Multiplicity-Free Products and Restrictions of Weyl Characters

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Contents

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0. Introduction

A module V is *multiplicity-free* if every irreducible submodule of V occurs with multiplicity one. Assuming complete reducibility, this means that the irreducible decomposition of V is canonical and that the centralizer of the action on V is commutative.

In this paper, we classify all multiplicity-free tensor products of finite-dimensional irreducible representations of every complex semisimple Lie algebra $\mathfrak{g}$ (or equivalently, all connected semisimple complex Lie groups G).[1] This completes the work started in [**St2**], where the case $\mathfrak{g} = sl(n)$ is treated. As a byproduct of the classification, we are also able to classify all irreducible representations of $\mathfrak{g}$ that are multiplicity-free when restricted to a reductive subalgebra of $\mathfrak{g}$ of "parabolic type."[2]

Thanks to the Weyl character formula, the task of identifying multiplicity-free representations may be viewed as a combinatorial problem, and indeed, we use a wide variety of exclusively combinatorial and computational tools to carry out the classification.

One interesting feature of the classification is that if $\mathfrak{g}$ is simple, a tensor product of irreducible $\mathfrak{g}$-modules, say $U \otimes V$, cannot be multiplicity-free unless the highest weight of U or V is a multiple of a fundamental weight. For restrictions to a reductive subalgebra $\mathfrak{g}'$ of parabolic type, this implies that if the highest weight of V is not a multiple of a fundamental weight, then V cannot be multiplicity-free as a $\mathfrak{g}'$-module unless $\mathfrak{g}'$ is maximal. We know of no short or conceptual explanation for either of these facts, although the Adjoint Rule (see Proposition 2.14) easily shows that $V \otimes V^*$ cannot be multiplicity-free unless the highest weight of V is a multiple of a fundamental weight.

A. Related Classifications.

Littelmann [**L2**] has classified all pairs of maximal parabolic subgroups $P, Q \subset G$ such that $G/P \times G/Q$ is spherical (i.e., has a dense orbit relative to the action of a Borel subgroup B). This in turn is equivalent to a classification of all pairs of fundamental weights ω, ω' such that $V_{m\omega} \otimes V_{m'\omega'}$ is multiplicity-free for all $m, m' \geqslant 0$, where V_λ denotes the irreducible $\mathfrak{g}$-module of highest weight λ.

In a few cases, we could have shortened the proof of our classification theorem by quoting some of the entries in [**L2**]; however, in many other cases, these tensor products are subsumed inside larger families of multiplicity-free products. In fact, the classification of multiplicity-free tensor products enables us conversely to classify all (not necessarily maximal) parabolic subgroups $P, Q \subset G$ such that $G/P \times G/Q$ is spherical. In addition to the maximal pairs, there are several non-maximal examples in $G = SL(n)$ and $G = \mathrm{Spin}(2n)$, and two in E_6. In all cases (assuming G is simple), one of the two parabolic subgroups is necessarily maximal; this reflects the previously mentioned fact that $V_\mu \otimes V_\nu$ cannot be multiplicity-free unless μ or ν is a multiple of a fundamental weight.

More recently, for $G = SL(n)$ (see [**MWZ1**]) and $G = Sp(2n)$ (see [**MWZ2**]), Magyar, Weyman and Zelevinsky have classified all parabolic subgroups $P_1, \dots, P_k \subset G$ such that the variety $G/P_1 \times \cdots \times G/P_k$ has finitely many G-orbits. In the special case $P_1 = B$, this is equivalent to $G/P_2 \times \cdots \times G/P_k$ being spherical, so their results include (for $G = SL(n)$ and $G = Sp(2n)$) the classification of all spherical varieties of the form $G/P \times G/Q$.

[1]In the exceptional cases, King and Wybourne have recently obtained the same classification [**KW**].

[2]By this we mean any proper subalgebra spanned by a Cartan subalgebra of $\mathfrak{g}$ and the root spaces of $\mathfrak{g}$ corresponding to some parabolic root subsystem.

We remark that since a Cartan subalgebra of $\mathfrak{g}$ is of parabolic type, another special case of our main result is the classification of irreducible $\mathfrak{g}$-modules with multiplicity-free weight spaces, a result previously obtained by Howe (see Theorem 4.6.3 of [**Ho**]). Indeed, one obtains that if $\mathfrak{g}$ is simple, then an irreducible $\mathfrak{g}$-module V_λ has one-dimensional weight spaces if and only if

(1) λ is minuscule,
(2) λ is quasi-minuscule and $\mathfrak{g}$ has only one short simple root,
(3) $\mathfrak{g} = sp(6)$ and $\lambda = \omega_1$ (following the numbering of Section 1), or
(4) $\mathfrak{g} = sl(n+1)$ and $\lambda = m\omega_1$ or $\lambda = m\omega_n$.

It is possible to give a relatively short proof of this result that bypasses our classification but uses the same methods. Another approach is to use the classification of one-dimensional weight spaces of $\mathfrak{g}$-modules due to Berenstein and Zelevinsky [**BZ**].

B. Notation.

Let n be a positive integer, and set $[n] := \{1, 2, \ldots, n\}$.

Throughout this paper, Φ shall denote a finite crystallographic root system spanning some real Euclidean space $\mathbf{E}$ with orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ and inner product $\langle \cdot, \cdot \rangle$. We let $\alpha_1, \ldots, \alpha_n \in \Phi$ denote a choice of simple roots and Φ^+ the corresponding positive roots. For $J \subset [n]$, the parabolic root subsystem generated by $\{\alpha_i : i \in J\}$ is denoted Φ_J.

The co-root corresponding to $\beta \in \Phi$ is $\beta^\vee = 2\beta/\langle \beta, \beta \rangle$.

We let $\omega_1, \ldots, \omega_n \in \mathbf{E}$ denote the fundamental weights (i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$), and

$$\Lambda = \{\lambda \in \mathbf{E} : \beta \in \Phi \Rightarrow \langle \lambda, \beta^\vee \rangle \in \mathbf{Z}\} = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n$$

the lattice of (integral) weights. A weight λ is *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$ for $1 \leqslant i \leqslant n$, and Λ^+ denotes the semigroup of dominant weights. We partially order Λ by the rule

$$\mu \preccurlyeq \nu \quad \text{if } \nu - \mu \in \Lambda^+.$$

This is *not* the usual partial ordering of the weight lattice; however, tensor product multiplicities are monotone relative to this order (see Proposition 2.9), so it is the correct ordering for our purposes.

The Weyl group W is the (finite) group of isometries of $\mathbf{E}$ generated by reflections $s_1, \ldots, s_n$ through the hyperplanes orthogonal to $\alpha_1, \ldots, \alpha_n$. For $J \subset [n]$, the subgroup generated by $\{s_i : i \in J\}$ is denoted W_J. The length of $w \in W$, denoted $\ell(w)$, is the minimum number of terms needed to express w as a product of the generators s_i, and the sign of w, denoted $\mathrm{sgn}(w)$, is $\det(w) = (-1)^{\ell(w)}$. There is a unique element w_0 of maximum length.

Let $\rho := \sum_{\alpha \in \Phi^+} \alpha/2$. One knows that $\rho = \omega_1 + \cdots + \omega_n$.

We now introduce a set of formal exponentials $\{e^\lambda : \lambda \in \Lambda\}$ satisfying $e^\mu e^\nu = e^{\mu+\nu}$, so that $\mathbf{Z}[e^\lambda : \lambda \in \Lambda]$ may be identified with the group ring $\mathbf{Z}[\Lambda]$. Since W permutes Φ (and hence Λ), this extends naturally to a W-action on $\mathbf{Z}[\Lambda]$. The alternating sums

$$\Delta(\lambda) := \sum_{w \in W} \mathrm{sgn}(w) e^{w\lambda} \qquad (\lambda \in \Lambda^+)$$

form a free $\mathbf{Z}$-basis for the skew-invariant part of $\mathbf{Z}[\Lambda]$, and the Weyl characters associated with Φ may be defined by setting

$$\chi(\lambda) := \Delta(\lambda+\rho)/\Delta(\rho) \qquad (\lambda \in \Lambda^+).$$

These form a free $\mathbf{Z}$-basis for the W-invariant subring $\mathbf{Z}[\Lambda]^W$.

The Weyl character $\chi(\lambda)$ is the (formal) character of the irreducible $\mathfrak{g}$-module V_λ. In particular, the coefficient $K_{\lambda,\mu}$ in the expansion

$$\chi(\lambda) = \sum_{\mu\in\Lambda} K_{\lambda,\mu} e^\mu$$

is the dimension of the μ-weight space of V_λ, and the coefficient $c(\lambda;\mu,\nu)$ in the expansion

$$\chi(\mu)\chi(\nu) = \sum_{\lambda\in\Lambda^+} c(\lambda;\mu,\nu)\chi(\lambda)$$

is the multiplicity of V_λ in $V_\mu \otimes V_\nu$.

It is important to note that the definition of $\chi(\lambda)$ makes sense even if λ is not dominant. In this case, $\chi(\lambda) = \mathrm{sgn}(w)\chi(\mu)$ if $w(\mu+\rho) = \lambda+\rho$. In particular, $\chi(\lambda) = 0$ if $\lambda+\rho$ has a nontrivial W-stabilizer.

C. Organization.

The general classification problem easily reduces to the simple case, so we restrict our attention to the irreducible root systems. In Section 1, we list the results—all pairs of Weyl characters with a multiplicity-free product, as well as all multiplicity-free restrictions of Weyl characters to reductive subalgebras of parabolic type. The fact that the latter is a corollary of the former is explained in Section 2B (see Corollary 2.5). We also list all pairs $I, J \subset [n]$ indexing parabolic subgroups P_I and P_J of G such that $G/P_I \times G/P_J$ is spherical; this amounts to identifying all sums of fundamental weights ω and ω' such that $\chi(m\omega)\chi(m\omega')$ is multiplicity-free for all sufficiently large m.

In Section 2, we assemble a large collection of tools for analyzing tensor product multiplicities. These tools are uniformly valid for all root systems, and are applied on a case-by-case basis in the subsequent sections to $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$, and the exceptional root systems. (For $\mathcal{A}_n$, there is a separate proof in [**St2**].)

The overall structure of the proof for each root system Φ is the same. The pairs of dominant weights that index multiplicity-free products form an order ideal of $\Lambda^+ \times \Lambda^+$ relative to $\preccurlyeq$ (Corollary 2.10), so once the proposed list of products has been proved multiplicity-free, we provide a list that includes all minimal elements in the complementary subset of $\Lambda^+ \times \Lambda^+$, and show that the corresponding products are not multiplicity-free. Since every order filter of $(\mathbf{Z}^{\geqslant 0})^{2n}$ is finitely generated, it follows that only a finite number of products needs to be shown not to be multiplicity-free.

In nearly every case, each proof that a product is (or is not) multiplicity-free is an application of one of the tools from Section 2, although this is not to say that the applications are uniformly easy. In some instances involving the exceptional root systems, we rely on machine computations. In the classical cases, in addition to the tools from Section 2, we also use some special rules for multiplication by the characters of the fundamental representations (see Propositions 4.1, 5.1 and 6.1).

1. Statement of Results

For each irreducible root system Φ, we list below all multiplicity-free products $\chi(\mu)\chi(\nu)$, followed by lists of all pairs J, μ such that the Φ_J-restriction[3] of $\chi(\mu)$ is multiplicity-free, and all pairs I, J such that $G/P_I \times G/P_J$ is spherical. In each case, a labeled Dynkin diagram is provided that indicates how the fundamental weights have been numbered.

The case $\Phi = \mathcal{A}_n$. Diagram labeling: $1{-}2{-}\cdots{-}n$.

THEOREM 1.1.A [**St2**]. *For $\Phi = \mathcal{A}_n$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j, k \leqslant n$, we have*

(i) $\mu = 0$ *or* $\mu = \omega_i$,
(ii) $\mu = m\omega_1$ *or* $\mu = m\omega_n$,
(iii) $\mu = 2\omega_i$ *and* $\nu \preccurlyeq m\omega_j + m\omega_k$,
(iv) $\mu \preccurlyeq m\omega_2$ *or* $\mu \preccurlyeq m\omega_{n-1}$ *and* $\nu \preccurlyeq m\omega_j + m\omega_k$,
(v) $\mu \preccurlyeq m\omega_i$ *and* $\nu \preccurlyeq \omega_j + m\omega_k$,
(vi) $\mu \preccurlyeq m\omega_i$, $\nu \preccurlyeq m\omega_j + m\omega_k$, *and* $k \in \{1, j+1, n\}$,

or the same with μ and ν interchanged.

COROLLARY 1.2.A. *For $\Phi = \mathcal{A}_n$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j \leqslant n$, we have*

(i) $\mu = 0$, $\mu = \omega_i$, $\mu = m\omega_1$ *or* $\mu = m\omega_n$,
(ii) $J^c = \{1\}$ *or* $J^c = \{n\}$,
(iii) $|J^c| = 2$ *and* $\mu = 2\omega_i$, $\mu = m\omega_2$, *or* $\mu = m\omega_{n-1}$,
(iv) $J^c = \{2\}$ *or* $J^c = \{n-1\}$ *and* $\mu \preccurlyeq m\omega_i + m\omega_j$,
(v) $|J^c| = 1$ *and* $\mu \preccurlyeq \omega_i + m\omega_j$,
(vi) $|J^c| = 1$, $\mu \preccurlyeq m\omega_i + m\omega_j$, *and* $i \in \{1, j+1, n\}$, *or*
(vii) $J^c = \{1, j\}$, $J^c = \{j, j+1\}$, *or* $J^c = \{j, n\}$ *and* $\mu = m\omega_i$.

COROLLARY 1.3.A [**MWZ1**]. *For $G = SL(n+1)$ and all (proper) $I, J \subset [n]$, the variety $G/P_I \times G/P_J$ is spherical if and only if up to interchanging I and J, we have*

(i) $I^c = \{1\}$ *or* $\{n\}$,
(ii) $I^c = \{2\}$ *or* $\{n-1\}$ *and* $|J^c| = 2$,
(iii) $|I^c| = |J^c| = 1$, *or*
(iv) $|I^c| = 1$ *and* $J^c = \{1, j\}, \{j, j+1\}$ *or* $\{j, n\}$ $(1 < j < n)$.

The case $\Phi = \mathcal{B}_n$. Diagram labeling: $1{\Leftarrow}2{-}3{-}\cdots{-}n$.

THEOREM 1.1.B. *For $\Phi = \mathcal{B}_n$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j \leqslant n$, we have*

(i) $\mu = 0$, $\mu = \omega_1$, *or* $\mu = \omega_n$,
(ii) $\mu = \omega_i$ *and* $\nu = m\omega_j$,
(iii) $\mu = 2\omega_1$ *and* $\nu = m\omega_j$,

[3] See Section 2B for an explanation of this terminology

(iv) $\mu \preccurlyeq m\omega_1$ and $\nu \preccurlyeq m\omega_1$,
(v) $\mu \preccurlyeq m\omega_n$ and $\nu \preccurlyeq \omega_1 + m\omega_i$,

or the same with μ and ν interchanged.

COROLLARY 1.2.B. *For $\Phi = \mathcal{B}_n$, a proper Φ_J-restriction $\chi(\mu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i \leqslant n$, we have*

(i) $\mu = 0$, $\mu = \omega_1$, or $\mu = \omega_n$,
(ii) $|J^c| = 1$ and $\mu = \omega_i$, $\mu = 2\omega_1$, or $\mu = m\omega_n$,
(iii) $J^c = \{1\}$ and $\mu = m\omega_1$, or
(iv) $J^c = \{n\}$ and $\mu \preccurlyeq \omega_1 + m\omega_i$.

COROLLARY 1.3.B. *For $G = Spin(2n+1)$ and all (proper) $I, J \subset [n]$, the variety $G/P_I \times G/P_J$ is spherical if and only if up to interchanging I and J, we have*

(i) $I^c = \{n\}$ and $|J^c| = 1$, or
(ii) $I^c = J^c = \{1\}$.

The case $\Phi = \mathcal{C}_n$. Diagram labeling: $1 \Rightarrow 2 - 3 - \cdots - n$.

THEOREM 1.1.C. *For $\Phi = \mathcal{C}_n$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j \leqslant n$, we have*

(i) $\mu = 0$ or $\mu = \omega_n$,
(ii) $\mu \preccurlyeq m\omega_n$ and $\nu \preccurlyeq m\omega_j$,
(iii) $\mu = \omega_i$ and $\nu \preccurlyeq m\omega_1 + m\omega_j$,
(iv) $\mu = 2\omega_1$ and $\nu = m\omega_j$,
(v) $\mu = \omega_i + \omega_j$ and $\nu = m\omega_1$,
(vi) $\mu = 3\omega_i$ and $\nu = m\omega_1$,
(vii) $\mu = \omega_1$ and $\nu \preccurlyeq m\omega_1 + m\omega_i + m\omega_j$,
(viii) $\mu \preccurlyeq m\omega_1$ and $\nu \preccurlyeq m\omega_1 + \omega_j$,

or the same with μ and ν interchanged.

COROLLARY 1.2.C. *For $\Phi = \mathcal{C}_n$, a proper Φ_J-restriction $\chi(\mu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j \leqslant n$, we have*

(i) $\mu = 0$ or $\mu = \omega_n$,
(ii) $|J^c| = 1$ and $\mu = \omega_i$, $\mu = 2\omega_1$, or $\mu = m\omega_n$,
(iii) $J^c = \{1\}$ and $\mu = \omega_i + \omega_j$, $\mu = 3\omega_i$, or $\mu \preccurlyeq m\omega_1 + \omega_j$,
(iv) $J^c = \{n\}$ and $\mu = m\omega_i$,
(v) $J^c = \{1, j\}$ and $\mu = \omega_i$, or
(vi) $J^c \subseteq \{1, i, j\}$ and $\mu = \omega_1$.

COROLLARY 1.3.C [**MWZ2**]. *For $G = Sp(2n)$ and all (proper) $I, J \subset [n]$, the variety $G/P_I \times G/P_J$ is spherical if and only if up to interchanging I and J, we have*

(i) $I^c = \{n\}$ and $|J^c| = 1$, or
(ii) $I^c = J^c = \{1\}$.

The case $\Phi = \mathcal{D}_n$. Diagram labeling: $\begin{array}{c} \quad 2 \\ \quad | \\ 1 - 3 - 4 - \cdots - n. \end{array}$

THEOREM 1.1.D. *For $\Phi = \mathcal{D}_n$ ($n \geqslant 4$), the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j, k \leqslant n$, we have*

(i) $\mu \in \{0, \omega_1, \omega_2, \omega_n\}$,
(ii) $\mu = \omega_i$, $\nu = m\omega_j$, *and* $i + j \geqslant n + 2$,
(iii) $\mu = 2\omega_1$ *or* $\mu = 2\omega_2$ *and* $\nu = m\omega_j$,
(iv) $\mu = \omega_i$ *and* $\nu = m\omega_1$ *or* $\nu = m\omega_2$,
(v) $\mu \preccurlyeq m\omega_n$ *and* $\nu \preccurlyeq m\omega_1 + m\omega_i$ *or* $\nu \preccurlyeq m\omega_2 + m\omega_i$,
(vi) $\mu \preccurlyeq m\omega_1$ *or* $\mu \preccurlyeq m\omega_2$ *and* $\nu \preccurlyeq m\omega_{n-2}$ *or* $\nu \preccurlyeq m\omega_{n-1} + m\omega_n$,
(vii) $\mu \preccurlyeq m\omega_i$ *and* $\nu \preccurlyeq m\omega_j + m\omega_n$ *or* $\nu \preccurlyeq m\omega_1 + m\omega_2$ $(i, j = 1, 2)$,
(viii) ($n = 4$ *only*) $\mu \preccurlyeq m\omega_i$ *and* $\nu \preccurlyeq m\omega_{3-i} + m\omega_3$ $(i = 1, 2)$,

or the same with μ and ν interchanged.

COROLLARY 1.2.D. *For $\Phi = \mathcal{D}_n$ ($n \geqslant 4$), a proper Φ_J-restriction $\chi(\mu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i, j, k \leqslant n$, we have*

(i) $\mu \in \{0, \omega_1, \omega_2, \omega_n\}$,
(ii) $J^c = \{i\}$, $\mu = \omega_j$, *and* $i + j \geqslant n + 2$,
(iii) $|J^c| = 1$ *and* $\mu = 2\omega_1$, $\mu = 2\omega_2$, *or* $\mu = m\omega_n$,
(iv) $J^c = \{i\}$, $\mu \preccurlyeq m\omega_j + m\omega_k$, *and* $i, j, k \in \{1, 2, n\}$,
(v) $J^c = \{i, j\}$, $\mu = m\omega_k$, *and* $i, j, k \in \{1, 2, n\}$,
(vi) $J^c = \{1, i\}$ *or* $J^c = \{2, i\}$ *and* $\mu = m\omega_n$,
(vii) $J^c = \{1\}$ *or* $J^c = \{2\}$ *and* $\mu = \omega_i$, $\mu = m\omega_{n-2}$, *or* $\mu \preccurlyeq m\omega_{n-1} + m\omega_n$,
(viii) $J^c = \{n\}$ *and* $\mu \preccurlyeq m\omega_1 + m\omega_i$ *or* $\mu \preccurlyeq m\omega_2 + m\omega_i$,
(ix) $J^c \subseteq \{n - 1, n\}$ *or* $J^c = \{n - 2\}$ *and* $\mu = m\omega_1$ *or* $\mu = m\omega_2$,
(x) ($n = 4$ *only*) $J^c = \{i\}$ *and* $\mu \preccurlyeq m\omega_{3-i} + m\omega_3$ $(i = 1, 2)$, *or*
(xi) ($n = 4$ *only*) $J^c = \{i, 3\}$ *and* $\mu = m\omega_{3-i}$ $(i = 1, 2)$.

COROLLARY 1.3.D. *For $G = Spin(2n)$ ($n \geqslant 4$) and all (proper) subsets $I, J \subset [n]$, the variety $G/P_I \times G/P_J$ is spherical if and only if up to interchanging I and J, we have*

(i) $I^c = \{n\}$ *and* $J^c = \{i\}, \{1, i\}$, *or* $\{2, i\}$ $(1 \leqslant i \leqslant n)$,
(ii) $I^c = \{1\}$ *or* $I^c = \{2\}$ *and* $J^c \subsetneq \{1, 2, n\}$, $J^c \subseteq \{n - 1, n\}$, *or* $J^c = \{n - 2\}$, *or*
(iii) ($n = 4$ *only*) $I^c = \{1\}$ *and* $J^c = \{2, 3\}$ *or* $I^c = \{2\}$ *and* $J^c = \{1, 3\}$.

The case $\Phi = \mathcal{E}_n$ ($n = 6, 7, 8$). Diagram labeling: $\begin{array}{c} 2 \\ | \\ 1-3-4-5-\cdots-n \end{array}$.

THEOREM 1.1.E6. *For $\Phi = \mathcal{E}_6$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i \leqslant 6$, we have*

(i) $\mu = 0$, $\mu = \omega_1$, *or* $\mu = \omega_6$,
(ii) $\mu = \omega_2$ *and* $\nu = m\omega_i$,
(iii) $\mu = \omega_3$ *or* $\mu = \omega_5$ *and* $\nu = m\omega_2$,
(iv) $\mu \preccurlyeq m\omega_1$ *or* $\mu \preccurlyeq m\omega_6$ *and* $\nu \preccurlyeq m\omega_i$ $(i \neq 4)$,
(v) $\mu \preccurlyeq m\omega_1$ *or* $\mu \preccurlyeq m\omega_6$ *and* $\nu \preccurlyeq m\omega_1 + m\omega_6$,

or the same with μ and ν interchanged.

Corollary 1.2.E6. *For $\Phi = \mathcal{E}_6$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if for some integer $m > 0$, we have*

(i) $\mu = 0$, $\mu = \omega_1$, or $\mu = \omega_6$,
(ii) $|J^c| = 1$ and $\mu = \omega_2$,
(iii) $J^c = \{2\}$ and $\mu \in \{\omega_3, \omega_5, m\omega_1, m\omega_6\}$,
(iv) $J^c = \{3\}$ or $J^c = \{5\}$ and $\mu = m\omega_1$ or $\mu = m\omega_6$,
(v) $J^c = \{1\}$ or $J^c = \{6\}$ and $\mu \in \{m\omega_2, m\omega_3, m\omega_5\}$ or $\mu \preccurlyeq m\omega_1 + m\omega_6$, or
(vi) $J^c = \{1, 6\}$ and $\mu = m\omega_1$ or $\mu = m\omega_6$.

Corollary 1.3.E6. *For $G = E_6$ and all (proper) $I, J \subset [6]$, the variety $G/P_I \times G/P_J$ is spherical if and only if $I^c = \{1\}$ or $\{6\}$ and $J^c = \{i\}$ $(i \neq 4)$ or $\{1, 6\}$ (or vice-versa).*

Theorem 1.1.E7. *For $\Phi = \mathcal{E}_7$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integers $m > 0$ and $1 \leqslant i \leqslant 7$, we have*

(i) $\mu = 0$ or $\mu = \omega_7$,
(ii) $\mu = \omega_1$ and $\nu = m\omega_i$,
(iii) $\mu = \omega_2$ and $\nu = m\omega_1$ or $\nu = m\omega_2$,
(iv) $\mu = \omega_6$ and $\nu = m\omega_1$ or $\nu = m\omega_7$,
(v) $\mu \preccurlyeq 2\omega_7$ and $\nu = m\omega_6$,
(vi) $\mu \preccurlyeq m\omega_7$ and $\nu \preccurlyeq m\omega_i$ $(i = 1, 2, 7)$,

or the same with μ and ν interchanged.

Corollary 1.2.E7. *For $\Phi = \mathcal{E}_7$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if for some integer $m > 0$, we have*

(i) $\mu = 0$ or $\mu = \omega_7$,
(ii) $|J^c| = 1$ and $\mu = \omega_1$,
(iii) $J^c = \{1\}$ and $\mu = \omega_2$, $\mu = \omega_6$, or $\mu = m\omega_7$
(iv) $J^c = \{2\}$ and $\mu = \omega_2$ or $\mu = m\omega_7$,
(v) $J^c = \{6\}$ and $\mu = \omega_7$ or $\mu = 2\omega_7$, or
(vi) $J^c = \{7\}$ and $\mu \in \{\omega_6, m\omega_1, m\omega_2, m\omega_7\}$.

Corollary 1.3.E7. *For $G = E_7$ and all (proper) $I, J \subset [7]$, the variety $G/P_I \times G/P_J$ is spherical if and only if $I^c = \{7\}$ and $J^c = \{1\}, \{2\}$ or $\{7\}$ (or vice-versa).*

Theorem 1.1.E8. *For $\Phi = \mathcal{E}_8$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integer $m > 0$, we have $\mu = 0$, or*

(i) $\mu = \omega_8$ and $\nu = m\omega_i$ $(1 \leqslant i \leqslant 8)$,
(ii) $\mu = \omega_1$ and $\nu = m\omega_1$ or $\nu = m\omega_8$,

or the same with μ and ν interchanged.

Corollary 1.2.E8. *For $\Phi = \mathcal{E}_8$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if $\mu = 0$, or*

(i) $|J^c| = 1$ and $\mu = \omega_8$, or
(ii) $J^c = \{1\}$ or $J^c = \{8\}$ and $\mu = \omega_1$.

The case $\Phi = \mathcal{F}_4$. Diagram labeling: 1−2⇐3−4.

THEOREM 1.1.F4. *For $\Phi = \mathcal{F}_4$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integer $m > 0$, we have $\mu = 0$, or*

(i) $\mu = \omega_1$ *and* $\nu \preccurlyeq m\omega_1 + m\omega_3 + m\omega_4$ *or* $\nu \preccurlyeq m\omega_2 + m\omega_3 + m\omega_4$,
(ii) $\mu = \omega_4$ *and* $\nu = m\omega_i$ $(1 \leqslant i \leqslant 4)$,
(iii) $\mu = 2\omega_1$ *or* $\mu = \omega_2$ *and* $\nu = m\omega_4$,

or the same with μ and ν interchanged.

COROLLARY 1.2.F4. *For $\Phi = \mathcal{F}_4$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if $\mu = 0$, or*

(i) $1 \in J$ *or* $2 \in J$ *and* $\mu = \omega_1$,
(ii) $|J^c| = 1$ *and* $\mu = \omega_4$, *or*
(iii) $J^c = \{4\}$ *and* $\mu = 2\omega_1$ *or* $\mu = \omega_2$.

The case $\Phi = \mathcal{G}_2$. Diagram labeling: 1⇚2.

THEOREM 1.1.G2. *For $\Phi = \mathcal{G}_2$, the product $\chi(\mu)\chi(\nu)$ is multiplicity-free if and only if for some integer $m > 0$, we have*

(i) $\mu = 0$ *or* $\mu = \omega_1$,
(ii) $\mu = \omega_2$ *and* $\nu = m\omega_1$ *or* $\nu = m\omega_2$,
(iii) $\mu = 2\omega_1$ *and* $\nu = m\omega_2$,

or the same with μ and ν interchanged.

COROLLARY 1.2.G2. *For $\Phi = \mathcal{G}_2$, a proper Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if*

(i) $\mu = 0$ *or* $\mu = \omega_1$,
(ii) $J = \{2\}$ *and* $\mu = \omega_2$, *or*
(iii) $J = \{1\}$ *and* $\mu = 2\omega_1$ *or* $\mu = \omega_2$.

COROLLARY 1.3.{E8,F4,G2} [**L2**]. *For $G = E_8$, F_4 and G_2, there are no nontrivial spherical varieties of the form $G/P_I \times G/P_J$.*

2. Basic Tools

In this section, we collect together several important properties of tensor product multiplicity that will be needed for our classification of multiplicity-free products.

A. The Brauer-Klimyk Rule.

The following rule for multiplication of Weyl characters may be traced to papers of Brauer [**B**] and Klimyk [**Kl**].

PROPOSITION 2.1. *For all $\mu, \nu \in \Lambda^+$, we have* $\chi(\mu)\chi(\nu) = \sum_{\xi \in \Lambda} K_{\mu,\xi}\, \chi(\nu + \xi)$.

Proof. Since $K_{\mu,\xi} = K_{\mu,w\xi}$ for all $w \in W$, it follows that

$$\begin{aligned}\chi(\mu)\Delta(\nu+\rho) &= \sum_{\xi\in\Lambda,\, w\in W} \operatorname{sgn}(w) K_{\mu,\xi}\, e^{\xi + w(\nu+\rho)} \\ &= \sum_{\xi\in\Lambda,\, w\in W} \operatorname{sgn}(w) K_{\mu,\xi}\, e^{w(\nu+\xi+\rho)} = \sum_{\xi\in\Lambda} K_{\mu,\xi} \Delta(\nu+\xi+\rho). \quad \square\end{aligned}$$

We remark that the above decomposition is not sign-free, since $\nu + \xi$ need not be dominant. In order to rewrite the sum in terms of Weyl characters $\chi(\lambda)$ with λ dominant, one needs to determine, for each weight ξ, the unique dominant weight in the orbit of $\nu + \xi + \rho$, say $\lambda + \rho = w(\nu + \xi + \rho)$, and replace $\chi(\nu + \xi)$ with $\mathrm{sgn}(w)\chi(\lambda)$. (If $\lambda + \rho$ is dominant, but λ is not, then $\chi(\lambda) = 0$.) For further discussion of using the Brauer-Klimyk Rule as the basis of a tensor product multiplicity algorithm, see [**St1**].

B. Branching.

Let Φ' be any (not necessarily parabolic) root subsystem of Φ, with W' denoting the corresponding Weyl group. For each weight $\lambda \in \Lambda$, let

$$\chi'(\lambda) = \Delta'(\lambda + \rho)/\Delta'(\rho) = \Delta'(\lambda + \rho')/\Delta'(\rho'),$$

where Δ' and ρ' denote the Φ'-analogues of Δ and ρ. These are essentially Weyl characters relative to Φ', aside from the fact that λ has been chosen from the weight lattice for Φ, rather than Φ'.

Letting Λ^* denote the set of weights in Λ that are dominant relative to Φ', it is easy to show that $\{\chi'(\mu) : \mu \in \Lambda^*\}$ forms a $\mathbf{Z}$-basis for the W'-invariant part of $\mathbf{Z}[\Lambda]$. In particular, for each $\mu \in \Lambda^+$, there exist integers $M'(\mu, \lambda)$ such that

$$\chi(\mu) = \sum_{\lambda \in \Lambda^*} M'(\mu, \lambda)\chi'(\lambda).$$

We call this decomposition the *Φ'-restriction of $\chi(\mu)$*.

If $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$ for all $\alpha, \beta \in \Phi'$ (e.g., if Φ' is parabolic), then a Cartan subalgebra of $\mathfrak{g}$ and the root subspaces of $\mathfrak{g}$ corresponding to Φ' span a reductive (but not necessarily semisimple) Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, and the integers $M'(\mu, \lambda)$ are nonnegative, being the multiplicities of the irreducible $\mathfrak{g}'$-submodules of the $\mathfrak{g}$-module V_μ.

The branching multiplicities $M'(\mu, \lambda)$ may be computed from knowledge of the weight multiplicities $K_{\mu,\xi}$ by means of the following analogue of the Brauer-Klimyk Rule.

Proposition 2.2. *For all $\mu \in \Lambda^+$ and $\Phi' \subset \Phi$, we have* $\chi(\mu) = \sum_{\xi \in \Lambda} K_{\mu,\xi}\, \chi'(\xi)$.

This is not a sign-free decomposition, since ξ need not be Φ'-dominant.

Proof. Using the fact that $K_{\mu,w\xi} = K_{\mu,\xi}$ for all $w \in W'$, we find

$$\chi(\mu)\Delta'(\rho') = \sum_{\xi \in \Lambda,\, w \in W'} \mathrm{sgn}(w) K_{\mu,\xi}\, e^{\xi + w\rho'} = \sum_{\xi \in \Lambda} K_{\mu,\xi} \Delta'(\xi + \rho'). \quad \square$$

Specializing to the parabolic case, suppose $\Phi' = \Phi_J$ and $W' = W_J$ for some $J \subset [n]$, and let $M_J(\mu, \lambda)$ denote the coefficient of $\chi'(\lambda)$ in the Φ_J-restriction of $\chi(\mu)$.

Proposition 2.3. *If $\mu, \nu \in \Lambda^+$ and $J \subseteq \{j : \langle \nu, \alpha_j^\vee \rangle = 0\}$, then*

$$\chi(\mu)\chi(\nu) = \sum_{\delta \in \Lambda^*} M_J(\mu, \delta)\chi(\nu + \delta).$$

Proof. For each $\xi \in \Lambda$, there is an element $w \in W_J$ such that $w(\xi + \rho') = \delta + \rho'$ is Φ_J-dominant. In that case, $\chi'(\delta) = \operatorname{sgn}(w)\chi'(\xi)$ is nonzero only if δ is also Φ_J-dominant. Similarly, since W_J fixes ν and $\langle \rho, \alpha_j^\vee \rangle = \langle \rho', \alpha_j^\vee \rangle$ for all $j \in J$, it follows that $w(\nu+\xi+\rho) = \nu + \delta + \rho$, and $\chi(\nu + \delta) = \operatorname{sgn}(w)\chi(\nu + \xi)$ is nonzero only if δ is Φ_J-dominant. Restricting our attention to those ξ for which δ is Φ_J-dominant, we see that the corresponding term in Proposition 2.2 contributes $\operatorname{sgn}(w)K_{\mu,\xi}$ to the coefficient of $\chi'(\delta)$ in $\chi(\mu)$, whereas in Proposition 2.1, it contributes $\operatorname{sgn}(w)K_{\mu,\xi}\chi(\nu + \delta)$ to the decomposition of $\chi(\mu)\chi(\nu)$. $\square$

Remark 2.4. (a) The Brauer-Klimyk Rule is the special case $J = \varnothing$.

(b) The above decomposition is not sign-free, since $\nu+\delta$ need not be dominant. However, there are only finitely many $\delta \in \Lambda^*$ such that $M_J(\mu, \delta) \neq 0$, so if ν is sufficiently deep in the wall of the dominant chamber indexed by J, then $\nu + \delta$ will be dominant for all such δ. In that case, $M_J(\mu, \delta) = c(\nu + \delta; \mu, \nu)$; this shows in particular that branching multiplicities are also tensor product multiplicities.

Corollary 2.5. *If $\mu, \nu \in \Lambda^+$ and $J = \{j : \langle \nu, \alpha_j^\vee \rangle = 0\}$, then the Φ_J-restriction of $\chi(\mu)$ is multiplicity-free if and only if $\chi(\mu)\chi(m\nu)$ is multiplicity-free for all sufficiently large m.*

Since tensor product multiplicities are monotone (see Proposition 2.9 below), one may replace "all sufficiently large m" with "all $m \geqslant 0$" in the above result.

C. Stability.

Fix a parabolic subsystem $\Phi_J \subseteq \Phi$. The inclusion $\Phi_J \to \Phi$ induces a natural homomorphism $\Lambda \to \Lambda_J$ between the corresponding weight lattices, denoted $\lambda \mapsto \bar{\lambda}$. Under this map, the fundamental weights for Φ_J are $\{\bar{\omega}_j : j \in J\}$, and we have $\bar{\omega}_j = 0$ for $j \notin J$. We remark that Λ and Λ_J both include copies of the root lattice $\mathbf{Z}\Phi_J$, and $\lambda \mapsto \bar{\lambda}$ restricts to the identity map between these two copies.

Given that the choice of Φ_J is understood, we let $\bar{\chi}(\mu)$ denote the Weyl character (relative to Φ_J) corresponding to $\mu \in \Lambda_J^+$, and let $\bar{c}(\lambda; \mu, \nu)$ denote the multiplicity of $\bar{\chi}(\lambda)$ in $\bar{\chi}(\mu)\bar{\chi}(\nu)$ (given that $\lambda, \mu, \nu \in \Lambda_J^+$).

Proposition 2.6. *If $\lambda, \mu, \nu \in \Lambda^+$ and $\mu + \nu - \lambda \in \mathbf{Z}\Phi_J$, then $c(\lambda; \mu, \nu) = \bar{c}(\bar{\lambda}; \bar{\mu}, \bar{\nu})$.*

Proof. For $\gamma \in \mathbf{Z}\Phi$, let $P(\gamma)$ denote the number of (unordered) partitions of γ into a sum of positive roots—the coefficient of e^γ in the formal series $\prod_{\alpha>0}(1 - e^\alpha)^{-1}$. By Steinberg's Formula (e.g., [**H**, §24]), one knows that

$$c(\lambda; \mu, \nu) = \sum_{w,w' \in W} \operatorname{sgn}(w) \operatorname{sgn}(w') P((\mu + \nu - \lambda) - \sigma_\mu(w) - \sigma_\nu(w')), \tag{2.1}$$

where $\sigma_\mu(w) := (\mu + \rho) - w(\mu + \rho)$. Now consider that

$$\sigma_\mu(s_i w) - \sigma_\mu(w) = \langle w(\mu + \rho), \alpha_i^\vee \rangle \alpha_i = \langle \mu + \rho, w^{-1}\alpha_i^\vee \rangle \alpha_i.$$

Furthermore, $\ell(s_i w) > \ell(w)$ implies that $w^{-1}\alpha_i$ is a positive root (e.g., [**H**, §10]), so in this case, $\sigma_\mu(s_i w) - \sigma_\mu(w)$ is a positive (integer) multiple of α_i. Proceeding by induction with respect to length, we deduce that $\sigma_\mu(w)$ is a positive $\mathbf{Z}$-linear combination of the simple roots α_j such that s_j occurs in some (equivalently, every) reduced expression for w.

It follows that if $\mu + \nu - \lambda \in \mathbf{Z}\Phi_J$, then the term in (2.1) corresponding to the pair $w, w' \in W$ is nonzero only if $w, w' \in W_J$. Furthermore, since $\Phi_J = \Phi \cap \mathbf{Z}\Phi_J$, the partition function for Φ_J is the restriction of $P(\cdot)$ to $\mathbf{Z}\Phi_J$. Hence, the expression for $\bar{c}(\bar{\lambda}; \bar{\mu}, \bar{\nu})$ analogous to (2.1) has exactly the same nonzero terms. $\square$

COROLLARY 2.7. *If $\mu, \nu \in \Lambda^+$ and $\bar{\chi}(\bar{\mu})\bar{\chi}(\bar{\nu})$ is not multiplicity-free relative to Φ_J, then $\chi(\mu)\chi(\nu)$ is not multiplicity-free relative to Φ.*

Proof. If $\bar{\chi}(\bar{\mu})\bar{\chi}(\bar{\nu})$ is not multiplicity-free, then $\bar{c}(\delta; \bar{\mu}, \bar{\nu}) \geqslant 2$ for some weight $\delta \in \Lambda_J^+$. Setting $\gamma = \bar{\mu} + \bar{\nu} - \delta \in \mathbf{Z}^{\geqslant 0}\Phi_J^+$, it makes sense to define $\lambda := \mu + \nu - \gamma \in \Lambda$, since Λ includes Φ (and therefore $\mathbf{Z}\Phi_J$). Furthermore, λ is dominant, since $\langle \lambda, \alpha_j^\vee \rangle = \langle \delta, \alpha_j^\vee \rangle \geqslant 0$ for $j \in J$ and $\langle \gamma, \alpha_j^\vee \rangle \leqslant 0$ for $j \notin J$. Hence $c(\lambda; \mu, \nu) = \bar{c}(\delta; \bar{\mu}, \bar{\nu}) \geqslant 2$ by Proposition 2.6. $\square$

D. Triple Symmetry.

For each dominant weight λ, let $\lambda^* = -w_0\lambda$ denote the unique dominant weight in the W-orbit of $-\lambda$, so that $\chi(\lambda^*)$ is the character of the dual module V_λ^*. It is easy to check that the map $\lambda \mapsto \lambda^*$ is induced by a diagram automorphism (the Weyl involution); it is trivial if and only if $w_0 = -1$.

The following 3-fold symmetry of tensor product multiplicities is well-known.

PROPOSITION 2.8. *The multiplicity of $\chi(0)$ in $\chi(\mu)\chi(\nu)\chi(\lambda^*)$ is $c(\lambda; \mu, \nu)$. In particular, $c(\lambda; \mu, \nu)$ is a symmetric function of λ^*, μ and ν.*

Proof. By Schur's Lemma, one knows that $V_\mu \otimes V_\nu^*$ includes a copy of the trivial $\mathfrak{g}$-module if and only if $\mu = \nu$; in that case, there is exactly one such copy. Hence, the multiplicity of $\chi(\lambda)$ in any character ϕ is the multiplicity of $\chi(0)$ in $\chi(\lambda^*)\phi$. $\square$

For a proof based solely on manipulations of the Weyl Character Formula, see [**St1**, §7C].

E. Combinatorial Models, Monotonicity, and the PRV Theorem.

We define a *combinatorial model* for the Weyl character corresponding to $\mu \in \Lambda^+$ to be a triple (X, ω, δ), where $X = X(\mu)$ is a set and ω (the *weight*) and δ (the *depth*) are functions $X \to \Lambda$ with the property that for all $\lambda, \nu \in \Lambda^+$, we have

$$c(\lambda; \mu, \nu) = \big|\{x \in X(\mu) : \omega(x) = \lambda - \nu,\ \nu + \delta(x) \in \Lambda^+\}\big|. \tag{2.2}$$

It is not immediately clear from this definition, but true (we claim), that (1) combinatorial models exist for all $\mu \in \Lambda^+$, and (2) the Weyl character $\chi(\mu)$ may in fact be viewed as a generating function for $X(\mu)$ with respect to the weight ω.

Postponing the proofs of these claims temporarily, note that the condition $\nu + \delta(x) \in \Lambda^+$ in (2.2) becomes progressively weaker as ν moves deeper in the dominant chamber. This yields the following monotonicity property of tensor product multiplicities.

PROPOSITION 2.9. *For all $\lambda, \mu, \nu, \delta \in \Lambda^+$, we have $c(\lambda + \delta; \mu, \nu + \delta) \geqslant c(\lambda; \mu, \nu)$.*

Recall that $\mu \preccurlyeq \nu$ means $\nu - \mu \in \Lambda^+$.

COROLLARY 2.10. *If $\chi(\mu)\chi(\nu)$ is multiplicity-free, then the same is true for all products $\chi(\mu')\chi(\nu')$ such that $\mu' \preccurlyeq \mu$ and $\nu' \preccurlyeq \nu$, and if $\chi(\mu)\chi(\nu)$ is not multiplicity-free, then the same is true for all products $\chi(\mu')\chi(\nu')$ such that $\mu \preccurlyeq \mu'$ and $\nu \preccurlyeq \nu'$.*

We remark that if $\nu \in \Lambda^+$ and $x \in X$ satisfy $\nu + \omega(x) \notin \Lambda^+$, then there is no $\lambda \in \Lambda^+$ such that x contributes to the multiplicity in (2.2). Furthermore, if $\langle \delta(x), \alpha_i^\vee \rangle \geqslant 0$, then the condition $\langle \nu + \delta(x), \alpha_i^\vee \rangle \geqslant 0$ is vacuous. Thus the validity of (2.2) is preserved if we replace δ with a new depth function δ' satisfying

$$\langle \delta'(x), \alpha_i^\vee \rangle = \min(0,\ \langle \omega(x), \alpha_i^\vee \rangle,\ \langle \delta(x), \alpha_i^\vee \rangle) \quad (1 \leqslant i \leqslant n).$$

Equivalently, we may add the requirement that combinatorial models satisfy $\delta(x) \preccurlyeq \omega(x)$ and $-\delta(x) \in \Lambda^+$ for all $x \in X$; we say that such a model is *standard*.

Turning now to the question of existence, let us first note that Kashiwara's Crystal Bases or Littelmann's Path Model may be used to explicitly construct combinatorial models for every Weyl character (see [**K**], [**L1**], [**L3**], or [**St3**]); we will discuss one particular construction in the next subsection. To give a non-constructive existence proof, let us extend the notation $c(\lambda; \mu, \nu)$ to all $\lambda, \mu, \nu \in \Lambda$ by setting $c(\lambda; \mu, \nu) = 0$ if any of λ, μ, ν fail to be dominant. Now define

$$N_\mu(\xi, \nu) = \sum_{J \subseteq [n]} (-1)^{|J|} c(\nu + \xi - \omega_J; \mu,\ \nu - \omega_J), \tag{2.3}$$

where $\omega_J = \sum_{j \in J} \omega_j$. Note that $N_\mu(\xi, \nu) = 0$ unless $\mu, \nu, \nu + \xi \in \Lambda^+$.

A simple inclusion-exclusion argument shows that for all $\lambda, \mu, \nu \in \Lambda$, we have

$$c(\lambda; \mu, \nu) = \sum_{\delta \in \Lambda^+} N_\mu(\lambda - \nu, \nu - \delta). \tag{2.4}$$

Furthermore, if $\mu \in \Lambda^+$ and X is a standard combinatorial model for $\chi(\mu)$, then it follows from (2.3) that $N_\mu(\xi, \nu) = |\{x \in X : \omega(x) = \xi,\ \delta(x) = -\nu\}|$ for all $\xi, \nu \in \Lambda$. Conversely, if $N_\mu(\xi, \nu) \geqslant 0$ for all $\xi, \nu \in \Lambda$, then any set consisting of $N_\mu(\xi, \nu)$ elements of weight ξ and depth $-\nu$ for all $\xi, \nu \in \Lambda$ constitutes a standard combinatorial model for $\chi(\mu)$.

The inequalities $N_\mu(\xi, \nu) \geqslant 0$ $(\xi, \nu \in \Lambda)$ are thus equivalent to the existence of combinatorial models for $\chi(\mu)$. Furthermore, the non-trivial inequalities (i.e., the cases with $\nu, \nu + \xi \in \Lambda^+$) are consequences of the PRV Theorem (see Theorem 2.1 of [**PRV**]); viz.,

$$c(\lambda; \mu, \nu) = \dim V_\mu(\lambda - \nu, \nu),$$

where $V_\mu(\xi)$ denotes the ξ-weight space of V_μ,

$$V_\mu(\xi, \nu) = \left\{ v \in V_\mu(\xi) : e_i^{\langle \nu, \alpha_i^\vee \rangle + 1}(v) = 0,\ 1 \leqslant i \leqslant n \right\},$$

and $e_1, \ldots, e_n$ denote standard generators for the nilpotent part of a Borel subalgebra of $\mathfrak{g}$. Indeed, it follows from (2.3) or (2.4) that

$$N_\mu(\xi, \nu) = \dim V_\mu(\xi, \nu) / \overline{V}_\mu(\xi, \nu) \geqslant 0,$$

where $\overline{V}_\mu(\xi, \nu)$ denotes the sum of all spaces $V_\mu(\xi, \delta)$ such that $\delta \prec \nu$.

Finally, to explain the connection between combinatorial models and generating functions, note that there are only finitely many nonzero terms $K_{\mu,\xi}\,\chi(\nu+\xi)$ in the Brauer-Klimyk Rule (Proposition 2.1), and hence $\nu+\xi$ is dominant in all such cases if ν is sufficiently deep in the dominant chamber. On the other hand, if $X(\mu)$ is a standard combinatorial model for $\chi(\mu)$, then for all $x \in X(\mu)$, we have that $\nu+\omega(x)$ is dominant whenever $\nu+\delta(x)$ is dominant, and

$$\chi(\mu)\chi(\nu) = \sum_{x\in X(\mu):\,\nu+\delta(x)\in\Lambda^+} \chi(\nu+\omega(x)).$$

Comparing this with the Brauer-Klimyk Rule (for ν sufficiently deep), we conclude that

$$K_{\mu,\xi} = \big|\{x \in X(\mu) : \omega(x) = \xi\}\big|,$$

or equivalently, $\chi(\mu) = \sum_{x\in X(\mu)} e^{\omega(x)}$. In particular, $X(\mu)$ is necessarily finite.

F. Lakshmibai-Seshadri Chains.

One particular way to realize combinatorial models for Weyl characters involves chains in the Bruhat order. This model was conjectured by Lakshmibai and Seshadri and proved by Littelmann [**L1**]. Here, we (mostly) follow the notation in Section 8 of [**St3**].

Given $\lambda \in \Lambda^+$, let '$<$' denote the Bruhat ordering of the W-orbit of λ; i.e., the transitive closure of the relations

$$t_\alpha\xi < \xi \quad \text{if} \quad \langle \xi, \alpha\rangle > 0 \quad (\xi \in W\lambda,\ \alpha \in \Phi^+),$$

where t_α denotes the reflection corresponding to α. We use the notation $\zeta \lessdot \xi$ to indicate that ξ covers ζ; for this, it is not sufficient (but obviously necessary) that $\zeta = t_\alpha\xi$ for some $\alpha \in \Phi^+$ such that $\langle \xi, \alpha\rangle > 0$.

Given any rational $b > 0$, we define the *b-Bruhat ordering* of $W\lambda$ to be the transitive closure of the relations

$$t_\alpha\xi <_b \xi \quad \text{if} \quad t_\alpha\xi \lessdot \xi \text{ and } b\langle \xi, \alpha^\vee\rangle \in \mathbf{Z} \quad (\xi \in W\lambda,\ \alpha \in \Phi^+).$$

Thus $\zeta <_b \xi$ is a covering relation of the b-Bruhat order if and only if ξ covers ζ in $(W\lambda, <)$ and $b(\xi - \zeta)$ is a positive integer multiple of a positive root. Notice also that if b is an integer, then the b-Bruhat order coincides with the original Bruhat order.

A *Lakshmibai-Seshadri chain* (or *LS chain*) x of *type* λ is a pair consisting of a chain in $(W\lambda, <)$ of any length $l \geqslant 0$, say $\xi_0 < \xi_1 < \cdots < \xi_l$, and an increasing sequence of rationals $0 = b_0 < b_1 < \cdots < b_l < b_{l+1} = 1$ such that

$$\xi_0 <_{b_1} \xi_1 <_{b_2} \cdots <_{b_l} \xi_l.$$

The *weight* of x is defined to be

$$\omega(x) = \sum_{0\leqslant j\leqslant l} (b_{j+1} - b_j)\xi_j \;=\; \xi_l - \sum_{1\leqslant j\leqslant l} b_j(\xi_j - \xi_{j-1}) \in \Lambda, \tag{2.5}$$

and the *depth* of x is defined to be the unique weight $\delta(x) \in \Lambda$ such that

$$\langle \delta(x), \alpha_i^\vee \rangle = \min_{0 \leqslant k \leqslant l+1} \sum_{0 \leqslant j < k} (b_{j+1} - b_j) \langle \xi_j, \alpha_i^\vee \rangle. \tag{2.6}$$

It is not obvious from the definition that $\langle \delta(x), \alpha_i^\vee \rangle$ is $\mathbf{Z}$-valued; however, this fact may be deduced from properties of the Bruhat order (e.g., see Section 8 of [**St3**]).

THEOREM 2.11 (Littelmann [**L1**]). *For each $\lambda \in \Lambda^+$, the set $C(\lambda)$ consisting of all LS chains of type λ is a (standard) combinatorial model for $\chi(\lambda)$.*

G. The Minuscule, Quasi-Minuscule, and Adjoint Rules.
A weight μ is *minuscule* if $\langle \mu, \alpha^\vee \rangle \leqslant 1$ for all $\alpha \in \Phi$.

PROPOSITION 2.12. *If $\mu, \nu \in \Lambda^+$ and μ is minuscule, then*

$$\chi(\mu)\chi(\nu) = \sum_{\xi \in W\mu:\, \nu+\xi \in \Lambda^+} \chi(\nu + \xi).$$

In particular, $\chi(\mu)\chi(\nu)$ is multiplicity-free.

Proof. Given that μ is minuscule, it follows that the b-Bruhat ordering of the orbit of μ is trivial unless b is an integer. Hence, the only LS chains of type μ are singletons $\xi \in W\mu$. Furthermore, we have $\omega(\xi) = \xi$ and

$$\langle \delta(\xi), \alpha_i^\vee \rangle = \min(0, \langle \xi, \alpha_i^\vee \rangle) = \begin{cases} -1 & \text{if } \langle \xi, \alpha_i^\vee \rangle < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now apply Theorem 2.11 and (2.2). $\square$

In an irreducible root system, there are at most two W-orbits of roots. If two orbits occur, the lengths of roots in these orbits differ ("long" and "short"); in cases where there is only one orbit, we adopt the convention that the roots are said to be short. In this way, every irreducible root system has exactly one orbit of short roots, denoted Φ_s.

We let I_s denote the set of indices of the short simple roots.

A weight μ is *quasi-minuscule* if $\langle \mu, \alpha^\vee \rangle \leqslant 2$ for all $\alpha \in \Phi$ and equality occurs for a unique root α. If Φ is irreducible, then there is a unique dominant quasi-minuscule weight; namely, the short dominant root $\bar{\alpha}$ (e.g., see the discussion in Section 4C of [**St3**]).

PROPOSITION 2.13. *If Φ is irreducible and $\nu \in \Lambda^+$, then*

$$\chi(\bar{\alpha})\chi(\nu) = k\chi(\nu) \ + \sum_{\beta \in \Phi_s:\, \nu+\beta \in \Lambda^+} \chi(\nu + \beta),$$

where $k = k(\nu) = |\{i \in I_s : \langle \nu, \alpha_i^\vee \rangle > 0\}|$. In particular, $\chi(\bar{\alpha})\chi(\nu)$ is multiplicity-free if and only if $k(\nu) \leqslant 1$.

Proof. Given $\beta \in \Phi_s$, we have $\langle \beta, \alpha^\vee \rangle \leqslant 2$ for all $\alpha \in \Phi$, and equality occurs if and only if $\alpha = \beta$. Hence, for $0 < b < 1$, there are no nontrivial relations in the b-Bruhat ordering of Φ_s except for those of the form $-\beta <_b \beta$ with $b = 1/2$ and $-\beta \lessdot \beta$. It is easy to check that $\beta \in \Phi_s$ covers $-\beta$ in the normal Bruhat ordering if and only if β is simple, so the LS chains of type $\bar{\alpha}$ consist of the singleton chains, one for each $\beta \in \Phi_s$, and the doubleton chains $-\alpha_j <_{1/2} \alpha_j$ ($j \in I_s$). Furthermore, from (2.5) and (2.6), we have

$$\omega(\beta) = \beta, \qquad \omega(-\alpha_j <_{1/2} \alpha_j) = 0,$$
$$\langle \delta(\beta), \alpha_i^\vee \rangle = \min(0, \langle \beta, \alpha_i^\vee \rangle), \quad \delta(-\alpha_j <_{1/2} \alpha_j) = -\omega_j.$$

Now apply Theorem 2.11 and (2.2). □

We remark that Propositions 2.12 and 2.13 are easy to deduce directly from the Brauer-Klimyk Rule (see Corollaries 7.2 and 7.3 of [**St1**]).

Continuing the hypothesis that Φ is irreducible, let $\widetilde{\alpha}$ denote the highest root; i.e., the highest weight of the adjoint representation. If Φ has only one W-orbit, then $\widetilde{\alpha} = \bar{\alpha}$; otherwise, $\widetilde{\alpha}$ is the long dominant root. We claim that $\chi(\widetilde{\alpha})$ has a (standard) combinatorial model $X(\widetilde{\alpha})$ whose objects are the roots $\alpha \in \Phi$, together with n objects 0_i ($1 \leqslant i \leqslant n$), the latter having weight 0 and depth $-\omega_i$. The weight and depth of the root α are defined to be α and $\delta(\alpha)$, where $\langle \delta(\alpha), \alpha_i^\vee \rangle = -k$ if k is the largest integer such that $\alpha + k\alpha_i \in \Phi$.

Proposition 2.14. *If Φ is irreducible with highest root $\widetilde{\alpha}$, then $X(\widetilde{\alpha})$ is a standard combinatorial model for $\chi(\widetilde{\alpha})$. Hence,*

$$\chi(\widetilde{\alpha})\chi(\nu) = k'\chi(\nu) \ + \sum_{\beta \in \Phi:\, \nu + \delta(\beta) \in \Lambda^+} \chi(\nu + \beta),$$

for all $\nu \in \Lambda^+$, where $k' = k'(\nu) = |\{i : \langle \nu, \alpha_i \rangle > 0\}|$. In particular, the product $\chi(\widetilde{\alpha})\chi(\nu)$ is multiplicity-free if and only if $k'(\nu) \leqslant 1$.

Proof. For each simple root α_i, the members of $X(\widetilde{\alpha})$ may by partitioned into strings; i.e., maximal sequences of roots of the form $\beta, \beta + \alpha_i, \dots, \beta + k\alpha_i$, along with the "degenerate" strings $-\alpha_i, 0_i, \alpha_i$ and singleton strings 0_j for each $j \neq i$. Any string with more than two elements is said to be *long*. No string has more than four elements, and long non-degenerate strings occur only if Φ has both long and short roots. Indeed, the first and last roots in such a string must be long, and the interior roots and α_i must be short.

A key observation to make is that for every object $x \in X(\widetilde{\alpha})$, there is at most one index i such that $\langle \delta(x), \alpha_i^\vee \rangle < 0$ and the α_i-string through x is long. This is easy to see for each of the objects 0_i and $-\alpha_i$ (in both cases, $\langle \delta(x), \alpha_j^\vee \rangle = 0$ for $j \neq i$), so we may restrict our attention to (long) non-degenerate strings. In particular, a counterexample could occur only if Φ has both long and short roots, including at least two short simple roots; i.e., only if $\Phi = \mathcal{C}_n$ ($n \geqslant 3$) or $\Phi = \mathcal{F}_4$. We leave to the reader the easy task of checking that there are no counterexamples in these cases. (It is also possible to give a longer, classification-free proof.)

Now consider the generating function

$$\phi = \sum_{x \in X(\widetilde{\alpha})} e^{\omega(x)} = ne^0 + \sum_{\beta \in \Phi} e^\beta.$$

It is clear that ϕ is W-invariant, so reasoning similar to the proof of Proposition 2.1 shows that for all $\nu \in \Lambda^+$, we have

$$\phi\chi(\nu) = \sum_{x \in X(\widetilde{\alpha})} \chi(\nu + \omega(x)). \tag{2.7}$$

Let $Y_i(\nu)$ denote the set of objects $x \in X(\widetilde{\alpha})$ such that the α_i-string through x is long and $\langle \nu + \delta(x), \alpha_i^\vee \rangle < 0$. The above discussion shows that $Y_1(\nu), \dots, Y_n(\nu)$ are disjoint. We also claim that for each $x \in Y_i(\nu)$, there is a unique $x' \in Y_i(\nu)$ such that

$$s_i(\nu + \omega(x) + \rho) = \nu + \omega(x') + \rho,$$

whence $\chi(\nu + \omega(x)) = -\chi(\nu + \omega(x'))$ and the net contribution of $Y_i(\nu)$ to (2.7) is 0.

To prove the claim, note that the last member of the α_i-string through x has weight $\omega(x) - \langle \delta(x), \alpha_i^\vee \rangle \alpha_i$ (by definition). Furthermore, the α_i-strings are stable under the action of s_i (in fact, s_i reverses the string), so the first member of the string has weight $\omega(x) - \langle \omega(x) - \delta(x), \alpha_i^\vee \rangle \alpha_i$. Hence, there is a (unique) member of the α_i-string through x of weight $\omega(x) - k\alpha_i$ if and only if

$$\langle \delta(x), \alpha_i^\vee \rangle \leqslant k \leqslant \langle \omega(x) - \delta(x), \alpha_i^\vee \rangle. \tag{2.8}$$

Since x is a member of this string (the case $k = 0$), it follows that $\delta(x) \preccurlyeq 0$ and $\delta(x) \preccurlyeq \omega(x)$. In other words, δ is a standard depth function.

For the claim, we seek an object x' of weight $\omega(x) - k\alpha_i$, where $k = \langle \nu + \omega(x) + \rho, \alpha_i^\vee \rangle$. The fact that this value of k satisfies (2.8) follows easily from the fact that ν is dominant, $\delta(x) \preccurlyeq \omega(x)$, and $\langle \nu + \delta(x), \alpha_i^\vee \rangle < 0$, so x' exists. Also, the calculation

$$\langle \nu + \delta(x'), \alpha_i^\vee \rangle = \langle \nu + \delta(x), \alpha_i^\vee \rangle - k = \langle \delta(x) - \omega(x),\, \alpha_i^\vee \rangle - 1 < 0$$

shows that $x' \in Y_i(\nu)$, proving the claim.

The only remaining terms in (2.7) are those for which $\nu + \delta(x)$ is dominant, or else $\langle \nu + \delta(x), \alpha_i^\vee \rangle < 0$ for some i for which the α_i-string through x is short. However, the latter case occurs only if x is the first element of a two-element string (whence $\langle \delta(x), \alpha_i^\vee \rangle = \langle \omega(x), \alpha_i^\vee \rangle = -1$) and $\langle \nu, \alpha_i^\vee \rangle = 0$. Hence s_i fixes $\nu + \omega(x) + \rho$ and $\chi(\nu + \omega(x)) = 0$. The corresponding terms may therefore be omitted from (2.7), yielding

$$\phi\chi(\nu) = \sum_{x \in X(\widetilde{\alpha}):\, \nu + \delta(x) \in \Lambda^+} \chi(\nu + \omega(x)).$$

This shows simultaneously that $\phi = \chi(\widetilde{\alpha})$ (take $\nu = 0$) and that $X(\widetilde{\alpha})$ is a combinatorial model for $\chi(\widetilde{\alpha})$. $\square$

H. The Chain Rule and the Multi-Minuscule Rule.

Following [**St4**], let $\leftarrow$ denote the binary relation on $C(\mu)$ obtained by setting

$$\big(\zeta_0 <_{b_1} \zeta_1 <_{b_2} \cdots <_{b_k} \zeta_k\big) \;\leftarrow\; \big(\xi_0 <_{c_1} \xi_1 <_{c_2} \cdots <_{c_l} \xi_l\big) \quad \text{if} \;\; \zeta_k \leqslant \xi_0.$$

This is a transitive, asymmetric ($x \leftarrow y$ and $y \leftarrow x$ implies $x = y$), but not necessarily reflexive relation—a "proset" in the terminology of [**St4**].

Proposition 2.15. *For all $\mu, \nu \in \Lambda^+$ and $m \geqslant 0$, we have*

$$\chi(m\mu)\chi(\nu) = \sum \chi(\nu + \omega(x_1) + \cdots + \omega(x_m)),$$

where the sum ranges over the set of all m-chains $x_1 \leftarrow \cdots \leftarrow x_m$ such that $x_k \in C(\mu)$ and $\nu + \omega(x_1) + \cdots + \omega(x_{k-1}) + \delta(x_k) \in \Lambda^+$ for $1 \leqslant k \leqslant m$.

Proof. As noted in Section 3 of [**St4**], there is a bijection between $C(m\mu)$ and the set of all m-tuples $(x_1, \ldots, x_m)$ such that $x_i \in C(\mu)$ and $x_1 \leftarrow \cdots \leftarrow x_m$. In this bijection, the LS chain x of type $m\mu$ corresponding to $(x_1, \ldots, x_m)$ is obtained by

(1) replacing each relation $\zeta <_b \xi$ in x_i with $m\zeta <_{b'} m\xi$, where $b' = (b + i - 1)/m$,
(2) inserting the relation $m\zeta <_{i/m} m\xi$ between the maximum element ζ of x_i and the minimum ξ of x_{i+1} (or identify the pair, if $\zeta = \xi$), and
(3) concatenating the resulting chains together.

It follows that the weight and depth of x satisfy

$$\omega(x) = \omega(x_1) + \cdots + \omega(x_m),$$
$$\langle \delta(x), \alpha_i^\vee \rangle = \min_{1 \leqslant k \leqslant m} \langle \omega(x_1) + \cdots + \omega(x_{k-1}) + \delta(x_k), \alpha_i^\vee \rangle.$$

Now apply Theorem 2.11. $\square$

Recall that in case μ is minuscule, $C(\mu)$ may be identified with the W-orbit of μ, and the relation $\leftarrow$ is simply the Bruhat ordering $\leqslant$. Hence,

Corollary 2.16. *If $\mu, \nu \in \Lambda^+$ and μ is minuscule, then*

$$\chi(m\mu)\chi(\nu) = \sum_{\xi_1 \leqslant \cdots \leqslant \xi_m :\, \xi_k \in W\mu} \chi(\nu + \xi_1 + \cdots + \xi_m),$$

where the sum is restricted to m-chains such that $\nu + \xi_1 + \cdots + \xi_k \in \Lambda^+$ for $1 \leqslant k \leqslant m$.

Continuing the hypothesis that $\mu, \nu \in \Lambda^+$ and μ is minuscule, we say that a chain $\xi_1 \leqslant \cdots \leqslant \xi_m$ in $W\mu$ is *ν-dominant* if it contributes a term in the above decomposition of $\chi(m\mu)\chi(\nu)$; i.e., $\nu + \xi_1 + \cdots + \xi_k \in \Lambda^+$ for $1 \leqslant k \leqslant m$. In addition, we say that a strict chain $\zeta_1 < \cdots < \zeta_l$ in $W\mu$ is *generically ν-dominant* if for every pair i, k such that $\langle \zeta_k, \alpha_i^\vee \rangle < 0$, there is some $0 \leqslant j < k$ such that $\langle \zeta_j, \alpha_i^\vee \rangle > 0$ (taking $\zeta_0 := \nu$). If a chain is ν-dominant, then the strict chain formed by the set of distinct weights that appear in the multichain must be generically ν-dominant. Hence,

Corollary 2.17. *If $\mu, \nu \in \Lambda^+$ and μ is minuscule, then $\chi(m\mu)\chi(\nu)$ is multiplicity-free for all $m \geqslant 0$, except possibly if there are generically ν-dominant chains $\xi_1 < \cdots < \xi_k$ and $\zeta_1 < \cdots < \zeta_l$ and integers $a_i, b_j \geqslant 0$ (not all zero) such that*

$$\xi_i = \zeta_j \text{ implies } a_i = 0 \text{ or } b_j = 0,$$
$$a_1 + \cdots + a_k = b_1 + \cdots + b_l,$$
$$a_1\xi_1 + \cdots + a_k\xi_k = b_1\zeta_1 + \cdots + b_l\zeta_l.$$

REMARK 2.18. (a) Relaxing the nonnegativity constraint on a_i and b_j, the above condition reduces to linear independence in an affine hyperplane; i.e., linear independence of the distinct vectors among $(\xi_1, 1), \ldots, (\xi_k, 1), (\zeta_1, 1), \ldots, (\zeta_l, 1)$. If every pair of generically ν-dominant chains is "affinely independent" in this sense, then Corollary 2.17 implies that $\chi(m\mu)\chi(\nu)$ is multiplicity-free for all $m \geqslant 0$.

(b) The Bruhat ordering of a minuscule orbit $W\mu$ is particularly simple. Indeed, by a theorem of Proctor [**P**], one knows that $\zeta \leqslant \xi$ if and only if $\xi - \zeta$ is in the nonnegative span of the simple roots (assuming $\zeta, \xi \in W\mu$), and ξ covers ζ if and only if $\zeta = s_i\xi = \xi - \alpha_i$ for some i $(1 \leqslant i \leqslant n)$.

I. The Twice-Quasi-Minuscule Rule.

Recall that $\bar{\alpha}$ denotes the short dominant root, assuming Φ is irreducible.

PROPOSITION 2.19. *Let $\nu \in \Lambda^+$ and $J = \{j : \langle \nu, \alpha_j^\vee \rangle = 0\}$. If Φ is irreducible, then the multiplicity of $\chi(\nu)$ in $\chi(2\bar{\alpha})\chi(\nu)$ is the number of W_J-orbits consisting entirely of short positive roots, except for those that contain a short simple root α_j such that $\langle \nu, \alpha_j^\vee \rangle = 1$.*

Proof. By Proposition 2.15, the multiplicity of $\chi(\nu)$ in $\chi(2\bar{\alpha})\chi(\nu)$ is the number of 2-chains $x \leftarrow y$ in $C(\bar{\alpha})$ with $\nu + \delta(x)$ and $\nu + \omega(x) + \delta(y)$ dominant and $\omega(x) + \omega(y) = 0$. Now recall from the proof of Proposition 2.13 that the LS chains of type $\bar{\alpha}$ are singletons (one for each $\beta \in \Phi_s$) and doubletons $-\alpha_i <_{1/2} \alpha_i$ $(i \in I_s)$. The latter have weight 0 and are incompatible with each other relative to $\leftarrow$, so the 2-chains of weight 0 all have the form $-\beta \leftarrow \beta$ $(\beta \in \Phi_s^+)$. Furthermore, since $\langle \delta(\beta), \alpha_i^\vee \rangle = \min(0, \langle \beta, \alpha_i^\vee \rangle)$, the dominance of $\nu + \delta(-\beta)$ and $\nu - \beta + \delta(\beta)$ is equivalent to $\nu - \beta \in \Lambda^+$, so

$$|\{\beta \in \Phi_s^+ : \ \nu - \beta \in \Lambda^+\}|$$

is the multiplicity of $\chi(\nu)$ in $\chi(2\bar{\alpha})\chi(\nu)$.

Given $\beta \in \Phi_s$, we have $\langle \beta, \alpha_j^\vee \rangle \leqslant 1$ unless $\beta = \alpha_j$. Since $\langle \nu, \alpha_j^\vee \rangle \geqslant 1$ for $j \notin J$, it follows that if β is not simple, then $\nu - \beta$ is dominant if and only if $\langle \beta, \alpha_j^\vee \rangle \leqslant 0$ for all $j \in J$. In other words, β must be anti-dominant relative to Φ_J, and there is exactly one such root in every W_J-orbit. However, some W_J-orbits in Φ_s may contain both positive and negative roots—these are precisely the orbits in Φ_J; their anti-dominant members are negative. The remaining W_J-orbits in Φ_s are either all positive or all negative. Now among the orbits of positive roots in Φ_s are those whose anti-dominant member is a simple root $\beta = \alpha_j$ (one for each $j \in I_s - J$). In these cases, we have $\langle \beta, \alpha_j^\vee \rangle = 2$, so $\nu - \beta$ is dominant if and only if $\langle \nu, \alpha_j^\vee \rangle \geqslant 2$. $\square$

Let c_j denote the coefficient of α_j in $\bar{\alpha}$.

COROLLARY 2.20. *The product $\chi(2\bar{\alpha})\chi(\nu)$ is not multiplicity-free if*

(i) *$\nu = \omega_j$ and $c_j \geqslant 3$, or α_j is long and $c_j \geqslant 2$,*
(ii) *$\nu = 2\omega_j$ and $c_j \geqslant 2$,*
(iii) *$\nu = \omega_i + \omega_j$ and $c_i + c_j \geqslant 3$ or α_i and α_j are long $(i \neq j)$, or*
(iv) *$\nu = 2\omega_i + \omega_j$ $(i \neq j)$ and α_i is short.*

Proof. Let $f_1, \ldots, f_n$ denote the basis of $\mathbf{E}^*$ dual to the simple roots (i.e., $f_j(\alpha_i) = \delta_{ij}$). For each $j \notin J$, f_j vanishes on Φ_J and is constant on W_J-orbits. It follows that a lower bound for the number of W_J-orbits of short positive roots is the number of nonzero tuples $(f_j(\beta) : j \notin J)$ that occur as β varies over Φ_s^+.

If $\beta \in \Phi_s^+$ is not simple, we must have $\langle \beta, \alpha_i^\vee \rangle = 1$ and $\beta - \alpha_i = s_i\beta \in \Phi_s^+$ for some i. It follows that there are W_J-orbits of short positive roots such that f_j $(j \notin J)$ achieves every integer value from 1 to $c_j = f_j(\beta)$. Furthermore, if α_j is long, then there is at least one orbit containing short roots β such that $f_j(\beta) = 1$, and this orbit contains no simple roots. Thus by Proposition 2.19, $\chi(2\bar{\alpha})\chi(\omega_j)$ cannot be multiplicity-free if $c_j \geqslant 3$, or if $c_j = 2$ and α_j is long, whence (i) follows. Analogous reasoning applies to (ii), except that in this case, we are also allowed to count the orbit of α_j, even if it is short.

Similarly, if $f = \sum_{j \notin J} f_j$, then f must achieve every integer value from 1 to $\sum_{j \notin J} c_j$. Since $f = 1$ on orbits that contain simple roots, Proposition 2.19 implies that $\chi(2\bar{\alpha})\chi(\nu)$ cannot be multiplicity-free if $\sum_{j \notin J} c_j \geqslant 3$, yielding the part of (iii) with $c_i + c_j \geqslant 3$. For the last of (iii), note that if α_i and α_j are both long and $c_i = c_j = 1$, then there must be orbits of short positive roots such that $f = 1$ and $f = 2$; neither of these orbits contains a simple root. Similar reasoning applies to (iv), except that in this case, we are also allowed to count the orbit containing α_i. □

3. The case $\Phi = \mathcal{A}_n$

Theorem 1.1.A is proved in [**St2**].

4. The case $\Phi = \mathcal{B}_n$

Realization: $\mathcal{B}_n = \{\pm\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\}$.

Simple roots: $\alpha_1 = \varepsilon_1,\ \ \alpha_i = \varepsilon_i - \varepsilon_{i-1}\ (1 < i \leqslant n)$.

Fundamental weights: $\omega_1 = (1/2)(\varepsilon_1 + \cdots + \varepsilon_n),\ \ \omega_i = \varepsilon_i + \cdots + \varepsilon_n\ (1 < i \leqslant n)$.

Minuscule weights: ω_1. Quasi-minuscule weight: ω_n.

A. The Exterior Powers Rule.

The quasi-minuscule weight $\omega_n = \varepsilon_n$ is the highest weight of the defining representation of $\mathfrak{g} = so(2n+1)$, and the exterior algebra of this representation has the (graded) character

$$\phi(t) = \sum_{k=0}^{2n+1} \phi_k t^k := (1+t)\prod_{i=1}^{n}(1+te^{\varepsilon_i})(1+te^{-\varepsilon_i}).$$

In the following, we will derive a rule for the Weyl character expansion of $\phi(t)\chi(\nu)$ for each $\nu \in \Lambda^+$; in turn this will yield a "Pieri Rule" for multiplication by $\chi(\omega_i)$ $(1 < i \leqslant n)$. Previous Pieri-type rules for the classical cases have been given by Weyman [**W**].

To describe the rule, let $B^+(\nu)$ denote the set consisting of all dominant weights of the form $\nu \pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$. For each $\mu \in B^+(\nu)$, define

$$p^B_{\mu,\nu}(t) = t^k(1+t^{2n_0+1})\prod_{r \geqslant 1}(1+t^2+\cdots+t^{2n_r}),$$

where $n_r = |\{i : \langle \mu, \varepsilon_i \rangle = \langle \nu, \varepsilon_i \rangle = r/2\}|$ and $k = n - (n_0 + n_1 + \cdots)$ (i.e., the number of coordinates where μ and ν differ). Note that the coefficients of $p^B_{\mu,\nu}(t)$ are multiplicity-free if and only if there is at most one $r > 0$ such that $n_r > 0$.

PROPOSITION 4.1. *For all $\nu \in \Lambda^+$, we have*

$$\phi(t)\chi(\nu) = \sum_{\mu \in B^+(\nu)} p^B_{\mu,\nu}(t)\chi(\mu).$$

In particular (taking $\nu = 0$), $\chi(\omega_i) = \phi_{n-i+1}$ $(1 < i \leqslant n)$ and $\chi(2\omega_1) = \phi_n$.

Proof. Since $\phi(t)$ is W-invariant, an argument similar to the proof of Proposition 2.1 may be used to show that

$$\phi(t)\chi(\nu) = \sum_{K \subseteq E} t^{|K|}\chi(\nu + \sigma(K)), \tag{4.1}$$

where $E = \{0, \pm\varepsilon_1, \ldots, \pm\varepsilon_n\}$, and $\sigma(K)$ denotes the sum of the members of K.

Given $K \subseteq E$, consider the largest $j > 1$ (if any) such that

(1) $-\varepsilon_j, \varepsilon_{j-1} \in K$, $\varepsilon_j, -\varepsilon_{j-1} \notin K$ and $\langle\nu, \varepsilon_j\rangle = \langle\nu, \varepsilon_{j-1}\rangle$, or
(2) $\varepsilon_j, -\varepsilon_j \in K$, $\varepsilon_{j-1}, -\varepsilon_{j-1} \notin K$ and $\langle\nu, \varepsilon_j\rangle = \langle\nu, \varepsilon_{j-1}\rangle$.

If (1) holds, then we have $\langle\nu + \sigma(K), \alpha_j^\vee\rangle = -2$ and $s_j(\nu + \sigma(K) + \rho) = \nu + \sigma(K) + \alpha_j + \rho$, whence $\chi(\nu + \sigma(K)) = -\chi(\nu + \sigma(K'))$, where $K' = K \cup \{\varepsilon_j\} - \{\varepsilon_{j-1}\}$. Furthermore, K' satisfies (2), and there is no larger j such that K' satisfies (1) or (2). Hence, the pairing $K \leftrightarrow K'$ is bijective (and preserves cardinality), so we may delete all terms from (4.1) satisfying (1) or (2) for any $j > 1$.

Among the remaining choices for K, suppose there exists $j \geqslant 1$ such that

(3) $0, -\varepsilon_1 \in K$, $\varepsilon_1 \notin K$ and $\langle\nu, \varepsilon_1\rangle = 0$,
(4) $\varepsilon_1, -\varepsilon_1 \in K$, $0 \notin K$ and $\langle\nu, \varepsilon_1\rangle = 0$,
(5) $-\varepsilon_1, \varepsilon_i, -\varepsilon_i \in K$ $(1 < i \leqslant j)$, $0, \varepsilon_1 \notin K$ and $\langle\nu, \varepsilon_i\rangle = 0$ $(1 \leqslant i \leqslant j)$, or
(6) $0, \varepsilon_i, -\varepsilon_i \in K$ $(1 \leqslant i < j)$, $\varepsilon_j, -\varepsilon_j \notin K$ and $\langle\nu, \varepsilon_i\rangle = 0$ $(1 \leqslant i \leqslant j)$.

If there is a choice for j in (5), we insist on the largest possible. If (3) holds, then we have $\chi(\nu+\sigma(K)) = -\chi(\nu+\sigma(K'))$, where $K' = K \cup \{\varepsilon_1\} - \{0\}$. In this case, K' satisfies (4) and does not satisfy (1) or (2) (and conversely), so the pairing $K \leftrightarrow K'$ is bijective. Similarly, if K satisfies (5), then $t^{|K|}\chi(\nu + \sigma(K))$ may be canceled with $t^{|K'|}\chi(\nu + \sigma(K'))$, where $K' = K \cup \{0, \varepsilon_1\} - \{\varepsilon_j, -\varepsilon_j\}$; here, K' satisfies (6) and not (1) or (2) (and conversely).

In all remaining cases, we have $\langle\nu+\sigma(K), \alpha_j^\vee\rangle \geqslant -1$ for all j, thanks to (the lack of) (1), (3) and (5). Moreover, if equality occurs for some j, then $\chi(\nu+\sigma(K)) = 0$, so we may insist that $\mu := \nu+\sigma(K)$ is dominant. Now consider the sets $N_r = \{i : \langle\nu, \varepsilon_i\rangle = \langle\mu, \varepsilon_i\rangle = r/2\}$ for each integer $r \geqslant 0$. The dominance of μ and ν forces N_r to consist of (say) n_r consecutive integers. Furthermore, we must have $\varepsilon_i, -\varepsilon_i \in K$ or $\varepsilon_i, -\varepsilon_i \notin K$ for all $i \in N_r$, so in order to avoid (2), it must be the case that for some l,

$$K \cap \{\pm\varepsilon_i : i \in N_r\} = \{\pm\varepsilon_j, \ldots, \pm\varepsilon_{j+l-1}\} \quad (0 \leqslant l \leqslant n_r),$$

where j denotes the smallest member of N_r. Note that the generating function for these possibilities is $1 + t^2 + \cdots + t^{2n_r}$. Furthermore, in the case $r = 0$, to avoid (4) and (6), we must have either $0 \notin K$ and $l = 0$, or else $0 \in K$ and $l = n_0$, a pair of choices with generating function $1 + t^{2n_0+1}$. Conversely, it is easy to see that every K of this form satisfies none of (1)–(6), so the coefficient of $\chi(\mu)$ in (4.1) is indeed $p^B_{\mu,\nu}(t)$. $\square$

B. Proof of Theorem 1.1.B.

The products listed in Theorem 1.1.B can be shown to be multiplicity-free as follows.

(i) Apply the Minuscule and Quasi-Minuscule Rules (Propositions 2.12 and 2.13).

(ii),(iii) The Exterior Powers Rule (Proposition 4.1) shows that the multiplicity of $\chi(\mu)$ in $\phi(t)\chi(m\omega_j)$ has the form $t^a(1+t^{2b+1})(1+t^2+\cdots+t^{2c})$ for various integers a, b, c. These polynomials are clearly multiplicity-free.

(iv) Apply the Multi-Minuscule Rule (Corollary 2.16) and the criterion of Corollary 2.17. The W-orbit of $\mu = \omega_1$ consists of all vectors of the form $\pm(1/2)\varepsilon_1 \pm \cdots \pm (1/2)\varepsilon_n$, and if we take $\nu = m'\omega_1$ ($m' > 0$), then the weights

$$\xi_k = (-1/2)(\varepsilon_1 + \cdots + \varepsilon_k) + (1/2)(\varepsilon_{k+1} + \cdots + \varepsilon_n) \quad (0 \leqslant k \leqslant n)$$

are the unique weights $\xi \in W\mu$ such that $\nu + \xi$ is dominant, and it is not hard to see that every generically ν-dominant chain in $W\mu$ is a subset of the chain $\xi_n < \cdots < \xi_1 < \xi_0 = \omega_1$. This chain is affinely independent in the sense of Remark 2.18, so by Corollary 2.17, the product $\chi(m\omega_1)\chi(m'\omega_1)$ is multiplicity-free.

(v) Apply the Chain Rule (Proposition 2.15). The LS chains of the quasi-minuscule type $\mu = \omega_n$ consist of the singletons $\pm\varepsilon_j$, together with one doubleton chain '0' of weight 0 (see the proof of Proposition 2.13), and we have

$$-\varepsilon_n \leftarrow \cdots \leftarrow -\varepsilon_1 \leftarrow 0 \leftarrow \varepsilon_1 \leftarrow \cdots \leftarrow \varepsilon_n.$$

In order for an m-chain of these elements, say $\xi_1 \leftarrow \cdots \leftarrow \xi_m$, to contribute to the expansion of $\chi(m\mu)\chi(\nu)$ in Proposition 2.15, a necessary condition is that the chain must be ν-dominant (i.e., $\nu + \xi_1 + \cdots + \xi_k \in \Lambda^+$ for $1 \leqslant k \leqslant n$), and 0 may appear at most once (since $0 \not\leftarrow 0$). Setting $\xi = \xi_1 + \cdots + \xi_m$, note that the sum of the coordinates in ξ has parity equal to the number of nonzero terms, so there must be either no 0 or one 0 in the chain, according to whether this parity agrees with m or $m-1$. In other words, the number of 0's in a chain that contributes to the multiplicity of $\chi(\nu+\xi)$ in $\chi(m\mu)\chi(\nu)$ depends only on m and ξ. If we now take $\nu = \omega_1 + m'\omega_i$ for some i ($1 \leqslant i \leqslant n$), then the remainder of a ν-dominant chain must consist of some number of copies of $-\varepsilon_i$, ε_{i-1} (if $i > 1$), ε_i, and ε_n (if $i < n$), in that order. Since $\{\pm\varepsilon_i, \varepsilon_{i-1}, \varepsilon_n\}$ is an affinely independent set (see Remark 2.18), it follows that $\chi(m\mu)\chi(\nu)$ is multiplicity-free.

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ, ν) such that

(N1) $\mu \in \{2\omega_1, \omega_2, \ldots, \omega_{n-1}\}$ and $\nu = \omega_i + \omega_j$ $(1 \leqslant i < j \leqslant n)$,
(N2) $(\mu, \nu) = (2\omega_i, 2\omega_j)$ $(1 < i, j < n)$,
(N3) $(\mu, \nu) = (\omega_1 + \omega_n, \omega_1 + \omega_n)$,
(N4) $(\mu, \nu) = (2\omega_n, \omega_i + \omega_j)$ $(1 < i < j \leqslant n)$,
(N5) $(\mu, \nu) = (2\omega_n, 2\omega_1 + \omega_i)$ $(1 < i \leqslant n)$, or
(N6) $(\mu, \nu) = (3\omega_1, 2\omega_i)$ $(1 < i < n)$.

These products can be shown not to be multiplicity-free as follows.

(N1) If $\nu = \omega_i + \omega_j$ and $\lambda = \varepsilon_k + \cdots + \varepsilon_{i-1} + \nu$, where $1 \leqslant k \leqslant i < j \leqslant n$ (with $\lambda = \nu$ when $k = i$), then by the Exterior Powers Rule (Proposition 4.1), $\chi(\lambda)$ has multiplicity

$$t^{i-k}(1 + t^{2k-1})(1 + t^2 + \cdots + t^{2j-2i})(1 + t^2 + \cdots + t^{2n-2j+2})$$

in $\phi(t)\chi(\nu)$. The coefficient of t^{l+r} in this polynomial is $\geqslant 2$ for $l = i - k$ or $l = i + k - 1$ and all $r = 2, 4, \dots, 2n - 2i$. As k varies from 1 to i, the values for l range from 0 to $2i - 1$, so the coefficient of t^r in $\phi(t)\chi(\nu)$ is not multiplicity-free for $2 \leqslant r \leqslant 2n - 1$.

(N2) Proceeding by induction with respect to n, it suffices by stability (Corollary 2.7) to restrict our attention to the case $(\mu, \nu) = (2\omega_{n-1}, 2\omega_j)$, where $1 < j < n$. Applying the Chain Rule (Proposition 2.15), it is easy to see that $C(\omega_{n-1})$ includes two singleton chains, $\beta = \varepsilon_j + \varepsilon_{j+1}$ and $\beta = \alpha_j = \varepsilon_j - \varepsilon_{j-1}$, such that $\nu - \beta \in \Lambda^+$. In both cases, β is a positive root, so it is clear that $-\beta \leftarrow \beta$. Therefore, the multiplicity of $\chi(2\omega_j)$ in $\chi(2\omega_{n-1})\chi(2\omega_j)$ is at least 2.

(N3) The Adjoint Rule (Proposition 2.14) implies that $c(\omega_1 + \omega_n\,;\, \widetilde{\alpha}, \omega_1 + \omega_n) = 2$, and hence $c(\widetilde{\alpha}\,;\, \omega_1 + \omega_n, \omega_1 + \omega_n) = 2$ by triple symmetry (Proposition 2.8).

(N4),(N5) The simple root coordinates of $\omega_n = \bar{\alpha}$ are $(1, \dots, 1)$ and α_1 is the only short simple root, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

(N6) Proceeding by induction with respect to n, it suffices by stability (Corollary 2.7) to restrict our attention to the case $(\mu, \nu) = (3\omega_1, 2\omega_{n-1})$. Applying the Multi-Minuscule Rule (Corollary 2.16), note that the Bruhat ordering of the W-orbit of the minuscule weight ω_1 includes the ν-dominant chains

$$\begin{aligned} &\omega_1 - \omega_{n-1} < -\omega_1 + \omega_2 - \omega_{n-1} + \omega_n < \omega_1, \\ &\omega_1 - \omega_{n-1} < \omega_1 - \omega_{n-1} + \omega_n < -\omega_1 + \omega_2, \end{aligned}$$

so the multiplicity of $\chi(\omega_1 + \omega_2 + \omega_n)$ in $\chi(3\omega_1)\chi(2\omega_{n-1})$ is at least 2.

5. The case $\Phi = \mathcal{C}_n$

Realization: $\mathcal{C}_n = \{\pm 2\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\}$.
Simple roots: $\alpha_1 = 2\varepsilon_1,\ \alpha_i = \varepsilon_i - \varepsilon_{i-1}\ (1 < i \leqslant n)$.
Fundamental weights: $\omega_i = \varepsilon_i + \cdots + \varepsilon_n\ (1 \leqslant i \leqslant n)$.
Minuscule weights: ω_n. Quasi-minuscule weight: ω_{n-1}.

A. The Rule for Fundamental Weights.

In order to derive a rule for decomposing the products $\chi(\omega_i)\chi(\nu)$, let us introduce

$$\psi(t) = \sum_{k=0}^{2n+2} \psi_k t^k := (1 - t^2) \prod_{i=1}^{n} (1 + te^{\varepsilon_i})(1 + te^{-\varepsilon_i}).$$

Aside from the factor of $1 - t^2$, this is the graded character of the exterior algebra of the defining representation of $\mathfrak{g} = sp(2n)$.

Since the weight lattice of $\mathcal{C}_n$ is a sublattice of the weight lattice of $\mathcal{B}_n$, it makes sense to re-use the notation from Section 4A. In particular, $B^+(\nu)$ shall continue to denote the set of dominant weights of the form $\nu \pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k}$. Given that ν is in the weight lattice of $\mathcal{C}_n$, then the same is true for every member of $B^+(\nu)$, so this notation is unambiguous. For each $\mu \in B^+(\nu)$, we define

$$p^C_{\mu,\nu}(t) = t^k (1 - t^2) \prod_{r \geqslant 0} (1 + t^2 + \cdots + t^{2n_r}),$$

where $n_r = |\{i : \langle \mu, \varepsilon_i \rangle = \langle \nu, \varepsilon_i \rangle = r\}|$ and $k = |\{i : \langle \mu, \varepsilon_i \rangle \neq \langle \nu, \varepsilon_i \rangle\}|$.

Proposition 5.1. *For all $\nu \in \Lambda^+$, we have*

$$\psi(t)\chi(\nu) = \sum_{\mu \in B^+(\nu)} p^C_{\mu,\nu}(t)\chi(\mu).$$

In particular (taking $\nu = 0$), $\chi(\omega_i) = \psi_{n-i+1} = -\psi_{n+i+1}$ $(1 \leqslant i \leqslant n)$.

Proof. Let $\chi'(\nu)$ and ρ' denote the $\mathcal{B}_n$-analogues of $\chi(\nu)$ and ρ, and set $\theta = \Delta(\rho)/\Delta(\rho')$. Since $\mathcal{B}_n$ and $\mathcal{C}_n$ share the same Weyl group, we have

$$\chi(\nu)\theta = \chi(\nu)\Delta(\rho)/\Delta(\rho') = \Delta(\nu+\rho)/\Delta(\rho') = \Delta(\nu+\omega+\rho')/\Delta(\rho') = \chi'(\nu+\omega),$$

where $\omega = \rho - \rho' = (\varepsilon_1 + \cdots + \varepsilon_n)/2$. By Proposition 4.1, it follows that

$$\phi(t)\chi(\nu)\theta = \phi(t)\chi'(\nu+\omega) = \sum_{\mu \in B^+(\nu+\omega)} p^B_{\mu,\nu+\omega}(t)\chi'(\mu) = \sum_{\mu \in B^+(\nu)} p^B_{\mu+\omega,\nu+\omega}(t)\chi(\mu)\theta,$$

so the coefficient of $\chi(\mu)$ in $\psi(t)\chi(\nu) = (1-t)\phi(t)\chi(\nu)$ is $p^C_{\mu,\nu}(t) = (1-t)p^B_{\mu+\omega,\nu+\omega}(t)$. $\square$

For each integer $m \geqslant 0$, the Laurent polynomial $\langle m\rangle := t^{-m}(1+t^2+\cdots+t^{2m})$ may be identified with the Weyl character for $\mathcal{A}_1$ whose highest weight is m times the fundamental weight. In this way, the coefficients of the polynomials $p^C_{\mu,\nu}(t)$ may be recognized as tensor product multiplicities for $\mathcal{A}_1$.

Corollary 5.2. *If $\nu \in \Lambda^+$, $\mu \in B^+(\nu)$, and $n_0, n_1, \ldots$ are defined as above, then the multiplicity of $\chi(\mu)$ in $\chi(\omega_i)\chi(\nu)$ is the same as the multiplicity of $\langle i-1\rangle$ in $\langle n_0\rangle\langle n_1\rangle\cdots$.*

Proof. By Proposition 5.1, the multiplicity of $\chi(\mu)$ in $\chi(\omega_i)\chi(\nu)$ is the same as coefficient of t^{n-i+1} in $p^C_{\mu,\nu}(t) = t^n(1-t^2)\langle n_0\rangle\langle n_1\rangle\cdots$. On the other hand, the coefficient of t^{n-i+1} in $t^n(1-t^2)\langle m\rangle$ is 1 or 0 according to whether $m = i-1$. $\square$

B. A Semi-Minuscule Chain Rule.

The weight ω_1 is not minuscule; however, $(1/2)\omega_1$ is a minuscule weight for $\mathcal{B}_n$. Since $\mathcal{B}_n$ and $\mathcal{C}_n$ share the same Weyl group W, it follows that the Bruhat ordering of the W-orbit of ω_1 is the standard order; i.e., $\zeta \leqslant \xi$ if and only if $\xi - \zeta$ is in the nonnegative span of the simple roots (cf. Remark 2.18(b)).

Let $W' \subset W$ denote the Weyl group of the root subsystem of Φ isomorphic to $\mathcal{A}_{n-1}$ generated by $\alpha_2, \ldots, \alpha_n$, and define an equivalence relation on the W-orbit of $(1/2)\omega_1$ by declaring $\zeta \sim \xi$ if ζ and ξ belong to the same W'-orbit, or equivalently, $\langle\zeta,\omega_1\rangle = \langle\xi,\omega_1\rangle$.

Proposition 5.3. *For all $\nu \in \Lambda^+$ and $m \geqslant 0$, we have*

$$\chi(m\omega_1)\chi(\nu) = \sum_{\xi_1 \leqslant \cdots \leqslant \xi_{2m}} \chi(\nu + \xi_1 + \cdots + \xi_{2m}),$$

where the sum is restricted to $2m$-chains in the W-orbit of $(1/2)\omega_1$ such that $\xi_{2i-1} \sim \xi_{2i}$ $(1 \leqslant i \leqslant m)$ and $\nu + \xi_1 + \cdots + \xi_i$ is dominant $(1 \leqslant i \leqslant 2m)$.

Proof. Since $\langle\omega_1, \alpha^\vee\rangle \leqslant 2$ for all roots α, it follows that every LS chain of type ω_1 is either a singleton, or a doubleton of the form $\zeta_1 <_{1/2} \zeta_2$. Moreover, a singleton ζ may be viewed as a "weak" doubleton in which $\zeta_1 = \zeta_2 = \zeta$; in this way, all LS chains are pairs (ζ_1, ζ_2) such that $\zeta_1 \leqslant_{1/2} \zeta_2$. If we set $\xi_1 = (1/2)\zeta_1$ and $\xi_2 = (1/2)\zeta_2$, then the weight of the pair (ζ_1, ζ_2) is $\xi_1 + \xi_2$, and the depth vector δ (see (2.6)) is given by

$$\langle \delta, \alpha_i^\vee \rangle = \min(0,\ \langle \xi_1, \alpha_i^\vee \rangle,\ \langle \xi_1 + \xi_2, \alpha_i^\vee \rangle).$$

Thus for $\nu \in \Lambda^+$, $\nu + \delta$ is dominant if and only if $\nu + \xi_1$ and $\nu + \xi_1 + \xi_2$ are dominant.

Now since $(1/2)\omega_1$ is minuscule relative to $\mathcal{B}_n$, it follows that the covering relations of the Bruhat orderings of the W-orbits of ω_1 or $(1/2)\omega_1$ are generated by simple reflections (see Remark 2.18(b)). Furthermore, for all $\zeta \in W\omega_1$, we have that $\langle \zeta, \alpha_i^\vee \rangle$ is even if and only if $i \neq 1$, so $\zeta_1 \leqslant_{1/2} \zeta_2$ if and only if $\xi_1 \leqslant \xi_2$ and $\xi_1 \sim \xi_2$. Now apply the Chain Rule (Proposition 2.15). $\square$

Remark 5.4. (a) Note that ξ is in the nonnegative span of the simple roots if and only if $\langle \xi, \omega_i \rangle \geqslant 0$ for $1 \leqslant i \leqslant n$. Thus for ξ_1, ξ_2 in the W-orbit of $(1/2)\omega_1$, we have $\xi_1 \leqslant \xi_2$ if and only if $N_i(\xi_1) \leqslant N_i(\xi_2)$ for $1 \leqslant i \leqslant n$, where $N_i(\xi)$ denotes the sum of the last i coordinates of ξ relative to $\varepsilon_1, \dots, \varepsilon_n$. Moreover, $\xi_1 \sim \xi_2$ if and only if $N_n(\xi_1) = N_n(\xi_2)$.

(b) For a chain $\xi_1 \leqslant \cdots \leqslant \xi_{2m}$ as in Proposition 5.3, the sequence formed by the coefficients of ε_n must be non-decreasing, and hence consists of (say) k copies of $-1/2$, followed by $2m - k$ copies of $1/2$. If this chain contributes to the multiplicity of $\chi(\lambda)$ in $\chi(m\omega_1)\chi(\nu)$, then the coefficient of ε_n in $\lambda - \nu$ must be $m - k$. Thus for fixed m, ν and λ, the last coordinate of each term in every contributing chain is completely determined.

(c) Let $\xi \mapsto \bar{\xi}$ denote orthogonal projection onto the span of $\varepsilon_1, \dots, \varepsilon_{n-1}$, a mapping that projects the weight lattice of $\mathcal{C}_n$ onto the weight lattice of $\mathcal{C}_{n-1}$. If (in the notation of (b)) we have $\langle \lambda - \nu, \varepsilon_n \rangle = m$, then $\langle \xi_i, \varepsilon_n \rangle = 1/2$ for all i, and $\bar{\xi}_1 \leqslant \cdots \leqslant \bar{\xi}_{2m}$ is a chain that contributes to the decomposition of $\chi(m\bar{\omega}_1)\chi(\bar{\nu})$, a product of $\mathcal{C}_{n-1}$-characters. If this product is known to be multiplicity-free (e.g., by induction), then we may prove the same for $\chi(m\omega_1)\chi(\nu)$ by considering only those λ for which $\langle \lambda - \nu, \varepsilon_n \rangle < m$. In these cases, we have $\langle \xi_1, \varepsilon_n \rangle = -1/2$ in every contributing chain.

C. Proof of Theorem 1.1.C.

The products listed in Theorem 1.1.C can be shown to be multiplicity-free as follows.

(i) Apply the Minuscule Rule (Proposition 2.13).

(ii) Apply the Multi-Minuscule Rule (Corollary 2.16) and the criterion of Corollary 2.17. The W-orbit of ω_n consists of the vectors $\pm\varepsilon_i$, and their Bruhat ordering is the chain

$$-\varepsilon_n < \cdots < -\varepsilon_1 < \varepsilon_1 < \cdots < \varepsilon_n.$$

If we take $\nu = m'\omega_j$ ($m' > 0$), then $-\varepsilon_j$ and ε_n are the unique weights $\xi \in W\omega_n$ such that $\nu + \xi$ is dominant, and it is not hard to see that every generically ν-dominant chain is a subset of the chain $-\varepsilon_j < \varepsilon_{j-1} < \varepsilon_j < \varepsilon_n$ (omitting ε_{j-1} if $j = 1$ and ε_j if $j = n$). This chain is affinely independent in the sense of Remark 2.18, so by Corollary 2.17, the product $\chi(m\omega_n)\chi(m'\omega_j)$ is multiplicity-free.

(iii) The Rule for Fundamental Weights (Proposition 5.1) shows that the multiplicity of $\chi(\lambda)$ in $\psi(t)\chi(m\omega_1 + m\omega_j)$ has the form $t^a(1 - t^{2b})(1 + t^2 + \cdots + t^{2c})$ for various sets of integers a, b, c. The coefficients of these polynomials are clearly $\leqslant 1$.

(iv) Applying the Semi-Minuscule Rule, let $\xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \xi_4$ be a chain contributing to the multiplicity of $\chi(\lambda)$ in $\chi(2\omega_1)\chi(m\omega_j)$, as in Proposition 5.3. We claim that the chain is uniquely determined by m, j and λ. By Remark 5.4(c), we may assume $\langle \xi_1, \varepsilon_n \rangle = -1/2$, in which case the condition that $m\omega_j + \xi_1$ is dominant forces $\xi_1 = (1/2)\omega_1 - \omega_j$, and hence the conditions $\xi_1 \leqslant \xi_2$, $\xi_1 \sim \xi_2$ and $m\omega_j + \xi_1 + \xi_2 \in \Lambda^+$ force

$$\begin{aligned} \xi_2 &= (-1/2)\omega_1 + \omega_{k+1} - \omega_j + \omega_{n-k+1}, \\ m\omega_j + \xi_1 + \xi_2 &= \omega_{k+1} + (m-2)\omega_j + \omega_{n-k+1} \end{aligned}$$

for some $k \leqslant \min(j-1, n-j+1)$ (including possibly $k = 0$, in which case $\omega_{n-k+1} = 0$). Since the coefficients of $\varepsilon_{n-k+1}, \ldots, \varepsilon_n$ in ξ_2 are all $1/2$, the same is true for every $\xi \geqslant \xi_2$, and in particular, for ξ_3 and ξ_4. Furthermore, k is the smallest index such that the coefficient of ε_{n-k} in $m\omega_j + \xi_1 + \xi_2$ is $\leqslant m-1$, so k is also the smallest index such that the coefficient of ε_{n-k} in λ is $\leqslant m$. Thus k and ξ_2 are uniquely determined. Also, reasoning similar to Remark 5.4(c) shows that if we project ξ_3 and ξ_4 onto the span of $\varepsilon_1, \ldots, \varepsilon_{n-k}$, we obtain a doubleton that contributes a term in the decomposition of a product of two $\mathcal{C}_{n-k}$-characters; namely, $\chi(\omega_1)\chi(\omega_{k+1} + (m-2)\omega_j)$. This product is multiplicity-free by (vii), so the projections of ξ_3 and ξ_4, and hence ξ_3 and ξ_4 themselves, are unique.

(v) Applying the Semi-Minuscule Rule, set $\nu = \omega_i + \omega_j$ $(i \leqslant j)$ and let $\xi_1 \leqslant \cdots \leqslant \xi_{2m}$ be a chain that contributes to the multiplicity of $\chi(\lambda)$ in $\chi(m\omega_1)\chi(\nu)$, as in Proposition 5.3. We claim that the chain is uniquely determined by m, ν and λ. By Remark 5.4(c), we may assume that $\langle \xi_1, \varepsilon_n \rangle = -1/2$. Setting $\nu_l = \nu + \xi_1 + \cdots + \xi_l$, the condition that ν_1 is dominant forces

$$\xi_1 = (1/2)\omega_1 - \omega_i + \omega_k - \omega_j, \quad \nu_1 = (1/2)\omega_1 + \omega_k \quad (i \leqslant k \leqslant j)$$

for some k. Furthermore, the condition that ν_2 is dominant forces

$$\xi_2 = (-1/2)\omega_1 + \omega_a - \omega_k + \omega_b, \quad \nu_2 = \omega_a + \omega_b \quad (1 \leqslant a \leqslant k \leqslant b \leqslant n+1)$$

for some a, b, with the convention that $\omega_{n+1} = 0$. By adding the condition $\xi_1 \sim \xi_2$, we also obtain that k (and hence ξ_1) is a function of a and b. We now claim that a and b are uniquely determined. Since the coefficients of $\varepsilon_b, \ldots, \varepsilon_n$ in ξ_2 are all $1/2$, the same must be true for $\xi_3, \ldots, \xi_{2m}$. Along with the condition that ν_3 is dominant, this forces

$$\xi_3 = (1/2)\omega_1 - \omega_a + \omega_c, \quad \nu_3 = (1/2)\omega_1 + \omega_c + \omega_b \quad (a \leqslant c \leqslant b)$$

for some c, and in turn, $\xi_3 \leqslant \xi_4$, $\xi_3 \sim \xi_4$ and $\nu_4 \in \Lambda^+$ force

$$\xi_4 = (-1/2)\omega_1 + \omega_{c-a+1}, \quad \nu_4 = \omega_{c-a+1} + \omega_c + \omega_b.$$

The coefficients of $\varepsilon_1, \ldots, \varepsilon_n$ are either 0 in ν_4 or $1/2$ in ξ_4, so every subsequent term in the chain must be $(1/2)\omega_1$. Therefore $\lambda = (m-2)\omega_1 + \omega_{c-a+1} + \omega_c + \omega_b$, so a, b and c (and hence the entire chain) are uniquely determined.

(vi) Applying the Semi-Minuscule Rule, set $\nu = 3\omega_i$ and let $\xi_1 \leqslant \cdots \leqslant \xi_{2m}$ be a chain that contributes to the multiplicity of $\chi(\lambda)$ in $\chi(m\omega_1)\chi(\nu)$, as in Proposition 5.3.

We claim that the chain is uniquely determined by m, i and λ. We may assume that $\langle\xi_1,\varepsilon_n\rangle = -1/2$ (Remark 5.4(c)), in which case the condition that $\nu+\xi_1$ is dominant forces $\xi_1 = (1/2)\omega_1 - \omega_i$ and $\nu+\xi_1 = (1/2)\omega_1 + 2\omega_i$. In turn, the conditions $\xi_1 \leqslant \xi_2$, $\xi_1 \sim \xi_2$ and $\nu+\xi_1+\xi_2 \in \Lambda^+$ force

$$\xi_2 = (-1/2)\omega_1 + \omega_{k+1} - \omega_i + \omega_{n-k+1}, \quad \nu+\xi_1+\xi_2 = \omega_{k+1} + \omega_i + \omega_{n-k+1}$$

for some $k \leqslant \min(i-1, n-i+1)$. Now since the coefficients of $\varepsilon_{n-k+1},\dots,\varepsilon_n$ in ξ_2 are all $1/2$, the same must be true for $\xi_3,\dots,\xi_{2m}$. Also, k is the smallest index such that the coefficient of ε_{n-k} in $\nu+\xi_1+\xi_2$ is $\leqslant 2$, so k is also the smallest index such that the coefficient of ε_{n-k} in λ is $\leqslant m+1$. Thus k and ξ_2 are uniquely determined. In addition, reasoning similar to Remark 5.4(c) shows that if we project $\xi_3,\dots,\xi_{2m}$ onto the span of $\varepsilon_1,\dots,\varepsilon_{n-k}$, we obtain a chain that contributes a term in the decomposition of a product of two $\mathcal{C}_{n-k}$-characters; namely, $\chi((m-1)\omega_1)\chi(\omega_{k+1}+\omega_i)$. This product is multiplicity-free by (v), so the projected chain, and hence the original chain, is unique.

(vii) The Rule for Fundamental Weights (Corollary 5.2) shows that the multiplicity of $\chi(\mu)$ in $\chi(\omega_1)\chi(m\omega_1+m\omega_i+m\omega_j)$ is the same as the multiplicity of $\langle 0\rangle$ in $\langle a\rangle\langle b\rangle\langle c\rangle$, or equivalently (Proposition 2.8) the multiplicity of $\langle c\rangle$ in $\langle a\rangle\langle b\rangle$, for various integers a,b,c. However, $\langle a\rangle\langle b\rangle = \langle a+b\rangle + \langle a+b-2\rangle + \cdots + \langle |a-b|\rangle$ is multiplicity-free.

(viii) Applying the Semi-Minuscule Rule, set $\nu = m'\omega_1 + \omega_j$ and let $\xi_1 \leqslant \cdots \leqslant \xi_{2m}$ be a chain that contributes to the multiplicity of $\chi(\lambda)$ in $\chi(m\omega_1)\chi(\nu)$, as in Proposition 5.3. We claim that the chain is uniquely determined by m, ν and λ. By Remark 5.4(c), we may assume $\langle\xi_1,\varepsilon_n\rangle = -1/2$. Since $\nu+\xi_1$ must be dominant, we have

$$\xi_1 = (-1/2)\omega_1 + \omega_i - \omega_j, \quad \nu+\xi_1 = (m'-1/2)\omega_1 + \omega_i$$

for some $i \leqslant j$. In that case, the conditions $\xi_1 \leqslant \xi_2$, $\xi_1 \sim \xi_2$ and $\nu+\xi_1+\xi_2 \in \Lambda^+$ force

$$\xi_2 = (-1/2)\omega_1 + \omega_{k+1}, \quad \nu+\xi_1+\xi_2 = (m'-1)\omega_1 + \omega_i + \omega_{k+1},$$

where $k = n-j+i \geqslant i$, following the convention that $\omega_{n+1} = 0$. Now since the coefficients of $\varepsilon_{k+1},\dots,\varepsilon_n$ in ξ_2 are all $1/2$, the same must be true for $\xi_3,\dots,\xi_{2m}$. Furthermore, k is the largest index such that the coefficient of ε_k in $\nu+\xi_1+\xi_2$ is $\leqslant m'$, so k is also the largest index such that the coefficient of ε_k in λ is $\leqslant m'+m-1$. Thus k, i, ξ_1 and ξ_2 are uniquely determined. In addition, reasoning similar to Remark 5.4(c) shows that if we orthogonally project $\xi_3,\dots,\xi_{2m}$ onto the span of $\varepsilon_1,\dots,\varepsilon_k$, we obtain a chain that contributes a term in the decomposition of a product of two $\mathcal{C}_k$-characters; namely, $\chi((m-1)\omega_1)\chi((m'-1)\omega_1+\omega_i)$. By induction, it follows that this projected chain, and hence the original chain, is unique.

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ,ν) such that

(N1) $(\mu,\nu) = (\omega_k, \omega_i+\omega_j)$ $(1<k<n, 1<i<j\leqslant n)$,
(N2) $(\mu,\nu) = (\omega_1+\omega_n, \omega_i+\omega_j)$ $(1<i<j\leqslant n)$,
(N3) $(\mu,\nu) = (\omega_1+\omega_i, 2\omega_j)$ $(1<i,j\leqslant n)$,
(N4) $(\mu,\nu) = (\omega_1+\omega_i, \omega_1+\omega_j)$ $(1<i,j\leqslant n)$,

(N5) $(\mu,\nu) = (2\omega_i, 2\omega_j)$ $(1 < i, j < n)$,
(N6) $(\mu,\nu) = (2\omega_n, \omega_i + \omega_j)$ $(1 \leqslant i < j \leqslant n)$,
(N7) $(\mu,\nu) = (\omega_1, \omega_i + \omega_j + \omega_k)$ $(1 < i < j < k \leqslant n)$,
(N8) $(\mu,\nu) = (2\omega_1, \omega_1 + \omega_i + \omega_j)$ $(1 < i \leqslant j \leqslant n)$,
(N9) $(\mu,\nu) = (2\omega_1, \omega_i + 2\omega_j)$ $(1 < i, j \leqslant n,\ i \neq j)$, or
(N10) $(\mu,\nu) = (3\omega_1, 4\omega_i)$ $(1 < i < n)$.

These products can be shown not to be multiplicity-free as follows.

(N1) Applying the Rule for Fundamental Weights (Corollary 5.2), the multiplicity of $\chi(\mu)$ in $\chi(\omega_k)\chi(\omega_i + \omega_j)$ is the same as the multiplicity of $\langle k-1\rangle$ in $\langle a\rangle\langle b\rangle\langle c\rangle$ for certain integers $a, b, c \geqslant 0$ such that $a \leqslant i-1$, $b \leqslant j-i$ and $c \leqslant n-j+1$. Moreover, as μ varies over $B^+(\omega_i + \omega_j)$, the corresponding a, b and c assume all possible values in these ranges. Now if a, b, c are positive, then $\langle a+b-2\rangle$ occurs in $\langle a\rangle\langle b\rangle$ and $\langle a+b+c-2\rangle$ occurs in both $\langle a+b\rangle\langle c\rangle$ and $\langle a+b-2\rangle\langle c\rangle$ and hence twice in $\langle a\rangle\langle b\rangle\langle c\rangle$. However, as a, b, c vary over the positive parts of their ranges, $a+b+c-2$ assumes every value from 1 to $n-2$.

(N2) It suffices to show that for $1 < i < j \leqslant n$, we have

$$c(\omega_1 + \omega_{n-j+i}\,;\, \omega_1 + \omega_n\,,\, \omega_i + \omega_j) \geqslant 2. \tag{5.1}$$

By the Minuscule Rule (Proposition 2.12), we have $\chi(\omega_n)\chi(\omega_1) = \chi(\omega_1 + \omega_n) + \chi(\omega_2)$, and we know that $\chi(\omega_2)\chi(\omega_1 + \omega_j)$ is multiplicity-free for all j (see item (iii) above), so it suffices by triple symmetry (Proposition 2.8) to show that $\chi(\omega_1 + \omega_{n-j+i})$ has multiplicity at least 3 in $\chi(\omega_n)\chi(\omega_1)\chi(\omega_i + \omega_j)$. For this, one can check that

$$\omega_1 + \omega_{n-j+i} + \omega_n,\ \omega_2 + \omega_{n-j+i},\ \omega_1 + \omega_{n-j+i+1} \in B^+(\omega_i + \omega_j),$$

and for each weight μ in this list, the Rule for Fundamental Weights (Corollary 5.2) shows that the multiplicity of $\chi(\mu)$ in $\chi(\omega_1)\chi(\omega_i + \omega_j)$ is the multiplicity of $\langle 0\rangle$ in $\langle m\rangle\langle m\rangle$, $\langle 1\rangle\langle m\rangle\langle m+1\rangle$ and $\langle m'\rangle\langle m'\rangle$ in these three respective cases, where $m = \min(j-i, n-j)$ and $m' = \min(j-i, n-j+1)$. Each of these multiplicities is positive (in fact, 1), and the Minuscule Rule shows that $\chi(\omega_1 + \omega_{n-j+i})$ occurs in each of the products $\chi(\omega_n)\chi(\mu)$, so the claim follows.

(N3) Proceeding by induction with respect to n, one may use stability (Corollary 2.7) to reduce this to the cases $(\mu,\nu) = (\omega_1 + \omega_i, 2\omega_n)$ and $(\mu,\nu) = (\omega_1 + \omega_n, 2\omega_i)$, where $1 < i \leqslant n$. The former is part of (N6), whereas for the latter, it suffices to show that

$$c(\omega_1 + \omega_n\,;\, 2\omega_i\,,\, \omega_1 + \omega_n) \geqslant 2. \tag{5.2}$$

Applying the Chain Rule (Proposition 2.15), note that $C(\omega_i)$ includes the singleton chains $\xi = \varepsilon_1 + \cdots + \varepsilon_{n-i+1}$ and $\xi = \varepsilon_1 + \cdots + \varepsilon_{n-i} + \varepsilon_n$. In both cases, it is easy to check that $\omega_1 + \omega_n - \xi$ is dominant and that $-\xi \leftarrow \xi$; hence (5.2) follows.

(N4) Proceeding by induction with respect to n, one may use stability (Corollary 2.7) to reduce this to the case $(\mu,\nu) = (\omega_1 + \omega_n, \omega_1 + \omega_k)$, where $1 < k \leqslant n$. For this, apply either (5.1) (if $k < n$) or (5.2) (if $k = n$) and use triple symmetry (Proposition 2.8).

(N5) Proceeding by induction with respect to n, one may use stability (Corollary 2.7) to reduce this to the case $(\mu,\nu) = (2\omega_{n-1}, 2\omega_j)$, where $1 < j < n$. For this, apply the

Twice-Quasi-Minuscule Rule (Corollary 2.20), bearing in mind that the quasi-minuscule weight $\omega_{n-1} = \bar{\alpha}$ has simple root coordinates $(1, 2, \ldots, 2, 1)$.

(N6) Apply the Adjoint Rule (Proposition 2.14).

(N7) If $\nu = \omega_i + \omega_j + \omega_k$ $(1 < i < j < k \leqslant n)$ and $\lambda = \omega_2 + \omega_{i-1} + \omega_{j-1} + \omega_{k-1}$, then by the Rule for Fundamental Weights (Proposition 5.1), the multiplicity of $\chi(\lambda)$ in $\chi(\omega_1)\chi(\nu)$ is the coefficient of t^n in $p^C_{\lambda,\nu}(t) = t^{n-4}(1-t^2)(1+t^2)^4$ (namely, 2).

(N8) Applying the Semi-Minuscule Rule (Proposition 5.3) with $\nu = \omega_1 + \omega_i + \omega_j$, it is easy to verify that for all $j \geqslant i > 1$, the chains

$$\begin{gathered}(-1/2)\omega_1 \leqslant (-1/2)\omega_1 \leqslant (1/2)\omega_1 - \omega_i + \omega_j \leqslant (-1/2)\omega_1 + \omega_{j-i+1},\\ (-1/2)\omega_1 + \omega_{i-1} - \omega_i \leqslant (-1/2)\omega_1 + \omega_{j-1} - \omega_j \\ \leqslant (1/2)\omega_1 - \omega_{i-1} + \omega_j \leqslant (-1/2)\omega_1 + \omega_{j-i+1} - \omega_{j-1} + \omega_j\end{gathered}$$

show that the multiplicity of $\chi(\omega_{j-i+1} + 2\omega_j)$ in $\chi(2\omega_1)\chi(\nu)$ is at least 2.

(N9) Setting $\nu = \omega_i + 2\omega_j$ or $\nu = 2\omega_i + \omega_j$ $(1 < i < j \leqslant n)$, the first case follows from triple symmetry (Proposition 2.8) and the calculation in (N8). For the second, apply the Semi-Minuscule Rule (Proposition 5.3), noting that the chains

$$\begin{gathered}(1/2)\omega_1 - \omega_i \leqslant (1/2)\omega_1 - \omega_i \leqslant (-1/2)\omega_1 + \omega_2 \leqslant (-1/2)\omega_1 + \omega_2,\\ (1/2)\omega_1 - \omega_i \leqslant (-1/2)\omega_1 + \omega_2 - \omega_i + \omega_{j-1} - \omega_j \leqslant (1/2)\omega_1 - \omega_{j-1} + \omega_j \leqslant (-1/2)\omega_1 + \omega_2\end{gathered}$$

show that the multiplicity of $\chi(2\omega_2 + \omega_j)$ in $\chi(2\omega_1)\chi(\nu)$ is at least 2.

(N10) Proceeding by induction with respect to n, one may use stability (Corollary 2.7) to reduce this to the case $(\mu, \nu) = (4\omega_{n-1}, 3\omega_1)$. Applying the Chain Rule (Proposition 2.15), there is a pair of chains of singletons from $C(\omega_{n-1})$; namely,

$$\begin{gathered}-\varepsilon_2 - \varepsilon_1 \leftarrow -\varepsilon_2 - \varepsilon_1 \leftarrow \varepsilon_n + \varepsilon_2 \leftarrow \varepsilon_n + \varepsilon_2,\\ -\varepsilon_2 - \varepsilon_1 \leftarrow \varepsilon_2 - \varepsilon_1 \leftarrow \varepsilon_n - \varepsilon_1 \leftarrow \varepsilon_n + \varepsilon_1,\end{gathered}$$

showing that the multiplicity of $\chi(\omega_1 + 2\omega_2 + 2\omega_n)$ in $\chi(4\omega_{n-1})\chi(3\omega_1)$ is at least 2.

6. The case $\Phi = \mathcal{D}_n$

Realization: $\mathcal{D}_n = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\}$.

Simple roots: $\alpha_1 = \varepsilon_1 + \varepsilon_2$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ $(1 < i \leqslant n)$.

Fundamental weights: $\omega_{1,2} = (1/2)(\pm\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$, $\omega_i = \varepsilon_i + \cdots + \varepsilon_n$ $(2 < i \leqslant n)$.

Minuscule weights: $\omega_1, \omega_2, \omega_n$. Quasi-minuscule weight: ω_{n-1}.

A. The Exterior Powers Rule.

The defining representation of $\mathfrak{g} = so(2n)$ is minuscule with highest weight $\omega_n = \varepsilon_n$, and the exterior algebra of this representation has the (graded) character

$$\theta(t) = \sum_{k=0}^{2n} \theta_k t^k := \prod_{i=1}^{n} (1 + te^{\varepsilon_i})(1 + te^{-\varepsilon_i}).$$

For each $\nu \in \Lambda^+$, let $D^+(\nu)$ denote the set of all $\mu \in \Lambda^+$ of the form $\nu \pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k}$, and for all such μ, define

$$p^D_{\mu,\nu}(t) = t^k \prod_{r \geqslant 1} (1 + t^2 + \cdots + t^{2n_r}) \cdot \begin{cases} 1 + t^{2n_0} & \text{if } n_0 > 0, \\ 1 & \text{if } n_0 = 0, \end{cases}$$

where $n_r = |\{i : \langle \mu, \varepsilon_i \rangle = \langle \nu, \varepsilon_i \rangle = \pm r/2\}|$ and $k = n - \sum n_r = |\{i : \langle \mu, \varepsilon_i \rangle \neq \langle \nu, \varepsilon_i \rangle\}|$. Note that $\langle \mu, \varepsilon_i \rangle = \langle \nu, \varepsilon_i \rangle = -r/2 < 0$ may occur only if $i = 1$.

Proposition 6.1. *For all $\nu \in \Lambda^+$, we have*

$$\theta(t)\chi(\nu) = \sum_{\mu \in D^+(\nu)} p^D_{\mu,\nu}(t)\chi(\mu).$$

In particular (taking $\nu = 0$), we have $\theta_i = \chi(\omega_{n-i+1})$ $(1 \leqslant i \leqslant n-2)$, $\theta_{n-1} = \chi(\omega_1 + \omega_2)$, and $\theta_n = \chi(2\omega_1) + \chi(2\omega_2)$.

Proof. Proceeding as in the proof of Proposition 4.1, we have

$$\theta(t)\chi(\nu) = \sum_{K \subseteq E} t^{|K|} \chi(\nu + \sigma(K)), \tag{6.1}$$

where $E = \{\pm\varepsilon_1, \ldots, \pm\varepsilon_n\}$ and $\sigma(K)$ denotes the sum of the members of K.

Given $K \subseteq E$, consider the largest $j > 1$ (if any) such that

(1) $-\varepsilon_j, \varepsilon_{j-1} \in K$, $\varepsilon_j, -\varepsilon_{j-1} \notin K$ and $\langle \nu, \varepsilon_j \rangle = \langle \nu, \varepsilon_{j-1} \rangle$, or
(2) $\varepsilon_j, -\varepsilon_j \in K$, $\varepsilon_{j-1}, -\varepsilon_{j-1} \notin K$ and $\langle \nu, \varepsilon_j \rangle = \langle \nu, \varepsilon_{j-1} \rangle$.

If (1) holds, then we have $\langle \nu + \sigma(K), \alpha_j^\vee \rangle = -2$ and $s_j(\nu + \sigma(K) + \rho) = \nu + \sigma(K) + \alpha_j + \rho$, whence $\chi(\nu + \sigma(K)) = -\chi(\nu + \sigma(K'))$, where $K' = K \cup \{\varepsilon_j\} - \{\varepsilon_{j-1}\}$. Furthermore, K' satisfies (2), and there is no larger j such that K' satisfies (1) or (2). Hence, the pairing $K \leftrightarrow K'$ is bijective (and preserves cardinality), so we may delete all terms from (6.1) satisfying (1) or (2) for any $j > 1$.

Among the remaining choices for K, consider the largest $j > 1$ (if any) such that

(3) $-\varepsilon_2, -\varepsilon_1 \in K$, $\varepsilon_2, \varepsilon_1 \notin K$ and $\langle \nu, \varepsilon_2 \rangle = -\langle \nu, \varepsilon_1 \rangle > 0$,
(4) $\varepsilon_2, -\varepsilon_2 \in K$, $\varepsilon_1, -\varepsilon_1 \notin K$ and $\langle \nu, \varepsilon_2 \rangle = -\langle \nu, \varepsilon_1 \rangle > 0$,
(5) $-\varepsilon_1, -\varepsilon_2, \varepsilon_i, -\varepsilon_i \in K$ $(3 \leqslant i \leqslant j)$, $\varepsilon_1, \varepsilon_2 \notin K$ and $\langle \nu, \varepsilon_i \rangle = 0$ $(1 \leqslant i \leqslant j)$, or
(6) $\varepsilon_i, -\varepsilon_i \in K$ $(1 \leqslant i < j)$, $\varepsilon_j, -\varepsilon_j \notin K$ and $\langle \nu, \varepsilon_i \rangle = 0$ $(1 \leqslant i \leqslant j)$.

If (3) holds, then we have $\langle \nu + \sigma(K), \alpha_1^\vee \rangle = -2$ and $\chi(\nu + \sigma(K)) = -\chi(\nu + \sigma(K'))$, where $K' = K \cup \{\varepsilon_2\} - \{-\varepsilon_1\}$. Furthermore, K' satisfies (4) and neither K nor K' satisfy (1) or (2), so the pairing of K and K' is bijective. Similarly, if (5) holds, then $\chi(\nu + \sigma(K))$ may be canceled with $\chi(\nu + \sigma(K'))$, where $K' = K \cup \{\varepsilon_1, \varepsilon_2\} - \{\varepsilon_j, -\varepsilon_j\}$ (a term that satisfies (6)).

In all remaining cases, we have $\langle \nu + \sigma(K), \alpha_j^\vee \rangle \geqslant -1$ for all j, whence $\mu := \nu + \sigma(K)$ is dominant, or $\chi(\nu + \sigma(K)) = 0$. The latter terms may also be omitted from (6.1), so we assume that μ is dominant. This given, consider the sets $N_r = \{i : \langle \nu, \varepsilon_i \rangle = \langle \mu, \varepsilon_i \rangle = \pm r/2\}$ for each integer $r \geqslant 0$. The dominance of μ and ν forces N_r to consist of (say) n_r

consecutive integers. Furthermore, we must have $\varepsilon_i, -\varepsilon_i \in K$ or $\varepsilon_i, -\varepsilon_i \notin K$ for all $i \in N_r$, so in order to avoid (2) and (4), it must be the case that for some l,

$$K \cap \{\pm\varepsilon_i : i \in N_r\} = \{\pm\varepsilon_j, \dots, \pm\varepsilon_{j+l-1}\} \quad (0 \leqslant l \leqslant n_r),$$

where j denotes the smallest member of N_r. Note that the generating function for these possibilities is $1 + t^2 + \cdots + t^{2n_r}$. In the case $r = 0$, to avoid (6), we must also have $l = 0$ or $l = n_0$, a pair of choices with generating function $1 + t^{2n_0}$ (if $n_0 > 0$) or 1 (if $n_0 = 0$), so the coefficient of $\chi(\mu)$ in (6.1) is indeed $p^D_{\mu,\nu}(t)$. $\square$

Let $\lambda \mapsto \lambda^\star$ denote reflection through the hyperplane orthogonal to ε_1. Thus $\omega_1^\star = \omega_2$, $\omega_2^\star = \omega_1$, and $\omega_i^\star = \omega_i$ for $i \geqslant 3$. This involution is a diagram automorphism of Φ, so it lifts to an automorphism of $\mathbf{Z}[\Lambda]$ in which $\chi(\lambda) \mapsto \chi(\lambda^\star)$ for all $\lambda \in \Lambda^+$.

COROLLARY 6.2. *If $\nu = \nu^\star \in \Lambda^+$ and the multiplicity of $\chi(\lambda)$ in $\theta_n\chi(\nu)$ is*

(a) *is at least 3 for some λ, then $\chi(2\omega_1)\chi(\nu)$ and $\chi(2\omega_2)\chi(\nu)$ are not multiplicity-free.*
(b) *at most 1 (respectively, at most 2) for all λ such that $\lambda \neq \lambda^\star$ (respectively, $\lambda = \lambda^\star$), then $\chi(2\omega_1)\chi(\nu)$ and $\chi(2\omega_2)\chi(\nu)$ are multiplicity-free.*

Proof. (a) By Proposition 6.1, one knows $\theta_n = \chi(2\omega_1) + \chi(2\omega_2)$. In particular, $\theta_n\chi(\nu)$ is *-invariant, so if the multiplicity of $\chi(\lambda)$ in $\theta_n\chi(\nu)$ is $\geqslant 3$, then the multiplicity of $\chi(\lambda)$ in $\chi(2\omega_1)\chi(\nu)$ or $\chi(2\omega_2)\chi(\nu)$ is $\geqslant 2$, and a similar statement holds for $\chi(\lambda^\star)$ with ω_1 and ω_2 interchanged.

(b) If the stated conditions hold, then $\chi(\lambda)$ could have multiplicity $\geqslant 2$ in $\chi(2\omega_1)\chi(\nu)$ only if $\lambda = \lambda^\star$, in which case the multiplicity of $\chi(\lambda)$ would also be $\geqslant 2$ in $\chi(2\omega_2)\chi(\nu)$ and hence $\geqslant 4$ in $\theta_n\chi(\nu)$. $\square$

B. Proof of Theorem 1.1.D.

The products listed in Theorem 1.1.D can be shown to be multiplicity-free as follows.

(i) Apply the Minuscule Rule (Proposition 2.12).

(ii) The Exterior Powers Rule (Proposition 6.1) shows that the multiplicity of $\chi(\lambda)$ in $\theta(t)\chi(m\omega_j)$ has the form $t^{n-a}(1 + t^2 + \cdots + t^{2a})$ or $t^{n-a-b}(1 + t^2 + \cdots + t^{2a})(1 + t^{2b})$, where $0 \leqslant a \leqslant n - j + 1$ and $1 \leqslant b < j$ (assuming $j \geqslant 3$). The coefficients of the former are multiplicity-free, whereas the coefficients of t^k in the latter are multiplicity-free for $k < n - a + b$. Since the minimum value of $n - a + b$ is j, it follows that $\theta_i\chi(m\omega_j)$ is multiplicity-free for $i < j$. (The cases with $j < 3$ are part of (iv).)

(iii) A calculation similar to the previous one shows that the multiplicity of $\chi(\lambda)$ in $\theta_n\chi(m\omega_j)$ is $\leqslant 2$, and equality occurs only if $\lambda = \lambda^\star$, so $\chi(2\omega_1)\chi(m\omega_j)$ and $\chi(2\omega_2)\chi(m\omega_j)$ are multiplicity-free by Corollary 6.2.

(iv) By the Exterior Powers Rule (Proposition 6.1), the multiplicity of $\chi(\lambda)$ in either $\theta(t)\chi(m\omega_1)$ or $\theta(t)\chi(m\omega_2)$ has the form $t^{n-a}(1+t^2+\cdots+t^{2a})$ for various a, so $\chi(\omega_i)\chi(m\omega_1)$ and $\chi(\omega_i)\chi(m\omega_2)$ are multiplicity-free for $i \geqslant 3$. (The cases with $i < 3$ are part of (i).)

(v) Apply the Multi-Minuscule Rule (Corollary 2.16) and the criterion of Corollary 2.17. The Bruhat ordering of the minuscule orbit $W\omega_n$ is the union of two chains; namely,

$$-\varepsilon_n < \cdots < -\varepsilon_2 < \begin{matrix} -\varepsilon_1 \\ \varepsilon_1 \end{matrix} < \varepsilon_2 < \cdots < \varepsilon_n.$$

By means of diagram automorphisms, we may assume $\nu = m_1\omega_1 + m_i\omega_i$ $(2 \leqslant i \leqslant n)$. If $i > 2$, every generically ν-dominant chain is a subset of $-\varepsilon_i < -\varepsilon_1 < \varepsilon_{i-1} < \varepsilon_i < \varepsilon_n$ (omitting ε_n if $i = n$), and this chain is affinely independent in the sense of Remark 2.18. If $i = 2$, there are two maximal generically ν-dominant chains; namely,

$$-\varepsilon_2 < -\varepsilon_1 < \varepsilon_2 < \varepsilon_n, \quad -\varepsilon_2 < \varepsilon_1 < \varepsilon_2 < \varepsilon_n.$$

The union of these two chains has a unique affine dependence relation, but it does not satisfy the nonnegativity constraints of Corollary 2.17.

(vi) Via diagram automorphisms, we may assume $\mu = m\omega_1$. Further automorphisms reduce the case $n = 4$ to (v), so we also assume $n \geqslant 5$. Applying the Multi-Minuscule Rule (Corollary 2.16) with $\nu = m_1\omega_{n-1} + m_2\omega_n$, there is a unique generically ν-dominant chain in $W\omega_1$ that is maximal with respect to inclusion; namely,

$$\omega_1 - \omega_{n-1} < \omega_2 - \omega_n < \omega_2 - \omega_{n-1} + \omega_n < -\omega_1 + \omega_3 < \omega_1.$$

This chain is affinely independent, so $\chi(m\omega_1)\chi(\nu)$ is multiplicity-free by Corollary 2.17. In the case $\nu = m'\omega_{n-2}$, there are two maximal generically ν-dominant chains in $W\omega_1$ when $n \geqslant 6$; namely,

$$\omega_2 - \omega_{n-2} < \omega_1 - \omega_{n-2} + \omega_n < \xi_1 < \begin{matrix}\xi_2 \\ \xi_3\end{matrix} < \xi_4 < -\omega_1 + \omega_3 < \omega_1,$$

where we set $\xi_1 = -\omega_2 + \omega_3 - \omega_{n-2} + \omega_{n-1}$, $\xi_2 = -\omega_2 + \omega_4$, $\xi_3 = \omega_2 - \omega_{n-2} + \omega_{n-1}$, and $\xi_4 = \omega_2 - \omega_3 + \omega_4$. The union of these chains has a unique affine dependence relation,

$$\xi_1 - \xi_2 - \xi_3 + \xi_4 = 0.$$

Since ξ_2 and ξ_3 occur with the same sign, Corollary 2.17 implies that $\chi(m\omega_1)\chi(\nu)$ is multiplicity-free. When $n = 5$, the above pair of chains degenerates into a single chain

$$\omega_2 - \omega_3 < \omega_1 - \omega_3 + \omega_5 < -\omega_2 + \omega_4 < \omega_2 - \omega_3 + \omega_4 < -\omega_1 + \omega_3 < \omega_1,$$

and this chain is affinely independent.

(vii) Applying the Multi-Minuscule Rule (Corollary 2.16) with $\nu = m_1\omega_1 + m_2\omega_2$, there is a unique maximal generically ν-dominant chain in $W\omega_1$; namely, $\xi_n < \cdots < \xi_2 < \omega_1$, where $\xi_k = -\omega_1 + \omega_{k+1}$ (k even) and $\xi_k = -\omega_2 + \omega_{k+1}$ (k odd), following the convention that $\omega_{n+1} = 0$. This chain is affinely independent, so $\chi(m\omega_1)\chi(\nu)$ is multiplicity-free by Corollary 2.17. For the remaining possibilities, it suffices by diagram symmetry to take $\nu = m_1\omega_1 + m_2\omega_n$ and $\mu = m\omega_1$ or $\mu = m\omega_2$. In these cases, a simple inductive argument shows that every term of a ν-dominant chain in $W\omega_1$ or $W\omega_2$ is of the form $\xi(k,l)$, where

$$\xi(k,l) := \begin{cases} -\omega_1 + \omega_k - \omega_l + \omega_{l+1} & \text{if } 3 \leqslant k \leqslant l \leqslant n, \\ \omega_k - \omega_l + \omega_{l+1} & \text{if } 1 \leqslant k < 3 \leqslant l \leqslant n, \end{cases}$$

$\xi(2,2) = -\omega_1 + \omega_3$, $\xi(1,2) = -\omega_2 + \omega_3$, $\xi(1,1) = \omega_2$, and $\xi(0,0) = \omega_1$. Note that $\xi(k,l) \in W\omega_1$ if k is even and $\xi(k,l) \in W\omega_2$ if k is odd, so the parity of k in every term $\xi(k,l)$ that occurs is always even or always odd. Furthermore, given that $\xi(k,l)$ and $\xi(k',l')$ are in the same W-orbit, we have $\xi(k,l) \leqslant \xi(k',l')$ if and only if $k \geqslant k'$ and $l \geqslant l'$. We also claim that if $\xi(k,l)$ appears, then l may have parity opposite to k only if $l = n$. Indeed, if k and l have opposite parity, then $l \geqslant 2$ and $-\omega_l$ appears in $\xi(k,l)$, and hence ω_l must appear in some previous term (if $l < n$); this is possible only if there is an $l' \geqslant l$ such that $\xi(l,l')$ appears, contradicting the hypothesis that k and l have opposite parity. Thus, every ν-dominant chain has the form $\xi(k_1,l_1) \leqslant \cdots \leqslant \xi(k_m,l_m)$, where $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$, $l_1 \geqslant \cdots \geqslant l_m \geqslant 0$, and $k_1, l_1, \ldots$ all have the same parity, except possibly the occurrences of n among $l_1,\ldots,l_m$. To prove that $\chi(m\omega_1)\chi(\nu)$ and $\chi(m\omega_2)\chi(\nu)$ are multiplicity-free, it therefore suffices to show that for fixed m, all such chains are uniquely determined by their total weight, say λ. To see this, note that the coefficient of ε_n in $\xi(k,l)$ is $-1/2$ for $l = n$ and $1/2$ for $l < n$, so the number of indices i such that $l_i = n$ is $m/2 - \langle \lambda, \varepsilon_n \rangle$. The remaining l_i's and k_j's are either all even or all odd. In these respective cases, the number of indices i such that $l_i = 2,4,\ldots$ (resp., $l_i = 1,3,\ldots$) are the coefficients of $\omega_3, \omega_5, \ldots$ (resp., $\omega_2, \omega_4, \ldots$) in λ. Since the length of the sequence is m, this also determines the number of indices with $l_i = 0$, so the entire l-sequence is determined. In particular, the number of occurrences of $\xi(0,0)$ and $\xi(1,1)$ is also determined, and it is easy to see, given the l-sequence, that the number of indices j such that $k_j = 4,6,\ldots$ (resp., $k_j = 3,5,\ldots$) may be inferred from the coefficients of $\omega_4, \omega_6, \ldots$ (resp., $\omega_3, \omega_5, \ldots$) in λ. Since the sequence has length m, this also determines the respective number of indices j such that $k_j = 2$ and $k_j = 1$, and hence the entire chain.

(viii) follows from (v) by diagram automorphisms.

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ,ν) such that

(N1) $(\mu,\nu) = (\omega_i, \omega_j)$ $(i,j \geqslant 3, i+j \leqslant n+1)$,
(N2) $(\mu,\nu) = (\omega_k, \omega_i + \omega_j)$ $(3 \leqslant k < n, 1 \leqslant i < j \leqslant n)$,
(N3) $(\mu,\nu) = (2\omega_i, 2\omega_j)$ $(3 \leqslant i,j < n)$,
(N4) $\mu,\nu \in \{\omega_1+\omega_2, \omega_1+\omega_n, \omega_2+\omega_n\}$,
(N5) $\mu = 2\omega_n$ and $\nu = \omega_i + \omega_j$ $(3 \leqslant i < j \leqslant n)$ or $\nu = \omega_1 + \omega_2 + \omega_j$ $(3 \leqslant j \leqslant n)$,
(N6) $\mu \in \{2\omega_1, 2\omega_2\}$ and $\nu = \omega_i + \omega_j$ $(i = 1,2,\ 3 \leqslant j < n)$,
(N7) $\mu \in \{2\omega_1, 2\omega_2\}$ and $\nu = \omega_i + \omega_j$ $(3 \leqslant i < j \leqslant n,\ (i,j) \neq (n-1,n))$,
(N8) $\mu \in \{2\omega_1, 2\omega_2\}$ and $\nu = \omega_1 + \omega_2 + \omega_n$, or
(N9) $\mu \in \{3\omega_1, 3\omega_2\}$ and $\nu = 2\omega_i$ $(3 \leqslant i \leqslant n-3)$.

When $n = 4$, (N6) should be replaced with

(N6′) $(\mu,\nu) = (2\omega_i, \omega_i + \omega_3)$ or $(\mu,\nu) = (2\omega_i, \omega_{3-i} + \omega_3 + \omega_4)$ $(i = 1,2)$.

These products can be shown not to be multiplicity-free as follows.

(N1) Applying the Exterior Powers Rule (Proposition 6.1), set $\nu = \omega_j$ $(3 \leqslant j < n)$, and note that $\chi(\omega_3)$ and $\chi(\omega_1+\omega_2)$ have multiplicity $t^{j-3}(1+t^4)(1+t^2+\cdots+t^{2(n-j+1)})$ and $t^{j-2}(1+t^2)(1+t^2+\cdots+t^{2(n-j+1)})$ in $\theta(t)\chi(\nu)$. For each i such that $j \leqslant i < n$, the coefficient of t^i in one of these expressions is 2, so $\chi(\omega_{n-i+1})\chi(\omega_j)$ (for $j \leqslant i < n-1$) and $\chi(\omega_1+\omega_2)\chi(\omega_j)$ are not multiplicity-free.

(N2) The case $\nu = \omega_1 + \omega_2$ is covered by the argument for (N1), so by applying diagram automorphisms if necessary, we may assume $\nu = \omega_1 + \omega_j$ $(3 \leqslant j \leqslant n)$ or $\nu = \omega_i + \omega_j$ $(3 \leqslant i < j \leqslant n)$. In the former case, the Exterior Powers Rule (Proposition 6.1) shows that $\chi(\omega_1+\omega_j)$ and $\chi(\omega_2+\omega_j)$ have multiplicity $(1+t^2+\cdots+t^{2(j-1)})(1+t^2+\cdots+t^{2(n-j+1)})$ and $t(1+t^2+\cdots+t^{2(j-2)})(1+t^2+\cdots+t^{2(n-j+1)})$ in $\theta(t)\chi(\nu)$. For each k such that $2 \leqslant k \leqslant 2n-2$, the coefficient of t^k in one of these expressions is at least 2, and hence $\chi(\omega_k)\chi(\nu)$ is not multiplicity-free for $3 \leqslant k < n$. These calculations also show

$$c(\omega_1+\omega_j;\omega_{n-1},\omega_1+\omega_j) \geqslant 2, \quad c(\omega_2+\omega_j;\omega_{n-2},\omega_1+\omega_j) \geqslant 2, \tag{6.2}$$

and depending on the parity of n, either

$$c(\omega_1+\omega_j;\omega_1+\omega_2,\omega_1+\omega_j) \geqslant 2 \text{ or } c(\omega_2+\omega_j;\omega_1+\omega_2,\omega_1+\omega_j) \geqslant 2. \tag{6.3}$$

In the case $\nu = \omega_i+\omega_j$, set $\lambda_l = \varepsilon_l+\cdots+\varepsilon_n+\omega_j$, and note that for $1 \leqslant l \leqslant i$, the multiplicity of $\chi(\lambda_l)$ in $\theta(t)\chi(\nu)$ is

$$t^{l-1}(1+t^{2(i-l)})(1+t^2+\cdots+t^{2(j-i)})(1+t^2+\cdots+t^{2(n-j+1)}).$$

The coefficients of $t^2, t^4, \ldots, t^{2n-2i}$ in the product of the last two of the these factors are all $\geqslant 2$. At the same time, as l runs from 1 to i, every power of t from t^0 to $t^{2(i-1)}$ occurs in $t^{l-1}(1+t^{2(i-l)})$, so for each k such that $2 \leqslant k \leqslant n$, there is an $l \leqslant k$ such that the multiplicity of $\chi(\lambda_l)$ in $\theta_k\chi(\nu)$ is $\geqslant 2$.

(N3) Proceeding by induction with respect to n, it suffices by stability (Corollary 2.7) to consider the case $(\mu,\nu) = (2\omega_i, 2\omega_{n-1})$. Since the simple root coordinates of $\omega_{n-1} = \bar{\alpha}$ are $(1,1,2,\ldots,2,1)$, this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

(N4) Up to diagram automorphisms, it suffices to show that $\chi(\omega_1+\omega_n)\chi(\omega_1+\omega_n)$, $\chi(\omega_1+\omega_n)\chi(\omega_2+\omega_n)$, $\chi(\omega_1+\omega_2)\chi(\omega_1+\omega_n)$, and $\chi(\omega_1+\omega_2)\chi(\omega_1+\omega_2)$ are not multiplicity-free. The first two of these follow from (6.2) and triple symmetry (Corollary 2.8), the third from (6.3), and the fourth from triple symmetry and the fact that $c(\omega_1+\omega_2;\bar{\alpha},\omega_1+\omega_2) = 2$ (Proposition 2.13).

(N5) Applying the Multi-Minuscule Rule (Corollary 2.16), set $\nu = \omega_i+\omega_j$ (where $3 \leqslant i < j \leqslant n$), and note that the Bruhat ordering of the W-orbit of the minuscule weight $\omega_n = \varepsilon_n$ includes the chains $-\varepsilon_i < \varepsilon_i$ and $-\varepsilon_j < \varepsilon_j$. These chains are ν-dominant, so the multiplicity of $\chi(\nu)$ in $\chi(2\omega_n)\chi(\nu)$ is at least 2. Similarly, if $\nu = \omega_1+\omega_2+\omega_j$ $(3 \leqslant j \leqslant n)$, then the chains $-\varepsilon_2 < \varepsilon_2$ and $-\varepsilon_j < \varepsilon_j$ are ν-dominant.

(N6′) follows from (N5) by diagram automorphisms.

(N6) Applying the Multi-Minuscule Rule (Corollary 2.16) and diagram automorphisms, we may assume $n \geqslant 5$, $\mu = 2\omega_1$ and $\nu = \omega_1+\omega_j$ or $\nu = \omega_2+\omega_j$ $(3 \leqslant j < n)$. In these cases, the Bruhat ordering of the minuscule orbit $W\omega_1$ includes the relations

$$\begin{aligned}
&\omega_1-\omega_j+\omega_{j+2} < -\omega_1+\omega_3, \quad -\omega_1+\omega_3-\omega_j+\omega_{j+2} < \omega_1,\\
&\omega_1-\omega_j+\omega_{j+2} < -\omega_2+\omega_4, \quad -\omega_2+\omega_4-\omega_j+\omega_{j+2} < \omega_1 \quad (\text{if } j \geqslant 4),
\end{aligned}$$

following the convention that $\omega_{n+1} = 0$. The first two chains are ν-dominant for $\nu = \omega_1+\omega_j$ and prove that $\chi(\omega_1+\omega_3+\omega_{j+2})$ has multiplicity $\geqslant 2$ in $\chi(2\omega_1)\chi(\nu)$, whereas the last two are ν-dominant for $\nu = \omega_2+\omega_j$ (assuming $j \geqslant 4$) and prove that $\chi(\omega_1+\omega_4+\omega_{j+2})$ has multiplicity $\geqslant 2$ in $\chi(2\omega_1)\chi(\nu)$. In the case $\nu = \omega_2+\omega_3$, we may reduce to $n = 5$ by stability

(Corollary 2.7), in which case the chains $\omega_2 - \omega_3 < -\omega_2 + \omega_4$ and $-\omega_2 < \omega_2 - \omega_3 + \omega_4$ may be used to prove that $\chi(\omega_2 + \omega_4)$ has multiplicity $\geqslant 2$ in $\chi(2\omega_1)\chi(\nu)$.

(N7) By induction with respect to n, it suffices by stability (Corollary 2.7) to assume $\nu = \omega_i + \omega_n$ $(3 \leqslant i \leqslant n-2)$ or $\nu = \omega_{n-2} + \omega_{n-1}$. In the first case, set $\lambda = \omega_1 + \omega_2 + \omega_{i+2}$ and note that the Exterior Powers Rule (Proposition 6.1) shows that the multiplicity of $\chi(\lambda)$ in $\theta(t)\chi(\nu)$ is $t^{n-4}(1+t^2)^2(1+t^2+t^4)$, so the multiplicity of $\chi(\lambda)$ in $\theta_n\chi(\nu)$ is 4, whence Corollary 6.2 is applicable. In the second case, set $\lambda = \omega_1 + \omega_2 + \omega_{n-1}$ and follow the same reasoning.

(N8) Applying the Exterior Powers Rule (Proposition 6.1), set $\lambda = \omega_1 + \omega_2 + \omega_4$, and note that the multiplicity of $\chi(\lambda)$ in $\theta(t)\chi(\nu)$ is $t^{n-4}(1+t^2)^2(1+t^2+t^4)$, so the multiplicity of $\chi(\lambda)$ in $\theta_n\chi(\nu)$ is 4. Now apply Corollary 6.2.

(N9) Applying the Multi-Minuscule Rule (Corollary 2.16) and diagram automorphisms, we may assume $\mu = 3\omega_1$ and $\nu = 2\omega_i$ $(3 \leqslant i \leqslant n-3)$. In this case, the Bruhat ordering of the minuscule orbit $W\omega_1$ includes the chains

$$\omega_1 - \omega_i + \omega_{i+4} < -\omega_1 + \omega_3 - \omega_i + \omega_{i+2} < \omega_1,$$
$$\omega_1 - \omega_i + \omega_{i+4} < \omega_1 - \omega_i + \omega_{i+2} < -\omega_1 + \omega_3,$$

following the convention that $\omega_{n+1} = 0$. Both of these chains are ν-dominant, so the multiplicity of $\chi(\omega_1 + \omega_3 + \omega_{i+2} + \omega_{i+4})$ in $\chi(3\omega_1)\chi(\nu)$ is at least 2.

7. The Exceptional Cases

A. The case $\Phi = \mathcal{E}_6$.

Minuscule weights: ω_1, ω_6. Quasi-minuscule weight: ω_2.

The products listed in Theorem 1.1.E6 can be shown to be multiplicity-free as follows.

(i) The Minuscule Rule (Proposition 2.12).

(ii) The Quasi-Minuscule Rule (Proposition 2.13).

(iii) A machine calculation based on branching (Proposition 2.2 and Corollary 2.5).

(iv),(v) Apply the Multi-Minuscule Rule (Corollary 2.16) and the criterion of Corollary 2.17. Taking $\mu = \omega_1$ (the case $\mu = \omega_6$ is equivalent by a diagram automorphism), one may check that for each of $\nu = m'\omega_2$, $\nu = m'\omega_3$ and $\nu = m'\omega_5$, there is exactly one generically ν-dominant chain in $W\mu$ that is maximal with respect to inclusion; viz.,

$$\begin{aligned}
\omega_1 - \omega_2 < \omega_5 - \omega_2 < \omega_3 - \omega_1 < \omega_1 \qquad & (\nu = m'\omega_2),\\
\omega_2 - \omega_3 < \omega_1 + \omega_6 - \omega_3 < \omega_5 - \omega_2 < \omega_4 - \omega_3 < \omega_3 - \omega_1 < \omega_1 \quad & (\nu = m'\omega_3),\\
\omega_6 - \omega_5 < \omega_3 - \omega_5 < \omega_2 - \omega_6 < \omega_2 + \omega_6 - \omega_5 < \omega_4 - \omega_3 < \omega_1 \quad & (\nu = m'\omega_5).
\end{aligned}$$

Furthermore, each of these chains is an affinely independent set, so by Corollary 2.17, each product $\chi(m\omega_1)\chi(\nu)$ is multiplicity-free (see Remark 2.18). In the case $\nu = m_1\omega_1 + m_6\omega_6$, there are two generically ν-dominant chains that are maximal; namely,

$$-\omega_6 < \omega_5 - \omega_1 - \omega_6 < \begin{matrix} \omega_6 - \omega_1 \\ \omega_2 - \omega_6 \end{matrix} < \omega_2 + \omega_6 - \omega_5 < \omega_3 - \omega_1 < \omega_1.$$

The union of these two chains has a unique affine dependence relation,

$$(\omega_5 - \omega_1 - \omega_6) - (\omega_6 - \omega_1) - (\omega_2 - \omega_6) + (\omega_2 + \omega_6 - \omega_5) = 0.$$

Since the coefficients of $\omega_6 - \omega_1$ and $\omega_2 - \omega_6$ have the same sign, there is no dependence relation that meets the nonnegativity constraints in Corollary 2.17.

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ, ν) such that

(N1) $\mu \in \{\omega_3, \omega_4, \omega_5\}$ and $\nu \in \{\omega_3, \omega_4, \omega_5, \omega_1 + \omega_2, \omega_1 + \omega_6, \omega_2 + \omega_6\}$,
(N2) $(\mu, \nu) = (\omega_2, \omega_i + \omega_j)$ $(1 \leqslant i < j \leqslant 6)$,
(N3) $(\mu, \nu) = (\omega_1 + \omega_6, \omega_1 + \omega_6)$,
(N4) $\mu = 2\omega_1$ or $\mu = 2\omega_6$ and $\nu = \omega_4$,
(N5) $\mu = 2\omega_1$ or $\mu = 2\omega_6$ and $\nu = \omega_i + \omega_j$ $(1 \leqslant i < j \leqslant 6,\ (i,j) \neq (1,6))$, or
(N6) $\mu = 2\omega_2$, $\nu \in \{2\omega_2, 2\omega_3, \omega_4, 2\omega_5\}$.

These products can be shown not to be multiplicity-free as follows.

(N1) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1).

(N2) The Quasi-Minuscule Rule (Proposition 2.13).

(N3) Use stability (Corollary 2.7) and Theorem 1.1.A in the case $n = 5$.

(N4),(N5) A machine calculation based either on the Brauer-Klimyk Rule (Proposition 2.1), or the Multi-Minuscule Rule (Corollary 2.16).

(N6) The simple root coordinates of $\omega_2 = \bar{\alpha}$ are $(1, 2, 2, 3, 2, 1)$, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

B. The case $\Phi = \mathcal{E}_7$.

Minuscule weights: ω_7. Quasi-minuscule weight: ω_1.

The products listed in Theorem 1.1.E7 can be shown to be multiplicity-free as follows.

(i) The Minuscule Rule (Proposition 2.12).

(ii) The Quasi-Minuscule Rule (Proposition 2.13).

(iii)–(v) A machine calculation based on branching (Proposition 2.2 and Corollary 2.5).

(vi) Apply the Multi-Minuscule Rule (Corollary 2.16) and the criterion of Corollary 2.17. Taking $\mu = \omega_7$, one may check that for each of $\nu = m'\omega_1$ and $\nu = m'\omega_7$, there is exactly one generically ν-dominant chain in $W\mu$ that is maximal; viz.,

$$\begin{aligned} \omega_7 - \omega_1 < \omega_2 - \omega_1 < \omega_6 - \omega_7 < \omega_7 \quad & (\nu = m'\omega_1), \\ -\omega_7 < \omega_1 - \omega_7 < \omega_6 - \omega_7 < \omega_7 \quad & (\nu = m'\omega_7). \end{aligned}$$

Furthermore, each of these chains is an affinely independent set, so by Corollary 2.17, each product $\chi(m\omega_1)\chi(\nu)$ is multiplicity-free (see Remark 2.18). In the case $\nu = m'\omega_2$, there are two generically ν-dominant chains that are maximal; namely,

$$\omega_1 - \omega_2 < \omega_6 - \omega_2 < \omega_4 - \omega_1 - \omega_2 < \begin{matrix} \omega_2 - \omega_1 \\ \omega_3 - \omega_2 \end{matrix} < \omega_2 + \omega_3 - \omega_4 < \omega_5 - \omega_6 < \omega_7.$$

The union of these two chains has a unique affine dependence relation,

$$(\omega_4 - \omega_1 - \omega_2) - (\omega_2 - \omega_1) - (\omega_3 - \omega_2) + (\omega_2 + \omega_3 - \omega_4) = 0.$$

Since the coefficients of $\omega_2 - \omega_1$ and $\omega_3 - \omega_2$ have the same sign, there is no dependence relation that meets the nonnegativity constraints in Corollary 2.17.

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ,ν) such that

(N1) $(\mu,\nu)=(\omega_i,\omega_j)$ $(2\leqslant i,j\leqslant 6,\ (i,j)\neq(2,2))$,
(N2) $(\mu,\nu)=(\omega_1,\omega_i+\omega_j)$ $(1\leqslant i<j\leqslant 7)$,
(N3) $(\mu,\nu)=(\omega_1+\omega_7,\omega_i)$ $(2\leqslant i\leqslant 6)$,
(N4) $(\mu,\nu)=(2\omega_7,\omega_i)$ $(i=3,4,5)$ or $(\mu,\nu)=(3\omega_7,2\omega_6)$,
(N5) $(\mu,\nu)=(2\omega_7,\omega_i+\omega_j)$ $(1\leqslant i<j\leqslant 7)$,
(N6) $\mu=2\omega_1$, $\nu\in\{2\omega_1,2\omega_2,\omega_3,\omega_4,\omega_5,2\omega_6\}$, or
(N7) $(\mu,\nu)\in\{(\omega_2,\omega_2+\omega_7),(\omega_2,\omega_1+\omega_2),(2\omega_2,2\omega_2)\}$.

These products can be shown not to be multiplicity-free as follows.

(N1),(N3) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1).

(N2) The Quasi-Minuscule Rule (Proposition 2.13).

(N4),(N5) A machine calculation based either on the Brauer-Klimyk Rule (Proposition 2.1), or the Multi-Minuscule Rule (Corollary 2.16).

(N6) The simple root coordinates of $\omega_1=\bar{\alpha}$ are $(2,2,3,4,3,2,1)$, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

(N7) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1); the last two of these also follow via Theorem 1.1.E6 and stability (Corollary 2.7).

C. The case $\Phi=\mathcal{E}_8$.

Minuscule weights: none. Quasi-minuscule weight: ω_8.

The products listed in Theorem 1.1.E8 can be shown to be multiplicity-free as follows.

(i) The Quasi-Minuscule Rule (Proposition 2.13).

(ii) A machine calculation based on branching (Proposition 2.2 and Corollary 2.5).

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ,ν) such that

(N1) $(\mu,\nu)=(\omega_i,\omega_j)$ $(1\leqslant i,j\leqslant 7,\ (i,j)\neq(1,1))$,
(N2) $(\mu,\nu)=(\omega_8,\omega_i+\omega_j)$ $(1\leqslant i<j\leqslant 8)$,
(N3) $(\mu,\nu)=(\omega_1,\omega_1+\omega_8)$ or $(2\omega_1,2\omega_1)$, or
(N4) $\mu=2\omega_8$ and $\nu\in\{2\omega_1,\omega_2,\dots,\omega_7,2\omega_8\}$.

These products can be shown not to be multiplicity-free as follows.

(N1) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1). Note that the cases with $2\leqslant i,j\leqslant 6$ except $(i,j)=(2,2)$ are also covered by Theorem 1.1.E7 and stability (Corollary 2.7).

(N2) The Quasi-Minuscule Rule (Proposition 2.13).

(N3) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1). The second case is also covered by Theorem 1.1.E7 and stability (Corollary 2.7).

(N4) The simple root coordinates of $\omega_8=\bar{\alpha}$ are $(2,3,4,6,5,4,3,2)$, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

D. The case $\Phi=\mathcal{F}_4$.

Minuscule weights: none. Quasi-minuscule weight: ω_1. Adjoint weight: ω_4.

The products listed in Theorem 1.1.F4 can be shown to be multiplicity-free as follows.

(i) The Quasi-Minuscule Rule (Proposition 2.13).

(ii) The Adjoint Rule (Proposition 2.14).

(iii) A machine calculation based on branching (Proposition 2.2 and Corollary 2.5).

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ, ν) such that

(N1) $(\mu, \nu) = (\omega_4, \omega_i + \omega_j)$ $(1 \leqslant i < j \leqslant 4)$,
(N2) $(\mu, \nu) = (\omega_2, \omega_2)$ or (ω_3, ω_3),
(N3) $(\mu, \nu) = (\omega_2, \omega_3)$, $(\omega_2, \omega_1 + \omega_4)$, or $(\omega_3, \omega_1 + \omega_4)$,
(N4) $(\mu, \nu) = (\omega_1, \omega_1 + \omega_2)$,
(N5) $\mu = 2\omega_1$ and $\nu \in \{2\omega_1, \omega_2, \omega_3, \omega_1 + \omega_4\}$, or
(N6) $\mu = 2\omega_4$ and $\nu \in \{3\omega_1, 2\omega_2, \omega_3, 2\omega_4\}$.

These products can be shown not to be multiplicity-free as follows.

(N1) The Adjoint Rule (Proposition 2.14).

(N2) Since $\nu = \omega_2$ and $\nu = \omega_3$ both satisfy the hypothesis of Corollary 2.20 (see the argument for (N5) below), it follows from Proposition 2.19 and triple symmetry (Proposition 2.8) that $c(2\bar{\alpha}; \nu, \nu) = c(\nu; 2\bar{\alpha}, \nu) \geqslant 2$.

(N3) Use a machine calculation to check that $\chi(\omega_1+\omega_4)$ has multiplicity 2 in $\chi(\omega_2)\chi(\omega_3)$ and use triple symmetry (Proposition 2.8).

(N4) The Quasi-Minuscule Rule (Proposition 2.13).

(N5) The simple root coordinates of $\omega_1 = \bar{\alpha}$ are $(2, 3, 2, 1)$ and α_3 is long, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

(N6) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1).

E. The case $\Phi = \mathcal{G}_2$.

Minuscule weights: none. Quasi-minuscule weight: ω_1. Adjoint weight: ω_2.

The products listed in Theorem 1.1.G2 can be shown to be multiplicity-free as follows.

(i) The Quasi-Minuscule Rule (Proposition 2.13).

(ii) The Adjoint Rule (Proposition 2.14).

(iii) A machine calculation based on branching (Proposition 2.2 and Corollary 2.5).

To prove that there are no other multiplicity-free products, it suffices via Monotonicity (Corollary 2.10) to show that $\chi(\mu)\chi(\nu)$ is not multiplicity-free for all (μ, ν) such that

(N1) $(\mu, \nu) = (\omega_2, \omega_1 + \omega_2)$,
(N2) $(\mu, \nu) = (2\omega_1, 2\omega_1)$ or $(\mu, \nu) = (2\omega_1, \omega_1 + \omega_2)$, or
(N3) $(\mu, \nu) = (2\omega_2, 3\omega_1)$ or $(\mu, \nu) = (2\omega_2, 2\omega_2)$.

These products can be shown not to be multiplicity-free as follows.

(N1) The Adjoint Rule (Proposition 2.14).

(N2) The simple root coordinates of $\omega_1 = \bar{\alpha}$ are $(2, 1)$, so this follows from the Twice-Quasi-Minuscule Rule (Corollary 2.20).

(N3) A machine calculation based on the Brauer-Klimyk Rule (Proposition 2.1).

References

[BZ] A. D. Berenstein and A. V. Zelevinsky, When is the multiplicity of a weight equal to 1?, *Funct. Anal. Appl.* **24** (1990) 259–269.

[B] R. Brauer, Sur la multiplication des charactéristiques des groupes continus et semi-simples, *C. R. Acad. Sci. Paris* **204** (1937), 1784–1786.

[Ho] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond: The Schur lectures (1992), *Israel Math. Conf. Proc.* **8** (1995), 1–182, Bar-Ilan Univ., Ramat Gan.

[H] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer-Verlag, Berlin-New York, 1972.

[K] M. Kashiwara, On crystal bases, *in* "Representations of Groups (Banff, AB, 1994)," *CMS Conf. Proc.* **16** (1995), 155–197.

[KW] R. C. King and B. G. Wybourne, Multiplicity-free tensor products of irreducible representations of the exceptional Lie groups, *J. Phys. A* **35** (2002), 3489–3513.

[Kl] A. U. Klimyk, Decomposition of a direct product of irreducible representations of a semisimple Lie algebra into a direct sum of irreducible representations, *Amer. Math. Soc. Transl., Series 2* **76** (1968), 63–73.

[L1] P. Littelmann, A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras, *Invent. Math.* **116** (1994), 329–346.

[L2] P. Littelmann, On spherical double cones, *J. Algebra* **166** (1994), 142–157.

[L3] P. Littelmann, Paths and root operators in representation theory, *Ann. of Math.* **142** (1995), 499–525.

[MWZ1] P. Magyar, J. Weyman and A. Zelevinsky, Multiple flag varieties of finite type, *Adv. Math.* **141** (1999), 97–118.

[MWZ2] P. Magyar, J. Weyman and A. Zelevinsky, Symplectic multiple flag varieties of finite type, *J. Algebra* **230** (2000), 245–265.

[P] R. A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, *Europ. J. Combin.* **5** (1984), 331–350.

[PRV] K. R. Parthasarathy, R. Ranga Rao and V. S. Varadarajan, Representations of complex semi-simple Lie groups and Lie algebras, *Ann. of Math.* **85** (1967), 383–429.

[St1] J. R. Stembridge, Computational aspects of root systems, Coxeter groups, and Weyl characters, *in* "Interactions of Combinatorics and Representation Theory," *MSJ Memoirs* **11** (2001), 1–38, Math. Soc. Japan, Tokyo.

[St2] J. R. Stembridge, Multiplicity-free products of Schur functions, *Ann. Comb.* **5** (2001), 113–121.

[**St3**] J. R. Stembridge, Combinatorial models for Weyl characters, *Advances in Math.* **168** (2002), 96–131.

[**St4**] J. R. Stembridge, A weighted enumeration of maximal chains in the Bruhat order, *J. Algebraic Combin.* **15** (2002), 291–301.

[**W**] J. Weyman, Pieri's formulas for classical groups, *Contemp. Math.* **88** (1989), 177–184.