On the Classification of Multiplicity-Free Exterior Algebras

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31 March 2003

Abstract. We uniformly describe the irreducible character decompositions of the exterior algebras of two families of representations of semisimple Lie algebras. This simplifies parts of Howe's classification of skew multiplicity-free actions.

0. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{C}$ and V an irreducible finite-dimensional $\mathfrak{g}$-module. As part of his study of multiplicity-free actions (see Section 4 of [**H**]), Howe classified those pairs $(\mathfrak{g}, V)$ for which the exterior algebra $\Lambda(V)$ is multiplicity-free in each degree; i.e., the pairs for which every exterior power $\Lambda^k(V)$ has no irreducible submodule occurring with multiplicity two or more.

The outline of Howe's proof is as follows. First, one shows that a necessary condition is that every weight space of V must be one-dimensional. Indeed, if v is a weight vector of V such that every higher weight space of V (relative to the usual ordering) is one-dimensional, then the wedge product of v with a set of representatives from all higher weight spaces will necessarily be a maximal vector, and hence generate an irreducible submodule of V. It follows that if there is another weight vector, one that is independent of v but having the same weight, then some exterior power of V will fail to be multiplicity-free.

This condition confines the possibilities for $(\mathfrak{g}, V)$ to a short list: the minuscule representations, the symmetric powers of the defining representation of $\mathfrak{sl}(n)$ and its dual, the defining representation of $\mathfrak{so}(2n+1)$, one of the 14-dimensional representations of $\mathfrak{sp}(6)$, and the 7-dimensional representation of $\mathfrak{g}(2)$. (See Theorem 4.6.3 of [**H**].)

In determining which of these representations yield multiplicity-free exterior algebras, Howe found on a case-by-case basis that it is sometimes possible to obtain a complete set of maximal vectors generating the irreducible decomposition of $\Lambda(V)$ by taking wedge products of weight vectors as above. This yields a family of multiplicity-free exterior algebras that Howe terms "geometrically decomposable." A second family, "symplectic geometrically decomposable," is obtained by combining wedge products of weight vectors

Partially supported by a grant from the NSF.

with powers of an invariant symplectic form; i.e., an element of $\Lambda^2(V)$ carrying a trivial $\mathfrak{g}$-action (if one exists). The remaining seven multiplicity-free exterior algebras, as well as the remaining cases that must be shown not to be multiplicity-free, are treated individually.

In this paper, our objective is to simplify parts of this classification. By twisting the Weyl Denominator Formula, we are able to deduce, in a simple and uniform way, explicit irreducible character decompositions for the exterior powers of two families of $\mathfrak{g}$-modules (see Theorems 2.1 and 3.2). It is easily shown that these expansions are multiplicity-free, and it turns out that the modules involved are precisely the geometrically and symplectic geometrically decomposable families of Howe (see Remarks 2.3 and 3.4). Of course this does not give the classification problem a completely conceptual solution, since we do not obtain the multiplicity-free exterior algebras in the exceptional third family, nor do we treat the cases that are not multiplicity-free.

We assume familiarity with the basics of Weyl groups and root systems as in [**B**].

1. Preliminaries

Let Φ be a finite crystallographic root system spanning some real Euclidean space E with inner product $\langle\,\cdot\,,\cdot\,\rangle$. We let $W \subset GL(E)$ denote the Weyl group of Φ, and $\Phi^+ \subset \Phi$ a choice of positive roots. Given $\alpha \in \Phi$, we set $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and write $\alpha > 0$ or $\alpha < 0$ according to whether $\alpha \in \Phi^+$ or $-\alpha \in \Phi^+$. Define $\rho = \sum_{\beta\in\Phi^+}\beta/2$.

Given $w \in W$, we let $\Phi(w)$ denote the set of w-*inversions*; i.e.,

$$\Phi(w) = \{\beta \in \Phi^+ : w\beta < 0\}.$$

It is well-known that the number of inversions is the length; i.e., $\ell(w) := |\Phi(w)|$ is the minimum length of an expression for w as a product of simple reflections [**B**, VI.1.6].

Lemma 1.1. *For all $w \in W$, $\sigma(w) := \rho - w^{-1}\rho$ is the sum of all w-inversions.*

Proof. Proceed by induction on the number of inversions. If there are no inversions, then $w = 1$ and $\sigma(w) = 0$. Otherwise, there must be a simple root α such that $w\alpha < 0$. However, the reflection s corresponding to α permutes $\Phi^+ - \{\alpha\}$, so $s\rho = \rho - \alpha$ and

$$\Phi(w) = s\Phi(ws) \cup \{\alpha\}, \tag{1.1}$$

whence by induction, the sum of $\Phi(w)$ is $s(\rho - (ws)^{-1}\rho) + \alpha = \rho - w^{-1}\rho$. $\square$

Let $\theta \in E$ be a dominant co-weight; i.e., suppose that $\langle\beta,\theta\rangle$ is a nonnegative integer for all $\beta \in \Phi^+$. We let W_0 denote the stabilizer of θ in W, and Φ_0 the root subsystem of Φ corresponding to W_0; i.e.,

$$\Phi_0 = \{\beta \in \Phi : \langle\beta,\theta\rangle = 0\}.$$

It is well-known that there is a distinguished set W^0 of coset representatives for W/W_0; the members of W^0 are those w such that $\ell(w) < \ell(ws)$ (i.e., $|\Phi(w)| < |\Phi(ws)|$) for all simple reflections s in W_0 (Exercise IV.1.3 of [**B**]).

LEMMA 1.2. *For all $w \in W$, we have $w \in W^0$ if and only if there is no w-inversion in Φ_0. In that case, we have $\langle \sigma(w), \beta^\vee \rangle \leqslant 0$ for all positive $\beta \in \Phi_0$; i.e., $-\sigma(w)$ is Φ_0-dominant.*

Proof. If α is a simple root in Φ_0 with corresponding reflection s, then either w or ws must have α as an inversion, so by (1.1), membership in W^0 is equivalent to having no simple root of Φ_0 in the inversion set. However, positive roots are in the positive linear span of simple roots, so having a w-inversion in Φ_0 forces a simple w-inversion in Φ_0.

If $-\sigma(w)$ failed to be Φ_0-dominant for some $w \in W^0$, then there would have to be a simple root $\alpha \in \Phi_0$ such that $\langle \sigma(w), \alpha^\vee \rangle > 0$. Therefore, $\langle \rho - w^{-1}\rho, \alpha^\vee \rangle = \langle \sigma(w), \alpha^\vee \rangle \geqslant 1$, thanks to Lemma 1.1 and the fact that $\langle \cdot\,, \alpha^\vee \rangle$ is integer-valued on Φ. However, one knows that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple α [**B**, VI.1.10], so a root β is positive if and only if $\langle \rho, \beta^\vee \rangle > 0$. Hence $\langle \rho, w\alpha^\vee \rangle \leqslant \langle \rho, \alpha^\vee \rangle - 1 = 0$, so $w\alpha$ must be a negative root, contradicting the fact that w has no inversions in Φ_0. $\square$

Let L denote the lattice of (integral) weights; i.e., those $\lambda \in E$ such that $\langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}$ for all $\alpha \in \Phi$. Given a ground ring $\mathbf{k}$, we let $\mathbf{k}[L]$ denote the group ring of L, viewed as $\mathbf{k}$-linear combinations of formal exponentials e^λ ($\lambda \in L$) subject to the relations $e^0 = 1$ and $e^\mu \cdot e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in L$. For each $\lambda \in L$, we define

$$\chi(\lambda) = \Delta(\lambda + \rho)/\Delta(\rho), \quad \text{where} \quad \Delta(\lambda) = \sum_{w \in W} \operatorname{sgn}(w) e^{w\lambda},$$

and let $\Delta_0(\cdot)$, $\chi^0(\cdot)$, and ρ_0 denote the Φ_0-analogues of $\Delta(\cdot)$, $\chi(\cdot)$ and ρ. It is easy to show that $\Delta(\rho)$ evenly divides $\Delta(\lambda + \rho)$; i.e., $\chi(\lambda) \in \mathbf{Z}[L]$ [**B**, VI.3.3]. By the Weyl Character Formula, one knows that if λ is dominant, then $\chi(\lambda)$ is the character of an irreducible $\mathfrak{g}$-module of highest weight λ, where $\mathfrak{g}$ is a semisimple Lie algebra with root system Φ.

Let f denote the index of the root lattice in L, so that $\langle \lambda, \theta \rangle \in (1/f)\mathbf{Z}$ for all $\lambda \in L$. In the following, we will use the Laurent polynomial ring $\mathbf{k} = \mathbf{Z}[t^{\pm 1/f}]$ as our ground ring.

LEMMA 1.3. *We have*

$$\prod_{\beta \in \Phi:\, \langle \beta, \theta \rangle > 0} \left(1 - t^{\langle \beta, \theta \rangle} e^{-\beta}\right) = \sum_{x \in W^0} \operatorname{sgn}(x) t^{\langle \sigma(x), \theta \rangle} \chi^0(-\sigma(x)).$$

It is important to note that in the above sum, $-\sigma(x)$ is Φ_0-dominant (Lemma 1.2).

Proof. Define an automorphism of $\mathbf{k}[L]$ by assigning $e^\lambda \mapsto t^{-\langle \lambda, \theta \rangle} e^\lambda$ for $\lambda \in L$. Applying this automorphism to the Weyl Denominator Formula [**B**, VI.3.3]; namely,

$$\prod_{\beta > 0} (1 - e^{-\beta}) = e^{-\rho} \Delta(\rho) = \sum_{w \in W} \operatorname{sgn}(w) e^{w\rho - \rho},$$

we obtain

$$\prod_{\beta > 0} \left(1 - t^{\langle \beta, \theta \rangle} e^{-\beta}\right) = \sum_{w \in W} \operatorname{sgn}(w) t^{\langle \rho - w\rho, \theta \rangle} e^{w\rho - \rho}.$$

If $\beta \in \Phi_0$, then the corresponding term on the left is $1 - e^{-\beta}$, so the factor $e^{-\rho_0}\Delta_0(\rho_0)$ may be isolated (by the Weyl Formula again, this time for Φ_0), yielding

$$\begin{aligned}\prod_{\beta\in\Phi:\,\langle\beta,\theta\rangle>0}\left(1 - t^{\langle\beta,\theta\rangle}e^{-\beta}\right) &= \frac{1}{\Delta_0(\rho_0)}\sum_{w\in W}\operatorname{sgn}(w)t^{\langle\rho-w\rho,\theta\rangle}e^{w\rho-\rho+\rho_0}\\ &= \frac{1}{\Delta_0(\rho_0)}\sum_{x\in W^0}\operatorname{sgn}(x)\sum_{w\in W_0}\operatorname{sgn}(w)t^{\langle\rho-wx^{-1}\rho,\theta\rangle}e^{wx^{-1}\rho-\rho+\rho_0}\\ &= e^{-\rho+\rho_0}\sum_{x\in W^0}\operatorname{sgn}(x)t^{\langle\rho-x^{-1}\rho,\theta\rangle}\Delta_0(x^{-1}\rho)/\Delta_0(\rho_0), \qquad (1.2)\end{aligned}$$

since $\langle\rho - wx^{-1}\rho, \theta\rangle$ is independent of $w \in W_0$.

Now W_0 permutes $\Phi^+ - \Phi_0$, so it fixes $\rho - \rho_0 = \sum_{\langle\beta,\theta\rangle>0}\beta/2$, hence

$$e^{-\rho+\rho_0}\Delta_0(x^{-1}\rho) = \Delta_0(x^{-1}\rho - \rho + \rho_0) = \Delta_0(-\sigma(x) + \rho_0).$$

It follows that the term indexed by x in (1.2) simplifies to $\operatorname{sgn}(x)t^{\langle\sigma(x),\theta\rangle}\chi^0(-\sigma(x))$. $\square$

2. The Minuscule Case

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ.

Continuing the hypothesis that θ is a dominant co-weight, let $\mathfrak{g}_0$ denote the (semisimple) Lie subalgebra of $\mathfrak{g}$ generated by the root spaces indexed by Φ_0. For integers $k \neq 0$, let

$$\Phi_k = \{\beta \in \Phi : \langle\beta,\theta\rangle = k\},$$

and define $\mathfrak{g}_k$ to be the $\mathfrak{g}_0$-submodule of $\mathfrak{g}$ spanned by the root spaces indexed by Φ_k.

Now assume that θ is minuscule; i.e.,

$$\langle\beta,\theta\rangle \leqslant 1 \quad \text{for all } \beta \in \Phi.$$

This is equivalent to Φ being the disjoint union of Φ_{-1}, Φ_0, and Φ_1. In this case, $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are abelian Lie subalgebras of $\mathfrak{g}$; if $[\mathfrak{g}_1, \mathfrak{g}_1]$ were nonzero, then Φ_2 would be nonempty.

The following result may be deduced from the abelian case of Kostant's work on the cohomology of nilradicals (see Theorem 5.14 of [**K**]). Indeed, the cohomology of an abelian Lie algebra (in our case, $\mathfrak{g}_{-1}$) is its exterior algebra. Our objective here is to obtain the result as a corollary of the Weyl Denominator Formula.

THEOREM 2.1. *If θ is minuscule, then*

$$\prod_{\beta\in\Phi:\,\langle\beta,\theta\rangle=1}\left(1 + te^{-\beta}\right) = \sum_{x\in W^0} t^{\ell(x)}\chi^0(-\sigma(x)).$$

In particular, each exterior power $\Lambda^k(\mathfrak{g}_{-1})$ is multiplicity-free as a $\mathfrak{g}_0$-module, and there is one irreducible constituent in $\Lambda^k(\mathfrak{g}_{-1})$ for each $x \in W^0$ of length k.

Note that a similar statement also holds for the exterior algebra of $\mathfrak{g}_1$.

Proof. We have $\Phi^+ - \Phi_0 = \Phi_1$, so Lemma 1.3 implies

$$\prod_{\beta \in \Phi_1} \left(1 - te^{-\beta}\right) = \sum_{x \in W^0} \operatorname{sgn}(x) t^{\langle \sigma(x), \theta \rangle} \chi^0(-\sigma(x)).$$

Furthermore, given that $x \in W^0$, every x-inversion β is in Φ_1 (Lemma 1.2), so $\sigma(x)$ is the sum of $\ell(x)$ members of Φ_1 (Lemma 1.1), and $\ell(x) = \langle \sigma(x), \theta \rangle$. Bearing in mind that $\operatorname{sgn}(x) = (-1)^{\ell(x)}$, the claimed formula follows.

To confirm that each of the exterior powers of $\mathfrak{g}_{-1}$ are multiplicity-free, note that as x varies over W^0, each of the expressions $\sigma(x) = \rho - x^{-1}\rho$ are clearly distinct. Moreover, fixing the degree and the $\mathfrak{g}_0$-character indexed by x in the above sum amounts to fixing $\langle \sigma(x), \theta \rangle$ and the projection of $\sigma(x)$ onto the hyperplane orthogonal to θ; i.e., it fixes $\sigma(x)$ and hence x. $\square$

Remark 2.2. Assuming $\mathfrak{g}$ is simple, every dominant minuscule (co-)weight is a fundamental (co-)weight (for the classification, see Exercise VI.4.15 of [**B**]), so the Dynkin diagram of $\mathfrak{g}_0$ may be obtained from the diagram of $\mathfrak{g}$ by deleting a single node. Furthermore, there is only one member of W^0 of length 1 (the simple reflection corresponding to the deleted node), so $\Lambda^1(\mathfrak{g}_{-1}) \cong \mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module, and the negative simple root corresponding to the deleted node is the highest weight of $\mathfrak{g}_{-1}$.

In the following, we list all instances of Theorem 2.1 with $\mathfrak{g}$ simple.

(a) In the case $\mathfrak{g} = \mathfrak{sl}(n)$, every fundamental co-weight is minuscule, and the corresponding subalgebras $\mathfrak{g}_0$ are of the form $\mathfrak{sl}(k) \oplus \mathfrak{sl}(n-k)$ $(1 \leqslant k < n)$, degenerating to $\mathfrak{g}_0 = \mathfrak{sl}(n-1)$ in the extreme cases ($k = 1$ or $k = n-1$). As a $\mathfrak{g}_0$-module, $\mathfrak{g}_{-1}$ is isomorphic to $(\mathbf{C}^k)^* \otimes \mathbf{C}^{n-k}$ (or $\mathbf{C}^{n-1}$ or its dual, in the extreme cases), where $\mathbf{C}^k$ denotes the defining representation of $\mathfrak{sl}(k)$. Furthermore, the expansion in Theorem 2.1 is equivalent to the Schur function identity

$$\prod_{i=1}^{k} \prod_{j=1}^{n-k} (1 + tx_i y_j) = \sum_{\lambda} t^{|\lambda|} s_\lambda(x_1, \ldots, x_k) s_{\lambda'}(y_1, \ldots, y_{n-k}),$$

where the sum ranges over partition diagrams that fit in a $k \times n-k$ rectangle, and λ' denotes the conjugate of λ. This is easily recognized as a well-known variation of the Cauchy identity; the fact that it may be deduced from the Weyl Denominator Formula is implicit in work of Macdonald (see Example I.4.7 of [**M**]).

(b) In the case $\mathfrak{g} = \mathfrak{so}(n)$, one of the end nodes of the diagram indexes a minuscule co-weight for which $\mathfrak{g}_0 = \mathfrak{so}(n-2)$. Here, it is easy to check that $\mathfrak{g}_{-1}$ is the defining representation of $\mathfrak{g}_0$, and Theorem 2.1 implies the familiar fact that the exterior powers of the defining representation are fundamental representations (with the exception of the largest—"middle"—exterior power).

(c) For $\mathfrak{g} = \mathfrak{sp}(2n)$, there is one minuscule co-weight, the corresponding subalgebra $\mathfrak{g}_0$ is $\mathfrak{sl}(n)$, and the modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are the symmetric square $S^2(\mathbf{C}^n)$ and its dual.

(d) In the case $\mathfrak{g} = \mathfrak{so}(2n)$, in addition to (b) there are two dominant minuscule co-weights whose corresponding subalgebra $\mathfrak{g}_0$ is $\mathfrak{sl}(n)$, and the modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic to $\Lambda^2(\mathbf{C}^n)$ and its dual.

(e) In the case $\mathfrak{g} = \mathfrak{e}(6)$, there are two minuscule co-weights that are dual two each other; in both cases, the subalgebra $\mathfrak{g}_0$ is $\mathfrak{so}(10)$, and the modules $\mathfrak{g}_{\pm 1}$ are isomorphic to the two spin representations of $\mathfrak{so}(10)$.

(f) In the case $\mathfrak{g} = \mathfrak{e}(7)$, there is one minuscule co-weight, the corresponding subalgebra $\mathfrak{g}_0$ is $\mathfrak{e}(6)$, and the modules $\mathfrak{g}_{\pm 1}$ are the two minuscule representations of $\mathfrak{e}(6)$.

It should be noted that the character expansions implied by Theorem 2.1 in the (c) and (d) cases are equivalent to Schur function identities due to Littlewood [**L**]. The fact that these may be deduced from the Weyl Denominator Formula has been noted by Macdonald (see Example I.5.9 of [**M**]).

REMARK 2.3. The $\mathfrak{g}_0$-modules $\mathfrak{g}_{\pm 1}$ that occur with $\mathfrak{g}_0$ simple are

(a) $\mathbf{C}^n$, $(\mathbf{C}^n)^*$, $S^2(\mathbf{C}^n)$, $S^2(\mathbf{C}^n)^*$, $\Lambda^2(\mathbf{C}^n)$, $\Lambda^2(\mathbf{C}^n)^*$ for $\mathfrak{sl}(n)$,
(b) the defining representation of $\mathfrak{so}(n)$,
(c) the spin representations of $\mathfrak{so}(10)$, and
(d) the minuscule representations of $\mathfrak{e}(6)$.

As noted in the introduction, these are precisely the modules that generate multiplicity-free exterior algebras that are "geometrically decomposable" in the sense of Howe [**H**].

3. The Quasi-Minuscule Case

Now assume that $\mathfrak{g}$ is simple, or equivalently, that Φ is irreducible, and that $\widetilde{\alpha}$ is the highest root. In this case, $\theta = \widetilde{\alpha}^\vee$ is quasi-minuscule in the sense that

$$\langle \beta, \theta \rangle \leqslant 2 \quad \text{for all } \beta \in \Phi,$$

with equality if and only if $\beta = \widetilde{\alpha}$ [**B**, VI.1.8]. In other words, Φ is the disjoint union of Φ_k ($-2 \leqslant k \leqslant 2$), and we have $\Phi_2 = \{\widetilde{\alpha}\}$. It follows that $\mathfrak{g}_2$ is a trivial $\mathfrak{g}_0$-module, and that $\mathfrak{g}_{\geqslant 1} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie subalgebra of $\mathfrak{g}$ of Heisenberg type; i.e., nilpotent with a one-dimensional derived subalgebra.

As in the previous section, W_0 denotes the stabilizer of θ; i.e., $\widetilde{\alpha}^\vee$ in this case.

Let $t_0 \in W$ denote the reflection corresponding to $\widetilde{\alpha}$, and define $m = \langle \rho, \theta \rangle = \langle \rho, \widetilde{\alpha}^\vee \rangle$. We say that $w \in W^0$ is *positive* if $w\widetilde{\alpha} > 0$; otherwise, $w\widetilde{\alpha} < 0$ and w is *negative.*

LEMMA 3.1. *The map $x \mapsto xt_0$ defines a bijection between the positive and negative members of W^0. Furthermore, if $x \in W^0$ is positive, then*

(a) $\langle \sigma(x), \theta \rangle = \ell(x) < m$,
(b) $\langle \sigma(xt_0), \theta \rangle = \ell(xt_0) + 1 = 2m - \ell(x)$, *and*
(c) $\sigma(xt_0) = \sigma(x) + (m - \ell(x))\widetilde{\alpha}$.

Proof. The map $x \mapsto xt_0$ is obviously an involution, and it is clear that exactly one of $x\widetilde{\alpha}$ and $xt_0\widetilde{\alpha} = -x\widetilde{\alpha}$ is a positive root, so to prove the first statement, it suffices to show that $x \in W^0$ implies $xt_0 \in W^0$. Arguing by contradiction, note that if $xt_0 \notin W^0$ then xt_0 must have an inversion in Φ_0 (Lemma 1.2). However, the roots in Φ_0 are fixed by t_0, so this would also produce an x-inversion in Φ_0, forcing $x \notin W^0$.

Now suppose that $x \in W^0$ is positive. Since no root in Φ_0 or Φ_2 can be an x-inversion (Lemma 1.2 and positivity of x), all x-inversions β are in Φ_1. Hence $\sigma(x)$ is a sum of

$\ell(x) = |\Phi(x)|$ roots in Φ_1 (Lemma 1.1), and we obtain $\langle\sigma(x),\theta\rangle = \ell(x)$. Nearly the same argument applies to xt_0; the only difference is that $\widetilde{\alpha}$ is now an inversion, so $\sigma(xt_0)$ is a sum of $\ell(xt_0) - 1$ roots in Φ_1 and the root $\widetilde{\alpha} \in \Phi_2$, whence $\langle\sigma(xt_0),\theta\rangle = \ell(xt_0) + 1$.

To prove (c), use (a) to obtain

$$\begin{aligned}\sigma(xt_0) &= \rho - t_0x^{-1}\rho = (\rho - t_0\rho) + t_0\sigma(x)\\ &= m\widetilde{\alpha} + \sigma(x) - \langle\sigma(x),\widetilde{\alpha}^\vee\rangle\widetilde{\alpha} = \sigma(x) + (m - \ell(x))\widetilde{\alpha}.\end{aligned}$$

This also implies $\langle\sigma(xt_0),\theta\rangle = \langle\sigma(x),\theta\rangle + 2(m - \ell(x)) = 2m - \ell(x)$, completing (b).

To prove the bound on $\ell(x)$ in (a), note that $2m = \langle 2\rho,\theta\rangle = \sum_{\beta>0}\langle\beta,\theta\rangle = |\Phi_1| + 2|\Phi_2|$, so $|\Phi_1| = 2(m-1)$. Moreover, for each x-inversion β, we have $\langle t_0\beta,\theta\rangle = -\langle\beta,\widetilde{\alpha}^\vee\rangle = -1$, so $-t_0\beta = \widetilde{\alpha} - \beta$ is also a root in Φ_1. However, this root cannot also be an x-inversion; otherwise $\widetilde{\alpha} = \beta + (\widetilde{\alpha} - \beta)$ would also be an x-inversion, contradicting positivity. Thus Φ_1 may be partitioned into $m-1$ pairs such that $\Phi(x)$ includes at most one member from each pair, and hence $\ell(x) \leqslant m - 1$. $\square$

Theorem 3.2. *If θ is quasi-minuscule (i.e., $\theta = \widetilde{\alpha}^\vee$), then*

$$\prod_{\beta:\,\langle\beta,\theta\rangle=1}\left(1 + te^{-\beta}\right) = \sum_{x\in W^0} t^{\ell(x)} \sum_{0\leqslant k<m-\ell(x)} t^{2k}\chi^0(-\sigma(x) - k\widetilde{\alpha}).$$

In particular, each exterior power $\Lambda^k(\mathfrak{g}_{-1})$ is multiplicity-free as a $\mathfrak{g}_0$-module.

Note that there are nonzero terms indexed by $x \in W^0$ in the above sum only if $\ell(x) < m$. By Lemma 3.1, this is equivalent to x being positive.

Proof. Applying Lemma 1.3, we obtain

$$\left(1 - t^2e^{-\widetilde{\alpha}}\right)\prod_{\beta\in\Phi_1}\left(1 - te^{-\beta}\right) = \sum_{x\in W^0}\operatorname{sgn}(x)t^{\langle\sigma(x),\theta\rangle}\chi^0(-\sigma(x)). \tag{3.1}$$

Now rearrange the sum over W^0 into a sum over pairs x and xt_0 with $x \in W^0$ positive. For such x, we know that $\langle\sigma(x),\theta\rangle = \ell(x)$ and $\langle\sigma(xt_0),\theta\rangle = 2m - \ell(x)$ (Lemma 3.1). Furthermore, $\chi^0(\mu + k\widetilde{\alpha}) = e^{k\widetilde{\alpha}}\chi^0(\mu)$ since $\widetilde{\alpha}$ is W_0-invariant, so we have

$$\chi^0(-\sigma(xt_0)) = \chi^0(-\sigma(x) - (m - \ell(x))\widetilde{\alpha}) = e^{-(m-\ell(x))\widetilde{\alpha}}\chi^0(-\sigma(x)),$$

again by Lemma 3.1. It follows that (3.1) may be rewritten in the form

$$\left(1 - t^2e^{-\widetilde{\alpha}}\right)\prod_{\beta\in\Phi_1}\left(1 - te^{-\beta}\right) = \sum_{x\in W^0:\,x\widetilde{\alpha}>0}\operatorname{sgn}(x)t^{\ell(x)}\left(1 - (t^2e^{-\widetilde{\alpha}})^{m-\ell(x)}\right)\chi^0(-\sigma(x)).$$

Dividing both sides by $1 - t^2e^{-\widetilde{\alpha}}$, we obtain

$$\prod_{\beta\in\Phi_1}\left(1 - te^{-\beta}\right) = \sum_{x\in W^0:\,x\widetilde{\alpha}>0}\operatorname{sgn}(x)t^{\ell(x)}\sum_{0\leqslant k<m-\ell(x)} t^{2k}\chi^0(-\sigma(x) - k\widetilde{\alpha}).$$

Since the inner sum is empty if $x \in W^0$ is negative, the condition $x\widetilde{\alpha} > 0$ is redundant. Replacing $t \to -t$, we obtain the claimed formula.

To confirm that every exterior power of $\mathfrak{g}_{-1}$ is multiplicity-free, note that for $x \in W^0$ and $0 \leqslant k < m - \ell(x)$, the degree of the corresponding term in the above expansion is $\ell(x) + 2k = \langle \sigma(x) + k\widetilde{\alpha}, \theta \rangle$. Thus, fixing the degree and the irreducible $\mathfrak{g}_0$-character fixes the projections of $\sigma(x) + k\widetilde{\alpha}$ onto θ and its orthogonal complement; i.e., it fixes $\sigma(x) + k\widetilde{\alpha}$.

Given distinct $x, y \in W^0$ and integers k, l such that $\sigma(x) + k\widetilde{\alpha} = \sigma(y) = l\widetilde{\alpha}$, it follows that $x^{-1}\rho$ and $y^{-1}\rho$ differ by a nonzero multiple of $\widetilde{\alpha}$. However, a given W-orbit (or more generally, any sphere) has at most two points on any line, so $x^{-1}\rho$ and $y^{-1}\rho$ are the unique points of $W\rho$ on the line they determine. However, this line also includes the point $t_0 x^{-1}\rho$, so we must have $y = xt_0$, and hence x is positive and y is negative, or vice-versa. Since the above expansion is limited to positive members of W^0, we conclude that each irreducible $\mathfrak{g}_0$-character appears at most once per degree in the above expansion. □

Remark 3.3. There is one instance of Theorem 3.2 for each simple $\mathfrak{g}$. In each case, the Dynkin diagram of $\mathfrak{g}_0$ may be obtained by deleting the nodes of the diagram of $\mathfrak{g}$ that are adjacent to the extra node of the extended diagram. Furthermore, the members of W^0 of length 1 are the simple reflections corresponding to the deleted nodes, so there is an irreducible component of $\Lambda^1(\mathfrak{g}_{-1}) \cong \mathfrak{g}_{-1}$ for each deleted node, and the highest weights of these components are the negative simple roots corresponding to these nodes.

In the following table, we list the possibilities for $\mathfrak{g}$, $\mathfrak{g}_0$, and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$.

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$
$\mathfrak{sl}(n)$	$\mathfrak{sl}(n-2)$	$\mathbf{C}^{n-2} \oplus (\mathbf{C}^{n-2})^*$
$\mathfrak{so}(n)$	$\mathfrak{so}(n-2) \oplus \mathfrak{sl}(2)$	$\mathbf{C}^{n-2} \otimes \mathbf{C}^2$
$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n-2)$	$\mathbf{C}^{2n-2}$
$\mathfrak{e}(6)$	$\mathfrak{sl}(6)$	$\Lambda^3(\mathbf{C}^6)$
$\mathfrak{e}(7)$	$\mathfrak{so}(12)$	spin representation
$\mathfrak{e}(8)$	$\mathfrak{e}(7)$	minuscule representation
$\mathfrak{f}(4)$	$\mathfrak{sp}(6)$	$\Lambda^3(\mathbf{C}^6)/\mathbf{C}^6$
$\mathfrak{g}(2)$	$\mathfrak{sl}(2)$	$S^3(\mathbf{C}^2)$

In the above table, the module $\mathbf{C}^n$ should be understood as the defining representation of the relevant classical algebra. In the case $\mathfrak{g} = \mathfrak{e}(7)$, it should be noted that $\mathfrak{so}(12)$ has two spin representations; one of them occurs as $\mathfrak{g}_{-1}$, the other as $\mathfrak{g}_1$. In the case $\mathfrak{g} = \mathfrak{f}(4)$, the module $\mathfrak{g}_{-1}$ is the unique irreducible $\mathfrak{sp}(6)$-module with one-dimensional weight spaces other than the defining representation.

Remark 3.4. The pairs $(\mathfrak{g}_{\pm 1}, \mathfrak{g}_0)$ such that $\mathfrak{g}_{\pm 1}$ is irreducible and $\mathfrak{g}_0$ is simple are

(a) $\Lambda^3(\mathbf{C}^6)$ for $\mathfrak{sl}(6)$ and $S^3(\mathbf{C}^2)$ for $\mathfrak{sl}(2)$,
(b) the defining representation of $\mathfrak{sp}(2n)$ and $\Lambda^3(\mathbf{C}^6)/\mathbf{C}^6$ for $\mathfrak{sp}(6)$,
(c) the spin representations of $\mathfrak{so}(12)$, and
(d) the minuscule representation of $\mathfrak{e}(7)$.

These are precisely the multiplicity-free exterior algebras that are defined by Howe to be "symplectic geometrically decomposable" [**H**]. Moreover, according to Howe's classification

(Theorem 4.7.3, *op. cit.*), the only multiplicity-free exterior algebras not covered by this result or the one in Section 2 are those of

(a) $S^4(\mathbf{C}^2)$, $S^5(\mathbf{C}^2)$, and $S^6(\mathbf{C}^2)$ for $\mathfrak{sl}(2)$ and $S^3(\mathbf{C}^3)$ for $\mathfrak{sl}(3)$,
(b) the spin representations of $\mathfrak{so}(7)$ and $\mathfrak{so}(9)$, and
(c) the 7-dimensional representation of $\mathfrak{g}(2)$.

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