Canonical Bases and Self-Evacuating Tableaux

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this paper, we prove that the action of a certain involution on canonical bases for irreducible $\mathfrak{g}$-modules defined by Lusztig agrees with the action of a special element of the associated simply connected Lie group, up to a scalar of unit absolute value. This leads to formulas for the number of fixed points of the involution by means of the Weyl character formula. In the $\mathfrak{g} = sl(n)$ case, Lusztig's involution has been proved by Berenstein and Zelevinsky to coincide with evacuation of semistandard tableaux. Thus we obtain as a corollary formulas for the number of self-evacuating semistandard tableaux of fixed shape. We also prove some refinements that keep track of the fixed points of the canonical basis belonging to each weight space. In particular, we obtain a formula for the number of self-evacuating standard tableaux of fixed shape.

Contents

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0. Introduction

The main purpose of this paper is to present some further examples of the "$q = -1$ phenomenon" introduced in [**Ste**]. The examples we present here rely partly on some recent work of Berenstein and Zelevinsky [**BZ**] that relates the evacuation operation on semistandard tableaux to the action of a fundamental involution on canonical bases for irreducible $U_q(sl_n)$-modules, as developed by Lusztig [**L**].

By the "$q = -1$ phenomenon" we mean a situation of the following type. One has a set of combinatorial objects B (such as tableaux), together with a generating function $F(q)$ that enumerates the objects in B according to some weight depending on q. (The use of q here is not intended to suggest any relationship to the quantum parameter q.) The $q = -1$ phenomenon occurs when there is a "natural" involution on B such that $F(-1)$ is the number of fixed points of the involution. If there is a closed formula for $F(q)$, then there is hence also a closed formula for the number of fixed points.

In this paper, the set B is a canonical basis for an irreducible $\mathfrak{g}$-module V, where $\mathfrak{g}$ is a (complex, semisimple) Lie algebra. The action of $\mathfrak{g}$ of course lifts naturally to the associated simply connected Lie group $\tilde{G}$. In this situation, there exist elements $x = x(q)$ in the Cartan subgroup of $\tilde{G}$ with the property that the trace of $x(q)$ on V can be computed via the q-analogue of the Weyl dimension formula. On the other hand, since the Cartan subgroup acts diagonally with respect to the basis B, the trace of $x(q)$ can be viewed as a generating function $F(q)$ for B. Building on previous work in [**Ste**], we prove (Theorem 1.2) that there is an element of $\tilde{G}$ that is conjugate to $x(-1)$ in $\tilde{G}$ and whose action on V, up to a scalar of unit absolute value, agrees with a certain fundamental involution on B studied by Lusztig [**L**, §3]. Thus, up to a scalar factor, $F(-1)$ is the number of fixed points of Lusztig's involution, and the net result is that we have obtained a large class of examples of the $q = -1$ phenomenon.

In Section 2, we analyze in detail the number of fixed points of Lusztig's involution. We characterize (Theorem 2.1) when it is nonzero, and show that in general (Theorem 2.2), it can be expressed in terms of dimensions of irreducible representations of a Lie algebra $\mathfrak{g}_{(2)}$ whose root system is contained in the root system of $\mathfrak{g}$. In the simply laced case, $\mathfrak{g}_{(2)}$ can be naturally realized as a Lie subalgebra of $\mathfrak{g}$, but not in general (see Remark 2.3(b)). In the case $\mathfrak{g} = sl(n)$, the situation is particularly striking—the number of fixed points is the dimension of an irreducible representation of the centralizer of the element $x(-1)$ whose trace counts the fixed points.

In Section 3, we restrict our attention to the case $\mathfrak{g} = sl(n)$. Since irreducible $\mathfrak{g}$-modules have bases in one-to-one correspondence with semistandard tableaux, it follows that the fixed points of Lusztig's involution must be interpretable as a set of tableaux.

Unfortunately, it is by no means clear how to find a natural correspondence between members of the canonical basis and semistandard tableaux; however, this has been settled by Berenstein and Zelevinsky [**BZ**], who also show that in terms of tableaux, Lusztig's involution corresponds to Schützenberger's evacuation operation.[1] Thus we obtain as a corollary (Theorem 3.1) an enumeration of semistandard tableaux fixed by evacuation. It is interesting to note (see Remark 3.2(d)) that the enumeration of self-complementary plane partitions can be obtained as a special case.

In Section 4, we refine the results for $sl(n)$ by giving (Theorem 4.1) an indirect computation of the number of members of the canonical basis fixed by the involution (i.e., the number of self-evacuating semistandard tableaux) belonging to each weight space (i.e., having a given content) in each irreducible representation. The special case of standard tableaux is particularly interesting, since in this case one can give an explicit closed formula (Theorem 4.3). In fact, the number of standard self-evacuating tableaux of a given shape turns out to be expressible as the dimension of an irreducible representation of the centralizer in S_n of w_0, the longest element of S_n.

In the last section, we provide a second method for enumerating self-evacuating standard tableaux. We show (Theorem 5.1) that with respect to the Kazhdan-Lusztig basis for irreducible representations of S_n, the action of w_0, or its negative, is identical to the action of evacuation on standard tableaux. Given the connections between Kazhdan-Lusztig bases and canonical bases elucidated by Grojnowski and Lusztig [**GL**] this conclusion is inevitable, although the proof we give is independent of this. In any case, one obtains as a corollary that the number of self-evacuating standard tableaux of given shape is (up to sign) the evaluation of an S_n-character at w_0.

We remark that the combinatorial results we obtain here have (or could have) been previously obtained by more elementary methods (see Remarks 3.2 and 4.4). On the other hand, although the machinery we employ is not strictly necessary, we believe that it offers deeper insight.

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1. Canonical Bases and Special Involutions

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, Weyl group W,

[1] The fact that the involution turns out to be evacuation should come as no surprise, since evacuation seems to be the only "natural" involution on tableaux.

root system $\Phi \subset \mathfrak{h}^*$, simple roots $\alpha_1, \ldots, \alpha_n$, weight lattice Λ, and enveloping algebra $U(\mathfrak{g})$. Choose a set of standard generators e_j, f_j, h_j $(1 \le j \le n)$ for $\mathfrak{g}$ and $U(\mathfrak{g})$ satisfying the Serre relations [**H**, §18]. Thus in particular, the h_j's form a basis for $\mathfrak{h}$, and e_j (resp., f_j) spans the α_j (resp., $-\alpha_j$) root space of $\mathfrak{g}$.

Let $\langle \cdot\,, \cdot \rangle$ denote the inner product on $\mathfrak{h}^*$ obtained from the Killing form, and let $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ denote the co-root corresponding to each $\alpha \in \Phi$. Define

$$\rho = (1/2) \textstyle\sum_{\alpha \in \Phi^+} \alpha, \qquad \rho^\vee = (1/2) \textstyle\sum_{\alpha \in \Phi^+} \alpha^\vee,$$

where Φ^+ denotes the set of roots in the positive span of the simple roots. It is well-known that $\langle \cdot\,, \rho^\vee \rangle$ is the height function on Φ; i.e., the linear functional on $\mathfrak{h}^*$ that assigns height one to the simple roots. Let h_0 denote the corresponding element of $\mathfrak{h}$, so that $\langle \lambda, \rho^\vee \rangle = \lambda(h_0)$ for all $\lambda \in \mathfrak{h}^*$.

Let $w_0 \in W$ denote the longest element of the Weyl group, and denote by δ the Weyl involution; i.e., the permutation of $[n] = \{1, \ldots, n\}$ defined by $w_0 \alpha_j = -\alpha_{\delta(j)}$. From the Serre relations, it is easy to check that the map

$$e_j \mapsto f_{\delta(j)}, \quad f_j \mapsto e_{\delta(j)}, \quad h_j \mapsto -h_{\delta(j)}$$

extends to an automorphism of $\mathfrak{g}$ and $U(\mathfrak{g})$; by abuse of notation, we will also denote this involution by δ.

Let G be the adjoint Lie group associated with $\mathfrak{g}$, with H the Cartan subgroup corresponding to $\mathfrak{h}$. Recall that there is a homomorphism from the normalizer $N(H)$ onto W; we will refer to any preimage in $N(H)$ of some $w \in W$ as an *extension* of w. In [**Ste**] (see Theorem 4.4(a)), we proved the existence of *special involutions* in G. These are involutions $x_0 \in G$ such that

- x_0 is an extension of w_0,
- x_0 is conjugate in G to $\mathrm{Exp}(i\pi h_0)$ $(i = \sqrt{-1})$.

LEMMA 1.1. *There is a special involution $x_0 \in G$ such that $x_0 = \delta$ as automorphisms of $\mathfrak{g}$.*

Proof. Let x_0 be a special involution. As an extension of w_0, x_0 must act as w_0 on $\mathfrak{h}$ (i.e., $x_0(h_j) = -h_{\delta(j)}$ for $1 \le j \le n$), and moreover, x_0 must interchange the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{w_0\alpha}$ for all $\alpha \in \Phi$. Since x_0 is an involution, it follows that there must exist nonzero scalars $c_j \in \mathbf{C}$ such that $x_0 e_j = c_j f_{\delta(j)}$ and $x_0 f_j = c_{\delta(j)}^{-1} e_{\delta(j)}$. We seek to prove that x_0 can be modified, if necessary, so that $c_j = 1$.

Applying x_0 to the relation $h_j = [e_j, f_j]$, we obtain

$$-h_{\delta(j)} = [c_j f_{\delta(j)}, c_{\delta(j)}^{-1} e_{\delta(j)}] = -c_j c_{\delta(j)}^{-1} h_{\delta(j)},$$

and therefore $c_j = c_{\delta(j)}$. Now for any nonzero scalars $b_j \in \mathbf{C}$, there is an automorphism of $\mathfrak{g}$ in H such that $e_j \mapsto b_j e_j$, $f_j \mapsto b_j^{-1} f_j$, and $h_j \mapsto h_j$. In particular, let $t \in H$ denote the automorphism corresponding to the choice $b_j = \sqrt{c_j}$. If we take $x_0' = t x_0 t^{-1}$, then x_0' will also be a special involution. However, using the relation $c_j = c_{\delta(j)}$, one can easily check that x_0' agrees with δ on $\mathfrak{g}$. $\square$

Henceforth, let us add to the list of defining properties of special involutions the condition that x_0 agree with δ as an automorphism of $\mathfrak{g}$.

Now choose a dominant weight $\lambda \in \Lambda$. Following Lusztig's work [**L**] on the Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$, there is a canonical basis B_λ for the irreducible $U_q(\mathfrak{g})$-module of highest weight λ. By taking the classical limit ($q \longrightarrow 1$), one also obtains a canonical basis for the irreducible $U(\mathfrak{g})$-module of highest weight λ. By abuse of notation we will denote this basis also by B_λ.

The members of B_λ are weight vectors for the action of $\mathfrak{h}$. In particular, included among them are a highest and a lowest weight vector, which we label as b_+ and b_-, respectively. In Lemma 3.2 and Theorem 3.3 of [**L**], Lusztig proves that there is an involution $\eta = \eta_\lambda$ on B_λ and an automorphism Δ of $U_q(\mathfrak{g})$ with the property that

$$\eta(E.b_+) = \Delta(E).b_- \tag{1.1}$$

for all $E \in U_q(\mathfrak{g})$.[2] (Here we have extended η linearly to the entire $U_q(\mathfrak{g})$-module.) In the limit $q \longrightarrow 1$, Δ specializes to the automorphism δ of $U(\mathfrak{g})$.

In the following, let $\tilde{G}$ denote the simply connected cover of G.

Theorem 1.2. *If $\tilde{x}_0 \in \tilde{G}$ is such that $\mathrm{Ad}(\tilde{x}_0) = x_0$ is a special involution, and V is an irreducible $\mathfrak{g}$-module of highest weight λ, then*

$$|\operatorname{tr}_V(\tilde{x}_0)| = \#\{b \in B_\lambda : b = \eta(b)\}; \tag{1.2}$$

i.e., the number of elements in the canonical basis fixed by η is, up to a scalar of unit absolute value, the trace of (the $\tilde{G}$-preimage of) a special involution acting on V.

Proof. The action of $\tilde{G}$ and $U(\mathfrak{g})$ on V are related in such a way that

$$x.(e.v) = \mathrm{Ad}(x)(e).(x.v)$$

for all $x \in \tilde{G}$, $e \in U(\mathfrak{g})$ and $v \in V$. (This is most easily seen by passing to a linear representation of $\tilde{G}$ in which $\mathrm{Ad}(x)(e) = xex^{-1}$.) In particular, since $\mathrm{Ad}(\tilde{x}_0) = x_0 = \delta$, it follows that

$$\tilde{x}_0.(e.b_+) = \delta(e).(\tilde{x}_0.b_+)$$

[2] In [**L**], both η and Δ are denoted by Λ.

for all $e \in U(\mathfrak{g})$. Comparing this with (1.1), bearing in mind that $\Delta \to \delta$ in the limit $q \to 1$, we see that

$$\tilde{x}_0.v = c\eta(v) \tag{1.3}$$

for all $v \in V$, where c is the nonzero scalar such that $\tilde{x}_0.b_+ = cb_-$. (Since x_0 is an extension of w_0, $\tilde{x}_0$ must permute the weight spaces of V according to w_0; in particular, $\tilde{x}_0$ must interchange the highest and lowest weight spaces. Thus c exists.)

It follows that $\mathrm{tr}_V(\tilde{x}_0) = c\,\mathrm{tr}_V(\eta)$. However, since η permutes the canonical basis, its trace is merely the number of fixed points. Thus to complete the proof, we need only to check that $|c| = 1$. For this, note that $\mathrm{Ad}(\tilde{x}_0)^2 = x_0^2 = 1$, so $(\tilde{x}_0)^2 \in \mathrm{Ker}(\mathrm{Ad}) = Z(\tilde{G})$. Since $Z(\tilde{G})$ is finite, it follows that $\tilde{x}_0$ must have finite order. However, $(\tilde{x}_0)^2$ acts as the scalar c^2 on V, so c must be a root of unity. $\square$

Remark 1.3. (a) The above argument (viz., (1.3) in the case of minuscule V) provides a classification-free derivation of Theorem 4.4(b) of [**Ste**].

(b) The $\mathfrak{g} = sl(n)$ case of the above result is implicit in the analysis of canonical bases by Berenstein and Zelevinsky (cf. the proof of Proposition 8.8 in [**BZ**].)

(c) Recall that by definition, x_0 is conjugate in G to $\mathrm{Exp}(i\pi h_0)$, where $\lambda(h_0) = \langle \lambda, \rho^\vee \rangle$ for all $\lambda \in \mathfrak{h}^*$. It follows that we can likewise choose $\tilde{x}_0$ so that it is conjugate in $\tilde{G}$ to $\mathrm{Exp}(i\pi h_0)$. (Here we are committing another abuse of notation, using $\mathrm{Exp}(\cdot)$ for the exponential maps for both G and $\tilde{G}$.) Since $2\rho^\vee \in \mathbf{Z}\Phi^\vee$ (the co-root lattice), it follows that $4\pi i\lambda(h_0) \in 2\pi i\mathbf{Z}$ for all weights $\lambda \in \Lambda$. Hence our choice of $\tilde{x}_0$ has order 2 or 4, and the scalar c appearing in (1.3) is a fourth root of unity.

(d) For some root systems (e.g., $\Phi = A_n$ with n even), it happens that $\rho^\vee \in \mathbf{Z}\Phi^\vee$; in such cases, the choice of $\tilde{x}_0$ we made in (c) will be an involution and $c = \pm 1$. In fact, in all cases we claim that either the trace of $\tilde{x}_0$ on V is 0, or else $\tilde{x}_0$ acts as an involution on V. This allows us to sharpen (1.2) by asserting that

$$\varepsilon_\lambda\,\mathrm{tr}_V(\tilde{x}_0) = \#\{b \in B_\lambda : b = \eta(b)\},$$

for some $\varepsilon_\lambda = \pm 1$. To prove this claim, recall that $\tilde{x}_0$ permutes the weight spaces of V according to the action of w_0 on Λ. It follows that if $\tilde{x}_0$ has nonzero trace, then there must be some w_0-invariant weight $\mu \in \Lambda$ with positive multiplicity in V. However in that case, we have $\langle \mu, \rho^\vee \rangle = \langle w_0\mu, w_0\rho^\vee \rangle = \langle \mu, -\rho^\vee \rangle$, so $\langle \mu, \rho^\vee \rangle = 0$. Hence $\langle \nu, \rho^\vee \rangle \in \mathbf{Z}$ for all weights ν occurring in V, since $\nu - \mu \in \mathbf{Z}\Phi$ for all such ν and $\langle\,\cdot\,, \rho^\vee \rangle$ is $\mathbf{Z}$-valued on Φ. Thus $2\pi i\nu(h_0) \in 2\pi i\mathbf{Z}$ for all such ν, so $\mathrm{Exp}(i\pi h_0)$ and $\tilde{x}_0$ are involutions on V.

2. Traces of Special Involutions

Having identified (up to sign) the number of fixed points of the involution η on B_λ as the trace of some element $\tilde{x}_0$ in the Lie group $\tilde{G}$, we now turn to the problem of explicitly evaluating this trace.

As in the previous section, let V denote an irreducible $\mathfrak{g}$-module of highest weight λ. Let $V = \bigoplus_{\mu\in\Lambda} V_\mu$ denote the weight space decomposition of V. For $h \in \mathfrak{h}$ and scalars $q, t \in \mathbf{C}$ such that $q = e^t$, we define

$$\varphi_\lambda(h;q) := \sum_{\mu\in\Lambda} (\dim V_\mu) e^{t\mu(h)} = \mathrm{tr}_V(\mathrm{Exp}(th)).$$

Note that if $\mu(h)$ is $\mathbf{Z}$-valued for all $\mu \in \Lambda$ appearing with nonzero multiplicity, this is a well-defined Laurent polynomial in q; otherwise, it is artificial for us to regard $\varphi_\lambda(h;q)$ as a function of q, rather than t.

By the q-analogue of the Weyl dimension formula (e.g., Lemma 2.5 of [**Ste**]), it is known that

$$\varphi_\lambda(h_0;q) = q^{-\langle\lambda,\rho^\vee\rangle} \prod_{\alpha\in\Phi+} \frac{1-q^{\langle\lambda+\rho,\alpha^\vee\rangle}}{1-q^{\langle\rho,\alpha^\vee\rangle}}, \tag{2.1}$$

where $h_0 \in \mathfrak{h}$ is such that $\mu(h_0) = \langle\mu,\rho^\vee\rangle$ for $\mu \in \mathfrak{h}^*$, as in Section 1.

Following Remark 1.3(c), we may and shall assume that $\tilde{x}_0$ is chosen to be conjugate in $\tilde{G}$ to $\mathrm{Exp}(i\pi h_0)$. Thus it follows that

$$\mathrm{tr}_V(\tilde{x}_0) = \varphi_\lambda(h_0;-1).$$

In view of (2.1), it is clear that one can give an explicit product formula for $\mathrm{tr}_V(\tilde{x}_0)$. In the following, we will provide a detailed analysis of this calculation, beginning with a characterization of the dominant weights λ such that $\varphi_\lambda(h_0;-1) \neq 0$.

For an arbitrary weight $\lambda \in \Lambda$, define

$$\Phi_\lambda := \{\alpha \in \Phi : \langle\alpha^\vee,\lambda\rangle \in 2\mathbf{Z}\}.$$

In particular, let us set

$$\Phi_{(2)} := \Phi_\rho = \{\alpha \in \Phi : \langle\alpha^\vee,\rho\rangle \in 2\mathbf{Z}\};$$

these are the roots in Φ whose co-roots have even height in $\Phi^\vee$. It is not hard to see that each Φ_λ is a root subsystem of Φ; indeed, more generally, if A is any subgroup of the root lattice $\mathbf{Z}\Phi$, then $A \cap \Phi$ is a root subsystem of Φ. In our case, we have $\Phi_\lambda = (A_\lambda \cap \Phi^\vee)^\vee$, where $A_\lambda = \{\alpha \in \mathbf{Z}\Phi^\vee : \langle \alpha, \lambda \rangle \in 2\mathbf{Z}\}$, a subgroup of $\mathbf{Z}\Phi^\vee$.

For example, consider $\Phi = A_{n-1} = \{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le n\}$. In this case Φ is simply laced, so there is no distinction between α and $\alpha^\vee$. The root $\varepsilon_i - \varepsilon_j$ has height $i-j$, so

$$\begin{aligned}\Phi_{(2)} &= \{\varepsilon_i - \varepsilon_j : i = j \bmod 2,\ 1 \le i \ne j \le n\} \\ &= \{\varepsilon_{2i} - \varepsilon_{2j} : 1 \le i \ne j \le n/2\} \cup \{\varepsilon_{2i-1} - \varepsilon_{2j-1} : 1 \le i \ne j \le (n+1)/2\} \\ &\cong A_{\lfloor n/2 \rfloor - 1} \oplus A_{\lceil n/2 \rceil - 1}.\end{aligned}$$

The isomorphism class of $\Phi_{(2)}$ for each irreducible root system Φ can be found in the Appendix.

Note that $\Phi_\lambda = \Phi_\mu$ iff $\lambda - \mu \in 2\Lambda$; thus the distinct root systems Φ_λ can be parameterized by $\Lambda/2\Lambda$, an elementary abelian group of order 2^n.

Theorem 2.1. *For dominant $\lambda \in \Lambda$, the following are equivalent.*

(a) $\varphi_\lambda(h_0; -1) \ne 0$.
(b) $\#\Phi_{\lambda+\rho} = \#\Phi_{(2)}$.
(c) $\Phi_{\lambda+\rho} \cong \Phi_{(2)}$.
(d) $\lambda + \rho \in W\rho \bmod 2\Lambda$ *(i.e., $\lambda + \rho = 2\mu + w\rho$ for some $\mu \in \Lambda$, $w \in W$).*

In other words, the set of all weights $\{\lambda + \rho : \varphi_\lambda(h_0; -1) \ne 0\}$ forms a single W-orbit in $\Lambda/2\Lambda$.

Proof. The trace of $\mathrm{Exp}(i\pi h_0)$ on V is nonzero if and only if the orders to which $q = -1$ is a zero of the numerator and denominator of (2.1) are equal. However, these orders are clearly $\#\Phi_{\lambda+\rho}$ and $\#\Phi_\rho = \#\Phi_{(2)}$, respectively. Thus (a) and (b) are equivalent.

Obviously, (c) implies (b). To prove (d) implies (c), suppose $w(\lambda + \rho) = 2\mu + \rho$. We then have

$$\begin{aligned}\alpha \in \Phi_{\lambda+\rho} \Leftrightarrow \langle \lambda + \rho, \alpha^\vee \rangle \in 2\mathbf{Z} &\Leftrightarrow \langle 2\mu + \rho, (w\alpha)^\vee \rangle \in 2\mathbf{Z} \\ &\Leftrightarrow \langle \rho, (w\alpha)^\vee \rangle \in 2\mathbf{Z} \Leftrightarrow w\alpha \in \Phi_{(2)}.\end{aligned}$$

Hence $\Phi_{(2)} = w\Phi_{\lambda+\rho}$, so $\Phi_{(2)} \cong \Phi_{\lambda+\rho}$.

To complete the proof, we need only to show that (b) implies (d). However, this is proved (case-by-case) in Lemma 5.4 of [**Ste**]. □

Let $\mathfrak{h}^*_{(2)}$ denote the subspace of $\mathfrak{h}^*$ spanned by $\Phi_{(2)}$. For $\mu \in \Lambda$ such that $\langle \mu, \alpha^\vee \rangle \in 2\mathbf{Z}$ for all $\alpha \in \Phi_{(2)}$, the orthogonal projection of $\mu/2$ onto $\mathfrak{h}^*_{(2)}$ belongs to the weight lattice of $\Phi_{(2)}$. As a particular case, consider the weight ρ. Since ρ is strongly dominant as a weight for Φ, then so too must (the projection of) $\rho/2$ be for $\Phi_{(2)}$. Thus

$$(1/2)\rho = \nu_0 + \rho_{(2)} \bmod (\mathfrak{h}^*_{(2)})^\perp$$

for some dominant weight ν_0 for $\Phi_{(2)}$, where $2\rho_{(2)} = \sum_{\alpha\in\Phi^+_{(2)}} \alpha$.

Let $\mathfrak{g}_{(2)}$ denote a semisimple Lie algebra with root system $\Phi_{(2)}$. On a case-by-case basis (see the Appendix), we have found that for each irreducible root system Φ, ν_0 is either 0 or a fundamental (in fact, minuscule) weight for $\Phi_{(2)}$. Moreover, the dimension of the irreducible $\mathfrak{g}_{(2)}$-module of highest weight ν_0 is a power of 2.

For any dominant weight ν for $\Phi_{(2)}$, let $d(\nu)$ denote the dimension of the corresponding irreducible $\mathfrak{g}_{(2)}$-module. Of course, by the Weyl dimension formula, one knows that

$$d(\nu) = \prod_{\alpha\in\Phi^+_{(2)}} \langle \nu + \rho_{(2)}, \alpha^\vee\rangle / \langle \rho_{(2)}, \alpha^\vee\rangle. \tag{2.2}$$

THEOREM 2.2. *If $\lambda \in \Lambda$ is dominant, then $\varphi_\lambda(h_0; -1)$ is either 0, or else there exists a dominant weight ν for $\Phi_{(2)}$ and $w \in W$ such that $w(\lambda+\rho)/2 = \nu + \rho_{(2)}$ mod $(\mathfrak{h}^*_{(2)})^\perp$ and*

$$\varphi_\lambda(h_0; -1) = \varepsilon_\lambda d(\nu)/d(\nu_0),$$

where $\varepsilon_\lambda = (-1)^{\langle\lambda,\rho^\vee\rangle} = \pm 1$.

Proof. If $\varphi_\lambda(h_0; -1) \neq 0$, then $w(\lambda+\rho) = \rho \bmod 2\Lambda$ for some $w \in W$, by Theorem 2.1. Recall (from the proof of the implication (d)$\Rightarrow$(c)) that this implies $w\Phi_{\lambda+\rho} = \Phi_{(2)}$. Now since $\lambda+\rho$ is strongly dominant, we have $\langle \lambda+\rho, \alpha\rangle \neq 0$ for all $\alpha \in \Phi$, and the same is therefore true for every weight in the W-orbit of $\lambda+\rho$. It follows that we can replace w with xw for some $x \in W_{(2)}$ (the Weyl group of $\Phi_{(2)}$) so that $w\Phi_{\lambda+\rho} = \Phi_{(2)}$ and $\langle w(\lambda+\rho), \alpha\rangle > 0$ for all $\alpha \in \Phi^+_{(2)}$. We therefore have

$$w(\lambda+\rho)/2 = \nu + \rho_{(2)} \bmod (\mathfrak{h}^*_{(2)})^\perp$$

for some dominant weight ν for $\Phi_{(2)}$.

Now consider using (2.1) to evaluate $\varphi_\lambda(h_0; -1)$. For any sets of integers a_j and b_j with equal numbers of even terms among the a_j's and b_j's, we have

$$\lim_{q\to -1} \frac{(1-q^{a_1})\cdots(1-q^{a_m})}{(1-q^{b_1})\cdots(1-q^{b_m})} = \frac{\prod_{j:a_j \text{ even}} a_j}{\prod_{j:b_j \text{ even}} b_j}.$$

Therefore,

$$\varphi_\lambda(h_0;-1) = (-1)^{\langle\lambda,\rho^\vee\rangle} \frac{\prod_{\alpha\in\Phi^+_{\lambda+\rho}} \langle\lambda+\rho,\alpha^\vee\rangle}{\prod_{\alpha\in\Phi^+_\rho}\langle\rho,\alpha^\vee\rangle}.$$

Note that the terms in the numerator of this product can be rewritten as $\langle w(\lambda+\rho),(w\alpha)^\vee\rangle$. However, since w was chosen so that $w\Phi_{\lambda+\rho}=\Phi_{(2)}$, the numerator can be rewritten as a product over $\Phi_{(2)}$, yielding

$$(-1)^{\langle\lambda,\rho^\vee\rangle}\varphi_\lambda(h_0;-1) = \prod_{\alpha\in\Phi^+_{(2)}} \frac{\langle w(\lambda+\rho),\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle} = \prod_{\alpha\in\Phi^+_{(2)}} \frac{\langle\nu+\rho_{(2)},\alpha^\vee\rangle}{\langle\nu_0+\rho_{(2)},\alpha^\vee\rangle} = \frac{d(\nu)}{d(\nu_0)}.$$

To complete the proof, we need to show that $\langle\lambda,\rho^\vee\rangle\in\mathbf{Z}$, so that $\varepsilon_\lambda=\pm1$. However, this is a consequence of Remark 1.3(d). Alternatively, one can verify this directly as follows. Recall that $\mu-w\mu\in\mathbf{Z}\Phi$ for arbitrary $\mu\in\Lambda$ and $w\in W$. Therefore since $\lambda+\rho$ is of the form $2\mu+w^{-1}\rho$, we have $\lambda=2\mu-(\rho-w^{-1}\rho)\in2\Lambda+\mathbf{Z}\Phi$. However, $\langle\cdot\,,\rho^\vee\rangle$ is $\mathbf{Z}$-valued on both 2Λ and $\mathbf{Z}\Phi$. $\square$

Remark 2.3. (a) There are other specializations of $\varphi_\lambda(h_0;q)$ that have been studied in the literature. For example, by a theorem of Kostant [**Ko**], it is known that if ω is a root of unity of order equal to the Coxeter number of W, then $\varphi_\lambda(h_0;\omega)=0$ or ±1. See also Exercise 10.15 of [**Ka**].

(b) Suppose that Φ is simply laced (i.e., of type A-D-E). In this case, $\alpha=\alpha^\vee$ for all $\alpha\in\Phi$, so $\Phi_{(2)}=\{\alpha\in\Phi:\langle\alpha,\rho^\vee\rangle\in2\mathbf{Z}\}$, and $\mathfrak{g}_{(2)}$ can be realized explicitly as the subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_\alpha$ for $\alpha\in\Phi_{(2)}$. Furthermore, the centralizer C of the element $t_0=\mathrm{Exp}(i\pi h_0)$ in $\tilde G$ is a Lie subgroup of $\tilde G$ whose Lie algebra is $\{e\in\mathfrak{g}:\mathrm{Ad}(t_0)e=e\}$ (e.g., see Section III.9.3 of [**B**]). However,

$$t_0e_\alpha=(-1)^{\langle\alpha,\rho^\vee\rangle}e_\alpha$$

for $e_\alpha\in\mathfrak{g}_\alpha$, so the Lie algebra of C is $\mathfrak{h}+\mathfrak{g}_{(2)}$, a reductive Lie algebra whose semisimple part is $\mathfrak{g}_{(2)}$. If Φ is not simply laced (e.g., $\Phi=C_n$ for $n\geq3$) then the subalgebra of $\mathfrak{g}$ generated by the root spaces indexed by $\alpha\in\Phi_{(2)}$ may be strictly larger than $\mathfrak{g}_{(2)}$.

(c) In some cases, the Weyl group W contains diagram automorphisms of $\Phi_{(2)}$. It follows that the choices for w and ν satisfying Theorem 2.2 need not be unique.

(d) Combining Theorem 1.2 and the above results, we conclude that Lusztig's involution η has at least one fixed point on B_λ if and only if $\lambda+\rho\in W\rho \bmod 2\Lambda$. In that case, we can find $w\in W$ and a dominant weight ν for $\Phi_{(2)}$ as described in Theorem 2.2 so that

$$\#\{b\in B_\lambda:\eta(b)=b\}=\varepsilon_\lambda\,\mathrm{tr}_V(\tilde x_0)=d(\nu)/d(\nu_0),$$

where $\varepsilon_\lambda = (-1)^{\langle \lambda, \rho^\vee \rangle}$.

3. Applications to Self-Evacuating Tableaux

We now specialize to the case $\mathfrak{g} = sl(n)$. Since we are interested primarily in the combinatorial applications, it will often be more convenient to work with $gl(n)$.

First consider the special involution x_0, and its extension $\tilde{x}_0$ to the covering group $SL(n)$ (or $GL(n)$). Let $v_1, \ldots, v_n$ denote a standard basis for the defining representation of $GL(n)$. The normalizer of the Cartan subgroup of diagonal matrices is the group of monomial matrices, and the most general possible extension of w_0 in $GL(n)$ is of the form $v_j \mapsto a_j v_{n+1-j}$ for arbitrary nonzero $a_j \in \mathbf{C}$. A simple calculation shows that for the adjoint action of this element on $sl(n)$ to agree with δ, it is necessary and sufficient that $a_1 = \cdots = a_n = a$ for some a. Furthermore, since

$$h_0 = \mathrm{diag}((n-1)/2, (n-3)/2, \ldots, -(n-1)/2),$$

the requirement that $\tilde{x}_0$ be conjugate to $\mathrm{Exp}(i\pi h_0)$ is satisfied if we take $a = i^{n-1}$. However, it is more convenient to take $a = 1$, so that $\tilde{x}_0 \in GL(n)$ (not necessarily $SL(n)$) is the transformation $v_j \mapsto v_{n+1-j}$; i.e., the usual permutation matrix for w_0. Note that a side effect of this choice is that the sign ε_λ appearing in Remark 1.3(d) will no longer be the same as the one in Theorem 2.2.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition[3] indexing an irreducible polynomial representation V of $GL(n)$. We can at the same time identify λ with the highest weight of V as an $sl(n)$-module. Define

$$s_\lambda(a_1, \ldots, a_n) = \det[a_i^{\lambda_j+n-j}] / \det[a_i^{n-j}]$$

to be the Schur function corresponding to λ, so that

$$\mathrm{tr}_V(\mathrm{diag}(a_1, \ldots, a_n)) = s_\lambda(a_1, \ldots, a_n). \tag{3.1}$$

We also define (cf. (2.2))

$$d(\lambda_1, \ldots, \lambda_n) = d(\lambda) := s_\lambda(1, \ldots, 1) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{3.2}$$

Let $D_\lambda := \{(i,j) \in \mathbf{Z}^2 : 1 \le i \le n,\ 1 \le j \le \lambda_i\}$ be the Young diagram attached to λ. It is well known that $\dim(V) = d(\lambda)$ is the number of semistandard tableaux of shape λ

[3] By this we mean an integer sequence of some fixed length, say n, satisfying $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$. This differs from the usual definition of a "partition" in which trailing zeros are either disallowed or ignored.

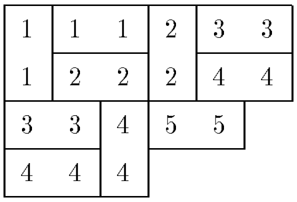

FIGURE 1.

with entries $\leq n$; i.e., the number of maps $T : D_\lambda \to [n]$ such that $T(i,j) \leq T(i,j+1)$ and $T(i,j) < T(i+1,j)$. More generally, one knows (e.g., [**M**, (5.12)]) that

$$s_\lambda(a_1, \ldots, a_n) = \sum_T a_1^{\mu_1(T)} \cdots a_n^{\mu_n(T)}, \tag{3.3}$$

summed over semistandard tableaux of shape λ, where $\mu_j(T) = \#T^{-1}(j)$. In particular, given any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbf{N}^n$, the number of semistandard tableaux of content μ (i.e., $\mu_j = \mu_j(T)$) is the dimension of the μ-weight space of V.

Let $\mathcal{S}_\lambda$ denote the set of semistandard tableaux of shape λ with entries $\leq n$. There is a natural involution $T \mapsto T^*$ on $\mathcal{S}_\lambda$, first defined by Schützenberger, known as *evacuation.* Although this involution has many equivalent definitions, one of the simplest involves the Robinson-Schensted correspondence.[4] Indeed, if $j = (j_1, \ldots, j_m)$ is any integer sequence whose insertion tableau under the Robinson-Schensted correspondence is the semistandard tableau T, then the insertion tableau for

$$j^* = (n+1-j_m, \ldots, n+1-j_1) \tag{3.4}$$

is T^*. In particular, the result is independent of j and has the same shape as T. If $T = T^*$, we will say that T is *self-evacuating.*

We define a *domino tiling* of shape λ to be a perfect matching of the cells in D_λ between nearest neighbors. More precisely, a domino tiling is an involution $M : D_\lambda \to D_\lambda$ such that $M(i,j) \in \{(i, j \pm 1), (i \pm 1, j)\}$ for all $(i,j) \in D_\lambda$. By a *semistandard domino tableau* with entries $\leq k$ and shape λ we mean a pair (M, T), consisting of a domino tiling M of D_λ and a map $T : D_\lambda \to [k]$ such that $T(i,j) \leq T(i,j+1)$, $T(i,j) \leq T(i+1,j)$, $T(M(i,j)) = T(i,j)$, and $T(i,j) = T(i+1,j)$ only if $\{(i,j), (i+1,j)\}$ forms a "domino" of M (i.e., $M(i,j) = (i+1,j)$). An example of a semistandard domino tableau is given in Figure 1. We remark that semistandard domino tableaux have also appeared recently in the work of Kirillov, Lascoux, Leclerc and Thibon [**KLLT**].

[4] For further details on both evacuation and the Robinson-Schensted correspondence, two useful references are [**K**, §5.1.4] and [**vL**]. See also [**G**] for a treatment of semistandard evacuation in particular.

From the preceding discussion it is clear that there must exist a weight-preserving bijection between $\mathcal{S}_\lambda$ and the canonical basis B_λ, although it is very unclear from the definitions how to construct such a bijection. In spite of this, the recent work of Berenstein and Zelevinsky has clarified the situation [**BZ**]. They construct a suitable bijection, and furthermore prove that under their bijection, Lusztig's involution η corresponds to evacuation. (See Theorem 8.2 of [**BZ**].)

As a corollary, it follows that the number of self-evacuating tableaux in $\mathcal{S}_\lambda$ can be counted. We also claim (when n is even) that it is fairly easy to show that the trace of the special involution $\tilde{x}_0$ on V counts semistandard domino tableaux of shape λ. To make these claims more explicit, we need some additional notation. We set

$$E_\lambda := \{\lambda_j + n - j : 1 \leq j \leq n\} \cap 2\mathbf{Z}, \quad O_\lambda := \{\lambda_j + n - j - 1 : 1 \leq j \leq n\} \cap 2\mathbf{Z},$$

and define λ^E (resp., λ^O) to be the partition $(\mu_1/2-(r-1), \mu_2/2-(r-2), \ldots, \mu_r/2)$, where $\mu_1, \ldots, \mu_r$ are the members of E_λ (resp., O_λ), listed in decreasing order. For example, if $\lambda = (6,4,2,2,1,1,0)$ (and hence $n = 7$), then $E_\lambda = \{12,6,2,0\}$, $O_\lambda = \{8,4,2\}$, $\lambda^E = (3,1,0,0)$, and $\lambda^O = (2,1,1)$.

THEOREM 3.1. *The following quantities are equal.*

(a) *The number of self-evacuating tableaux in* $\mathcal{S}_\lambda$.

(b) $(-1)^{\kappa(\lambda)} s_\lambda((-1)^0, (-1)^1, \ldots, (-1)^{(n-1)})$, *where* $\kappa(\lambda) = \lambda_2 + \lambda_4 + \cdots$.

(c) $\begin{cases} d(\lambda^E) d(\lambda^O) & \textit{if } |\#E_\lambda - \#O_\lambda| \leq 1, \\ 0 & \textit{otherwise.} \end{cases}$

Furthermore, if n is even, these quantities are also equal to

(d) *The number of semistandard domino tableaux with entries $\leq n/2$ and shape λ.*

Proof. By Theorem 1.2 and the result of Berenstein and Zelevinsky, we know that the number of self-evacuating tableaux in $\mathcal{S}_\lambda$ is, up to sign, the trace of $\tilde{x}_0$ on V. On the other hand, as a member of $GL(n)$, our choice of $\tilde{x}_0$ has eigenvalues $(-1)^0, (-1)^1, \ldots, (-1)^{n-1}$. In view of (3.1), this proves that quantities (a) and (b) are equal, modulo verification that (b) is nonnegative.

Perhaps the simplest way to prove that (b) is nonnegative is to recall the well-known identity (e.g., [**M**, I.3]; also a special case of (2.1))

$$s_\lambda(1, q, q^2, \ldots, q^{n-1}) = q^{n(\lambda)} \prod_{1 \leq i < j \leq n} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{j-i}}, \tag{3.5}$$

where $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$. It follows that if this expression is nonzero in the limit $q \to -1$, the sign must be the sign of the leading term; i.e., $(-1)^{n(\lambda)} = (-1)^{\kappa(\lambda)}$.

To prove that (b) and (c) are equal, we apply Theorem 2.2. Recall that in the $\mathfrak{g} = sl(n)$ case, we have $\mathfrak{g}_{(2)} \cong sl(\lfloor n/2 \rfloor) \oplus sl(\lceil n/2 \rceil)$. The embedding of $\mathfrak{g}_{(2)}$ in $\mathfrak{g}$ (see Remark 2.3(b)) is such that a weight $\mu = (\mu_1, \ldots, \mu_n) \in \Lambda$ belongs to the dominant chamber of $\Phi_{(2)}$ if and only if $\mu_1 > \mu_3 > \cdots$ and $\mu_2 > \mu_4 > \cdots$. It is also easy to check that $\nu_0 = 0$; i.e., $\rho = (n-1, \ldots, 1, 0)$ projects to $2\rho_{(2)}$ (cf. the Appendix).

Next observe that modulo 2Λ, ρ is equivalent to $(0,1,0,1,\ldots) = (1,0,1,0,\ldots)$. It follows that $\lambda + \rho$ is in the S_n-orbit of ρ mod 2Λ if and only if the number of even and odd terms in the set $\{\lambda_j + n - j : 1 \le j \le n\}$ differ by at most one; i.e., $|\#E_\lambda - \#O_\lambda| \le 1$. By Theorem 2.1, this characterizes when (b) is nonzero. (One can also deduce this directly from (3.5).) Assuming this condition holds, then as in Theorem 2.2, we can find $w \in S_n$ such that $w(\lambda+\rho)$ belongs to the dominant chamber of $\Phi_{(2)}$ and agrees with ρ mod 2Λ. It is not hard to see that the resulting dominant weight ν for $\Phi_{(2)}$ satisfying Theorem 2.2 has components λ^E and λ^O when projected onto the two irreducible factors of $\Phi_{(2)}$. Thus the equality of (b) and (c) follows.

Finally, to prove that (b) and (d) are equal (when n is even), let $\mathcal{D}_\lambda$ denote the set of semistandard domino tableaux (M,T) with entries $\le n/2$ and shape λ. Note that if we delete the cells of $T^{-1}(n/2)$, we obtain a domino tableau in $\mathcal{D}_\mu$ for some partition $\mu = (\mu_1, \ldots, \mu_{n-2})$. Conversely, a given $(M', T') \in \mathcal{D}_\mu$ with $D_\mu \subseteq D_\lambda$ is the restriction of some $(M,T) \in \mathcal{D}_\lambda$ if and only if $D = D_\lambda - D_\mu$ has at most two cells in any column (i.e., of the form $(\cdot, j)$) and has a domino tiling with vertical dominos in every column with two cells. If this condition holds, then semistandardness forces each pair of cells $(i,j), (i+1,j) \in D$ to be matched, so there is a unique way to extend (M', T') to D_λ. Thus we obtain the recurrence

$$\#\mathcal{D}_\lambda = \sum_\mu \#\mathcal{D}_\mu, \tag{3.6}$$

summed over μ such that $D_\lambda - D_\mu$ has at most two cells in any column and has a domino tiling with vertical dominos in each two-cell column.

On the other hand, let us analyze (3.3) under the specialization $a_i = (-1)^{i-1}$. Note that if we delete the cells of $T^{-1}(\{n-1, n\})$ from some $T \in \mathcal{S}_\lambda$, we obtain $T' \in \mathcal{S}_\mu$ for some partition $\mu = (\mu_1, \ldots, \mu_{n-2})$. Conversely, a given $T' \in \mathcal{S}_\mu$ with $D_\mu \subseteq D_\lambda$ is the restriction of some $T \in \mathcal{S}_\lambda$ if and only if $D = D_\lambda - D_\mu$ contains at most two cells in any column. Given two such cells $(i,j), (i+1,j) \in D$, semistandardness forces $T(i,j) = n-1$ and $T(i+1,j) = n$; the net contribution of such pairs to (3.3) (given that $a_i = (-1)^{i-1}$) is $(-1)^d$, where d denotes the number of columns of D with two cells.

Now suppose that there are k remaining cells of D in some row, say the ith, that are not in two-cell columns. These cells must form a contiguous block; say $(i,j), \ldots, (i, j+k-1)$.

The values of T on these cells can be assigned independently of the other cells, subject only to the condition that $n-1 \leq T(i,j) \leq \cdots \leq T(i,j+k-1) \leq n$. From this it is not hard to see that the net contribution of these cells to (3.3) is 0 for odd k, and 1 for even k. Hence, for a fixed $T' \in \mathcal{S}_\mu$, the part of (3.3) contributed by extensions of T' to D_λ is nonzero if and only if D has at most two cells in each column and has an even number of cells in each row that is not in a two-cell column. The latter is equivalent to the existence of a domino tiling of D with vertical dominos in the two-cell columns, so we obtain

$$s_\lambda((-1)^0,(-1)^1,\ldots,(-1)^{n-1}) = \sum_\mu (-1)^d s_\mu((-1)^0,(-1)^1,\ldots,(-1)^{n-3}),$$

with the range of summation being identical to (3.6).

To complete the proof, we note that deletion of a horizontal domino from a domino tiling of shape λ preserves the parity of the row lengths, whereas deleting a vertical domino changes the parity of two consecutive row lengths, and hence changes the parity of $\kappa(\lambda)$. Therefore, the quantities (b) and (d) satisfy the same recurrence. $\square$

Remark 3.2. (a) It is possible to formulate a variation of domino tableaux, allowing "monominos" in the first row, so that part (d) of Theorem 3.1 remains valid for odd n.

(b) A bijective proof of the equality of (c) and (d) has been given by Carré and Leclerc [**CL**, §6]. Also, in the case that every part of λ is even, there is a bijective proof of the equality of (a) and (d) described in [**KLLT**].

(c) Assume $n = 2r$ is even. Define the content of a semistandard domino tableau $(M,T) \in \mathcal{D}_\lambda$ to be $\mu = (\mu_1,\ldots,\mu_r)$, where $2\mu_j = \#T^{-1}(j)$. For fixed μ, let $\mathcal{D}_\lambda(\mu)$ denote the subset of such tableaux in $\mathcal{D}_\lambda$ of content μ. We note that the above proof of the equality of (b) and (d) actually proves something slightly stronger; namely,

$$(-1)^{\kappa(\lambda)} s_\lambda(a_1,-a_1,\ldots,a_r,-a_r) = \sum_\mu a_1^{2\mu_1}\cdots a_r^{2\mu_r}\cdot \#\mathcal{D}_\lambda(\mu). \tag{3.7}$$

(d) Fix nonnegative integers a and b ($b \leq n$), and let $\lambda = (a,\ldots,a,0\ldots,0)$ (b copies of a). There is a well known bijection in this case between $\mathcal{S}_\lambda$ and the set of plane partitions that fit in a box of dimensions $a \times b \times (n-b)$, given by the map $T \mapsto P$, where $P(i,j) = T(i,j) - i$. Furthermore, evacuation of tableaux corresponds under this map to complementation of plane partitions (i.e., the involution $P \mapsto P^c$, where $P^c(i,j) := n-b-P(b+1-i,a+1-j)$). Thus a special case of Theorem 3.1 provides an enumeration of self-complementary plane partitions, a result first obtained by Stanley [**St**]. It should be noted that we enumerated self-complementary plane partitions in [**Ste**] by essentially the same argument used above, the main difference being that we used bases for $sl(n)$-modules provided by standard monomial theory, not Lusztig's canonical bases. (See also [**Ku**].)

(e) Any basis for an irreducible $sl(n)$-module that is permuted either by $\tilde{x}_0$ or $-\tilde{x}_0$ yields another set of objects equal in number to those in Theorem 3.1 (namely, the fixed points of the involution). However, among the usual constructions of bases for irreducible $sl(n)$-modules, the only one we are aware of with the property that $\pm\tilde{x}_0$ permutes the basis in *every* irreducible $sl(n)$-module is Lusztig's canonical basis (or equivalent bases).

4. Weight Space Refinements

Temporarily returning to the general case, recall that special involutions $\tilde{x}_0$ permute the weight spaces of an irreducible $\mathfrak{g}$-module V according to w_0. It follows that if $b \in B_\lambda$ is a fixed point of the involution η, then $b \in V_\mu$ for some weight μ such that $w_0\mu = \mu$. On the other hand, since $\tilde{x}_0$ stabilizes V_μ in such cases, its trace on V_μ is meaningful. Thus as a corollary of the proof of Theorem 1.2, we have

$$|\operatorname{tr}_{V_\mu}(\tilde{x}_0)| = \#\{b \in B_\lambda \cap V_\mu : b = \eta(b)\}. \tag{4.1}$$

We can also compute the trace of $\tilde{x}_0$ on V_μ indirectly as follows. Consider the most general possible extension of w_0 in $\tilde{G}$; namely, $\tilde{x}_0 \cdot \operatorname{Exp}(h)$ for some $h \in \mathfrak{h}$. Since only the weight spaces V_μ such that $w_0\mu = \mu$ can contribute to the trace (and since $\operatorname{Exp}(h)$ acts diagonally on V_μ), it follows that

$$\operatorname{tr}_V(\tilde{x}_0 \cdot \operatorname{Exp}(h)) = \sum_{\mu:\, w_0\mu=\mu} e^{\mu(h)} \operatorname{tr}_{V_\mu}(\tilde{x}_0). \tag{4.2}$$

As h varies over $\mathfrak{h}$, the above expansion implicitly determines the traces of $\tilde{x}_0$ on the weight spaces it stabilizes.

Specializing now to the case $\mathfrak{g} = sl(n)$, let $K^*_\lambda(\mu)$ denote the number of self-evacuating tableaux in $\mathcal{S}_\lambda$ of content μ.

THEOREM 4.1. *(a) If $n = 2r$ is even, then*

$$(-1)^{\kappa(\lambda)} s_\lambda(a_1, -a_1, \ldots, a_r, -a_r) = \sum_\mu K^*_\lambda(\mu_1, \ldots, \mu_r, \mu_r, \ldots, \mu_1) a_1^{2\mu_1} \cdots a_r^{2\mu_r}.$$

(b) If $n = 2r+1$ is odd, then

$$(-1)^{\kappa(\lambda)} s_\lambda(a_1, -a_1, \ldots, a_r, -a_r, a_{r+1})$$
$$= \sum_\mu K^*_\lambda(\mu_1, \ldots, \mu_r, \mu_{r+1}, \mu_r, \ldots, \mu_1) a_1^{2\mu_1} \cdots a_r^{2\mu_r} a_{r+1}^{\mu_{r+1}}.$$

Proof. For (a), let $d = \mathrm{diag}(a_1, \ldots, a_r, a_r, \ldots, a_1) \in GL(n)$. Taking $\tilde{x}_0 \in GL(n)$ to be the permutation matrix corresponding to w_0 (as in Section 3), it is clear that the eigenvalues of $\tilde{x}_0 d$ are $\pm a_1, \ldots, \pm a_r$. Thus by (3.1) we have

$$\mathrm{tr}_V(\tilde{x}_0 d) = s_\lambda(a_1, -a_1, \ldots, a_r, -a_r).$$

On the other hand, by (4.2), the terms in the expansion of this expression are indexed by w_0-invariant weights; i.e., weights of the form $\bar{\mu} = (\mu_1, \ldots, \mu_r, \mu_r, \ldots, \mu_1)$. The coefficient of $a_1^{2\mu_1} \cdots a_r^{2\mu_r}$ must therefore be the trace of $\tilde{x}_0$ on $V_{\bar{\mu}}$. Up to a factor of ± 1, this is by (4.1) the number of fixed points of η on $B_\lambda \cap V_{\bar{\mu}}$, or equivalently (by the result of Berenstein and Zelevinsky) the number of self-evacuating tableaux in $\mathcal{S}_\lambda$ of content $\bar{\mu}$. To determine the correct sign, recall that $\tilde{x}_0$ acts as a scalar multiple of η, so the sign depends only on λ (not μ) and hence can be determined by setting $a_1 = \cdots = a_r = 1$ and applying Theorem 3.1. For (b), we use $d = \mathrm{diag}(a_1, \ldots, a_r, a_{r+1}, a_r, \ldots, a_1)$; the proof is otherwise identical. $\square$

An immediate consequence of (3.7) and the above result is the following refinement of the (a)=(d) portion of Theorem 3.1.

COROLLARY 4.2. *If $n = 2r$ is even, then $\#\mathcal{D}_\lambda(\mu) = K^*_\lambda(\mu_1, \ldots, \mu_r, \mu_r, \ldots, \mu_1)$.*

A semistandard tableau or domino tableau is said to be *standard* if it has content $(1, \ldots, 1)$. The implications of the preceding results are particularly interesting in the case of standard tableaux, since we can explicitly compute $K^*_\lambda(1, \ldots, 1)$, the number of self-evacuating standard tableaux of shape λ. Henceforth, we will assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is of size n (i.e., $\sum_i \lambda_i = n$) since standard tableaux of shape λ do not otherwise exist.

It is well known that the total number of standard tableaux of shape λ is given by Frobenius' formula (e.g., [**JK**, (2.3.22)])

$$f(\lambda) = f(\lambda_1, \ldots, \lambda_n) := n! \prod_{i=1}^{n} \frac{1}{(\lambda_i + n - i)!} \prod_{j=i+1}^{n} (\lambda_i - \lambda_j + j - i).$$

We sketch a derivation of this formula that will be needed for what follows.

For nonnegative integers k, define $\lambda 0^k$ to be the partition $(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$ (k zeros), a dominant weight for $gl(n+k)$. The number of semistandard tableaux of shape $\lambda 0^k$ with entries $\leq n + k$ can be expressed as

$$d(\lambda 0^k) = \sum_{m=1}^{n} c_m \binom{n+k}{m},$$

where c_m is the number of *surjective* semistandard tableaux $T : D_\lambda \to [m]$. In particular, c_n is the number of standard tableaux of shape λ. As a function of k, it is clear that

$$d(\lambda 0^k) = \frac{1}{n!} c_n k^n + O(k^{n-1})$$

in the limit $k \to \infty$. However, it is a routine computation using (3.2) to show that

$$d(\lambda 0^k) = \frac{1}{n!} f(\lambda) k^n + O(k^{n-1}), \tag{4.3}$$

thus proving Frobenius' formula.

Let χ^λ denote the irreducible character of S_n corresponding to the partition λ of size n. The following result is a "standard" counterpart to Theorem 3.1.

THEOREM 4.3. *Let λ be a partition of size n. Set $r = \lfloor n/2 \rfloor$ and suppose that λ^E is of size s. The following quantities are equal.*

(a) *The number of self-evacuating standard tableaux of shape λ.*

(b) $(-1)^{\kappa(\lambda)} \chi^\lambda(w_0)$.

(c) $\begin{cases} \binom{r}{s} f(\lambda^E) f(\lambda^O) & \text{if } |\#E_\lambda - \#O_\lambda| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$

Furthermore, if n is even, these quantities are also equal to

(d) *The number of standard domino tableaux of shape λ.*

Proof. Let T be a semistandard tableau of shape λ, with $i_1 < \cdots < i_m$ the set of distinct entries occurring in T. If T is self-evacuating, then we must have $i_j = n + 1 - i_{m+1-j}$ for $1 \leq j \leq m$. Also, i_j and i_{m+1-j} must occur equally often in T, which forces $n = m \bmod 2$. It is easy to see that we can substitute any other set of entries $i'_1 < \cdots < i'_m$ for those of T without affecting the self-evacuating property, provided that $i'_j = n + 1 - i'_{m+1-j}$ for $1 \leq j \leq m$. Setting $s = \lfloor m/2 \rfloor$, the number of such choices for $i'_1, \ldots, i'_m$ is $\binom{r}{s}$, assuming that $n = m \bmod 2$.

It follows that for integers $k \geq 0$, the number of self-evacuating semistandard tableaux of shape $\lambda 0^{2k}$ with entries $\leq n + 2k$ can be expressed as

$$F_k(\lambda) = \sum_{m=0}^{n} b_m \binom{k+r}{\lfloor m/2 \rfloor},$$

where b_m denotes the number of surjective semistandard tableaux $T : D_\lambda \to [m]$ that are self-evacuating. Since $b_{n-1} = 0$, it follows that

$$F_k(\lambda) = \frac{1}{r!} b_n k^r + O(k^{r-1}) \tag{4.4}$$

in the limit $k \to \infty$. However, from the definitions it is clear that $(\lambda 0^{2k})^E = \lambda^E 0^k$ and $(\lambda 0^{2k})^O = \lambda^O 0^k$. Hence by Theorem 3.1, we have that $F_k(\lambda)$ is nonzero if and only if $|\#E_\lambda - \#O_\lambda| \leq 1$, and in that case

$$F_k(\lambda) = d(\lambda^E 0^k) d(\lambda^O 0^k).$$

Using (4.3), we can obtain a second asymptotic expansion of $F_k(\lambda)$; comparing it with (4.4) (assuming $|\#E_\lambda - \#O_\lambda| \leq 1$) yields

$$b_n = \binom{r}{s} f(\lambda^E) f(\lambda^O),$$

where s (resp., $r-s$) is the size of λ^E (resp., λ^O). Since b_n is the number of self-evacuating standard tableaux of shape λ, this proves the equality of (a) and (c).

To prove the equality of (a) and (b), we use the fact (e.g., Remark 4.1 of [**Ko**]) that if the action of $GL(n)$ on V (which carries an irreducible representation of highest weight λ) is restricted to the subgroup S_n of permutation matrices, then the weight space V_μ for the weight $\mu = (1, \ldots, 1)$ (which S_n, being the Weyl group, stabilizes) carries the irreducible representation of S_n indexed by λ. Hence the trace of $\tilde{x}_0$ on V_μ is the value of the S_n-character χ^λ at w_0. However by (4.1), this trace is, up to a factor of ± 1, the number of fixed points of η in $B_\lambda \cap V_\mu$; i.e., the number of self-evacuating standard tableaux of shape λ. The fact that $(-1)^{\kappa(\lambda)}$ is the correct sign follows by the same reasoning used in the proof of Theorem 4.1.

The equality of (a) and (d) is a special case of Corollary 4.2. $\square$

Remark 4.4. (a) Various parts of the above result can be pieced together from elsewhere in the literature. For example, in the case of even n, it is easy to see that the equality of (b) and (d) is a special case of the Murnaghan-Nakayama recurrence for the irreducible characters of S_n (e.g., [**JK**, §2.4]). A bijective proof of the equality of (c) and (d) (actually, a more general result involving rim-hook tableaux) has been given by Stanton and White [**SW**] (see also [**JK**, §2.7]). Also, a bijection between the quantities in (a) and (d) is described in [**vL**, §4].

(b) We find it striking that the equality of (b) and (c) shows that the trace of w_0 on any irreducible representation of S_n is, up to sign, the dimension of an irreducible representation of the centralizer of w_0 (a Weyl group of type B). This is analogous to the fact (see Remark 2.3(b)) that the trace of $\tilde{x}_0$ on any irreducible representation of $GL(n)$ is, up to sign, the dimension of an irreducible representation of the centralizer of $\tilde{x}_0$. It would be interesting to have "natural" explanations of either of these phenomena.

(c) Similarly, it is known that if w is a Coxeter element for S_n (i.e., an n-cycle), then for any integer k, the trace of w^k on any irreducible S_n-module is, up to sign, the dimension of an irreducible module for the centralizer of w^k (e.g., see Corollary 2.7.33 of [**JK**]). This partially generalizes the previous remark, since w_0 is a power of an n-cycle when n is even. Also (cf. Remark 2.3(a)), regarding w as a permutation matrix in $GL(n)$, one can show that the trace of w^k on any irreducible representation of $GL(n)$ is, up to sign, the dimension of an irreducible representation of the centralizer of w^k in $GL(n)$.

5. Kazhdan-Lusztig Bases and Evacuation

The previous section demonstrates that the study of special involutions has interesting implications for the representation theory of the symmetric group. To recap, choose an irreducible $gl(n)$-module V whose highest weight corresponds to a partition λ of size n. Passing to the action of $GL(n)$ on V, recall that one obtains a copy of the irreducible S_n-module U indexed by λ by restricting to the subgroup of permutation matrices, acting on the weight space V_μ indexed by $\mu = (1, \ldots, 1)$. In particular, the action of $\tilde{x}_0$ on V_μ amounts to the action of w_0 on U as an S_n-module.

Now recall that $\tilde{x}_0$ (or its negative) permutes the canonical basis for V according to the involution η. By restriction, we similarly obtain a canonical basis for U in which w_0 (or its negative) acts as a permutation. By the result of Berenstein and Zelevinsky, we know that this permutation is evacuation of (in this special case) standard tableaux. The goal of this section is to provide a shorter, more direct proof of this fact using the Kazhdan-Lusztig basis for irreducible S_n-modules. More precisely, we prove the following.

THEOREM 5.1. *For each irreducible representation of S_n, the representing matrix for w_0 with respect to the Kazhdan-Lusztig basis is a permutation matrix, or the negative of one. Furthermore, if the basis elements are suitably labeled by standard tableaux, the permutation is evacuation.*

Of course, a corollary of this result (modulo the criterion for choosing the sign) is the equality of (a) and (b) in Theorem 4.3.

Before giving the proof, we first need to review the construction of the Kazhdan-Lusztig basis for S_n-modules, as found in [**KL**], [**GM**], or [**Sh**]. Let $s_1, \ldots, s_{n-1}$ denote the usual Coxeter generators for S_n in which s_i interchanges i and $i+1$. For $w \in S_n$, let $D(w)$ denote the (left) descent set of w; i.e., $D(w) = \{i : s_i w < w\}$, where '$<$' denotes the Bruhat ordering of S_n. Note that the map $w \mapsto w_0 w w_0$ is an automorphism of the Bruhat order. Moreover, since $w_0 s_i w_0 = s_{\delta(i)}$ (where $\delta(i) = n+1-i$ denotes the Weyl involution), it follows that

$$i \in D(w) \Leftrightarrow \delta(i) \in D(w_0 w w_0). \tag{5.1}$$

Let $A = \mathbf{C}[q^{\pm 1/2}]$ denote the ring of Laurent polynomials in the variable $q^{1/2}$, and let $\mathcal{H}_n$ denote the Iwahori-Hecke algebra of type A_{n-1}; i.e., the associative A-algebra with unit and with generators $T_1, \ldots, T_{n-1}$ and defining relations

$$T_i^2 = (q-1)T_i + q; \qquad T_iT_j = T_jT_i \quad |i-j| > 1; \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}.$$

For $w \in S_n$, the elements $T_w := T_{i_1} \ldots T_{i_l}$, obtained by choosing any minimum-length expression $w = s_{i_1} \ldots s_{i_l}$, provide a well-defined basis for $\mathcal{H}_n$. One obtains the group algebra $\mathbf{C}S_n$ via the map $q \mapsto 1$, $T_i \mapsto s_i$.

The definition of the Kazhdan-Lusztig representations of $\mathcal{H}_n$ (and hence also S_n) relies on knowledge of the "top coefficients" $\mu(z, w)$ of the Kazhdan-Lusztig polynomials $P_{z,w}$, defined for $z, w \in S_n$ such that $z < w$. It is more convenient for what follows to work with a symmetrized version of this data, denoted $\mu\{z, w\}$, which is defined to be $\mu(z, w)$ if $z < w$, $\mu(w, z)$ if $w < z$, and 0 otherwise.

A second item necessary for the construction of the representations is knowledge of the structure of the left cells of the Weyl group. Rather than repeat the definition here, it will suffice for our purposes merely to note that $w, z \in S_n$ belong to the same left cell if and only if they have the same "recording tableau" (also known as the left-, or Q-tableau) under the Robinson-Schensted correspondence. (For a proof, e.g. see Fact 8 of [**GM**]). Also of crucial importance is the fact that

$$\mu\{z, w\} \text{ depends only on } P(z) \text{ and } P(w), \tag{5.2}$$

where $P(w)$ denotes the "insertion tableau" for w (also known as the right-, or P-tableau). This property can be inferred from the results in [**KL**], although it is explicitly demonstrated in [**GM**] (see Fact 9).

Once the above data is available, one can construct a representation of $\mathcal{H}_n$ as follows. Choose a left cell $L \subset S_n$, and form the free A-module U on the symbols $C_w : w \in L$. Kazhdan and Lusztig prove that

$$T_i(C_w) = \begin{cases} qC_w + q^{1/2} \sum_{z \in L:\, i \in D(z)} \mu\{z, w\} C_z & \text{if } i \notin D(w) \\ -C_w & \text{if } i \in D(w) \end{cases} \tag{5.3}$$

defines an irreducible representation of $\mathcal{H}_n$, and moreover, all irreducible representations of $\mathcal{H}_n$ arise in this fashion.

Since the elements of a left cell are indexed by the set $\mathcal{F}^\lambda$ of standard tableaux of some shape λ, we can rewrite the above construction as follows. First, note that by (5.2), the quantity $\mu\{P, Q\}$ is well-defined for $P, Q \in \mathcal{F}^\lambda$. Second, a well-known property of the

Robinson-Schensted correspondence is the fact that $D(w)$ depends only on $P(w)$; indeed, $k \in D(w)$ if and only if k appears in a column strictly to the left of $k+1$ in $P(w)$. Thus $D(P)$ is similarly well-defined for $P \in \mathcal{F}^\lambda$. We can now redefine U to be the free A-module generated by the tableaux $P \in \mathcal{F}^\lambda$, and replace (5.3) with an equivalent action in which

$$T_i(P) = \begin{cases} qP + q^{1/2} \sum_{Q \in \mathcal{F}^\lambda : i \in D(Q)} \mu\{P, Q\} Q & \text{if } i \notin D(P) \\ -P & \text{if } i \in D(P), \end{cases}$$

for $P \in \mathcal{F}^\lambda$. By setting $q = 1$, $T_i = s_i$, we obtain an irreducible S_n-module.

Proof of Theorem 5.1. Let E denote the A-linear transformation on U induced by evacuation; i.e., $E(P) = P^*$ for $P \in \mathcal{F}^\lambda$. We have

$$ET_i(P) = \begin{cases} qP^* + q^{1/2} \sum_{Q \in \mathcal{F}^\lambda : i \in D(Q)} \mu\{P, Q\} Q^* & \text{if } i \notin D(P) \\ -P^* & \text{if } i \in D(P), \end{cases} \tag{5.4}$$

for $P \in \mathcal{F}^\lambda$, whereas

$$T_iE(P) = \begin{cases} qP^* + q^{1/2} \sum_{Q \in \mathcal{F}^\lambda : i \in D(Q)} \mu\{P^*, Q\} Q & \text{if } i \notin D(P^*) \\ -P^* & \text{if } i \in D(P^*). \end{cases} \tag{5.5}$$

Since $w \mapsto w_0ww_0$ is an automorphism of the Bruhat order, it follows that $\mu\{z, w\} = \mu\{w_0zw_0, w_0ww_0\}$ for $w, z \in S_n$. However by the definition of evacuation (cf. (3.4)), one knows that if $P = P(w)$, then $P^* = P(w_0ww_0)$. Thus we have

$$\mu\{P, Q\} = \mu\{P^*, Q^*\}$$

for all $P, Q \in \mathcal{F}^\lambda$. Furthermore, by (5.1), we see that

$$i \in D(P) \Leftrightarrow \delta(i) \in D(P^*).$$

Using these two properties of evacuation to compare (5.4) and (5.5), we see that

$$ET_i = T_{\delta(i)}E \tag{5.6}$$

as operators on U.

Now consider the action of T_{w_0} on U. For each index i, one can find a reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$ such that $i = i_1$. In that case, one obtains

$$T_iT_{w_0} = [(q-1)T_i + q]T_{i_2} \ldots T_{i_l} = (q-1)T_{w_0} + qT_{s_iw_0}.$$

Similarly, for each index j, one can find a reduced expression for w_0 ending with s_j, yielding

$$T_{w_0}T_j = (q-1)T_{w_0} + qT_{w_0 s_j}.$$

However, $s_i w_0 = w_0 s_{\delta(i)}$, so it follows that

$$T_{w_0}T_i = T_{\delta(i)}T_{w_0}.$$

In view of (5.6), we see that ET_{w_0} commutes with the action of $\mathcal{H}_n$ on the irreducible module U. By Schur's lemma, T_{w_0} is therefore a scalar multiple of E on U. Passing to S_n-modules, it follows that $w_0 = \pm E$ on U; the scalar factor here must be ± 1 since both w_0 and E are involutions. $\square$

REMARK 5.2. Independently, Mathas has recently proved a more general result (see Theorem 3.1 of [**Mth**]); namely, that in all left cell representations of the Iwahori-Hecke algebra $\mathcal{H}$ of any Weyl group, the representing matrix for T_{w_0} relative to the Kazhdan-Lusztig basis is of the form $\pm q^{k/2}$ times a permutation matrix. (Note that cell representations are not irreducible in general.) The proof is less elementary than the above argument, since it relies on the nonnegativity of both the Kazhdan-Lusztig polynomials as well as the structure constants of the Kazhdan-Lusztig basis for $\mathcal{H}$. In the A_n case, Mathas attributes the result to J. Graham [**Gr**].

Appendix

We list the following data for each irreducible root system Φ:

- The structure of $\Phi_{(2)}$.
- The weight ν_0 appearing in Theorem 2.2.
- The dimension $d(\nu_0)$ of the associated $\mathfrak{g}_{(2)}$-module.

Φ	$\Phi_{(2)}$	ν_0	$d(\nu_0)$
A_{2n}	$A_n \oplus A_{n-1}$	0	1
A_{2n+1}	$A_n \oplus A_n$	0	1
B_n	A_{n-1}	0	1
C_{2n}	$C_n \oplus D_n$	$\omega_1(D_n)$	2^{n-1}
C_{2n+1}	$C_n \oplus D_{n+1}$	$\omega_1(D_{n+1})$	2^n
D_{2n}	$D_n \oplus D_n$	$\omega_1(D_n)$	2^{n-1}
D_{2n+1}	$D_n \oplus D_{n+1}$	$\omega_1(D_n)$	2^{n-1}
E_6	$A_5 \oplus A_1$	$\omega_1(A_1)$	2
E_7	A_7	$\omega_7(A_7)$	8
E_8	D_8	$\omega_8(D_8)$	16
F_4	$B_3 \oplus A_1$	$\omega_1(A_1)$	2
G_2	$A_1 \oplus A_1$	$\omega_1(A_1)$	2

Here, $\omega_n(D_n)$ and $\omega_n(A_n)$ are the highest weights of the defining representations of $so(2n)$ and $sl(n+1)$, and $\omega_1(D_n)$ is the highest weight of the basic spin representation of $so(2n)$.

References

[BZ] A. Berenstein and A. Zelevinsky, Canonical bases for quantum groups of type A_r and piecewise-linear combinatorics, preprint.

[B] N. Bourbaki, "Lie Groups and Lie Algebras," Chps. 1–3, Springer-Verlag, Berlin-New York, 1989.

[CL] C. Carré and B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, *J. Algebraic Combinatorics* **4** (1995), 201–231.

[G] E. R. Gansner, On the equality of two plane partition correspondences, *Discrete Math.* **30** (1980), 121–132.

[GM] A. M. Garsia and T. J. McLarnan, Relations between Young's natural and the Kazhdan-Lusztig representations of S_n, *Adv. in Math.* **69** (1988), 32–92.

[**Gr**] J. Graham, Ph. D. thesis, University of Sydney, in preparation.

[**GL**] I. Grojnowski and G. Lusztig, On bases of irreducible representations of quantum GL_n, *Contemporary Math.* **139** (1992), 167–174.

[**H**] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer-Verlag, Berlin-New York, 1972.

[**JK**] G.D. James and A. Kerber, "The Representation Theory of the Symmetric Group," Addison-Wesley, Reading, MA, 1981.

[**Ka**] V. G. Kac, "Infinite Dimensional Lie Algebras," Cambridge Univ. Press, Cambridge, 1990.

[**KL**] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[**KLLT**] A. N. Kirillov, A. Lascoux, B. Leclerc and J.-Y. Thibon, Séries génératrices pour les tableaux de dominos, *C. R. Acad. Sci. Paris, Ser. I* **318** (1994), 395–400.

[**K**] D. E. Knuth, "The Art of Computer Programming," Vol. 3 (2nd ed.), Addison-Wesley, Reading, MA, 1973.

[**Ko**] B. Kostant, On Macdonald's η-function formula, the Laplacian, and generalized exponents, *Adv. in Math.* **20** (1976), 179–212.

[**Ku**] G. Kuperberg, Self-complementary plane partitions by Proctor's minuscule method, *Europ. J. Combin.* **15** (1994), 545–553.

[**L**] G. Lusztig, Canonical bases arising from quantized enveloping algebras. II. *Progr. Theoret. Phys. Suppl.* **102** (1991), 175–201.

[**M**] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, Oxford, 1979.

[**Mth**] A. Mathas, On the left cell representations of Iwahori-Hecke algebras of finite Coxeter groups, preprint.

[**Sh**] J. Y. Shi, "The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups," Lect. Notes in Math. 1179, Springer-Verlag, Berlin-New York, 1986.

[**St**] R. P. Stanley, Symmetries of plane partitions, *J. Combin. Theory Ser. A* **43** (1986), 103–113.

[**SW**] D. W. Stanton and D. E. White, A Schensted algorithm for rim hook tableaux, *J. Combin. Theory Ser. A* **40** (1985), 211–247.

[**Ste**] J. R. Stembridge, On minuscule representations, plane partitions and involutions in complex Lie groups, *Duke Math. J.* **73** (1994), 469–490.

[**vL**] M. A. A. van Leeuwen, The Robinson-Schensted and Schützenberger Algorithms Part I: New Combinatorial Proofs, CWI Technical Report AM-R9208, 1992.