Multiplicity-Free Products of Schur Functions

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Abstract. We classify all multiplicity-free products of Schur functions and all multiplicity-free products of characters of $SL(n, \mathbf{C})$.

0. Introduction

In this paper, we classify the products of Schur functions that are multiplicity-free; i.e., products for which every coefficient in the resulting Schur function expansion is 0 or 1. We also solve the slightly more general classification problem for Schur functions in any finite number of variables. The latter is equivalent to a classification of all multiplicity-free tensor products of irreducible representations of $GL(n)$ or $SL(n)$.

Multiplicity-free representations have many applications, typically based on the fact that their centralizer algebras are commutative, or that their irreducible decompositions are canonical; see the survey article by Howe [**H**]. We find it surprising that such a natural classification problem seems not to have been considered before.

Two well-known examples of multiplicity-free products are the Pieri rules (which correspond to a tensor product in which one of the factors is a symmetric or exterior power of the defining representation), and the rule for multiplying Schur functions of rectangular shape. The fact that the latter is multiplicity-free was first noticed by Kostant, and has played an important role in several applications. For example, Stanley used the rule to count self-complementary plane partitions [**St**]. More recently, the fact that these products are multiplicity-free has been a key property needed for explicit bijections constructed by Schilling-Warnaar [**SWa**], Shimozono [**S**], and Shimozono-White [**SWh**]

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related to q-analogues of Littlewood-Richardson coefficients. Further identities involving classical group characters of rectangular and near-rectangular shape (the latter also appear in the classification) have been investigated by Okada [**O**] and Krattenthaler [**K**].

A related problem that has been considered recently by Roger Howe [private communication] and Magyar-Weyman-Zelevinsky [**MWZ**] is the classification of products of flag varieties with finitely many $GL(V)$-orbits (or more generally, for any reductive group). Via coordinate rings, these yield multiplicity-free products of $GL(V)$-representations. Moreover, a comparison of our classification with that of [**MWZ**] shows that every multiplicity-free product of $GL(n)$-characters arises either in this fashion, or else as the conjugate of such a product. For example, the product of two Grassmannians and a full flag variety has finitely many orbits, and this is equivalent to the fact that the product of any two rectangular Schur functions is multiplicity-free.

1. Symmetries of Littlewood-Richardson coefficients

We (mostly) follow the standard notation for tableaux and Schur functions in [**M**]. In particular, a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ denotes a non-increasing sequence of finitely many positive integers; we write $|\lambda|$ for the sum and $\ell(\lambda)$ for the length. The *diagram* of λ is

$$D_\lambda := \{(i,j) \in \mathbf{Z}^2 : 1 \leqslant j \leqslant \lambda_i,\ 1 \leqslant i \leqslant \ell(\lambda)\},$$

a set of points in the plane with matrix-style coordinates. It is a convenient abuse of notation to identify λ with D_λ.

The conjugate of λ, denoted λ', is the partition whose diagram is the transpose of λ. Given partitions μ and ν, we write $\mu \cup \nu$ for the partition obtained by sorting the multiset union of the parts of μ and ν, and $\mu + \nu$ for the partition $(\mu_1 + \nu_1, \mu_2 + \nu_2, \dots)$ (pad μ or ν with trailing 0's, if necessary). Note that $(\mu + \nu)' = \mu' \cup \nu'$.

If $D_\mu \subseteq D_\lambda$, the difference $D_\lambda - D_\mu$ is called a *skew diagram*, and abbreviated λ/μ. A *semistandard tableau* of shape λ/μ is an array of positive integers $T(i,j) : (i,j) \in D_\lambda - D_\mu$ such that $T(i,j) \leqslant T(i,j+1)$ and $T(i,j) < T(i+1,j)$ for all i,j for which these expressions are defined. The *content* of T defined to be $\eta(T) = (\eta_1(T), \eta_2(T), \dots)$, where

$$\eta_k(T) := |\{(i,j) \in D_\lambda - D_\mu : T(i,j) = k\}|,$$

and the *(skew) Schur function* of shape λ/μ is the generating function

$$s_{\lambda/\mu} = \sum_T x_1^{\eta_1(T)} x_2^{\eta_2(T)} \cdots ,$$

summed over all semistandard tableaux of shape λ/μ. Since these generating functions are invariant under translation of the corresponding diagrams, it is convenient to identify any two skew diagrams that differ only by a translation.

The skew Schur functions are easily shown to be symmetric in the variables x_i, and the ordinary Schur functions s_λ (i.e., $s_{\lambda/\mu}$ with $\mu = \varnothing$) form a **Z**-basis for the ring of symmetric functions. (Proofs of this and all other facts mentioned in this section can be found in Chapter I of [**M**].) The *Littlewood-Richardson coefficients* $c(\lambda;\mu,\nu)$ are the structure constants for the multiplication of Schur functions; i.e.,

$$s_\mu s_\nu = \sum_\lambda c(\lambda;\mu,\nu) s_\lambda,$$

and the same coefficients also appear in the decomposition of skew Schur functions:

$$s_{\lambda/\mu} = \sum_\nu c(\lambda;\mu,\nu) s_\nu,$$

interpreting $s_{\lambda/\mu}$ as 0 when $D_\mu \not\subseteq D_\lambda$. This coincidence can also be expressed as

$$\langle s_\mu s_\nu, s_\lambda\rangle = c(\lambda;\mu,\nu) = \langle s_\nu, s_{\lambda/\mu}\rangle,$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product relative to which the ordinary Schur functions are orthonormal.

There are a number of symmetries involving Littlewood-Richardson coefficients. The most obvious one arises from the commutativity of multiplication: $c(\lambda;\mu,\nu) = c(\lambda;\nu,\mu)$. Less obvious is *conjugation symmetry*:

$$c(\lambda;\mu,\nu) = c(\lambda';\mu',\nu'),$$

which is equivalent to the existence of an automorphism of the ring of symmetric functions in which $s_\lambda \mapsto s_{\lambda'}$.

Given a skew diagram λ/μ, let $(\lambda/\mu)^\circ$ denote the skew diagram obtained by a 180° rotation of λ/μ, an operation that is well-defined up to translation. There is an easy content-reversing bijection between the semistandard tableaux of shapes λ/μ and $(\lambda/\mu)^\circ$. Since the Schur functions are symmetric, this implies

$$s_{\lambda/\mu} = s_{(\lambda/\mu)^\circ},$$

and this yields a further symmetry of Littlewood-Richardson coefficients.

If the skew diagram λ/μ can be partitioned into two subsets $\theta^{(1)}$ and $\theta^{(2)}$ with disjoint rows and columns then both subsets must themselves be skew diagrams, and the set of semistandard tableaux of shape λ/μ is effectively the Cartesian product of the semistandard tableaux of shapes $\theta^{(1)}$ and $\theta^{(2)}$. It follows that

$$s_{\lambda/\mu} = s_{\theta^{(1)}} s_{\theta^{(2)}}, \tag{1.1}$$

and this implies a cubic relation involving Littlewood-Richardson coefficients. If $\theta^{(1)}$ and $\theta^{(2)}$ happen to be translations of partition (i.e., non-skew) diagrams, the relation is linear.

Finally, let us mention the explicit description of $c(\lambda;\mu,\nu)$ known as the *Littlewood-Richardson rule*. Totally order $\mathbf{Z}^2$ so that (i,j) precedes (i',j') if and only if $i < i'$, or $i = i'$ and $j > j'$. A tableau T yields a word $w(T)$ when the entries $T(i,j)$ are read in this order. This word is said to be a *lattice permutation* if for all $n, k \geqslant 1$, the number of occurrences of k among the first n terms is a non-increasing function of k. In these terms, $c(\lambda;\mu,\nu)$ can be described as the number of semistandard tableaux T of shape λ/μ and content ν such that $w(T)$ is a lattice permutation. We call these *LR fillings.*

2. Products of Schur functions

A partition μ with at most one part size (i.e., empty, or of the form (c^r) for suitable $c, r > 0$) is said to be a *rectangle.* If it has either k rows or columns (i.e., $k = r$ or $k = c$), then we say that μ is a *k-line rectangle.*

A partition μ with exactly two part sizes (i.e., $\mu = (b^r c^s)$ for suitable $b > c > 0$ and $r, s > 0$) is said to be a *fat hook.* If it is possible to obtain a rectangle by deleting a single row or column from the fat hook μ, then we say that μ is a *near-rectangle.* For example, $(442), (533), (3311)$ and (4433) are all near-rectangles.

THEOREM 2.1. *The product $s_\mu s_\nu$ is multiplicity-free if and only if*

(i) *μ or ν is a one-line rectangle, or*
(ii) *μ is a two-line rectangle and ν is a fat hook (or vice-versa), or*
(iii) *μ is a rectangle and ν is a near-rectangle (or vice-versa), or*
(iv) *μ and ν are rectangles.*

The main fact about Littlewood-Richardson coefficients that we need in order to prove this result is the following.

LEMMA 2.2. *For all triples of partitions λ, μ, ν and all integers $r \geqslant 0$, we have*

$$c(\lambda + 1^r; \mu + 1^r, \nu) \geqslant c(\lambda;\mu,\nu), \tag{2.1}$$

$$c(\lambda \cup (r); \mu \cup (r), \nu) \geqslant c(\lambda;\mu,\nu). \tag{2.2}$$

Proof. Given one of the $c(\lambda;\mu,\nu)$ LR fillings of λ/μ having content ν, one may shift the first r rows to the right one column without violating the lattice permutation and semistandard conditions, thereby creating an LR filling of shape $(\lambda+1^r)/(\mu+1^r)$. This proves the first inequality; the second is equivalent to the first by conjugation symmetry. $\square$

COROLLARY 2.3. *If $s_\mu s_\nu$ is not multiplicity-free, then $s_{\mu+1^r}s_\nu$ and $s_{\mu\cup(r)}s_\nu$ are not multiplicity-free.*

Define a partial order on partitions by taking the transitive closure of the relations

$$\lambda \prec \lambda + 1^r, \quad \lambda \prec \lambda \cup (r)$$

for all partitions λ and all integers $r > 0$. It follows from Corollary 2.3 that the non-multiplicity-free pairs of partitions form an order filter relative to the product order induced by $\prec$ on pairs of partitions. However, it is a surprisingly delicate problem to determine in general when two pairs of partitions are related under this order. (If only part-unions are allowed, or dually, only column-additions, the relation is very simple.)

Perhaps even more surprising is the fact that the order filter of non-multiplicity-free pairs of partitions has only the following three generators, as will become evident in the proof of Theorem 2.1.

LEMMA 2.4. *The following are not multiplicity-free: (a) $s_{21}s_{21}$; (b) $s_{22}s_{321}$; (c) $s_{333}s_{4422}$.*

Proof. To prove that $s_\mu s_\nu$ is not multiplicity-free, it suffices to exhibit a pair of LR fillings of content μ and some common shape of the form λ/ν. In the cases at hand, the following are suitable:

$$\begin{array}{lll} * & * & 1 \\ * & 1 & \\ 2 & & \end{array} \quad \begin{array}{lll} * & * & 1 \\ * & 2 & \\ 1 & & \end{array}, \qquad \begin{array}{llll} * & * & * & 1 \\ * & * & 1 & \\ * & 2 & & \\ 2 & & & \end{array} \quad \begin{array}{llll} * & * & * & 1 \\ * & * & 2 & \\ * & 1 & & \\ 2 & & & \end{array},$$

$$\begin{array}{llllll} * & * & * & * & 1 & 1 \\ * & * & * & * & 2 & \\ * & * & 1 & 2 & & \\ * & * & 3 & & & \\ 2 & 3 & & & & \\ 3 & & & & & \end{array} \quad \begin{array}{llllll} * & * & * & * & 1 & 1 \\ * & * & * & * & 2 & \\ * & * & 1 & 3 & & \\ * & * & 2 & & & \\ 2 & 3 & & & & \\ 3 & & & & & \end{array}. \quad \square$$

Proof of Theorem 2.1. If neither μ nor ν is a rectangle, then both have at least two part sizes and $\mu, \nu \succcurlyeq (21)$, whence Lemma 2.4(a) and Corollary 2.3 imply that $s_\mu s_\nu$ cannot be multiplicity-free. Thus we may assume that μ (say) is a rectangle.

If μ has more than one row and column and ν has at least three part sizes, then $\mu \succcurlyeq (22)$ and $\nu \succcurlyeq (321)$, whence Lemma 2.4(b) and Corollary 2.3 imply that $s_\mu s_\nu$ cannot be multiplicity-free. Thus if μ is not a one-line rectangle, then ν must be a fat hook or a rectangle.

Assuming that $\nu = (b^r c^s)$ is a fat hook, the lengths of the horizontal and vertical segments that comprise the boundary of D_ν are $b-c, c, r, s$. Any fat hook whose corresponding dimensions are greater than these is $\succ \nu$. In particular, the only fat hooks not greater than (4422) are near-rectangles. It follows that if μ has more than two rows and columns, and ν is a fat hook but not a near-rectangle, then $\mu \succcurlyeq (333)$ and $\nu \succcurlyeq (4422)$, whence Lemma 2.4(c) and Corollary 2.3 imply that $s_\mu s_\nu$ cannot be multiplicity-free.

The above reasoning eliminates all products except for those explicitly listed in (i)–(iv) of the statement of Theorem 2.1. To finish the proof, we need to establish that each of these products are indeed multiplicity-free. Interestingly, it turns out that Corollary 2.3 (via its contrapositive) is crucial for this part of the argument as well: if $s_\mu s_\nu$ is multiplicity-free, then the same is true for any product corresponding to a pair of partitions *below* μ and ν relative to $\prec$. The advantage is that it is possible to find higher multiplicity-free products with simpler combinatorial structure.

(i) It is well-known that $s_\mu s_\nu$ is multiplicity-free whenever μ is a one-line rectangle. For example, if $\mu = (r)$, then the corresponding Littlewood-Richardson coefficients all involve tableaux filled with r identical entries, so there is obviously at most one LR filling of a given shape. Conjugation symmetry yields the analogous result for the case when $\mu = (1^r)$. (There are also much simpler proofs, independent of the LR rule, starting from the bi-alternant or tableau definitions of Schur functions.)

(iv) If μ and ν are both rectangles, we may replace ν with a new rectangle $\succcurlyeq \nu$ with more than $|\mu|$ rows and columns. In that case, every skew shape λ/ν of size $|\mu|$ is a disconnected union of two partition diagrams $\theta^{(1)}$ and $\theta^{(2)}$. In particular, (1.1) implies

$$c(\lambda;\mu,\nu) = \langle s_{\lambda/\nu}, s_\mu \rangle = \langle s_{\theta^{(1)}} s_{\theta^{(2)}}, s_\mu \rangle = \langle s_{\theta^{(1)}}, s_{\mu/\theta^{(2)}} \rangle.$$

However since μ is a rectangle, $\mu/\theta^{(2)}$ is rotationally equivalent to the diagram of some partition ϕ; i.e., $s_{\mu/\theta^{(2)}} = s_\phi$, whence $c(\lambda;\mu,\nu) = 1$ or 0, the former occurring if and only if $\phi = \theta^{(1)}$; i.e., $(\mu/\theta^{(2)})^\circ = \theta^{(1)}$.

(ii) If $\mu = (n,n)$ is a two-rowed rectangle and $\nu = (b^r c^s)$ is a fat hook, we may replace ν with a new fat hook $\succcurlyeq \nu$ whose dimensions (i.e., $b-c, c, r, s$) are all greater than $2n$. In that case, every skew shape λ/ν of size $2n$ is a disconnected union of three partition diagrams $\theta^{(i)}$ $(1 \leqslant i \leqslant 3)$. By a calculation similar to the previous case, it follows that

$$c(\lambda;\mu,\nu) = \langle s_{\theta^{(1)}} s_{\theta^{(2)}}, s_{\mu/\theta^{(3)}} \rangle = \langle s_{\theta^{(1)}} s_{\theta^{(2)}}, s_\phi \rangle = c(\phi;\theta^{(1)},\theta^{(2)}),$$

where ϕ denotes the partition for which $(\mu/\theta^{(3)})^{\circ} = \phi$. However μ has two rows, so ϕ and each $\theta^{(i)}$ must each have at most two rows (otherwise $c(\lambda;\mu,\nu) = 0$). Thus any LR filling of shape $\phi/\theta^{(2)}$ and content $\theta^{(1)}$ is a skew tableaux with a fixed number of 1's and 2's and at most two rows. However, there is at most one such filling: the first row must consist entirely of 1's, and the second row must have the 1's and 2's in increasing order. Hence $c(\phi;\theta^{(1)},\theta^{(2)}) \leqslant 1$ and $s_\mu s_\nu$ is multiplicity-free.

(iii) If μ is a rectangle and $\nu = (b^r c^s)$ is a near-rectangle, then one of the four dimensions $b-c, c, r, s$ is equal to 1, and the remaining three are at least 2. Thus we may replace ν with a new near-rectangle $\succcurlyeq \nu$ for which these three dimensions are all greater than $|\mu|$ (the fourth must remain fixed at 1). In that case, every skew shape λ/ν of size $|\mu|$ is a disconnected union of three partition diagrams $\theta^{(i)}$ $(1 \leqslant i \leqslant 3)$, one of which is a one-line rectangle. Since $c(\lambda;\mu,\nu) = \langle s_{\theta^{(1)}} s_{\theta^{(2)}} s_{\theta^{(3)}}, s_\mu\rangle$ is symmetric in the $\theta^{(i)}$'s we may assume that $\theta^{(1)}$ is a one-line rectangle. By the reasoning of the previous case, it follows that

$$c(\lambda;\mu,\nu) = c(\phi;\theta^{(1)},\theta^{(2)}), \quad (\mu/\theta^{(3)})^{\circ} = \phi.$$

However since $\theta^{(1)}$ is a row or column, this multiplicity is at most 1 by (i). $\square$

REMARK 2.5. (a) Corollary 2.3 and the fact that $s_1^3 = s_3 + 2s_{21} + s_{111}$ is not multiplicity-free together imply that no product of three nontrivial Schur functions is multiplicity-free.

(b) If the proof of Theorem 2.1 is examined carefully, it can be seen as constructive. More precisely, given a pair (μ,ν) such that $s_\mu s_\nu$ is not multiplicity-free, it provides an algorithm for constructing a partition λ such that $c(\lambda;\mu,\nu) \geqslant 2$.

3. Products of $GL(n)$ or $SL(n)$ characters

Let Λ_n denote the ring of symmetric polynomials in the variables $x_1,\dots,x_n$. By specializing the ordinary Schur functions s_λ to these variables (i.e., set $x_m = 0$ for $m > n$), one obtains a **Z**-basis for Λ_n by taking only those λ with $\ell(\lambda) \leqslant n$, whereas s_λ specializes to 0 if $\ell(\lambda) > n$. It follows that the Littlewood-Richardson coefficients, restricted to partitions of length at most n, are structure constants for Λ_n; i.e.,

$$s_\mu(x_1,\dots,x_n)s_\nu(x_1,\dots,x_n) = \sum_{\ell(\lambda)\leqslant n} c(\lambda;\mu,\nu)s_\lambda(x_1,\dots,x_n).$$

In particular, if $s_\mu s_\nu$ is multiplicity-free, then the same is true in Λ_n, but not conversely, since it might be the case that there are partitions λ of length $> n$ such that $c(\lambda;\mu,\nu) > 1$, but none of length $\leqslant n$.

In this section, we classify the products of Schur functions that are multiplicity-free in Λ_n. In view of the well-known relationship between Schur functions and irreducible representations of $SL(n,\mathbf{C})$ (or polynomial representations of $GL(n,\mathbf{C})$; e.g., see Appendix A of Chapter I in [**M**]), this amounts to a classification of the multiplicity-free tensor products of irreducible representations of $SL(n,\mathbf{C})$.

One immediate simplification can be deduced from the fact that

$$s_{\lambda+1^n}(x_1,\ldots,x_n)=(x_1\cdots x_n)s_\lambda(x_1,\ldots,x_n)\quad(\ell(\lambda)\leqslant n),$$

which is easy to see from any definition of the Schur functions. This reduces the classification of multiplicity-free products in Λ_n to the cases $s_\mu s_\nu$ with $\ell(\mu),\ell(\nu)<n$. At the level of representations, this corresponds to the fact that the partitions λ with $\ell(\lambda)\leqslant n$ index the irreducible polynomial $GL(n,\mathbf{C})$-modules V_λ, but $V_\lambda\cong V_{\lambda+1^n}$ as $SL(n,\mathbf{C})$-modules.

It will also be useful to exploit a symmetry of Littlewood-Richardson coefficients that is valid only in the n-variate context. This additional symmetry derives from the identity

$$s_\lambda(x_1^{-1},\ldots,x_n^{-1})=(x_1\cdots x_n)^{-\lambda_1}s_{\lambda^*}(x_1,\ldots,x_n),$$

where (assuming $\ell(\lambda)\leqslant n$)

$$\lambda^*:=(\lambda_1-\lambda_n,\lambda_1-\lambda_{n-1},\ldots,\lambda_1-\lambda_1).$$

Again, this is easy to see from any definition of the Schur function, but is perhaps best understood in terms of the corresponding modules: the dual of V_λ is isomorphic to V_{λ^*} (as an $SL(n,\mathbf{C})$-module). From either point of view, it follows that the same multiplicities occur in $s_\mu s_\nu$ and $s_{\mu^*}s_{\nu^*}$. In particular, $s_\mu s_\nu$ is multiplicity-free in Λ_n if and only if the same is true of $s_{\mu^*}s_{\nu^*}$.

Note that for the set of partitions λ with $\ell(\lambda)<n$, the operation $\lambda\mapsto\lambda^*$ is an involution, and that it preserves the number of columns of λ (i.e., λ_1). Furthermore, if λ is a rectangle or fat hook, then the same is true of λ^*.

THEOREM 3.1. *If $\ell(\mu),\ell(\nu)<n$, then $s_\mu s_\nu$ is multiplicity-free in Λ_n if and only if*

(i) *μ or ν or μ^* or ν^* is a one-line rectangle, or*
(ii) *μ or μ^* is a two-line rectangle and ν is a fat hook (or vice-versa), or*
(iii) *μ is a rectangle and ν or ν^* is a near-rectangle (or vice-versa), or*
(iv) *μ and ν are rectangles.*

LEMMA 3.2. *If $n \geqslant 3$ and $\mu = (21^{n-2})$, then s_μ^2 is not multiplicity-free in Λ_n.*

Proof. As in Lemma 2.4, it suffices to exhibit a pair of LR fillings of content μ and some common shape of the form λ/μ with at most n rows. For simplicity, we illustrate this only for the case $n = 5$:

$$\begin{array}{ccc} * & * & 1 \\ * & 2 & \\ * & 3 & \\ * & 4 & \\ 1 & & \end{array} \qquad \begin{array}{ccc} * & * & 1 \\ * & 1 & \\ * & 2 & \\ * & 3 & \\ 4 & & \end{array} \; . \quad \square$$

Proof of Theorem 3.1. Duality and Theorem 2.1 show that each of the products listed is multiplicity-free in Λ_n, so it suffices merely to show that there are no others.

First we argue that if μ and ν are not rectangles, then there is a partition λ such that $c(\lambda; \mu, \nu) \geqslant 2$ and $\ell(\lambda) \leqslant n$. By repeated application of (2.1) one sees that it suffices to prove this assertion when μ and ν both have exactly two columns. Of course to be non-rectangular, the columns must also have unequal lengths.

Now proceed by induction on n. If both $\ell(\mu), \ell(\nu) < n - 1$, then replace n by $n - 1$ and continue the induction; otherwise, $\ell(\mu) = n - 1$ or $\ell(\nu) = n - 1$. If (say) the latter occurs, but $\ell(\mu) < n - 1$, then ν must have at least two repeated parts.[1] Letting ν^- denote the partition obtained from ν by deleting one of the repeated parts, say k ($k = 1$ or 2), it follows by induction and (2.2) that there is a partition λ with $\ell(\lambda) \leqslant n - 1$ and

$$c(\lambda \cup (k); \mu, \nu) \geqslant c(\lambda; \mu, \nu^-) \geqslant 2. \tag{3.1}$$

Thus we may assume $\ell(\mu) = \ell(\nu) = n - 1$. Replacing (μ, ν) with (μ^*, ν^*) if necessary, we may also assume $\ell(\mu^*) = \ell(\nu^*) = n - 1$; this amounts to having 2 occur without multiplicity in both μ and ν, whence $\mu = \nu = (21^{n-2})$ and $n \geqslant 3$. However in that case, Lemma 3.2 proves the existence of a suitable λ, so the induction is complete.

Henceforth, we may assume that μ is a rectangle; say $\mu = (a^l)$; note that μ^* is the rectangle (a^{n-l}). We may further assume that $a, l, n - l \geqslant 2$; otherwise, (i) applies.

We claim that if ν (and therefore also ν^*) has three or more distinct column lengths, then $s_\mu s_\nu$ cannot be multiplicity-free in Λ_n. As in the previous argument, repeated application of (2.1) shows that it suffices to prove this assertion when $a = 2$ and ν consists of exactly three columns, all having different lengths. If ν has any pair of rows of the same length, possibly including rows of length 0 (i.e., $\ell(\nu) < n - 1$), then $n \geqslant 5$ and therefore $l \geqslant 3$ or $n - l \geqslant 3$. Replacing $(\mu, \nu) \to (\mu^*, \nu^*)$ if necessary, we may assume $n - l \geqslant 3$; i.e.,

[1] As a non-rectangle, μ has at least two rows, so $n \geqslant 4$. On the other hand, if ν were a non-rectangle with two columns and no repeated parts, then $n - 1 = \ell(\nu) = 2$.

$\ell(\mu) < n-2$. Proceeding by induction with respect to n, let ν^- denote the partition obtained by deleting some repeated part k from ν (possibly $k=0$). Since the dual of μ relative to Λ_{n-1} has more than one row (recall that $\ell(\mu) < n-2$), the induction hypothesis and (2.2) imply the existence of a partition λ of length at most $n-1$ as in (3.1). Since $\ell(\lambda \cup (k)) \leqslant n$, this shows that $s_\mu s_\nu$ cannot be multiplicity-free in Λ_n. The remaining possibility is that ν has three distinct column lengths, and no repeated row lengths (including 0). In that case, $\nu = (321)$, $n=4$, $\mu = (22)$, and the calculation used to prove Lemma 2.4(b) shows that $s_\mu s_\nu$ is not multiplicity-free in Λ_4.

Henceforth, we may assume that $\nu = (b^r c^s)$ is a fat hook and $a, l, n-l \geqslant 3$; otherwise (ii) or (iv) applies. We claim that if each of $b-c, c, r, s$, and $n-r-s$ are at least 2, then $s_\mu s_\nu$ cannot be multiplicity-free in Λ_n. As in the previous cases, repeated application of (2.1) shows that it suffices to prove this assertion when $a=3$, $b=4$, and $c=2$. Regarding 0 as a part of ν having multiplicity $n-r-s$, note that if any part of ν is repeated more than twice, then $n \geqslant 7$ and $l \geqslant 4$ or $n-l \geqslant 4$. Replacing $(\mu, \nu) \to (\mu^*, \nu^*)$ if necessary, we may assume $n-l \geqslant 4$; i.e., $\ell(\mu) < n-3$. The latter condition guarantees that μ satisfies the induction hypothesis when we pass from Λ_n to Λ_{n-1}. Letting ν^- denote the partition obtained by deleting some part k repeated more than twice from ν ($k = 4, 2$ or 0), we obtain by induction and (2.2) the existence of a partition λ of length at most $n-1$ satisfying (3.1), whence $s_\mu s_\nu$ cannot be multiplicity-free in Λ_n. The remaining possibility is that each part of ν occurs exactly twice; i.e., $\nu = (4422)$, $n=6$, and $\mu = (333)$. In that case, the calculation in Lemma 2.4(c) shows that $s_\mu s_\nu$ is not multiplicity-free in Λ_6.

The above argument eliminates all fat hooks ν except for those for which $b-c, c, r, s$ or $n-r-s = 1$. In the first four of these cases, ν is a near-rectangle; in the fifth case, ν^* is a near-rectangle. Either way, (iii) applies. $\square$

4. Final remarks

It would be interesting to generalize Theorem 3.1 to cover products of Weyl characters, or equivalently, tensor products of irreducible characters of semisimple complex Lie groups. There is an obvious analogue of (2.1) in this setting; namely,

$$c(\lambda + \omega; \mu + \omega, \nu) \geqslant c(\lambda; \mu, \nu), \tag{4.1}$$

where $\lambda, \omega, \mu, \nu$ range over dominant integral weights, and $c(\cdot\,;\cdot\,,\cdot)$ denotes tensor product multiplicity. (Without loss of generality, one may take ω to be a fundamental weight.) This inequality follows easily from Littelmann's path model [**L**]. Alternatively, A. Zelevinsky has pointed out to us that the inequality is also an immediate consequence of the PRV

Theorem (see Theorem 2.1 of [**PRV**]), which says

$$c(\lambda;\mu,\nu) = \dim\bigl(\{v \in V_\nu(\lambda-\mu) : e_i^{\mu(h_i)+1}(v) = 0,\ i = 1,2,\dots\}\bigr),$$

where V_ν is an irreducible representation of highest weight ν, $V_\nu(\gamma)$ is its γ-weight space, and $e_1, e_2, \dots, h_1, h_2, \dots$ denote standard generators for a corresponding Borel subalgebra of the Lie algebra in question.

On the other hand, we know of no analogue for (2.2), even in the other classical cases. Having both (2.1) and (2.2) available greatly reduced the number of cases that we needed to check in type A. In any case, for a fixed Lie group G of rank n, (4.1) implies that the set of non-multiplicity-free products forms an order filter in a product of $2n$ chains, and a simple combinatorial argument shows that it must be finitely generated.

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