorem of Björner [**Bj**], it is known that any linear extension of the right weak order provides a shelling order for the Coxeter complex $\Sigma(\Phi)$. The above result shows that the chambers of any Coxeter cone may be obtained by truncating a suitable linear extension of the weak order, and thus Coxeter cones are shellable.

3. Labeled Coxeter cones

We define $\lambda \in V$ to be a *reference point* or *labeling* of Ψ if $\langle \lambda, \beta \rangle \neq 0$ for all $\beta \in \Psi$. In such cases, we associate to the pair (Ψ, λ) the *labeled Coxeter cone*

$$\Delta(\Psi, \lambda) := \{\mu \in \Delta(\Psi) : \langle \lambda, \beta \rangle < 0 \text{ implies } \langle \mu, \beta \rangle > 0 \text{ for all } \beta \in \Psi\}.$$

This convex cone is obtained from $\Delta(\Psi)$ by deleting the faces orthogonal to one or more of the roots $\beta \in \Psi$ such that $\langle \lambda, \beta \rangle < 0$. These faces form a subcomplex of $\Delta(\Psi)$, so $\Delta(\Psi, \lambda)$ is a relative complex (i.e., a difference of two simplicial complexes).

Note that $\Delta(\Psi)$ is the closure of $\Delta(\Psi,\lambda)$; in particular, every Weyl chamber contained in $\Delta(\Psi)$ is also contained in $\Delta(\Psi,\lambda)$.

As a function of λ, the cone $\Delta(\Psi,\lambda)$ depends only on where λ resides among the regions defined by the arrangement of hyperplanes orthogonal to the roots in Ψ. Moreover, assuming Ψ has no redundancies, labelings from distinct regions produce distinct cones. One such region is the interior of the Coxeter cone itself (i.e., $\langle\lambda,\beta\rangle > 0$ for all $\beta\in\Psi$), in which case $\Delta(\Psi,\lambda)=\Delta(\Psi)$. We say that such labelings are *natural*.

In the poset context of Example 2.2, the labeling regions correspond bijectively with acyclic re-orientations of the covering relation of a poset P, and a labeled Coxeter cone corresponds to the set of order-preserving maps $f : P \to \mathbf{R}$ with the extra constraint that $f(i) < f(j)$ whenever $i < j$ and $g(i) > g(j)$ for some fixed labeling g. Aside from integrality constraints (to be discussed further in Example 4.2 below), these are essentially the "(P,g)-partitions" introduced by Stanley in [**S1**].

Remark 3.1. Labeled Coxeter cones arise naturally as soon as one requires a category of complexes that includes ordinary Coxeter cones and yet allows members to be cut by root hyperplanes in a way that leaves the pieces in the same category. Indeed, given a root β such that $\langle\lambda,\beta\rangle > 0$, we have $\langle\mu,\beta\rangle \geqslant 0$ or $\langle\mu,-\beta\rangle > 0$ for every $\mu\in V$, and hence

$$\Delta(\Psi,\lambda)=\Delta(\Psi\cup\{\beta\},\lambda)\ \cup\ \Delta(\Psi\cup\{-\beta\},\lambda) \qquad \text{(disjoint union).}$$

Similar reasoning applies if $\langle\lambda,\beta\rangle < 0$. Given that we always assume Ψ is nondegenerate, it should be noted that $\Psi\cup\{\beta\}$ and $\Psi\cup\{-\beta\}$ are nondegenerate if and only if $\Delta(\Psi)$ includes points (or chambers) on both sides of the hyperplane orthogonal to β.

Consider any cell of the Coxeter complex, say wC_J. We may assume that w is chosen from the set W^J of minimal coset representatives for W/W_J; i.e., $D(w)\subseteq J^c$. Note that membership in W^J requires that $w\alpha_j$ is positive for all $j\in J$, or equivalently, that $w\alpha$ is positive for all positive roots α in $\mathrm{Span}\{\alpha_j : j\in J\}$.

Lemma 3.2. *If $w\in W^J$ and λ is dominant, then*

$$wC_J\subset\Delta(\Psi,\lambda)\ \text{ if and only if }\ w\in X(\Psi).$$

Proof. Suppose $w\in X(\Psi)$. Since wC_J is on the boundary of $wC_\varnothing$, it follows that $\langle\mu,\beta\rangle\geqslant 0$ for all $\beta\in\Psi$ and all $\mu\in wC_J$, so $\Delta(\Psi,\lambda)$ could fail to include wC_J only if $\langle\mu,\beta\rangle = 0$ for some $\beta\in\Psi$ such that $\langle\lambda,\beta\rangle<0$. Given that λ is dominant, this forces β to be negative, and $wC_\varnothing\subset\Delta(\Psi)$ forces $w^{-1}\beta$ to be positive. However, having $\langle\mu,\beta\rangle=0$ for some $\mu\in wC_J$ forces $w^{-1}\beta$ into $C_J^\perp=\mathrm{Span}\{\alpha_j : j\in J\}$, and thus $\alpha := w^{-1}\beta$ is a positive root in $\mathrm{Span}\{\alpha_j : j\in J\}$ such that $w\alpha$ is negative, contradicting $w\in W^J$.

Conversely, suppose $wC_J \subset \Delta(\Psi,\lambda)$. If $w \notin X(\Psi)$, then we must have $\langle \mu, \beta\rangle < 0$ for some $\mu \in wC_\varnothing$ and $\beta \in \Psi$, in which case $w^{-1}\beta$ is a negative root. However, wC_J is part of $\Delta(\Psi,\lambda)$ so there must be some $\nu \in wC_J$ such that $\langle \nu, \beta\rangle = 0$ and $\langle \lambda, \beta\rangle > 0$, in which case the former implies $w^{-1}\beta \in \mathrm{Span}\{\alpha_j : j \in J\}$ and the latter implies that β is positive. Thus $\alpha := -w^{-1}\beta$ is a positive root in $\mathrm{Span}\{\alpha_j : j \in J\}$ such that $w\alpha$ is negative, contradicting $w \in W^J$. $\square$

Note that $\Delta(\Psi,\lambda)$ and $\Delta(w\Psi, w\lambda)$ are isomorphic (as complexes) for all $w \in W$, so there is no loss of generality in assuming that λ is dominant.

PROPOSITION 3.3 (Reiner [**R2**]). *If λ is dominant, then*

$$h_J(\Delta(\Psi,\lambda)) = |\{w \in X(\Psi) : D(w) = J\}|.$$

In particular, $h(\Delta(\Psi,\lambda); t) = \sum_{w\in X(\Psi)} t^{|D(w)|}$.

Proof. Lemma 3.2 shows that the chambers-to-subsets map $wC_\varnothing \mapsto D(w)$ induces a partition of the cells of $\Delta(\Psi,\lambda)$ that is compatible with (1.3). $\square$

The following result can be viewed as a root system analogue of Stanley's reciprocity theorem for (P,g)-partitions [**S1**].

PROPOSITION 3.4. *For all labelings λ, we have*

$$h_J(\Delta(\Psi,-\lambda)) = h_{J^c}(\Delta(\Psi,\lambda)).$$

In particular, $h(\Delta(\Psi,-\lambda); t) = t^n h(\Delta(\Psi,\lambda); 1/t)$.

Proof. By replacing (Ψ,λ) with $(w\Psi, w\lambda)$ if necessary, we may assume λ is dominant. Recalling that w_0 denotes the longest element of W, it follows that $-w_0\lambda$ is also dominant. Proposition 3.3 therefore implies

$$h_J(\Delta(\Psi,-\lambda)) = h_J(\Delta(w_0\Psi,-w_0\lambda)) = |\{w \in X(\Psi) : D(w_0w) = J\}|.$$

Now use the fact that $D(w_0w) = D(w)^c$ (see the discussion following (1.7)). $\square$

We refer to the labeling $-\lambda$ and the cone $\Delta(\Psi,-\lambda)$ as *opposite* to λ and $\Delta(\Psi,\lambda)$.

4. Hilbert functions and Ehrhart polynomials

For any pair (Ψ,λ) specifying a labeled Coxeter cone $\Delta(\Psi,\lambda)$, let us define a sequence of nonnegative integers $\Omega(\Psi,\lambda;m)$ for $m \geqslant 0$ by means of the expansion

$$\begin{aligned}\sum_{m\geqslant 0} \Omega(\Psi,\lambda;m)t^m &= \frac{h(\Delta(\Psi,\lambda);t)}{(1-t)^n} = f\Big(\Delta(\Psi,\lambda); \frac{t}{1-t}\Big) \\ &= \sum_{J\subseteq[n]} f_J(\Delta(\Psi,\lambda))(t+t^2+\cdots)^{|J|}. \qquad (4.1)\end{aligned}$$

Thus $\Omega(\Psi, \lambda; 0)$ is 0 or 1 according to whether λ is natural, and

$$\Omega(\Psi, \lambda; m) = \sum_{J \neq \varnothing} f_J(\Delta(\Psi, \lambda)) \binom{m-1}{|J|-1} \qquad (m > 0).$$

In particular, $\Omega(\Psi, \lambda; m)$ is a polynomial function of m for $m > 0$. One may also describe $\Omega(\Psi, \lambda; m)$ as the number of m-element multisets of vertices in the Coxeter complex whose support is a face of $\Delta(\Psi; \lambda)$. If λ is a natural labeling, so that $\Delta(\Psi, \lambda)$ is the complete Coxeter cone $\Delta(\Psi)$ (and therefore a simplicial complex), one may recognize $\Omega(\Psi, \lambda; m)$ as the Hilbert function for the face ring of $\Delta(\Psi)$ (see [**S4**]).

In case Φ is crystallographic, so that the roots generate a lattice, there is an alternative description of $\Omega(\Psi, \lambda; m)$ as the Ehrhart polynomial of a polyhedral complex naturally associated to $\Delta(\Psi, \lambda)$. Indeed, letting L denote the lattice dual to the root lattice; i.e.,

$$L = L(\Phi) = \{\lambda \in \operatorname{Span} \Phi : \langle \lambda, \beta \rangle \in \mathbf{Z} \text{ for all } \beta \in \Phi\},$$

there is a basis $\theta_1, \ldots, \theta_n$ for L dual to the simple roots; i.e., $\langle \theta_i, \alpha_j \rangle = \delta_{ij}$.

We remark that in some contexts (e.g., representation theory), the lattice L is known as the *co-weight lattice*, and $\theta_1, \ldots, \theta_n$ are the *fundamental co-weights*. In this case, W is a Weyl group, and W together with L (acting as a group of translations of V) generates the extended affine Weyl group $\widehat{W}$.

Noting that θ_i is an i-colored extreme ray of the dominant chamber $C_\varnothing$, each lattice point in the closure of $C_\varnothing$, say $\mu = \sum m_i \theta_i$ ($m_i \geqslant 0$), may be viewed as a multiset M of vertices of $C_\varnothing$ in which the i-colored vertex has multiplicity m_i. Defining δ to be the sum of the simple roots; i.e.,

$$\delta := \alpha_1 + \cdots + \alpha_n,$$

one sees that the height[2] function $\operatorname{ht}(\mu) := \langle \mu, \delta \rangle = \sum m_i$ yields the cardinality of M.

We remark that it is not hard to show that δ is a root if and only if Φ is irreducible, although this fact will not be significant in what follows.

The lattice L is W-stable, so if we use the W-action to extend the identification between lattice points and multisets to every face of the Coxeter complex, we obtain a bijection between m-element multisets of vertices supported on faces of $\Sigma(\Phi)$ and lattice points μ of height m, where

$$\operatorname{ht}(\mu) := \langle \mu^+, \delta \rangle \tag{4.2}$$

[2]This is not the usual height function for (co-)weights one encounters in representation theory.

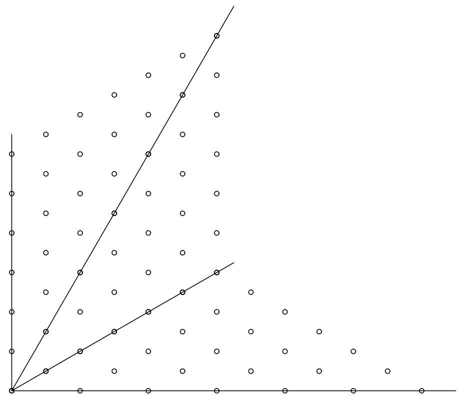

FIGURE 1: Lattice points of height $\leqslant 6$ in a $\mathcal{G}_2$ Coxeter cone.

and μ^+ denotes the dominant member of the W-orbit of μ. In this way we obtain

$$\Omega(\Psi, \lambda; m) = |\{\mu \in L \cap \Delta(\Psi, \lambda) : \mathrm{ht}(\mu) = m\}|, \tag{4.3}$$

and thus $\Omega(\Psi, \lambda; m)$ is the Ehrhart polynomial of the (bounded) polyhedral complex

$$\{\mu \in \Delta(\Psi, \lambda) : \mathrm{ht}(\mu) = 1\}.$$

If we relax the height constraint and let $\Omega_{\leqslant}(\Psi, \lambda; m)$ denote the number of lattice points in $\Delta(\Psi, \lambda)$ of height $\leqslant m$, then

$$\sum_{m \geqslant 0} \Omega_{\leqslant}(\Psi, \lambda; m) t^m = \frac{h(\Delta(\Psi, \lambda); t)}{(1-t)^{n+1}},$$

and $\Omega_{\leqslant}(\Psi, \lambda; m)$ may be recognized as the Ehrhart polynomial of

$$\{\mu \in \Delta(\Psi, \lambda) : \mathrm{ht}(\mu) \leqslant 1\}. \tag{4.4}$$

For an introduction to Ehrhart polynomials, see Section 4.6 of [**EC1**].

REMARK 4.1. If δ happens to be dominant, then the height function given in (4.2) has the alternative description

$$\mathrm{ht}(\mu) = \max\{\langle \mu, w\delta \rangle : w \in W\}.$$

Thus the polyhedral complex given in (4.4) is

$$\{\mu \in \Delta(\Psi, \lambda) : \langle \mu, w\delta \rangle \leqslant 1 \text{ for all } w \in W\},$$

and therefore it is evidently convex. However, it is not hard to check that $\mathcal{A}_n$ and $\mathcal{B}_n$ are the only irreducible crystallographic root systems for which δ is dominant, and the polyhedral complex defined by (4.4) is not generally convex (see Figure 1).

Example 4.2. (See also Example 4.6.34 in [**EC1**].) Let P be a partial ordering of $[n]$ corresponding to some $\Psi \subset \mathcal{A}_{n-1}$, as in Example 2.2. Let $g : P \to \mathbf{R}$ be a map defining some labeling λ of Ψ. Following Stanley [**S1**], a (P, g)-partition is an order-*reversing* map $f : P \to \mathbf{Z}^{\geqslant 0}$ such that $f(i) > f(j)$ whenever $i < j$ and $g(i) > g(j)$.

In our terms, this means that $-f$ corresponds to a point in the cone $\Delta(\Psi, \lambda)$ with nonpositive integer coordinates. However, the lattice $L = L(\mathcal{A}_{n-1})$ only has rank $n - 1$ (not n) and may be identified with the lattice of $\mathbf{Z}$-valued functions on P modulo the constant functions. Moreover, noting that $\delta = \varepsilon_n - \varepsilon_1$ in the case $\Phi = \mathcal{A}_{n-1}$, one sees that the "height" of any map $f : P \to \mathbf{R}$ is $\max(f) - \min(f)$. Thus one may interpret $\Omega(\Psi, \lambda; m)$ or $\Omega_{\leqslant}(\Psi, \lambda; m)$ as the number of (P, g)-partitions f with $\min(f) = 0$ and (respectively) $\max(f) = m$ or $\max(f) \leqslant m$. These are variations of Stanley's traditional order polynomial, which counts (P, g)-partitions f with $\min(f) \geqslant 1$ and $\max(f) \leqslant m$.

5. Rankings and lattice equivalence

From now on, we assume that Φ is crystallographic, so that the lattice $L = L(\Phi)$ is available, and (as in (4.3)) we are free to interpret $\Omega(\Psi, \lambda; m)$ as the number of lattice points in $\Delta(\Psi, \lambda)$ of height m.

Consider two labelings λ, λ' of Ψ and a lattice point $\nu \in L$. If the map $\mu \mapsto \mu + \nu$ is a bijection from $L \cap \Delta(\Psi, \lambda)$ to $L \cap \Delta(\Psi, \lambda')$, then we say that translation by ν is a *lattice equivalence* from $\Delta(\Psi, \lambda)$ to $\Delta(\Psi, \lambda')$.

It is clear that lattice equivalence defines an equivalence relation among the labelings of a given Coxeter cone. More generally, one could permit W-action as well, so that two cones are equivalent if their lattice points are $\widehat{W}$-transformations of each other.

Proposition 5.1. *If Ψ has no redundancies, then translation by $\nu \in L$ is a lattice equivalence from $\Delta(\Psi, \lambda)$ to $\Delta(\Psi, \lambda')$ if and only if for all $\beta \in \Psi$, we have*

$$\begin{aligned} \langle \nu, \beta \rangle &= +1 \quad \text{if } \langle \lambda, \beta \rangle > 0 \text{ and } \langle \lambda', \beta \rangle < 0, \\ \langle \nu, \beta \rangle &= -1 \quad \text{if } \langle \lambda, \beta \rangle < 0 \text{ and } \langle \lambda', \beta \rangle > 0, \\ \langle \nu, \beta \rangle &= 0 \qquad \text{otherwise.} \end{aligned} \tag{5.1}$$

Proof. To show that the stated conditions are necessary, fix a choice of $\beta \in \Psi$. Given that Ψ has no redundancies, there must be some Weyl chamber $wC_{\varnothing} \subset \Delta(\Psi)$ that has a facet orthogonal to β; i.e., $\beta = w\alpha_i$ for some i. Taking $\mu \in L$ to be any lattice point in the relative interior of this facet, so that μ is on only one root hyperplane, we must have $\langle \mu, \beta \rangle = 0$ and $\langle \mu, \beta' \rangle > 0$ for all other $\beta' \in \Psi$.

If $\langle \lambda, \beta \rangle > 0$, then $\mu \in L \cap \Delta(\Psi, \lambda)$. Given that translation by ν is a lattice equivalence, we must have $\mu + \nu \in L \cap \Delta(\Psi, \lambda')$. Since $\langle \mu + \nu, \beta \rangle = \langle \nu, \beta \rangle$, we obtain

$$\langle \lambda, \beta \rangle > 0,\ \langle \lambda', \beta \rangle > 0 \ \Rightarrow\ \langle \nu, \beta \rangle \geqslant 0,$$
$$\langle \lambda, \beta \rangle > 0,\ \langle \lambda', \beta \rangle < 0 \ \Rightarrow\ \langle \nu, \beta \rangle \geqslant 1.$$

Now suppose $\langle \lambda, \beta \rangle < 0$. The lattice point $\theta = w\theta_i$ satisfies $\langle \theta, \beta \rangle = 1$, and since $\Delta(\Psi)$ includes the boundary of $wC_{\varnothing}$, we also have $\langle \theta, \beta' \rangle \geqslant 0$ for all $\beta' \in \Psi$. In particular, $\mu + \theta$ is a lattice point in $\Delta(\Psi, \lambda)$, so $\mu + \theta + \nu$ must be a lattice point in $\Delta(\Psi, \lambda')$. Since $\langle \mu + \theta + \nu, \beta \rangle = \langle \nu, \beta \rangle + 1$, we obtain

$$\langle \lambda, \beta \rangle < 0,\ \langle \lambda', \beta \rangle > 0 \ \Rightarrow\ \langle \nu, \beta \rangle \geqslant -1,$$
$$\langle \lambda, \beta \rangle < 0,\ \langle \lambda', \beta \rangle < 0 \ \Rightarrow\ \langle \nu, \beta \rangle \geqslant 0.$$

Since translation by $-\nu$ must be a lattice equivalence from $\Delta(\Psi, \lambda')$ to $\Delta(\Psi, \lambda)$, it follows that the four implications above must also hold for $-\nu$, with the roles of λ and λ' reversed. This yields the stated conditions.

For the converse, assume ν satisfies the stated conditions and that $\mu \in L \cap \Delta(\Psi, \lambda)$. For each $\beta \in \Psi$, the following table summarizes the possible values for $\langle \mu + \nu, \beta \rangle$.

$\langle \lambda, \beta \rangle$	$\langle \lambda', \beta \rangle$	$\langle \mu, \beta \rangle$	$\langle \nu, \beta \rangle$	$\langle \mu + \nu, \beta \rangle$
> 0	> 0	$\geqslant 0$	0	$\geqslant 0$
> 0	< 0	$\geqslant 0$	1	$\geqslant 1$
< 0	> 0	$\geqslant 1$	-1	$\geqslant 0$
< 0	< 0	$\geqslant 1$	0	$\geqslant 1.$

Thus $\mu + \nu$ is a lattice point in $\Delta(\Psi, \lambda')$. By symmetry, the same reasoning shows that $\mu \in L \cap \Delta(\Psi, \lambda')$ implies $\mu - \nu \in L \cap \Delta(\Psi, \lambda)$, so the map $\mu \mapsto \mu + \nu$ is a bijection. □

As a special case, note that translation by $\nu \in L$ provides a lattice equivalence from $\Delta(\Psi, \lambda)$ to the oppositely labeled cone $\Delta(\Psi, -\lambda)$ if and only if for all $\beta \in \Psi$, we have $\langle \nu, \beta \rangle = +1$ if $\langle \lambda, \beta \rangle > 0$ and $\langle \nu, \beta \rangle = -1$ if $\langle \lambda, \beta \rangle < 0$. In such cases, ν itself could be used to provide the labeling of Ψ; i.e., $\Delta(\Psi, \lambda) = \Delta(\Psi, \nu)$. Thus, a labeled Coxeter cone is lattice-equivalent to its opposite if and only if it has the form $\Delta(\Psi, \lambda)$, where $\lambda \in L$ and

$$\langle \lambda, \beta \rangle = \pm 1 \quad \text{for all } \beta \in \Psi. \tag{5.2}$$

In this case, translation by λ provides the lattice equivalence. We define λ to be a *ranking* of Ψ when (5.2) holds, and say that the pair (Ψ, λ) is *ranked*.

Note that in a natural ranking, we have $\langle \lambda, \beta \rangle = 1$ for all $\beta \in \Psi$.

EXAMPLE 5.2. (a) Continuing Example 2.2, the naturally ranked $\mathcal{A}_{n-1}$ Coxeter cones correspond to ranked partial orderings of $[n]$; i.e., posets P for which there is a labeling $r : P \to \mathbf{Z}$ such that $r(i) - r(j) = 1$ for each covering relation $i > j$ in P. If we drop the requirement of naturality, we obtain posets for which there is a labeling r such that $r(i) - r(j) = \pm 1$ for each covering relation $i > j$. This category includes the "sign-graded" posets introduced by Brändén [**B2**] (see also the discussion in Example 6.4 below).

(b) Recall from Example 2.3 that there is a $\mathcal{B}_n$ Coxeter cone corresponding to any partial ordering P of $\pm[n]$ such that $i \mapsto -i$ is an anti-automorphism. In this context, the lattice L may be identified with $\mathbf{Z}^n$ so that P has a ranking if and only if there is a function $r : P \to \mathbf{Z}$ such that $r(-i) = -r(i)$ for all i, and

$$r(i) - r(j) = \begin{cases} \pm 1 & \text{if } j \neq -i, \\ \pm 2 & \text{if } j = -i. \end{cases}$$

for all covering relations $i > j$ in P. Note that $r(i) = \pm 1$ is forced when i covers $-i$.

(c) The root systems $\mathcal{B}_n$ and $\mathcal{C}_n$ have the same hyperplanes, so their Coxeter cones are indistinguishable. However, the lattice $L(\mathcal{C}_n)$ is larger than $L(\mathcal{B}_n)$, so $\mathcal{C}_n$-rankings are distinct from $\mathcal{B}_n$-rankings. Indeed, a $\mathcal{C}_n$-ranking for the poset P in the previous example is a $\mathbf{Z}$-valued or $(\mathbf{Z} + 1/2)$-valued function r on P such that $r(-i) = -r(i)$ for all i and $r(i) - r(j) = \pm 1$ for covering pairs $i > j$. Here, $r(i) = \pm 1/2$ is forced when i covers $-i$.

(d) If P has no covering pair $i > j$ such that $j = -i$, then the corresponding $\mathcal{B}_n$ cone is also a $\mathcal{D}_n$ cone (and conversely). Moreover, the lattices $L(\mathcal{D}_n)$ and $L(\mathcal{C}_n)$ coincide, so the conditions for a rank function r in this case are the same as those for $\mathcal{C}_n$.

6. Gradings and shift equivalence

Suppose that translation by $\nu \in L$ is a lattice equivalence from $\Delta(\Psi, \lambda)$ to $\Delta(\Psi, \lambda')$. Motivated by a previous construction in the special case of posets (see Section 4 of [**St2**]), we define this lattice equivalence to be a *shift equivalence* if translation by ν changes the height of each lattice point in $\Delta(\Psi, \lambda)$ by some constant k. Recalling (4.2), this means

$$\langle (\mu + \nu)^+, \delta \rangle - \langle \mu^+, \delta \rangle = k \quad \text{for all } \mu \in L \cap \Delta(\Psi, \lambda), \tag{6.1}$$

where μ^+ denotes the dominant member of the W-orbit of μ.

We say that k is the *value* of the shift.

PROPOSITION 6.1. *If there is a shift equivalence* $\Delta(\Psi, \lambda) \to \Delta(\Psi, \lambda')$ *of value* k*, then*

$$h(\Delta(\Psi, \lambda'); t) = t^k h(\Delta(\Psi, \lambda); t).$$

Proof. Given that translation by ν provides a bijection between the lattice points in $\Delta(\Psi, \lambda)$ and $\Delta(\Psi, \lambda')$, one sees that (4.3) and (6.1) imply

$$\Omega(\Psi, \lambda; m) = \Omega(\Psi, \lambda'; m+k) \quad \text{for all } m \in \mathbf{Z},$$

following the convention that $\Omega(\Psi, \lambda; m) = 0$ for $m < 0$. Applying (4.1), we obtain

$$\begin{aligned} \frac{t^k h(\Delta(\Psi, \lambda); t)}{(1-t)^n} &= \sum_{m \in \mathbf{Z}} \Omega(\Psi, \lambda; m) t^{m+k} \\ &= \sum_{m \in \mathbf{Z}} \Omega(\Psi, \lambda'; m+k) t^{m+k} = \frac{h(\Delta(\Psi, \lambda'); t)}{(1-t)^n}, \end{aligned}$$

and the result follows. □

PROPOSITION 6.2. *If translation by ν is a lattice equivalence $\Delta(\Psi, \lambda) \to \Delta(\Psi, \lambda')$, then it is a shift equivalence of value k if and only if $\langle \nu, w\delta \rangle = k$ for all $w \in X(\Psi)$.*

Proof. To show that the stated condition is necessary, consider $w \in X(\Psi)$. We may choose $\mu \in L$ sufficiently deep in the interior of the chamber $wC_\varnothing$ so that $\mu + \nu$ belongs to the same chamber. Hence $\mu^+ = w^{-1}\mu$ and $(\mu + \nu)^+ = w^{-1}(\mu + \nu)$, and therefore

$$\langle \nu, w\delta \rangle = \langle w^{-1}\nu, \delta \rangle = \langle (\mu + \nu)^+, \delta \rangle - \langle \mu^+, \delta \rangle = k.$$

For the converse, choose a lattice point μ in $\Delta(\Psi, \lambda)$, and consider $f(t) = \langle (\mu + t\nu)^+, \delta \rangle$, a continuous, piecewise linear function of t. Since $\mu + \nu$ is necessarily in $\Delta(\Psi)$ (in fact, a lattice point in $\Delta(\Psi, \lambda')$) and $\Delta(\Psi)$ is convex, it follows that the interval $0 \leqslant t \leqslant 1$ may be partitioned into subintervals where $\mu + t\nu$ is confined to single chambers $wC_\varnothing$ in $\Delta(\Psi)$. In any such subinterval, we have $(\mu + t\nu)^+ = w^{-1}(\mu + t\nu)$, and the given hypothesis implies $f(t) = \langle \mu + t\nu, w\delta \rangle = \langle \mu, w\delta \rangle + kt$. The continuity of f forces $\langle \mu, w\delta \rangle$ to be independent of w; i.e., f is linear, $k = f(1) - f(0) = \mathrm{ht}(\mu + \nu) - \mathrm{ht}(\mu)$, and translation by ν is a shift equivalence of value k. □

We define a labeled Coxeter cone to be *graded* if it is shift-equivalent to its opposite; the *rank* of the cone is defined to be the value of the shift. This requires in particular that the cone must be lattice-equivalent to its opposite; thus by Proposition 6.2 and the discussion surrounding (5.2), a labeled Coxeter cone is graded of rank k if and only if it has the form $\Delta(\Psi, \lambda)$ where $\lambda \in L$ is a ranking of Ψ (i.e., $\langle \lambda, \beta \rangle = \pm 1$ for all $\beta \in \Psi$) and

$$\langle \lambda, w\delta \rangle = k \quad \text{for all } w \in X(\Psi). \tag{6.2}$$

In this case, translation by λ provides the shift equivalence. We define the labeling λ to be a *grading* of Ψ of *rank* k when these two conditions hold (i.e., (5.2) and (6.2)), and say that the pair (Ψ, λ) is *graded* (and of rank k).

Note that if Ψ is in standard form (i.e., $\Psi \subseteq \Phi^+$), the rank of a grading λ is $\langle \lambda, \delta \rangle$.

Remark 6.3. (a) Consider any $\delta' \in V$ such that $\langle \delta', \alpha_i \rangle = 0$ if and only if $\langle \delta, \alpha_i \rangle = 0$. (This is not equivalent to δ and δ' having the same W-stabilizer, since neither δ nor δ' is necessarily dominant.) We claim that a ranking λ is a grading (i.e., (6.2) holds for some k) if and only if $\langle \lambda, w\delta' \rangle$ is constant as w varies over $X(\Psi)$. Indeed, $\langle \lambda, w\delta' \rangle$ is not constant if and only if there is an adjacent pair $w, ws_i \in X(\Psi)$ such that $\langle \lambda, w\delta' \rangle \neq \langle \lambda, ws_i\delta' \rangle$. However, $\delta' - s_i\delta'$ is a multiple of α_i, so this is equivalent to the multiple being nonzero (i.e., $\langle \delta', \alpha_i \rangle \neq 0$) and $\langle \lambda, w\alpha_i \rangle \neq 0$, neither of which depends on the choice of δ'.

(b) If we choose δ' in (a) so that it is dominant, then $\{w\delta' : w \in X(\Psi)\}$ is simply the intersection of $\Delta(\Psi)$ with the W-orbit of δ'; i.e., a ranking λ is a grading if and only if $\langle \lambda, \mu \rangle$ is constant as μ varies over $W\delta' \cap \Delta(\Psi)$. As noted previously (see Remark 4.1), it happens that δ is already dominant in $\mathcal{A}_n$ and $\mathcal{B}_n$, so one may take $\delta' = \delta$ in such cases.

Example 6.4. (a) Recall from Example 5.2(a) that a ranked Coxeter cone for $\mathcal{A}_{n-1}$ is a partial ordering P of $[n]$ with a labeling $r : P \to \mathbf{Z}$ such that $r(i) - r(j) = \pm 1$ for each covering relation $i > j$. In this case, δ is the dominant root of $\mathcal{A}_{n-1}$, so $W\delta \cap \Delta(\Psi)$ is the set of roots in $\Delta(\Psi)$; these correspond to maps $f : P \to \{0, \pm 1\}$ such that $f^{-1}(1) = \{i\}$ and $f^{-1}(-1) = \{j\}$ for some maximal i and minimal j in P. Pairing f against the ranking r yields $r(i) - r(j)$, so (following Remark 6.3) we conclude that r is a grading of P if and only if the rank difference between all pairs of maximal and minimal elements is constant. Equivalently, all maximal elements of P have the same rank, and all minimal elements of P have the same rank. These are precisely the sign-graded posets introduced by Brändén in [**B2**]. The natural gradings are gradings of posets in the usual sense (e.g., [**EC1**]).

(b) Let P be a "type $\mathcal{B}$" partial ordering of $\pm[n]$ with a ranking r as in Example 5.2(b). In this case, δ is the short dominant root, so $W\delta \cap \Delta(\Psi)$ is the set of short roots in $\Delta(\Psi)$; these correspond to maps $f : P \to \{0, \pm 1\}$ such that $f^{-1}(1) = \{i\}$ and $f^{-1}(-1) = \{-i\}$ for some maximal element i in P. Following Remark 6.3, we conclude that r is a grading of P if and only if all maximal elements of P have the same rank.

(c) If P is a type $\mathcal{B}$ partial ordering of $\pm[n]$ as above, but with a $\mathcal{C}_n$-ranking r as in Example 5.2(c), then the conditions for r to be a $\mathcal{C}_n$-grading are complicated by the fact that δ is not dominant in $\mathcal{C}_n$ for $n \geqslant 3$. Let us declare a subset C of $\pm[n]$ to be *central* if it is obtained by deleting a maximal and minimal pair from P of the form $\pm i$, or a sequence of such deletions. If C has $2k$ elements, we call it a $2k$-*center*. Following Remark 6.3, one can show that for $n \geqslant 3$, r is a $\mathcal{C}_n$-grading if and only if all maximal elements P have the same rank, and the maximal elements of all 2-centers of P have the same rank. Moreover, the latter rank is necessarily either 0 or $\pm 1/2$, depending on whether r is $\mathbf{Z}$-valued or $(\mathbf{Z}+1/2)$-valued, and the integral case occurs if and only if there is at least one incomparable pair of the form $i, -i$.

(d) Let P be a type $\mathcal{B}$ partial ordering of $\pm[n]$ as above, but with a $\mathcal{D}_n$-ranking r and no covering pair $i > j$ such that $j = -i$, as in Example 5.2(d). Assuming $n \geqslant 4$, one can use Remark 6.3 to show that r is a $\mathcal{D}_n$-grading if and only if all maximal elements of P have the same rank, and all maximal elements of all 4-centers of P have the same rank. Moreover, the latter rank is necessarily 0 or $\pm 1/2$, depending on whether r is $\mathbf{Z}$-valued or $(\mathbf{Z}+1/2)$-valued, and the integral case occurs if and only if some 4-center is an antichain (or equivalently, all 4-centers are antichains), or there is a unique 2-center.

The $\mathcal{A}_n$ case of the following is due to Brändén (see Corollary 2.4 of [**B2**]).

Proposition 6.5. *If λ is a grading of Ψ of rank k, then*

$$h(\Delta(\Psi,\lambda);t) = t^{n-k} h(\Delta(\Psi,\lambda);1/t), \tag{6.3}$$

and if λ' is a second grading of Ψ of rank k' and $\lambda - \lambda' \in 2L$, then

$$t^{k/2} h(\Delta(\Psi,\lambda);t) = t^{k'/2} h(\Delta(\Psi,\lambda');t). \tag{6.4}$$

Proof. The first identity follows from Propositions 3.4 and 6.1 and the fact that translation by λ is a shift equivalence from $\Delta(\Psi,\lambda)$ to $\Delta(\Psi,-\lambda)$ of value k.

For the second, set $\nu = (\lambda-\lambda')/2 \in L$ and observe that since λ and λ' are both rankings of Ψ (see (5.2)), it follows that for all $\beta \in \Psi$,

$$\langle \nu,\beta\rangle = +1 \;\text{ if } \langle \lambda,\beta\rangle = +1 \text{ and } \langle \lambda',\beta\rangle = -1,$$
$$\langle \nu,\beta\rangle = -1 \;\text{ if } \langle \lambda,\beta\rangle = -1 \text{ and } \langle \lambda',\beta\rangle = +1,$$

and $\langle \nu,\beta\rangle = 0$ otherwise. Therefore (Proposition 5.1) translation by ν is a lattice equivalence from $\Delta(\Psi,\lambda)$ to $\Delta(\Psi,\lambda')$. In addition, the fact that λ and λ' are both gradings of Ψ implies $\langle \lambda, w\delta\rangle = (k-k')/2$ for all $w \in X(\Psi)$ (see (6.2)), so Proposition 6.2 implies that translation by ν is a shift equivalence of value $(k-k')/2$. Now apply Proposition 6.1. $\square$

If λ and λ' are any two gradings of Ψ, the above reasoning shows that $\langle \lambda-\lambda',\beta\rangle \in 2\mathbf{Z}$ for all $\beta \in \Psi$, and (6.2) implies $\langle \lambda-\lambda', x\delta - y\delta\rangle = 0$ for all $x, y \in X(\Psi)$. Thus whenever Ψ and $\{x\delta - y\delta : x, y \in X(\Psi)\}$ generate the root lattice (a category that includes all $\mathcal{A}_n$ Coxeter cones), the hypothesis $\lambda - \lambda' \in 2L$ in (6.4) is redundant.

7. The h-polynomials of graded Coxeter cones

If (Ψ, λ) is graded of rank k, then Proposition 6.5 implies that $\Delta(\Psi, \lambda)$ has an h-vector that is symmetric and centered at $(n-k)/2$. Moreover, if we define

$$\bar{h}(\Delta(\Psi, \lambda); t) := t^{k/2} h(\Delta(\Psi, \lambda); t) \in \mathbf{Z}[t^{\pm 1/2}]$$

to be the *normalized* h-polynomial of $\Delta(\Psi, \lambda)$, then (6.4) implies

$$\bar{h}(\Delta(\Psi, \lambda); t) = \bar{h}(\Delta(\Psi, \lambda'); t) \quad \text{if } \lambda = \lambda' \bmod 2L. \tag{7.1}$$

We are now in a position to prove our main result: the determination of an explicit set of generators for the semigroup $\mathcal{H} = \mathcal{H}(\Phi) \subset \mathbf{Z}[t^{\pm 1/2}]$ consisting of all sums of normalized h-polynomials $\bar{h}(\Delta(\Psi, \lambda); t)$ for all graded pairs (Ψ, λ). As a corollary, we will deduce that graded Coxeter cones have nonnegative γ-vectors and unimodal h-vectors.

Note that since labeled Coxeter cones have nonnegative h-vectors, it is clear that $\mathcal{H}$ has a unique minimal generating set.

A. Saturation and irreducibility.

If λ is a ranking of Ψ, define the pair (Ψ, λ) to be *saturated* if for all $\beta \in \Phi$ such that $\langle \lambda, \beta \rangle = \pm 1$, either $\langle \mu, \beta \rangle \geqslant 0$ for all $\mu \in \Delta(\Psi)$, or $\langle \mu, \beta \rangle \leqslant 0$ for all $\mu \in \Delta(\Psi)$. If Ψ is in standard form, this is equivalent to having $\langle \mu, \beta \rangle \geqslant 0$ for all $\mu \in \Delta(\Psi)$ and all positive roots β with $\langle \lambda, \beta \rangle = \pm 1$.

We should note that in the context of posets (i.e., $\mathcal{A}_n$ Coxeter cones), the concept of saturation is due to Brändén [**B2**].

If λ is a ranking of Ψ that is not saturated, then $\Delta(\Psi)$ includes points on both sides of the hyperplane orthogonal to some root β with $\langle \lambda, \beta \rangle = \pm 1$. It follows that λ is a ranking of $\Psi \cup \{\beta\}$ and $\Psi \cup \{-\beta\}$, and (see Remark 3.1) both are nondegenerate and

$$h(\Delta(\Psi, \lambda); t) = h(\Delta(\Psi \cup \{\beta\}, \lambda); t) + h(\Delta(\Psi \cup \{-\beta\}, \lambda); t).$$

Furthermore, if λ is a grading of Ψ of rank k, then it follows easily from (6.2) that λ is also a rank k grading of both $\Psi \cup \{\beta\}$ and $\Psi \cup \{-\beta\}$. Thus in the graded case, the same additive relationship holds for the normalized h-polynomial; i.e.,

$$\bar{h}(\Delta(\Psi, \lambda); t) = \bar{h}(\Delta(\Psi \cup \{\beta\}, \lambda); t) + \bar{h}(\Delta(\Psi \cup \{-\beta\}, \lambda); t). \tag{7.2}$$

For $\mathcal{A}_n$ Coxeter cones, this observation is roughly equivalent to Theorem 3.2 of [**B2**].

Now define a graded pair (Ψ, λ) to be *irreducible* if (Ψ, λ') is saturated for every grading λ' of Ψ such that $\lambda = \lambda' \bmod 2L$. In particular, (Ψ, λ) must be saturated.

Proposition 7.1. *The semigroup $\mathcal{H}$ is generated by the normalized h-polynomials corresponding to all irreducible pairs (Ψ, λ).*

Proof. If (Ψ, λ) is graded but not irreducible, then there is a grading λ' of Ψ such that (Ψ, λ') is not saturated and $\lambda = \lambda' \bmod 2L$. Combining the latter with (6.4) or (7.1), we know that $\bar{h}(\Delta(\Psi, \lambda); t) = \bar{h}(\Delta(\Psi, \lambda'); t)$. Combining the former with (7.2), we deduce that $\bar{h}(\Delta(\Psi, \lambda); t)$ is a sum of normalized h-polynomials corresponding to two graded Coxeter cones of the form $\Delta(\Psi_1, \lambda')$ and $\Delta(\Psi_2, \lambda')$, so $\bar{h}(\Delta(\Psi, \lambda); t)$ does not belong to a minimal set of generators for $\mathcal{H}$. $\square$

B. Parabolic irreducibles.

For each $I \subseteq [n]$, let Φ_I denote the root subsystem of Φ generated by $\{\alpha_i : i \in I\}$.

Lemma 7.2. *If Ψ is in standard form and ranked by λ, then we have $W_I \subseteq X(\Psi)$, where $I = \{i : \langle \lambda, \alpha_i \rangle$ is even$\}$.*

Proof. Proceed by induction on the length of $w \in W_I$. In the case $w = 1$, the hypothesis of standard form implies $w \in X(\Psi)$. In the remaining cases, we may assume that there is some $i \in I$ such that $ws_i \in X(\Psi)$. The chambers indexed by ws_i and w are separated by exactly one root hyperplane; namely, the one orthogonal to $w\alpha_i$. It follows that $w \in X(\Psi)$ unless one of the facets of $\Delta(\Psi)$ is on this hyperplane. However λ is a ranking of Ψ, so all of the roots β normal to facets of $\Delta(\Psi)$ satisfy $\langle \lambda, \beta \rangle = \pm 1$, whereas $w\alpha_i$ is a root in Φ_I and hence $\langle \lambda, w\alpha_i \rangle$ is even. $\square$

For each $I \subseteq [n]$, let θ_I denote the unique member of L such that

$$\langle \theta_I, \alpha_j \rangle = \begin{cases} 0 & \text{if } j \in I, \\ 1 & \text{if } j \notin I. \end{cases}$$

Note that θ_I is dominant and has stabilizer W_I.

Proposition 7.3. *If $\Psi_I = \{w\alpha_j : w \in W_I,\ j \notin I\}$, then*

(a) $X(\Psi_I) = W_I$,

(b) *θ_I is a grading of Ψ_I of rank $n - |I|$,*

(c) $h(\Delta(\Psi_I, \theta_I); t) = h(\Sigma(\Phi_I); t) = h_I(t) := \sum_{w \in W_I} t^{|D(w)|}$, *and*

(d) *(Ψ_I, θ_I) is irreducible.*

Proof. (a) Note that W_I permutes the positive roots not in Φ_I, so Ψ_I is in standard form and ranked by θ_I. Thus Lemma 7.2 implies $W_I \subseteq X(\Psi_I)$. If equality fails, then by Proposition 2.6, one could find $w \in W_I$ and $j \notin I$ such that $ws_j \in X(\Psi_I)$. However,

w and ws_j are separated by the hyperplane orthogonal to the root $w\alpha_j \in \Psi_I$, so the corresponding chambers cannot both belong to $\Delta(\Psi_I)$, a contradiction.

(b) We have $\langle \theta_I, w\delta \rangle = \langle \theta_I, \delta \rangle = n - |I|$ for all $w \in W_I = X(\Psi_I)$, so it follows immediately from (6.2) that (Ψ_I, θ_I) is graded of rank $n - |I|$.

(c) This is a corollary of (a) and Proposition 3.3.

(d) If (Ψ_I, θ_I) failed to be irreducible, then there would exist some grading λ of Ψ_I such that (Ψ_I, λ) is not saturated and $\lambda = \theta_I \bmod 2L$. The former implies that $\Delta(\Psi_I)$ must contain points on either side of the hyperplane orthogonal to some root β such that $\langle \lambda, \beta \rangle = \pm 1$, and hence the latter forces $\langle \theta_I, \beta \rangle$ to be odd. However, (a) implies that the only root hyperplanes that separate chambers of $\Delta(\Psi_I)$ are those of Φ_I, whence $\beta \in \Phi_I$ and $\langle \theta_I, \beta \rangle = 0$, a contradiction. $\square$

Note that Proposition 2.4 implies that Ψ_I has no redundancies.

C. The classification of irreducible gradings.

For any ranking λ of Ψ, consider

$$\begin{aligned} A &= \{i : |\langle \lambda, w\alpha_i \rangle| \geqslant 2 \text{ for some } w \in X(\Psi)\}, \\ B &= \{i : \langle \lambda, w\alpha_i \rangle = \pm 1 \text{ for all } w \in X(\Psi)\}, \\ C &= \{i : \langle \lambda, w\alpha_i \rangle = 0 \text{ for all } w \in X(\Psi)\}. \end{aligned}$$

These represent disjoint sets of nodes in the Dynkin diagram of Φ.

LEMMA 7.4. *If (Ψ, λ) is saturated, then*

(a) *every node belongs to A, B, or C,*

(b) *there is no adjacent pair of nodes i, j such that $i \in A$ and $j \in C$,*

(c) *if λ is a grading of Ψ, then W_A fixes δ, and*

(d) *if Ψ is in standard form, then $X(\Psi) \subseteq W_{A \cup C} \cong W_A \times W_C$.*

Proof. (a) Assume $i \notin A$, so that $|\langle \lambda, w\alpha_i \rangle| \leqslant 1$ for all $w \in X(\Psi)$. If $|\langle \lambda, w\alpha_i \rangle|$ were not constant (as a function of w), then by following a sequence of simple reflections connecting chambers in $\Delta(\Psi)$, one could find a pair $w, ws_j \in X(\Psi)$ such that

$$\langle \lambda, ws_j\alpha_i \rangle - \langle \lambda, w\alpha_i \rangle = 1.$$

Since $s_j\alpha_i = \alpha_i + a\alpha_j$ for some integer a, it follows that $a\langle \lambda, w\alpha_j \rangle = 1$, hence $a = \pm 1$ and $\langle \lambda, w\alpha_j \rangle = \pm 1$, since both quantities are integers. However, given that (Ψ, λ) is saturated, every chamber of $\Delta(\Psi)$ must be on the same side of the hyperplane orthogonal to $w\alpha_j$, contradicting the fact that the chambers indexed by w and ws_j are on opposite sides.

Thus $|\langle\lambda, w\alpha_i\rangle|$ is constant as a function of w, and hence $i \in B$ or $i \in C$, according to whether the constant is 1 or 0.

(b) Suppose $i \in A$ and $j \in C$. There must exist some $w \in X(\Psi)$ such that $\langle\lambda, w\alpha_i\rangle = c$ for some integer c with $|c| \geqslant 2$. The chambers indexed by ws_i and w are separated only by the hyperplane orthogonal to $w\alpha_i$, and $w\alpha_i \notin \Psi$ (because λ ranks Ψ and $\langle\lambda, w\alpha_i\rangle \neq \pm 1$), so $ws_i \in X(\Psi)$. Since $s_i\alpha_j = \alpha_j + b\alpha_i$ for some integer b, we obtain

$$0 = \langle\lambda, ws_i\alpha_j\rangle - \langle\lambda, w\alpha_j\rangle = b\langle\lambda, w\alpha_i\rangle = bc,$$

and hence $b = 0$; i.e., i and j cannot be adjacent.

(c) Given $i \in A$, there exists some $w \in X(\Psi)$ such that $|\langle\lambda, w\alpha_i\rangle| \geqslant 2$. By the same reasoning used to prove (b), we must also have $ws_i \in X(\Psi)$. Given that λ is a grading (see (6.2)), the fact that $s_i\delta = \delta + c\alpha_i$ for some integer c implies

$$0 = \langle\lambda, ws_i\delta\rangle - \langle\lambda, w\delta\rangle = c\langle\lambda, w\alpha_i\rangle.$$

Thus $c = 0$ and s_i fixes δ.

(d) The fact that $W_{A\cup C}$ is a direct product follows from (b). Thus if the claimed result were false, then by Proposition 2.6 there would have to exist some pair $w, ws_i \in X(\Psi)$ such that $w \in W_{A\cup C}$ and $i \notin A \cup C$. However, the former implies that $\Delta(\Psi)$ includes chambers on both sides of the hyperplane orthogonal to $w\alpha_i$, and the latter implies (via (a)) that $i \in B$ and $\langle\lambda, w\alpha_i\rangle = \pm 1$, contradicting the fact that (Ψ, λ) is saturated. $\square$

Lemma 7.5. *If (Ψ, λ) is saturated, graded, and has no redundancies, then Ψ has a grading λ' such that $\lambda = \lambda' \bmod 2L$ and $|\langle\lambda', w\alpha_i\rangle| \leqslant 2$ for all i and all $w \in X(\Psi)$.*

Proof. Let A, B, and C denote the sets of nodes introduced for Lemma 7.4. Replacing (Ψ, λ) with $(w\Psi, w\lambda)$ if necessary, we may assume that Ψ is in standard form.

Let λ_0 be a shortest element of the coset $\lambda + 2L$.

We claim that $|\langle\lambda_0, \beta\rangle| \leqslant 2$ for all $\beta \in \Phi$. Indeed, the reflection of λ_0 through the hyperplane orthogonal to β is $\lambda_0 - \langle\lambda_0, \beta\rangle\beta^\vee$, where $\beta^\vee = 2\beta/\langle\beta, \beta\rangle$ denotes the co-root corresponding to β. Since L includes all co-roots (Φ is crystallographic), one sees that if (say) $\langle\lambda_0, \beta\rangle = c \geqslant 3$, then $\lambda_0 - 2\beta^\vee$ would be an interior point of the line segment from λ_0 to its reflection $\lambda_0 - c\beta^\vee$, and thus a strictly shorter element of the coset $\lambda + 2L$.

Let $L_C = \{\mu \in L : \langle\mu, \alpha_i\rangle = 0 \text{ for all } i \notin C\}$, a sublattice of L. Noting that $\langle\lambda, \alpha_i\rangle$ is even (in fact, 0) for all $i \in C$, it follows that there is a unique element $\omega \in L_C$ such that $\lambda' = \lambda_0 + 2\omega$ satisfies $\langle\lambda', \alpha_i\rangle = 0$ for all $i \in C$.

We claim that λ' has the desired properties. Certainly $\lambda' = \lambda \bmod 2L$.

Notice that W_C fixes λ' (and λ) by construction. It follows from Lemma 7.4(d) that for every $w \in X(\Psi)$ there is an element $w' \in W_A$ such that

$$\langle \lambda', w\alpha_i \rangle = \langle \lambda', w'\alpha_i \rangle \quad \text{for all } i.$$

In particular, if $i \in C$, then W_A fixes α_i (Lemma 7.4(b)), whence $\langle \lambda', w\alpha_i \rangle = \langle \lambda', \alpha_i \rangle = 0$ for all $i \in C$ and all $w \in X(\Psi)$. On the other hand, if $i \notin C$, then $i \in A \cup B$ (Lemma 7.4(a)) and $W_A \alpha_i \subseteq \Phi_{A \cup B}$ is orthogonal to ω, whence

$$\langle \lambda', w\alpha_i \rangle = \langle \lambda', w'\alpha_i \rangle = \langle \lambda_0 + 2\omega, w'\alpha_i \rangle = \langle \lambda_0, w'\alpha_i \rangle.$$

Thus the shortness of λ_0 forces $|\langle \lambda', w\alpha_i \rangle| \leqslant 2$.

To complete the proof, we must show that λ' is a grading of Ψ. For this, recall that W_C fixes λ' and W_A fixes δ (Lemma 7.4(c)), so (6.2) is clear for λ' via Lemma 7.4(d). Thus we need only to show that λ' is a ranking; i.e., $\langle \lambda', \beta \rangle = \pm 1$ for all $\beta \in \Psi$. However, given that Ψ has no redundancies, Proposition 2.4 implies that every $\beta \in \Psi$ is of the form $w\alpha_i$ for some index i and some $w \in X(\Psi)$. Since λ is a ranking of Ψ and $\lambda' = \lambda \bmod 2L$, it must be the case that $\langle \lambda, w\alpha_i \rangle = \pm 1$ and $\langle \lambda', w\alpha_i \rangle$ is odd. Having shown that $|\langle \lambda', w\alpha_i \rangle| \leqslant 2$, we conclude that $\langle \lambda', w\alpha_i \rangle = \pm 1$ and the result follows. $\square$

Remark 7.6. The above proof cannot be simplified by taking λ' to be λ_0 (i.e., a shortest element of the coset $\lambda + 2L$), since this need not produce a grading of Ψ. For example, consider $\lambda = \theta_7$ in $\mathcal{E}_8$ with the Dynkin diagram numbered in the form

$$\begin{array}{c} 2 \\ | \\ 1-3-4-5-6-7-8. \end{array}$$

Proposition 7.3 shows that the pair $(\Psi_{\langle 7 \rangle}, \theta_7)$ fits the hypotheses of Lemma 7.5, whereas from the diagram one sees that $\langle \alpha_8^\vee, \alpha_j \rangle = 2$, -1, or 0, according to whether $j = 8$, $j = 7$, or $j \leqslant 6$. Thus $\alpha_8^\vee = \theta_7 \bmod 2L$. Since the shortest nonzero elements of $L(\mathcal{E}_8)$ are the co-roots $\beta^\vee$, we conclude that $\alpha_8^\vee$ is a shortest element of the coset $\theta_7 + 2L$, and it is not hard to show that it is unique as such, up to sign. However $s_8 \in W_{\langle 7 \rangle}$ and $s_8\delta = \delta - \alpha_8$, so $\langle \alpha_8^\vee, \delta - s_8\delta \rangle = 2$ and $\pm\alpha_8^\vee$ is not a grading of $\Psi_{\langle 7 \rangle}$ (cf. (6.2)).

We are now in a position to classify the Coxeter cones with an irreducible grading.

Theorem 7.7. *If (Ψ, λ) is irreducible, in standard form, and has no redundancies, then $\Psi = \Psi_I$ and $X(\Psi) = W_I$, where $I = \{i : \langle \lambda, \alpha_i \rangle$ is even$\}$. In particular, θ_I is a grading of Ψ, and the normalized h-polynomial of $\Delta(\Psi, \lambda)$ is*

$$\bar{h}_I(t) := t^{(n-|I|)/2} h_I(t) = t^{(n-|I|)/2} \sum_{w \in W_I} t^{|D(w)|}.$$

Proof. Given that (Ψ, λ) is irreducible, we may replace λ with any other grading λ' equivalent to λ mod $2L$ without changing the normalized h-polynomial (Proposition 6.5). Thus by Lemma 7.5, we may assume $|\langle \lambda, w\alpha_i \rangle| \leqslant 2$ for all i and all $w \in X(\Psi)$. Since $X(\Psi)$ includes W_I (Lemma 7.2) and W_I permutes $\lambda + 2L$, it follows that for all $\beta \in \Psi_I$ (i.e., when $\beta = w\alpha_i$ for some $w \in W_I$ and $i \notin I$), the quantity $|\langle \lambda, \beta \rangle|$ is odd and $\leqslant 2$, and hence $\langle \lambda, \beta \rangle = \pm 1$. However, (Ψ, λ) is saturated and in standard form, so all chambers in $\Delta(\Psi)$ must be on the positive side of the hyperplanes orthogonal to each $\beta \in \Psi_I$; i.e., $X(\Psi) \subseteq X(\Psi_I) = W_I$ (Proposition 7.3). Thus $X(\Psi) = W_I$. As noted previously, Proposition 2.4 shows that there are no redundancies in Ψ_I, so we must have $\Psi = \Psi_I$. The remaining claims now follow from Proposition 7.3. $\square$

The above result (with Proposition 7.1) yields the promised description of a set of generators for the semigroup of normalized h-polynomials.

Corollary 7.8. *The semigroup $\mathcal{H}(\Phi)$ is generated by $\{\bar{h}_I(t) : I \subseteq [n]\}$.*

Thus if $\Delta(\Psi, \lambda)$ is graded of rank k, then the h-polynomial $h(t) = h(\Delta(\Psi, \lambda); t)$ is in the nonnegative integer span of $\{t^{-k/2}\bar{h}_I(t) : I \subseteq [n]\}$. On the other hand, $h(t)$ is a true polynomial (i.e., no fractional or negative powers) of degree $\leqslant n$, and the degrees of the highest and lowest terms of $t^{-k/2}\bar{h}_I(t)$ are $(n - k \pm |I|)/2$. Hence,

Corollary 7.9. *If (Ψ, λ) is graded of rank k, there exist integers $a_I \geqslant 0$ such that*

$$h(\Delta(\Psi, \lambda); t) = \sum_I a_I t^{(n-k-|I|)/2} h_I(t),$$

where I ranges over subsets of $[n]$ such that $|I| \leqslant n - |k|$ and $|I| = n - k$ mod 2.

As the h-polynomial of the Coxeter complex $\Sigma(\Phi_I)$, we know that $h_I(t)$ is centered at $|I|/2$ and has a nonnegative γ-vector (Theorem 1.2). Thus generalizing Brändén's result for posets (Theorem 4.2 of [**B2**]), we have

Corollary 7.10. *If (Ψ, λ) is graded, then $\Delta(\Psi, \lambda)$ has a nonnegative γ-vector. In particular, the h-vectors of graded Coxeter cones are symmetric and unimodal.*

Remark 7.11. The h-vectors of *labeled* Coxeter cones are not unimodal in general, even when they are symmetric. Although no examples of this are known in $\mathcal{A}_n$, there is a simple example in $\mathcal{D}_4$ due to Reiner [**R1**]. Numbering the Dynkin diagram of $\mathcal{D}_4$ so that s_1, s_2, s_4 commute, consider the elements $w = s_1 s_2 s_4$ and ws_3. It is easy to see that the closure of $wC_\varnothing \cup ws_3 C_\varnothing$ is a $\mathcal{D}_4$ Coxeter cone $\Delta(\Psi)$ whose six facets are the non-shared walls of the two chambers. Moreover, it is clear that $D(w) = \{1, 2, 4\}$ and $D(ws_3) = \{3\}$. Thus relative to a dominant labeling λ, the h-polynomial of $\Delta(\Psi, \lambda)$ is $t + t^3$.

Appendix: Some flag h-polynomials

One may identify the elements of $W(\mathcal{B}_n)$ with n-tuples of the form

$$w = (w_1, \dots, w_n) = (\sigma_1 u_1, \dots, \sigma_n u_n),$$

where $\sigma_i = \pm 1$ and $u = (u_1, \dots, u_n)$ is a permutation of $[n]$. Relative to a natural choice of simple reflections, one has $D(w) = \{i : w_{i-1} > w_i\}$, using the convention $w_0 = 0$.

In the following formula for the flag h-polynomial of the Coxeter complex for $\mathcal{B}_n$, we use $\chi(\,\cdot\,)$ as an indicator function; i.e., $\chi(A) = 1$ if A is true; 0 if A is false.

Proposition A.1. *We have*

$$h(\Sigma(\mathcal{B}_n); t_1, \dots, t_n) = \sum_{u \in S_n} \prod_{i=1}^{n} \left(t_i^{\chi(u_{i-1} < u_i)} + t_{i+1}^{\chi(u_i > u_{i+1})} \right),$$

following the convention that $u_0 = 0$ and $u_{n+1} = n + 1$.

Proof. The presence of $i \in D(w)$ is controlled by exactly one of σ_{i-1} or σ_i. Indeed,

(1) if $u_{i-1} > u_i$, then $i \in D(w)$ if and only if $\sigma_{i-1} = 1$,
(2) if $u_{i-1} < u_i$, then $i \in D(w)$ if and only if $\sigma_i = -1$,

and this is valid even in the degenerate case $i = 1$. Thus for fixed u, the effects of $\sigma_1, \dots, \sigma_n$ on the descent set $D(w)$ are mutually independent. More explicitly, σ_i negatively controls i when $u_{i-1} < u_i$ (i.e., $i \in D(w)$ if and only if $\sigma_i = -1$), and positively controls $i+1$ when $u_i > u_{i+1}$ (i.e., $i + 1 \in D(w)$ if and only if $\sigma_i = 1$), even in the degenerate case $i = n$. Thus we obtain the factorization

$$\sum_{\sigma \in \mathbf{Z}_2^n} \prod_{i \in D(\sigma u)} t_i \;=\; \big(t_1 + t_2^{\chi(u_1 > u_2)}\big)\big(t_2^{\chi(u_1 < u_2)} + t_3^{\chi(u_2 > u_3)}\big) \cdots \big(t_n^{\chi(u_{n-1} < u_n)} + 1\big).$$

Now sum over all $u \in S_n$ and apply (1.6). $\square$

Let $p(u) := |\{i : u_{i-1} < u_i > u_{i+1}\}|$ denote the number of peaks (or local maxima) in the sequence $(u_0, u_1, \dots, u_{n+1})$.

Corollary A.2 (Petersen [**P**, Proposition 4.15]). *We have*

$$h(\Sigma(\mathcal{B}_n); t) = \sum_{u \in S_n} (4t)^{p(u)} (1 + t)^{n - 2p(u)}.$$

In particular, $\Sigma(\mathcal{B}_n)$ has a nonnegative γ-vector.

Proof. Setting $t_1 = \cdots = t_n = t$ in Proposition A.1, one sees that the i-th factor specializes to either $2t$ (if $u_{i-1} < u_i > u_{i+1}$), 2 (if $u_{i-1} > u_i < u_{i+1}$), or $1 + t$ (otherwise). However, any sequence that begins and ends with an increase must necessarily have an equal number of peaks and valleys (local extrema), so the first two possibilities both occur $p(u)$ times, and the last must occur $n - 2p(u)$ times. $\square$

Remark A.3. It is well-known that $\Sigma(\mathcal{B}_n)$ is isomorphic (as a simplicial complex) to the barycentric subdivision of an n-dimensional cube, With this in mind, one can recognize Proposition A.1 as a disguised formula for the *cd*-index of the cube due to Billera-Ehrenborg-Readdy (see Proposition 8.1 of [**BER**]). More precisely, one should parse the factors in each summand of Proposition A.1 in the order $i = 1, 2, \ldots, n$, and replace each factor of the form $1 + t_i$ or $1 + t_{i+1}$ with c, and each factor of the form $t_i + t_{i+1}$ with d (and leave the remaining factors of 2 untouched). In this way, each summand corresponds to a non-commutative monomial in the variables c and $2d$, and the resulting non-commutative polynomial is the *cd*-index of the n-cube. For more about the *cd*-index, including definitions, see [**BER**], [**S3**], or [**S4**]; note also that by a theorem of Stanley [**S3**], one knows that the *cd*-index of every convex polytope has nonnegative coefficients.

Turning to the Coxeter complex of $\mathcal{D}_n$, recall that $W(\mathcal{D}_n)$ is the subgroup of $W(\mathcal{B}_n)$ consisting of the elements $w = (w_1, \ldots, w_n) = (\sigma_1 u_1, \ldots, \sigma_n u_n)$ ($\sigma_i = \pm 1$, $u \in S_n$) such that $\sigma_1 \cdots \sigma_n = 1$. Relative to a natural choice of simple reflections, one has

$$i \in D(w) \;\Leftrightarrow\; \begin{cases} w_{i-1} > w_i & \text{if } i > 1, \\ -w_1 > w_2 & \text{if } i = 1. \end{cases} \tag{A.1}$$

Proposition A.4. *We have*

$$h(\Sigma(\mathcal{D}_n); t_1, \ldots, t_n) = \frac{1}{2} \sum_{u \in S_n} \big(t_1^{\chi(u_1 > u_2)} + t_2^{\chi(u_1 > u_2)}\big)\big((t_1 t_2)^{\chi(u_1 < u_2)} + t_3^{\chi(u_2 > u_3)}\big) \times \prod_{i=3}^{n} \big(t_i^{\chi(u_{i-1} < u_i)} + t_{i+1}^{\chi(u_i > u_{i+1})}\big),$$

following the convention that $u_{n+1} = n + 1$.

Proof. Extending $D(w)$ as defined in (A.1) to all of $W(\mathcal{B}_n)$, note that $D(w)$ is unchanged if we replace 1 with -1 or vice-versa wherever it occurs in $(w_1, \ldots, w_n)$. Thus we may let $\sigma_1, \ldots, \sigma_n$ vary independently over $\mathbf{Z}_2$, and via (1.6), we have

$$h(\Sigma(\mathcal{D}_n); t_1, \ldots, t_n) = \frac{1}{2} \sum_{u \in S_n} \sum_{\sigma \in \mathbf{Z}_2^n} \prod_{i \in D(\sigma u)} t_i.$$

Here, the presence of $1 \in D(w)$ is controlled by exactly one of σ_1 or σ_2. Indeed,

(1) if $u_1 > u_2$, then $1 \in D(w)$ if and only if $\sigma_1 = -1$,

(2) if $u_1 < u_2$, then $1 \in D(w)$ if and only if $\sigma_2 = -1$.

For $i > 1$, the presence of $i \in D(w)$ is controlled by exactly one of σ_{i-1} or σ_i in the same way as in the proof of Proposition A.1. Thus again for fixed u, the effects of $\sigma_1, \dots, \sigma_n$ on the descent set $D(w)$ are mutually independent, and the summands factor into n parts, one for each generator σ_i. The remainder of the proof now proceeds as before, aside from the exceptional factors contributed by σ_1 and σ_2. Indeed, σ_1 either has no effect on $D(w)$ and contributes a factor of 2 (when $u_1 < u_2$), or forces 1 or 2 into $D(w)$ (when $u_1 > u_2$) depending on the sign, and contributes the factor t_1+t_2. Similarly, σ_2 controls the presence of both 1 and 2 in $D(w)$ (when $u_1 < u_2$) and 3 (when $u_2 > u_3$), which accounts for the second exceptional factor. □

Corollary A.5. *We have*

$$h(\Sigma(\mathcal{D}_n); t) = \frac{1}{2} \sum_{u \in S_n} \phi(u)(4t)^{p(u)}(1+t)^{n-2p(u)},$$

where

$$\phi(u) = \begin{cases} 2 & \text{if } u_1 < u_2 < u_3, \\ 0 & \text{if } u_2 < u_1 < u_3, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $\Sigma(\mathcal{D}_n)$ has a nonnegative γ-vector.

Proof. Let $D_i(u)$ and $B_i(u)$ denote the i-th factors of the summands indexed by $u \in S_n$ in Propositions A.1 and A.4, respectively. Clearly $D_i(u) = B_i(u)$ for $i > 2$; in the following table, we compare $D_1(u)D_2(u)$ and $B_1(u)B_2(u)$ after setting $t_1 = \cdots = t_n = t$.

	$D_1(u)D_2(u)$	$B_1(u)B_2(u)$
$u_1 < u_2,\ u_2 < u_3$	$2(1+t^2)$	$(1+t)^2$
$u_1 < u_2,\ u_2 > u_3$	$2t(1+t)$	$2t(1+t)$
$u_1 > u_2,\ u_2 < u_3$	$4t$	$4t$
$u_1 > u_2,\ u_2 > u_3$	$2t(1+t)$	$2t(1+t)$

Since interchanging u_1 and u_2 provides a bijection between permutations u that satisfy $u_1 < u_2 < u_3$ and $u_2 < u_1 < u_3$, it follows that we may eliminate the latter from the sum by absorbing their contributions into the former, at the expense of replacing the weight $D_1(u)D_2(u) = 2(1+t^2)$ with $2B_1(u)B_2(u) = 2(1+t)^2$. The result now follows by the same reasoning used to prove Corollary A.2. □

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