Appendix to: "A Curious Identity and the Volume of the Root Spherical Simplex " by C. De Concini and C. Procesi

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In this appendix, we provide an explanation for the "curious identity" (Theorem 1.2) of [**1**] without any case-by-case considerations. The proof is based on two elegant formulas, one due to L. Solomon, the other due to R. Steinberg. Both of these results deserve to be better known.

If W is a finite group generated by reflections in a real Euclidean space E, consider the class function on W defined by

$$\delta_W(q,t)(w) := \frac{\det(1-qw)}{\det(1-tw)} \qquad (w \in W),$$

where the determinants are evaluated as endomorphisms of E, and q, t are indeterminates. This may be viewed as a bi-graded character for $S(E)\otimes\Lambda(E)$, the tensor product of the symmetric and exterior algebras of E.

In his 1963 paper on invariants of finite reflection groups [**2**], Solomon explicitly determined the structure of the W-invariants of $S(E)\otimes\Lambda(E)$. At the level of characters, his structure theorem implies

$$\langle 1_W, \delta_W(q,t)\rangle_W = \prod_{i=1}^{\ell} \frac{1-qt^{d_i-1}}{1-t^{d_i}}, \tag{1}$$

where $d_1,\ldots,d_\ell$ are the degrees ($\ell = \dim E$), 1_W denotes the trivial character of W, and $\langle f,g\rangle_W := |W|^{-1}\sum_{w\in W} f(w)g(w)$ is the usual pairing of real-valued class functions f and g.

Henceforth, assume that W is a Weyl group with an irreducible root system $R\subset E$ of rank ℓ and simple reflections $S=\{s_1,\ldots,s_\ell\}$. Note that by setting $q=1$ and letting $t\to 1$ in (1), we obtain the quantity $\nu(R)$ in [**1**].

We let $s_0 \in W$ denote the reflection corresponding to the highest root and set $S_0 = S\cup\{s_0\}$. One may interpret S_0 as the W-image of the simple reflections of the associated affine Weyl group $\widehat{W}$.

Following Steinberg (see Section 3 of [**3**]), the action of $\widehat{W}$ on E descends to a W-action on the ℓ-torus E/Q (where Q denotes the root lattice), and the decomposition of E into simplicial alcoves by the arrangement of affine hyperplanes associated to R induces a simplicial decomposition of E/Q with a compatible W-action. Moreover, the W-stabilizers of the faces of E/Q are (up to conjugacy) generated by the various proper subsets of S_0.

Work supported by NSF grant DMS–0245385.

Given $w \in W$, Steinberg computes the Euler characteristic of the w-fixed subcomplex of E/Q in two different ways (see Theorem 3.12 of [**3**]), thereby obtaining the identity

$$\det(1-w) = \sum_{J \subset S_0} (-1)^{|S|-|J|} 1_{W_J}^W(w), \tag{2}$$

where W_J denotes the reflection subgroup generated by J, and $1_{W_J}^W$ denotes the permutation character of the action of W on W/W_J. It is important to note that J ranges over *proper* subsets of S_0.

Steinberg actually proves a more general identity that involves twisting by an involution; the above instance corresponds to the trivial involution. One may also recognize (2) as a companion to the more familiar identity

$$\det(w) = \sum_{J \subseteq S} (-1)^{|J|} 1_{W_J}^W(w).$$

Now consider the evaluation of

$$\lim_{t \to 1} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W.$$

First, notice that $\delta_W(1,t) \to 1_W$ as $t \to 1$, so we obtain

$$\lim_{t \to 1} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W = \langle \delta_W(1,0), 1_W \rangle_W = 1 \tag{3}$$

by setting $(q,t) = (1,0)$ in (1).

Second, notice that $\delta_W(1,0)(w) = \det(1-w)$, so (2) implies

$$\begin{aligned} \langle \delta_W(1,0), \delta_W(1,t) \rangle_W &= \sum_{J \subset S_0} (-1)^{|S|-|J|} \langle 1_{W_J}^W, \delta_W(1,t) \rangle_W \\ &= \sum_{J \subset S_0} (-1)^{|S|-|J|} \langle 1_{W_J}, \delta_{W_J}(1,t) \rangle_{W_J}, \end{aligned} \tag{4}$$

by Frobenius reciprocity. We can evaluate each of these terms by applying Solomon's formula to the reflection group W_J. But we need to be careful, because the action of W_J on E will have linear invariants if the rank of W_J is less than $\ell = |S|$. In such cases, this means that some of the degrees of W_J will equal 1, which introduces factors of $(1-q)/(1-t)$ in (1). Since we have set $q = 1$, these factors vanish.

Thus (4) should be restricted to ℓ-subsets of S_0, and we obtain

$$\langle \delta_W(1,0), \delta_W(1,t) \rangle_W = \sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{1 - t^{d_i^{(j)}-1}}{1 - t^{d_i^{(j)}}},$$

where $d_1^{(j)}, \ldots, d_l^{(j)}$ are the degrees of W_J for $J = S_0 - \{s_j\}$. Comparing this with (3) in the limit $t \to 1$, we obtain the "curious identity"

$$\sum_{j=0}^{\ell} \prod_{i=1}^{\ell} \frac{d_i^{(j)} - 1}{d_i^{(j)}} = 1.$$

References.

[**1**] C. De Concini and C. Procesi, A curious identity and the volume of the root spherical simplex, preprint.

[**2**] L. Solomon, Invariants of finite reflection groups, *Nagoya Math. J.* **22** (1963) 57–64.

[**3**] R. Steinberg, Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* (1968), no. 80.

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