On Minuscule Representations, Plane Partitions and Involutions in Complex Lie Groups

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1. Introduction

The main objective of this paper is to explain in an interesting way some of the combinatorial identities in [**Ste1**] (labeled collectively therein as the "$q = -1$ phenomenon") that arise in connection with the enumeration of symmetry classes of plane partitions. The explanation that we provide here requires a lengthy digression into the representation theory of semisimple Lie groups and their Lie algebras (and this digression will lead us further astray into some interesting questions about conjugacy classes of involutions in such groups), even though the basic reasoning that underpins what we intend to do is quite simple. Because these simple ideas may have broader applicability, and also because there is a danger that the simple ideas may be obscured by the elaborate framework that is constructed around them, we first give a brief indication of the nature of our explanations.

We should point out that Greg Kuperberg [**Ku**] has recently also found explanations of some instances of the $q = -1$ phenomenon in symmetry classes of plane partitions. His approach follows the same philosophy but it differs in the details.

First let us clarify what is meant by the $q = -1$ phenomenon. Suppose that we have a finite collection of combinatorial objects X with an associated generating function $F(q)$. Let us further suppose that there is some "natural" involution $x \mapsto x^*$ on the set X (e.g., a type of duality). We would then regard this setting as providing an instance of the $q = -1$ phenomenon if the number of fixed points of the involution (i.e., the number of self-dual objects) equals $F(-1)$. The utility of this situation is clear in cases for which one knows a closed formula for $F(q)$—by setting $q = -1$, one obtains a closed formula for $F(-1)$, and hence for the number of self-dual objects. Proofs of the $q = -1$ phenomenon thus have the potential for "explaining" the existence of closed formulas as special cases of other closed formulas. This is especially valuable in the context of symmetry classes of plane partitions, since this is a subject in which there are numerous difficult-to-prove closed formulas, but few explanations of such formulas.

To prove the instances of the $q = -1$ phenomenon we consider here, the approach we take is to first linearize the problem. That is, we replace the set X with the vector space V freely generated by X. We replace the involution with the linear transformation $A : x \mapsto x^*$ it induces on V, and we let $D(q) : x \mapsto q^{r(x)}x$ denote the diagonal transformation on V in which $r(\cdot)$ is the weight function on X whose generating function is $F(q)$. Since A is merely a permutation matrix, it is obvious that its trace is the number of self-dual objects, so in linear algebraic terms, the identity we seek to prove is equivalent to

$$\operatorname{tr} A = \operatorname{tr} D(-1).$$

We could prove that A and $D(-1)$ have the same trace by proving that they are conjugate as linear transformations. Although at first this would seem to be more difficult than the original problem, it is here that representation theory can be advantageous. Indeed, let us suppose that there is a group G acting on V. Let us further suppose that there are elements a and d in G for which A and $D(-1)$ are the representing matrices. In such a context, one could infer conjugacy of A and $D(-1)$ from conjugacy of a and d in G. Hence, the seemingly difficult task of proving conjugacy of two linear transformations on V might be reducible to a relatively easy question of conjugacy in a highly structured group G.

The plane partition identities that we (re-)prove in this paper are actually corollaries of a more general (new) result about minuscule posets. These posets are closely related to minuscule representations of Lie algebras; their combinatorial properties were first investigated from a unified point of view by Proctor [**P**], although the two most important special cases, the ones directly related to the enumeration of plane partitions, were considered earlier by Stanley.

Minuscule posets are special instances of Gaussian posets, a class of posets defined (by Stanley) roughly by the property that there should exist a "nice" product formula for the generating function for m-multichains of order ideals of the poset, for every m. An intriguing open question is whether or not there exist any non-minuscule Gaussian posets.

In Sections 2 and 3 of this paper, we assemble the necessary background material. Included is a discussion (Section 2.1) of Gaussian posets and their connection with symmetry classes of plane partitions. Of particular significance for us is the weight poset associated to any representation of a semisimple Lie algebra (Section 2.2). A key property of these posets is the fact that they are equipped with order-reversing involutions, given by the action of the longest element w_0 of the Weyl group. This general construction provides an order-reversing involution in any minuscule poset.

In Section 4, we prove the main result (Theorem 4.1): it amounts to the assertion that the $q = -1$ phenomenon occurs in any minuscule poset. More precisely, we prove that the generating function $F_m(q)$ for m-multichains of order ideals of a minuscule poset has the property that $F_m(-1)$ counts the number of such multichains that are fixed by the order-reversing involution. Corollaries of this result include enumerations of (1) self-complementary, and (2) symmetric self-complementary plane partitions. The proof, following the procedure outline above, relies on some basic aspects of Standard Monomial Theory in the minuscule case due to Seshadri [**S**] (see Theorem 3.1 below), as well as a new result (Theorem 4.4) concerning the properties of a special conjugacy class of involutions in the adjoint form of a semisimple Lie group G. This result is closely related to (and relies on) Kostant's work on the conjugacy class of so-called principal elements of G [**K**]. In the final section, we prove (Theorem 5.3) that the special involutions in the adjoint group are distinguished among all involutions by their trace in the adjoint representation, which turns out to be the minimum possible. Our proof is unsatisfying, however, since it is partially case-by-case, and computer calculations were used for the exceptional groups.

There are a total of six instances of the $q = -1$ phenomenon that are identified in [**Ste1**]. Two of these instances are explained here, and Kuperberg's approach also explains two instances (one of which coincides with one of our cases; see Examples 4.2 and 4.3 below for a more detailed discussion). Thus in combination, we have succeeded in explaining three of the six cases. In the remaining three cases, it is far from clear how one might construct a representation-theoretic explanation. A necessary first step would be to identify an algebraic structure possessing "natural" representations on vector spaces with bases indexed by cyclically symmetric or totally symmetric plane partitions.

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2. Preliminaries

If $P = (X, \leq)$ is a partially ordered set, let $J(P)$ denote the set of order ideals of P (i.e., subsets $I \subseteq X$ such that $x \leq y$, $y \in I$ imply $x \in I$), partially ordered by inclusion. The poset P will be said to be *ranked* if there is a real-valued function r on X such that x covers y (i.e., $x > y$ and there is no z such that $x > z > y$) implies $r(x) - r(y) = 1$. All of our rank functions will either be integer or half-integer valued. Note that $J(P)$ is always ranked (if P is finite), with rank function given by $r(I) = |I|$.

By a *complemented poset* we shall mean a triple $P = (X, \leq, c)$ consisting of a partial order $\leq$ and an order-reversing involution $x \mapsto x^c$ on X. Note that if P is complemented, then so too is $J(P)$, via the map $I \mapsto I^c := \{x \in X : x^c \notin I\}$.

For integers $n \geq 0$, let $[n] = \{1, 2, \ldots, n\}$, regarded as a totally ordered set.

2.1 Gaussian Posets

Let P be a finite poset and m a nonnegative integer. We define $F_m(P, q)$ to be the rank generating function for $J(P \times [m])$. Since order ideals of $P \times [m]$ can be identified with multichains $I_1 \subseteq \cdots \subseteq I_m$ of order ideals of P, it follows that

$$F_m(P, q) = \sum_{I_1 \subseteq \cdots \subseteq I_m : I_j \in J(P)} q^{|I_1| + \cdots + |I_m|}. \tag{2.1}$$

The poset P is said to be *Gaussian* if there exist integers $a_1, \ldots, a_n > 0$ such that

$$F_m(P, q) = \prod_{i=1}^{n} \frac{1 - q^{m + a_i}}{1 - q^{a_i}} \tag{2.2}$$

for all $m \geq 0$. It can be shown (see Exercise 4.25 of [**St1**]) that if $P = (X, \leq)$ is a Gaussian poset, then $|X| = n$, P is ranked, and the rank function can be defined so that minimal elements have rank 1 and the parameters $a_1, \dots, a_n$ are the ranks of the elements of P. Note that if we define $E = \{i : a_i \text{ is even}\}$, then (2.2) specializes to

$$F_m(P, -1) = \begin{cases} \prod_{i \in E}(m + a_i)/a_i & \text{if } m \text{ is even} \\ \prod_{i \notin E}(m + a_i)/\prod_{i \in E} a_i & \text{if } m \text{ is odd and } |E| = n/2, \end{cases} \tag{2.3}$$

and otherwise, $F_m(P, -1) = 0$.

The classification of Gaussian posets is an open problem. However, by a theorem of Proctor [**P**], one can construct a Gaussian poset from any minuscule representation of a semisimple Lie algebra, and every known Gaussian poset is obtained in this way. The details of this construction will be given in Section 3.

The two most important examples of Gaussian posets are the following.

Example 2.1. If $P = [a] \times [b]$, then the order ideals of $P \times [c]$ can be identified with the plane partitions whose three-dimensional diagrams fit in a box whose dimensions are $a \times b \times c$. The rank of an element of $J(P \times [c])$ corresponds to the size of the plane partition, so $F_c(P, q)$ amounts to the usual generating function for such plane partitions. By a result of MacMahon (e.g., see Example I.5.13 of [**M**]), there is a known product formula for $F_c(P, q)$, and it is routine to verify that it is compatible with (2.2). Thus P is Gaussian.

Example 2.2. Let $P = ([a] \times [a])/S_2$ (i.e., the quotient of $[a] \times [a]$ by the action of the involution $(i, j) \mapsto (j, i)$). We can regard $J(P \times [b])$ as the subset of $J([a] \times [a] \times [b])$ consisting of plane partitions invariant under transposition (i.e., the symmetry $(i, j, k) \mapsto (j, i, k)$). In this case, note that the rank function on $J(P \times [b])$ counts orbits of points under the action of transposition. In other words, for any transposition-invariant plane partition π in $J([a] \times [a] \times [b])$, the rank of the corresponding element of $J(P \times [b])$ is $|\{(i, j, k) \in \pi : i \leq j\}|$. A product formula for the corresponding generating function was conjectured by Bender and Knuth, and first proved by Andrews, Gordon and Macdonald (e.g., see Example I.5.19 of [**M**]). Again, it is routine to verify that the known product formula agrees with (2.2), so P is Gaussian.

Remark 2.3. If $P = (X, \leq, *)$ is a complemented poset, then so too is $P \times [m]$, by means of the involution $(x, i) \mapsto (x^*, m+1-i)$. In such cases, $J(P \times [m])$ will be naturally complemented as well (cf. the remarks at the beginning of this section). In Example 2.1, it is clear that the map $(i, j) \mapsto (a + 1 - i, b + 1 - j)$ is an order-reversing involution on P; the fixed points of the involution it induces on $J(P \times [c])$ are known in the literature as *self-complementary* plane partitions (e.g., [**MRR**], [**St2**]). It was first noted in [**Ste1**] that the $q = -1$ phenomenon occurs here: it turns out that $F_c(P, -1)$ is the number of self-complementary plane partitions that fit in a box whose dimensions are $a \times b \times c$. In particular, there is a simple product formula for this number, a special case of (2.3).

Similar remarks apply to Example 2.2. In that case, the order-reversing involution on P is given by $\{i, j\} \mapsto \{a+1-i, a+1-j\}$. As noted in [**Ste1**], the $q = -1$ phenomenon also occurs here, since $F_b(P, -1)$ turns out to be the number of self-complementary, transposition-invariant plane partitions that fit in a box whose dimensions are $a \times a \times b$.

It is unknown whether every Gaussian poset has an order-reversing involution, or even whether such posets are self-dual.

2.2 Weight Posets

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{C}$ with Cartan subalgebra $\mathfrak{h}$, root system $\Phi \subset \mathfrak{h}^*$, and Weyl group W. (For definitions and further details, see [**H1**].) Let $\langle \cdot\,, \cdot \rangle$ denote the inner product on $\mathfrak{h}^*$, and for each root $\alpha \in \Phi$, let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ denote the corresponding co-root. Let $\Lambda = \{\lambda \in \mathfrak{h}^* : \alpha \in \Phi \Rightarrow \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}\}$ denote the weight lattice.

Suppose that V is a finite-dimensional $\mathfrak{g}$-module. For each $\lambda \in \Lambda$, let

$$V_\lambda = \{v \in V : h \in \mathfrak{h} \Rightarrow h.v = \lambda(h)v\}$$

denote the weight space indexed λ. Let $\Lambda_V = \{\lambda \in \Lambda : V_\lambda \neq 0\}$ (a finite set) denote the weights of V. Recall that V is the direct sum of its weight spaces. Let us also recall that the weight multiplicities $\dim(V_\lambda)$ are W-invariant; in particular, W permutes Λ_V.

Let Φ^+ be a choice of positive roots, and let $\alpha_1, \ldots, \alpha_n$ denote the corresponding set of simple roots. Let $\omega_1, \ldots, \omega_n \in \Lambda$ denote the corresponding set of fundamental weights; i.e., the weights defined by the condition $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Recall that a weight λ is *dominant* if $\lambda \in \mathbf{N}\omega_1 + \cdots + \mathbf{N}\omega_n$, or equivalently, if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$.

Once the choice of Φ^+ is given, Λ is partially ordered by the rule

$$\lambda \geq \mu \quad \text{if} \quad \lambda - \mu \in \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_n;$$

i.e., $\lambda - \mu$ must be a nonnegative, integral linear combination of simple (or equivalently, positive) roots. It is easy to see that $(\Lambda, \leq)$ is a ranked poset; for the rank function r it is convenient to use the (restriction to Λ of the) unique linear functional on $\mathfrak{h}^*$ such that $r(\alpha_i) = 1$ for $1 \leq i \leq n$. By Proposition 29 in §VI.1.10 of [**B**], one has

$$r(\lambda) = \langle \lambda, \rho^\vee \rangle, \tag{2.4}$$

where $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$. Note that $2r(\lambda) \in \mathbf{Z}$ for all $\lambda \in \Lambda$.

Returning our attention to the $\mathfrak{g}$-module V, we define the *weight poset* Q_V to be the subposet of $(\Lambda, \leq)$ formed by Λ_V. Recall that if V is irreducible, then Q_V has a unique maximal element, the so-called highest weight of V, which is necessarily dominant. Since dominant weights parameterize the isomorphism classes of irreducible $\mathfrak{g}$-modules, it is reasonable to also use Q_λ as a notation for the weight poset associated with any irreducible $\mathfrak{g}$-module whose highest weight is the dominant weight λ.

The following property of weight posets may be well-known, but it is difficult (perhaps impossible) to find in the literature.

PROPOSITION 2.4. *Q_V is a ranked poset, with rank function r.*

Proof. Let $\lambda > \mu$ be a relation in Q_V. By definition, we have $\lambda - \mu = c_1\alpha_1 + \cdots + c_n\alpha_n$ with $c_i \in \mathbf{N}$. Since $0 < \langle \lambda - \mu, \lambda - \mu \rangle = \sum_i c_i \langle \lambda - \mu, \alpha_i \rangle$, it follows that there must exist an index i such that $c_i > 0$ and $\langle \lambda - \mu, \alpha_i \rangle > 0$. For such i, we must have either $\langle \lambda, \alpha_i \rangle > 0$ or $\langle \mu, \alpha_i \rangle < 0$.

Assuming the former case, it follows that $\sigma_i(\lambda) = \lambda - c\alpha_i$ for some integer $c > 0$, where $\sigma_i \in W$ denotes reflection through the hyperplane $\alpha_i^\perp$. Hence $\lambda - c\alpha_i \in \Lambda_V$, since the weights of V are W-stable. Using the fact that weight strings of representations must be unbroken (e.g., see Section 21.3 of [**H1**]), it follows that $\lambda - \alpha_i \in \Lambda_V$. Since $c_i > 0$, it follows that $\lambda > \lambda - \alpha_i \geq \mu$ in Q_V; therefore if λ covers μ, we must have $\lambda - \alpha_i = \mu$ and $r(\lambda) - r(\mu) = 1$, as needed.

There is a similar argument if $\langle \mu, \alpha_i \rangle < 0$. One deduces that $\mu + c\alpha_i \in \Lambda_V$ for some integer $c > 0$. We therefore also have $\mu + \alpha_i \in \Lambda_V$, since it belongs to the α_i-string through μ. Hence $\lambda \geq \mu + \alpha_i > \mu$ in Q_V, so if λ covers μ in Q_V, we must have $\lambda = \mu + \alpha_i$ and $r(\lambda) - r(\mu) = 1$. $\square$

Recall that the longest element w_0 of W is an involution that maps Φ^+ to $-\Phi^+$. It follows that $\lambda \mapsto w_0\lambda$ is an order-reversing involution on Λ. In particular, since Λ_V is W-stable, it follows that $Q_V = (\Lambda_V, \leq, w_0)$ is a ranked, complemented poset.

2.3 Characters

Let G be the simply connected (semisimple, complex) Lie group with Lie algebra $\mathfrak{g}$. The character of a $\mathfrak{g}$-module V is the formal sum

$$\varphi_V = \sum_{\mu \in \Lambda} (\dim V_\mu) e^\mu,$$

where e^μ denotes a formal exponential. If V is irreducible of highest weight λ, we shall also use φ_λ to denote the character of V. The $\mathfrak{g}$-module structure of V induces an action of G on V such that

$$\mathrm{tr}_V(\mathrm{Exp}(h)) = \sum_{\mu \in \Lambda} (\dim V_\mu) e^{\mu(h)} \tag{2.5}$$

for all $h \in \mathfrak{h}$, where $\mathrm{Exp} : \mathfrak{g} \to G$ denotes the exponential map.

For any nonzero scalar $q = e^t \in \mathbf{C}$ and $\beta \in \mathfrak{h}^*$, let us also define

$$\varphi_V(\beta; q) = \sum_{\mu \in \Lambda} (\dim V_\mu) e^{t\langle \mu, \beta \rangle},$$

so that if $h \in \mathfrak{h}$ is the unique element such that $\langle \mu, \beta \rangle = \mu(h)$ for all $\mu \in \Lambda$, then $\mathrm{tr}_V(\mathrm{Exp}(th)) = \varphi_V(\beta; q)$. If β belongs to the root lattice $\mathbf{Z}\Phi$, then $\varphi_V(\beta; q)$ is a well-defined Laurent polynomial in q. However in general, $\langle \mu, \beta \rangle$ need not be $\mathbf{Z}$-valued, so it is a slight abuse of notation for us to regard $\varphi_V(\beta; q)$ as a function of q, rather than t.

The following q-analogue of the Weyl dimension formula is well-known.

Lemma 2.5. *If $\lambda \in \Lambda$ is any dominant weight, then*

$$\varphi_\lambda(\rho^\vee; q) = q^{-\langle \lambda, \rho^\vee \rangle} \prod_{\alpha \in \Phi^+} \frac{1 - q^{\langle \lambda + \rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}},$$

where $2\rho = \sum_{\alpha \in \Phi^+} \alpha$ and $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$.

Proof. By the Weyl denominator formula (e.g., [**H1**, §24]), one knows that

$$\sum_{w \in W} \varepsilon(w) e^{w\rho} = e^{-\rho} \prod_{\alpha \in \Phi^+} (e^\alpha - 1),$$

where $\varepsilon(w) = \det_{\mathfrak{h}^*}(w)$ denotes the sign character of W. By an application of the ring homomorphism induced by $e^\alpha \mapsto q^{\langle \alpha, \beta \rangle}$, we thus obtain

$$\sum_{w \in W} \varepsilon(w) q^{\langle w\rho, \beta \rangle} = q^{-\langle \rho, \beta \rangle} \prod_{\alpha \in \Phi^+} (q^{\langle \alpha, \beta \rangle} - 1) \tag{2.6}$$

for any $\beta \in \mathfrak{h}^*$. Hence by the Weyl character formula (e.g., [**H1**, §24])

$$\varphi_\lambda = \left[\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)} \right] \cdot \left[\sum_{w \in W} \varepsilon(w) e^{w\rho} \right]^{-1},$$

it follows that

$$\begin{aligned}\varphi_\lambda(\rho^\vee;q) &= \Big[\sum_{w\in W}\varepsilon(w)q^{\langle w(\lambda+\rho),\rho^\vee\rangle}\Big]\cdot\Big[\sum_{w\in W}\varepsilon(w)q^{\langle w\rho,\rho^\vee\rangle}\Big]^{-1}\\ &= \Big[\sum_{w\in W}\varepsilon(w)q^{\langle \lambda+\rho,w\rho^\vee\rangle}\Big]\cdot\Big[\sum_{w\in W}\varepsilon(w)q^{\langle \rho,w\rho^\vee\rangle}\Big]^{-1}\\ &= q^{-\langle\lambda,\rho^\vee\rangle}\prod_{\alpha\in\Phi+}\frac{q^{\langle\lambda+\rho,\alpha^\vee\rangle}-1}{q^{\langle\rho,\alpha^\vee\rangle}-1},\end{aligned}$$

using two applications of (2.6) with $\Phi^\vee$ and $\rho^\vee$ substituted for Φ and ρ. $\square$

It is easy to see that the same reasoning can be used to provide a second q-analogue of the Weyl dimension formula; namely,

$$\varphi_\lambda(\rho;q) = q^{-\langle\lambda,\rho\rangle}\prod_{\alpha\in\Phi+}\frac{1-q^{\langle\lambda+\rho,\alpha\rangle}}{1-q^{\langle\rho,\alpha\rangle}}. \tag{2.7}$$

Of course, this is identical to the previous case if $\Phi=\Phi^\vee$.

3. Minuscule Representations and Posets

Continuing the setting of the previous section, let V be a nontrivial irreducible $\mathfrak{g}$-module with highest weight λ. One says that V (and hence also λ) is *minuscule* if the action of W on Λ_V is transitive. It can be shown (Exercise VI.1.24 of [**B**]) that if $\mathfrak{g}$ is simple, then a minuscule weight must be one of the fundamental weights ω_i. However, not all fundamental weights are necessarily minuscule. A list of the minuscule weights for every simple Lie algebra is provided in the Appendix.

For convenience, let us assume that $\mathfrak{g}$ is simple, and suppose that $\lambda=\omega_i$ is a minuscule (hence fundamental) weight. Let V be the corresponding irreducible $\mathfrak{g}$-module. Note that the dual $\mathfrak{g}$-module V^* and its highest weight λ^* must also be minuscule. In *any* irreducible $\mathfrak{g}$-module, the highest weight space is one-dimensional, thus since V is assumed to have only one orbit of weights, it follows that all of its (nonzero) weight spaces are one-dimensional. Hence, V has a distinguished basis (unique up to scalar multiples) consisting of weight vectors; similar remarks apply to V^*. In the following, it will be useful to explicitly choose a weight vector basis for V^*; say $V^*_\mu=\mathbf{C}e_\mu$ for each $\mu\in\Lambda_{V^*}=-\Lambda_V$.

Now let G be the simply connected (simple, complex) Lie group with Lie algebra $\mathfrak{g}$, and let P_i denote the maximal parabolic subgroup corresponding to the simple root α_i. There is a natural identification of V^* with the space of global sections of a line bundle on the projective variety G/P_i. In particular, the homogeneous coordinate ring $R = \oplus_{m\geq 0} R^{(m)}$ of G/P_i can be viewed as a quotient of the symmetric algebra $S(V^*) = \oplus_{m\geq 0} S^m(V^*)$ of polynomial functions on V. Since $R^{(1)} \cong S^1(V^*) = V^*$, we can thus regard R as a quotient of the polynomial ring $\mathbf{C}[e_\mu : \mu \in \Lambda_{V^*}]$. By Standard Monomial Theory (see the first installment in the series of papers by Seshadri, Lakshmibai, et. al. [**S**]), one knows that there is a distinguished basis of R consisting of certain monomials in the variables e_μ (i.e., so-called "standard" monomials), together with certain straightening laws for converting nonstandard monomials into standard ones. As noted below, it turns out that the condition of standardness is related to the structure of the weight poset Q_{λ^*}.

Since the implications of Standard Monomial Theory (in the minuscule case) will be crucial for what we intend to do, we summarize what is needed in the following theorem, all of which is taken from [**S**].

THEOREM 3.1 (Seshadri). *Let V be a minuscule $\mathfrak{g}$-module of highest weight $\lambda = \omega_i$, let $V^* = \oplus_{\mu\in\Lambda_{V^*}} \mathbf{C} e_\mu$ be the weight space decomposition of V^*, and identify the homogeneous coordinate ring $R = \oplus_{m\geq 0} R^{(m)}$ of G/P_i as a quotient of $\mathbf{C}[e_\mu : \mu \in \Lambda_{V^*}]$.*

(a) *The monomials $e_{\mu_1} \cdots e_{\mu_m}$ such that $\mu_1 \leq \cdots \leq \mu_m$ in Q_{λ^*} form a basis for $R^{(m)}$.*
(b) *As a $\mathfrak{g}$-module, $R^{(m)}$ is irreducible of highest weight $m\lambda^*$.*

An immediate consequence of part (a) is the fact that $\dim(R^{(m)}) = Z(Q_\lambda, m+1)$, where $Z(P, m)$ denotes the number of multichains of length m in a poset P. (Since Q_{V^*} and Q_V are isomorphic posets, we can replace λ^* with λ.) Furthermore, by Proposition 4.1 of [**P**], one knows that the weight poset Q_λ of any minuscule representation is a distributive lattice. In other words there is a (unique) poset P_λ such that $Q_\lambda \cong J(P_\lambda)$. In such circumstances, a multichain $\mu_1 \leq \cdots \leq \mu_m$ in Q_λ corresponds to an order ideal of $P_\lambda \times [m]$, so we have

$$\dim(R^{(m)}) = Z(Q_\lambda, m+1) = F_m(P_\lambda, 1), \tag{3.1}$$

using the notation of Section 2.

The posets P_λ that arise in this fashion (i.e., posets P such that $J(P)$ is isomorphic to the weight poset of a minuscule representation) are said to be *minuscule*. It should be noted that our requirement that $\mathfrak{g}$ be simple has caused no genuine loss of generality. The (minuscule) weight posets that arise in the general semisimple case are merely direct products of (minuscule) weight posets in the simple cases. The corresponding minuscule posets are thus merely disjoint unions of the "irreducible" minuscule posets arising from the simple cases. A description of every irreducible minuscule poset is provided in the Appendix.

EXAMPLE 3.2. Let $\mathfrak{g} = sl(n)$, and let $\mathfrak{h}$ be the usual Cartan subalgebra consisting of diagonal matrices. Let us define $\varepsilon_j(\mathrm{diag}(a_1, \ldots, a_n)) = a_j$ so that $\alpha_j = \varepsilon_{j+1} - \varepsilon_j \in \mathfrak{h}^*$ $(1 \leq j < n)$ is a choice of simple roots. In this case, the irreducible $\mathfrak{g}$-module V of highest weight ω_{n-k} is isomorphic to $\wedge^k(\mathbf{C}^n)$. Furthermore, is easy to show that

$$\Lambda_V = \{\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}.$$

The Weyl group acts by permuting $\varepsilon_1, \ldots, \varepsilon_n$; from this it is clear that the action of W on Λ_V is transitive, so in fact *every* fundamental weight of $sl(n)$ is minuscule.

There is an obvious correspondence between the weights of V and Young diagrams that fit in a $k \times (n-k)$ rectangle. The weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}$ corresponds to the Young diagram whose rows lengths are $i_1 - 1, i_2 - 2, \ldots, i_k - k$. It is not hard to show from the definition that under this correspondence, the weight poset Q_V amounts to the ordering of Young diagrams by inclusion; i.e., $Q_V \cong J([k] \times [n-k])$. Thus, the minuscule poset associated with $V = \wedge^k(\mathbf{C}^n)$ is $[k] \times [n-k]$ (cf. Example 2.1).

EXAMPLE 3.3. Let $\mathfrak{g} = so(2n+1)$ be the Lie algebra of skew-symmetric matrices of order $2n+1$, and let $\mathfrak{h}$ be the Cartan subalgebra consisting of the matrices

$$[0] \oplus \begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & a_n \\ -a_n & 0 \end{bmatrix}.$$

Let $\varepsilon_j \in \mathfrak{h}^*$ be the functional whose value on the above matrix is ia_j $(i = \sqrt{-1})$. For the simple roots one can take $\alpha_1 = \varepsilon_1$ and $\alpha_j = \varepsilon_{j+1} - \varepsilon_j$ for $j > 1$. The Weyl group acts via signed permutations of $\varepsilon_1, \ldots, \varepsilon_n$. The spin representation of $\mathfrak{g}$ is of dimension 2^n and has highest weight $\omega = (\varepsilon_1 + \cdots + \varepsilon_n)/2$. Since the W-orbit of ω clearly has order 2^n, it follows that the spin representation is minuscule and that its weights are $(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n)/2$.

There is a one-to-one correspondence between Q_ω and self-conjugate Young diagrams that fit in an $n \times n$ square. The weight whose positive coordinates with respect to $\varepsilon_1, \ldots, \varepsilon_n$ occur in positions $i_1, \ldots, i_r$ corresponds to the self-conjugate Young diagram having r boxes on the main diagonal, with the first r rows (and hence also the first r columns) having lengths $i_1, \ldots, i_r$. One can check that the partial order on the weights corresponds to inclusion of Young diagrams. Hence, as posets, $Q_\omega \cong J(([n] \times [n])/S_2)$, so the minuscule poset associated with the spin representation is $([n] \times [n])/S_2$ (cf. Example 2.2).

Returning to the general case, suppose λ is an arbitrary minuscule weight and P_λ the associated minuscule poset. By (3.1) and part (b) of Theorem 3.1, we know that $F_m(P_\lambda, 1)$ is the dimension of an irreducible $\mathfrak{g}$-module. By means of Weyl's dimension formula, it follows that one can give an explicit product formula for this number. In fact, it turns out that the q-analogue of the Weyl dimension formula (recall Lemma 2.5) is essentially $F_m(P_\lambda, q)$, the rank generating function for $J(P_\lambda \times [m])$. Since it involves only a short digression, we include here the details of the proof.

THEOREM 3.4 (Proctor). *If λ is a minuscule weight and P_λ is the associated minuscule poset, then*

$$F_m(P_\lambda, q) = \prod_{\alpha \in \Phi^+} \frac{1 - q^{\langle m\lambda+\rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}}.$$

By Exercise VI.1.24(a) of [**B**], one knows that if λ is minuscule, then $\langle \lambda, \alpha^\vee \rangle$ is 0 or 1 for all $\alpha \in \Phi^+$. It follows that the above formula is compatible with (2.2), so we obtain

COROLLARY 3.5. *Minuscule posets are Gaussian.*

Proof of Theorem 3.4. Let V be a minuscule $\mathfrak{g}$-module of highest weight λ. If e_{μ_i} belongs to the μ_i weight space of V $(1 \le i \le m)$, then the monomial $e_{\mu_1} \cdots e_{\mu_m} \in S^m(V)$ belongs to the weight space indexed by $\mu_1 + \cdots + \mu_m$. Thus by Theorem 3.1, we have

$$\varphi_{m\lambda} = \sum_{\mu_1 \le \cdots \le \mu_m : \mu_i \in Q_\lambda} e^{\mu_1 + \cdots + \mu_m}.$$

Here we have replaced the dual weight λ^* with λ; this is justified since both are minuscule and their weight posets are isomorphic. In view of (2.4), it follows that

$$\varphi_{m\lambda}(\rho^\vee; q) = \sum_{\mu_1 \le \cdots \le \mu_m : \mu_i \in Q_\lambda} q^{r(\mu_1) + \cdots + r(\mu_m)}.$$

Except for the difference in normalization of the rank function, this expression is of the same form as (2.1). To adjust for the difference, we need to shift $r(\cdot)$ so that the minimum element of Q_λ (namely, $w_0\lambda$) has rank 0. For this, note that $w_0(\Phi^+) = -\Phi^+$ implies $w_0\rho^\vee = -\rho^\vee$, and therefore by (2.4), $r(w_0\lambda) = \langle w_0\lambda, \rho^\vee \rangle = -\langle \lambda, \rho^\vee \rangle$. Thus we conclude

$$F_m(P_\lambda, q) = q^{m\langle \lambda, \rho^\vee \rangle} \varphi_{m\lambda}(\rho^\vee; q). \tag{3.2}$$

Now apply the q-analogue of the Weyl dimension formula (Lemma 2.5). $\square$

For example, consider the poset $P = [a] \times [b]$ of Example 2.1. By means of Example 3.2, we know that P is minuscule, and thus Theorem 3.4 provides a proof of MacMahon's generating function for plane partitions in an $a \times b \times c$ box. Similarly, Example 3.3 shows that the poset $P = ([a] \times [a])/S_2$ considered in Example 2.2 is minuscule; in this case, Theorem 3.4 yields the product formula for the generating function for transposition-invariant plane partitions in an $a \times a \times b$ box.

REMARK 3.6. The latter of these two applications of Theorem 3.4 has a second variation that is noteworthy. Let ω denote the minuscule weight corresponding to the spin representation of $so(2n+1)$, as in Example 3.3. Using essentially the same argument, Proctor shows in [**P**, §7] that $q^{2m\langle\omega,\rho\rangle}\varphi_{m\omega}(\rho;q^2)$ is also a generating function for transposition-invariant plane partitions. In this case, the weight function counts the total number of points in the three-dimensional diagram of the plane partition, rather than the number of orbits of points as in Example 2.2. By the second q-analogue of the Weyl dimension formula (see (2.7)), one can thus give an explicit product formula for this generating function. The product formula was conjectured by MacMahon and first proved by Andrews and Macdonald.

4. The Main Results

Let λ be a minuscule weight, and let P_λ be the associated minuscule poset. Recall that $w_0 \in W$ acts as an order-reversing involution on the weight poset $Q_\lambda \cong J(P_\lambda)$; this induces a natural order-reversing involution on P_λ. Indeed, if $P = (X, \leq)$ is an arbitrary poset with the property that $J(P)$ is equipped with an order-reversing involution, say $I \mapsto I^c$, then for every $x \in X$, the order ideals $I_{\leq x} = \{y \in X : y \leq x\}$ and $I_{<x} = \{y \in X : y < x\}$ have the property that $I^c_{<x} - I^c_{\leq x} = \{x'\}$ for some $x' \in X$. One can easily check that $x \mapsto x'$ is indeed an order-reversing involution on P. Thus every minuscule poset is complemented.

Conversely, we have previously noted (Remark 2.3) that an order-reversing involution on a poset P induces order-reversing involutions on $P \times [m]$, $J(P)$, and $J(P \times [m])$. If we use multichains $I_1 \subseteq \cdots \subseteq I_m$ of order ideals of P to index the elements of $J(P \times [m])$, then the complement induced on $J(P \times [m])$ amounts to

$$I_1 \subseteq \cdots \subseteq I_m \longmapsto I^c_m \subseteq \cdots \subseteq I^c_1,$$

where $I \mapsto I^c$ denotes the complement induced on $J(P)$. In the case $P = P_\lambda$, the order ideals I_j correspond to weights $\mu_j \in Q_\lambda$. In this notation, the complement amounts to

$$\mu_1 \leq \cdots \leq \mu_m \longmapsto w_0\mu_m \leq \cdots \leq w_0\mu_1.$$

THEOREM 4.1. *If $P = (X, \leq, c)$ is a minuscule poset, then $F_m(P, -1)$ is the number of self-complementary order ideals of $P \times [m]$, or equivalently, the number of multichains $I_1 \subseteq \cdots \subseteq I_m$ $(I_j \in J(P))$ such that $I^c_j = I_{m+1-j}$.*

Since minuscule posets are Gaussian, it follows that there is an explicit product formula (namely, (2.3)) for the number of self-complementary order ideals of $P \times [m]$.

Example 4.2. Let $n = a+b$ and $P = [a]\times[b]$. Recall from Example 3.2 that the weight poset of the $sl(n)$-module $\wedge^a(\mathbf{C}^n)$ is isomorphic to $J(P)$; i.e., the set of Young diagrams in an $a \times b$ rectangle, ordered by inclusion. If we translate the action of w_0 (which in this case interchanges ε_i and ε_{n+1-i}) into an action on Young diagrams, we find that it is the same as the one induced by the order-reversing involution $(i,j) \mapsto (a+1-i, b+1-j)$ discussed in Remark 2.3. (As indeed it must be, since P has only one order-reversing involution unless $a{=}b$.) Thus a corollary of Theorem 4.1 is the fact that $F_c(P,-1)$ is the number of self-complementary plane partitions that fit in an $a \times b \times c$ box. Kuperberg's approach [**Ku**] also explains this identity.

Example 4.3. Let $P = ([n]\times[n])/S_2$. Recall from Example 3.3 that the weight poset of the spin representation of $so(2n+1)$ is isomorphic to $J(P)$, or equivalently, the set of self-conjugate Young diagrams in an $n \times n$ square, ordered by inclusion. Since P has only one order-reversing involution (namely, $\{i,j\} \mapsto \{n+1-j, n+1-i\}$), it must induce the same involution on $J(P)$ as the one induced by w_0. Thus a second corollary of Theorem 4.1 (see Remark 2.3) is the fact that $F_m(P,-1)$ is the number of transposition-invariant, self-complementary plane partitions that fit in an $n \times n \times m$ box.

It should also be noted that there is a second instance of the $q=-1$ phenomenon that occurs in this context. Recall (Remark 3.6) that there is a second generating function for transposition-invariant plane partitions for which there is a closed form; let us denote it by $F'_m(P,q)$. As noted in [**Ste1**], it turns out that $F'_m(P,-1)$ is the number of plane partitions invariant under transposition composed with complementation. Kuperberg's approach [**Ku**] proves this fact that via the linear algebra/group action technique we outlined in the introduction, using representations of $\mathbf{Z}_2 \ltimes SL(2n,\mathbf{C})$ with $\mathbf{Z}_2$ acting via the diagram automorphism of A_{2n-1}.

To outline our proof of Theorem 4.1, let us first note that it suffices to restrict our attention to minuscule posets that are irreducible; i.e., we may (continue to) assume that the Lie algebra $\mathfrak{g}$, and the associated simply connected Lie group G are simple.

Now let $H = \operatorname{Exp}\mathfrak{h}$ be the Cartan subgroup of G corresponding to our choice of $\mathfrak{h}$, and recall that the Weyl group W can be realized in this context as $N_G(H)/H$. We will say that an element $x \in N_G(H)$ is an *extension* of some $w \in W$ if its image mod H (which amounts to its action as an automorphism of $\mathfrak{h}$) is w.

Let $\operatorname{Ad} G \cong G/Z(G)$ denote the adjoint form of G. Under the identification between $\mathfrak{h}$ and $\mathfrak{h}^*$ provided by $\langle\cdot\,,\cdot\rangle$, there is an element $h_0 \in \mathfrak{h}$ that corresponds to $\rho^\vee \in \mathfrak{h}^*$; thus $\mu(h_0) = \langle \mu, \rho^\vee\rangle$ for all $\mu \in \Lambda$. In the language of [**K**, §5.2], h_0 is a principal regular element of $\mathfrak{g}$. Now consider the element $t_0 = \operatorname{Exp}(i\pi h_0) \in H$. If v is a vector of weight μ in some

$\mathfrak{g}$-module V, then the action of H on v is such that

$$t_0.v = e^{i\pi\mu(h_0)}v. \tag{4.1}$$

Therefore, since $2\mu(h_0) = \langle \mu, 2\rho^\vee \rangle \in \mathbf{Z}$ for all $\mu \in \Lambda$, it follows that $t_0^4 = 1$, and if V is such that $\langle \mu, \rho^\vee \rangle \in \mathbf{Z}$ for all $\mu \in \Lambda_V$ (as happens, e.g., in the adjoint representation), then t_0 acts as an involution on V. In particular, $\mathrm{Ad}(t_0)^2 = 1$. Since $\mathrm{Ker}(\mathrm{Ad}) = Z(G)$, we therefore have $t_0^2 \in Z(G)$.

We will say that any element of $\mathrm{Ad}\, G$ that is conjugate to $\mathrm{Ad}(t_0)$ is a *special involution*. The remainder of this paper is largely devoted to the study of properties of the conjugacy class of special involutions in $\mathrm{Ad}\, G$. In particular, we claim that Theorem 4.1 is a consequence of the following.

THEOREM 4.4. *Let G be a simple, simply connected complex Lie group. There is an extension $x_0 \in G$ of $w_0 \in W$ such that*

(a) *$\mathrm{Ad}(x_0)$ is a special involution.*
(b) *If V is a minuscule representation, then there is a scalar c and a basis for V consisting of weight vectors $e_\mu \in V_\mu$ such that $x_0.e_\mu = ce_{w_0\mu}$.*

We remark that if $x_0 \in G$ is any extension of w_0, then by definition, x_0 maps the μ-weight space of *any* $\mathfrak{g}$-module to its $w_0\mu$-weight space. The crucial point of part (b) is the existence of the scalar c whose value does not depend on the choice of weight space.

Before proceeding further, we show how to deduce Theorem 4.1 from this result.

Proof of Theorem 4.1. We may assume that P is a minuscule poset such that $J(P) \cong Q_V$, where V is a minuscule representation of the simple Lie algebra $\mathfrak{g}$. Now let $x_0 \in G$, $c \in \mathbf{C}$, and $e_\mu \in V_\mu$ be the data provided by Theorem 4.4. Since $\mathrm{Ad}(x_0)$ is conjugate to $\mathrm{Ad}(t_0)$, there exists an element $z \in Z(G) = \mathrm{Ker}(\mathrm{Ad})$ such that zx_0 is conjugate to t_0. Since $Z(G)$ is a finite group (and $t_0^4 = 1$) it follows that $z^{-1}t_0$ and hence x_0 must have finite order. In particular, c must be a root of unity. Also, by Schur's Lemma, in any irreducible $\mathfrak{g}$-module U, z must act as a scalar c' (which again must be a root of unity), so we have $\mathrm{tr}_U(t_0) = c' \,\mathrm{tr}_U(x_0)$.

Now by (2.5) and (4.1), we have

$$\mathrm{tr}_U(t_0) = \sum_{\mu \in \Lambda_U} (\dim U_\mu) e^{i\pi\langle \mu, \rho^\vee \rangle} = \varphi_U(\rho^\vee; -1)$$

for any $\mathfrak{g}$-module U. If we take U to be the irreducible $\mathfrak{g}$-module of highest weight $m\lambda$, where λ is the highest weight of V, then by (3.2) we have

$$F_m(P, -1) = c''\varphi_U(\rho^\vee; -1) = c'' \,\mathrm{tr}_U(t_0) = c'c'' \,\mathrm{tr}_U(x_0),$$

where c'' is some fourth root of unity.

On the other hand, by Theorem 3.1, we know that the monomials $e_{\mu_1} \cdots e_{\mu_m}$ such that $\mu_1 \le \cdots \le \mu_m$ in Q_V can be used to form a basis for U as a quotient of $S^m(V)$. By part (b) of Theorem 4.4, it follows that the action of x_0 is such that

$$x_0(e_{\mu_1} \cdots e_{\mu_m}) = c^m e_{w_0\mu_m} \cdots e_{w_0\mu_1},$$

so $c^{-m}\,\mathrm{tr}_U(x_0)$ is the number of self-complementary order ideals of $P \times [m]$. Comparing our two expressions for $\mathrm{tr}_U(x_0)$, we conclude that this is, aside from a possible scalar factor of absolute value 1, also equal to $F_m(P, -1)$. However by (2.3), we know that $F_m(P, -1)$ is nonnegative, so the scalar factor must be trivial. $\square$

Before starting the proof of Theorem 4.4, it will be useful to first examine in detail the simplest (but most important) special case, $G = SL(n, \mathbf{C})$. In this case, the most general possible extension of w_0 is of the form

$$x_0 = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix},$$

where the $a_j \in \mathbf{C}$ are such that $a_1 \cdots a_n = (-1)^{n(n-1)/2}$. (Here we have temporarily chosen $n = 4$ for clarity.) On the other hand, using the choice of simple roots introduced in Example 3.2 (and the coordinates chosen therein), the elements h_0 and $t_0 = \mathrm{Exp}(i\pi h_0)$ defined above are given by

$$\begin{aligned} h_0 &= \mathrm{diag}(-(n-1)/2, -(n-3)/2, \ldots, (n-1)/2), \\ t_0 &= \mathrm{diag}(i^{-(n-1)}, i^{-(n-3)}, \ldots, i^{n-1}). \end{aligned}$$

If we choose $a_1 = \cdots = a_n = i^{-(n-1)}$, then it is easy to see that x_0 and t_0 have the same eigenvalues, and thus are conjugate in G. In particular, $\mathrm{Ad}(x_0)$ will be a special involution. (Note also that $x_0^2 = (-1)^{n-1}$, so x_0 need not itself be an involution; it is only its Ad-image that acts as an involution.)

To satisfy part (b) of Theorem 4.4, let us continue the notation of Example 3.2 and consider one of the minuscule representations $V = \wedge^k(\mathbf{C}^n)$ of G. Using $e_1, \ldots, e_n$ to denote the standard basis of $\mathbf{C}^n$, it is easy to see that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$$

forms a weight vector basis for V; the vector $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is of weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}$. Furthermore, for the above choice of x_0, we have $x_0.e_j = i^{-(n-1)}e_{n+1-j}$. Hence

$$x_0.(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (-1)^{k(k-1)/2} i^{-k(n-1)} e_{n+1-i_k} \wedge \cdots \wedge e_{n+1-i_1},$$

which confirms part (b).

We are now ready to treat the general case.

Proof of Theorem 4.4. Let $\sigma_1, \ldots, \sigma_n \in W$ denote the reflections corresponding to the simple roots $\alpha_1, \ldots, \alpha_n$, and let $w = \sigma_1 \cdots \sigma_n$ denote a Coxeter element of W. Recall that the Coxeter number s is defined to be the order of w.

Now consider the element

$$t = \mathrm{Exp}(2\pi i h_0/s) \in H,$$

where $h_0 \in \mathfrak{h}$ remains as it was defined prior to the statement of Theorem 4.4. Following Kostant [**K**, §6.7], the Ad-image of t and its conjugates are said to be *principal elements* of $\mathrm{Ad}\, G$. Note that $\mathrm{Ad}(t)$ has order s. In Kostant's study of principal elements (see Theorem 8.6 of [**K**]), it develops that any extension $x \in G$ of any Coxeter element w has the property that $\mathrm{Ad}(x)$ is principal (i.e., conjugate to $\mathrm{Ad}(t)$). Now if s is even, then by Proposition 2 in Section V.6.2 of [**B**], it is possible to order the simple reflections σ_j so that the Coxeter element $w = \sigma_1 \cdots \sigma_n$ satisfies $w_0 = w^{s/2}$. It follows that if we define $x_0 = x^{s/2}$, where $x \in G$ is any extension of w, then x_0 will be an extension of w_0 that is Ad-conjugate to $t^{s/2} = \mathrm{Exp}(i\pi h_0) = t_0$.

Thus we have demonstrated the existence of an extension of w_0 whose Ad-image is a special involution, but only for the case of even s. To treat the case of odd s turns out to be much simpler; by inspecting the tables (e.g., in [**B**]) one finds that the Coxeter number is odd only for the root systems A_{2n}, and we have already verified Theorem 4.4 in this case. Thus the proof of (a) is complete.

We claim that (b) is satisfied by any extension of w_0 that satisfies (a). To prove this, let $x_0 \in G$ be such an extension, let V be a minuscule representation, and let $e_\mu \in V_\mu$ be a basis of weight vectors. Since x_0 is assumed to extend w_0, we must have $x_0 : V_\mu \to V_{w_0\mu}$, so there must exist scalars $c_\mu \in \mathbf{C}$ such that

$$x_0.e_\mu = c_\mu e_{w_0\mu}$$

for all $\mu \in \Lambda_V$. To prove (b), we must show that it is possible to adjust the basis e_μ so that the scalars c_μ do not depend on μ. Now since $\mathrm{Ad}(t_0)^2 = 1$ and $\mathrm{Ad}(x_0)$ is assumed to be conjugate to $\mathrm{Ad}(t_0)$, we have $\mathrm{Ad}(x_0)^2 = 1$ and thus $x_0^2 \in Z(G)$. Therefore by Schur's Lemma, x_0^2 must act as some scalar c on V, and hence $c_\mu c_{w_0\mu} = c$ for all $\mu \in \Lambda_V$.

Let us now partition Λ_V into $\Lambda_1 = \{\mu \in \Lambda_V : \mu = w_0\mu\}$ and $\Lambda_2 = \Lambda_V \setminus \Lambda_1$. If we choose a representative μ from each pair $\{\mu, w_0\mu\} \subset \Lambda_2$, then it is possible to replace e_μ by a suitable scalar multiple so that $c_\mu = c_{w_0\mu} = \sqrt{c}$, or $c_\mu = c_{w_0\mu} = -\sqrt{c}$, whichever is preferred. For weights $\mu \in \Lambda_1$, replacing e_μ by a scalar multiple will not affect c_μ, but in this case we do have $c_\mu^2 = c$, so $c_\mu = \pm\sqrt{c}$. It follows that if $|\Lambda_1| \leq 1$, we are done. In particular, this completes the proof for Weyl groups in which w_0 acts as the scalar -1, since only the weight 0 can be fixed by w_0 in such cases.

For the remaining cases, it suffices to calculate $\operatorname{tr}_V(x_0)$. Only the weight spaces indexed by $\mu \in \Lambda_1$ can contribute to the trace, and the amount each contributes is $\pm\sqrt{c}$. Therefore,

$$|\operatorname{tr}_V(x_0)| \leq |\Lambda_1|,$$

and equality occurs if and only if $c_\mu = \sqrt{c}$ for all $\mu \in \Lambda_1$, or $c_\mu = -\sqrt{c}$ for all $\mu \in \Lambda_1$. (Here we are using the fact, as noted in the proof of Theorem 4.1, that c must be a root of unity.) On the other hand, since x_0 is Ad-conjugate to t_0, we know that x_0 and t_0 differ up to conjugacy in G only by a central element, and therefore $|\operatorname{tr}_V(x_0)| = |\operatorname{tr}_V(t_0)|$. Bearing (4.1) and (2.4) in mind, it thus suffices to show

$$\Big|\sum_{\mu \in \Lambda_V} e^{i\pi r(\mu)}\Big| = |\Lambda_1|, \tag{4.2}$$

which amounts to the case $m = 1$ of Theorem 4.1. Note that the validity of (4.2) depends only on the abstract structure of Q_V as a complemented poset.

Now the root systems A_n ($n > 1$), D_n (n odd), and E_6 are the only ones for which $w_0 \neq -1$. For A_n, we have already explicitly verified (in the discussion preceding this proof) that in each minuscule representation, a weight vector basis e_μ can be chosen so that the scalars c_μ do not depend on μ. For D_n (with n odd), using the coordinates chosen in the Appendix, w_0 acts on $\mathfrak{h}^*$ as the linear transformation in which $\varepsilon_1 \mapsto \varepsilon_1$ and $\varepsilon_j \mapsto -\varepsilon_j$ for $j > 1$. Hence the only w_0-invariant weights are multiples of ε_1. It follows that if V is a minuscule representation for which Λ_1 is nonempty, its highest weight must be a multiple of ε_n. The only such multiple which is minuscule is easily seen to be ε_n, the highest weight of the defining representation of $so(2n)$. In this case, (4.2) is routine to verify, using the fact that $\Lambda_V = \{\pm\varepsilon_1, \ldots, \pm\varepsilon_n\}$, $\Lambda_1 = \{\pm\varepsilon_1\}$, and $\sum_{\mu \in \Lambda_V} q^{r(\mu)} = \sum_{j=0}^{n-1}(q^j + q^{-j})$. For E_6 there are two minuscule representations; they are 27-dimensional and dual to each other. The associated minuscule poset is depicted in Figure 2, and the weight poset in Figure 3. By inspection of Figure 3, one sees that $|\Lambda_1| = 3$ and

$$\sum_{\mu \in \Lambda_V} q^{r(\mu)} = 1 + \sum_{j=-4}^{4} q^j + \sum_{j=-8}^{8} q^j.$$

Setting $q = -1$ in this expression yields 3, thus verifying (4.2). $\square$

5. A Characterization of Special Involutions

Continuing the notation previously established, our final goal is to show that the conjugacy class of special involutions in $\mathrm{Ad}\, G$ can be characterized by its trace in the adjoint representation. Before presenting this characterization, we will first show that the trace can be computed in several different ways from data contained in the root system.

In the following, δ denotes the Weyl involution. By this we mean the permutation of the simple roots induced by $-w_0$; i.e., $\alpha_j = -w_0\alpha_{\delta(j)}$.

Proposition 5.1. *The following quantities are equal.*

(a) $\mathrm{tr}_{\mathfrak{h}^*}(w_0)$.

(b) $-|\{j : \delta(j) = j\}|$.

(c) $\sum_j (-1)^{e_j}$, *where* $e_1, e_2, \ldots$ *denote the exponents of* Φ.

(d) $\mathrm{tr}_{\mathfrak{g}}(x_0)$, *where* $x_0 \in G$ *is such that* $\mathrm{Ad}(x_0)$ *is a special involution.*

Proof. With respect to the basis of simple roots, the matrix representing the action of $-w_0$ on $\mathfrak{h}^*$ is the permutation matrix corresponding to δ. Its trace is therefore the number of fixed points of δ, and thus (a)=(b).

Now let $\theta = e^{2\pi i/s}$, where s denotes the Coxeter number. It is well-known (e.g., see [**B**, §V.6]) that the eigenvalues on $\mathfrak{h}^*$ for any Coxeter element of W are given by $\theta^{e_1}, \ldots, \theta^{e_n}$. Recall (as noted during the proof of Theorem 4.4) that if s is even, then there is a Coxeter element $w \in W$ such that $w_0 = w^{s/2}$. The eigenvalues of w_0 must then be $(-1)^{e_1}, \ldots, (-1)^{e_n}$, which proves (a)=(c) for even s. For odd s, note that since the inverse of any Coxeter element is also a Coxeter element, the exponents can be ordered so that $e_j + e_{n+1-j} = s$. Since s is assumed to be odd, this forces n to be even and $\sum_j (-1)^{e_j} = 0$. However in this case, it is known (Exercise V.6.2(b) of [**B**]) that one can choose a Coxeter element $w = ab$ so that a and b are each products of $n/2$ pairwise commuting reflections, and $w_0 = w^{(s-1)/2} a w^{-(s-1)/2}$. Hence $\mathrm{tr}_{\mathfrak{h}^*}(w_0) = \mathrm{tr}_{\mathfrak{h}^*}(a) = 0$, since any product of $n/2$ pairwise orthogonal reflections in an n-dimensional space has trace 0.

For part (d), it suffices to choose $x_0 = t_0 = \mathrm{Exp}(i\pi h_0) \in H$, where h_0 is the principal regular element we defined in Section 4. Since $\alpha(h_0) = r(\alpha)$ is the rank function on the weight poset of $\mathfrak{g}$, it follows upon exponentiation that

$$\mathrm{tr}_{\mathfrak{g}}(t_0) = n + \sum_{\alpha \in \Phi} (-1)^{r(\alpha)} = n - 2r_1 + 2r_2 - \cdots = \sum_{j \geq 1} (-1)^j (r_j - r_{j+1}),$$

where r_j denotes the number of roots α such that $r(\alpha) = j$. (The last equality follows from the fact that $r_1 = n = \dim \mathfrak{h}$.) However, by a theorem first proved in a uniform way

Φ	A_{2n}	A_{2n+1}	D_{2n+1}	E_6	$I_2(2m+1)$
κ	0	-1	$-(2n-1)$	-2	0

Table 1: Exceptional values of $\kappa(\Phi)$.

by Kostant [**K**], it is known that $r_j - r_{j+1}$ is the multiplicity of j as an exponent of Φ, and thus (c)=(d). $\square$

Let us define $\kappa = \kappa(\Phi)$ to be the common value of the above four expressions. We remark that the above proof of the equality (a)=(b)=(c) is equally valid for non-crystallographic root systems, so it is reasonable to extend the notation $\kappa(\Phi)$ to include these cases. Note that for "most" irreducible root systems, we have $w_0 = -1$ and therefore $\kappa(\Phi) = -\dim\mathfrak{h}$. Table 1 lists all of the instances for which this is not the case.

Remark 5.2. There is an alternative proof of the fact that $\mathrm{tr}_{\mathfrak{g}}(x_0) = \kappa(\Phi)$ whenever $\mathrm{Ad}(x_0)$ is a special involution. By Theorem 4.4(a), we can choose x_0 to be an extension of w_0. In that case, since w_0 interchanges Φ^+ and $-\Phi^+$, it follows that x_0 interchanges the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ ($\alpha \in \Phi$). Hence, only the Cartan subalgebra $\mathfrak{h}$ can contribute to the trace of x_0 on $\mathfrak{g}$. However, since x_0 is an extension of w_0, this means that the action of x_0 on $\mathfrak{h}$ is identical to the action of w_0 on $\mathfrak{h}$. Hence $\mathrm{tr}_{\mathfrak{g}}(x_0) = \mathrm{tr}_{\mathfrak{h}}(w_0) = \kappa(\Phi)$.

The following result is similar in spirit to the characterizations of principal elements of $\mathrm{Ad}\, G$ given by Kostant in [**K**].

Theorem 5.3. *Let G, $\mathfrak{g}$, and Φ be as above. If $x \in G$ is such that $\mathrm{Ad}(x)$ is an involution, then*

$$\mathrm{tr}_{\mathfrak{g}}(x) \geq \kappa(\Phi),$$

and equality occurs if and only if $\mathrm{Ad}(x)$ is a special involution.

Let us define $\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda, \alpha^\vee \rangle \in 2\mathbf{Z}\}$ for any $\lambda \in \Lambda$. Note that $\Phi_\lambda = \Phi_\mu$ if and only if $\lambda - \mu \in 2\Lambda$, so the distinct possibilities for Φ_λ are parameterized by $\Lambda/2\Lambda$, an elementary abelian group of order 2^n.

Lemma 5.4. *We have $|\Phi_\lambda| \geq |\Phi_\rho|$, and equality occurs if and only if λ and ρ belong to the same W-orbit of $\Lambda/2\Lambda$.*

Before proving this lemma, we first deduce Theorem 5.3 from it.

Proof of Theorem 5.3. If $x \in G$ is any Ad-involution, then x has finite order, since $\mathrm{Ker}(\mathrm{Ad})$ is finite. In particular, x belongs to some maximal torus of G, and must therefore be conjugate to some element of the Cartan subgroup H (e.g., [**H2**, §21.3]). Thus we may assume $x = \mathrm{Exp}(i\pi h)$ for some $h \in \mathfrak{h}$. The condition $\mathrm{Ad}(x)^2 = 1$ is equivalent to having $\alpha(h) \in \mathbf{Z}$ for all $\alpha \in \Phi$. It follows that there is an element λ in the co-weight lattice $\Lambda^\vee = \{\mu \in \mathfrak{h}^* : \alpha \in \Phi \Rightarrow \langle \mu, \alpha \rangle \in \mathbf{Z}\}$ such that $\langle \lambda, \alpha \rangle = \alpha(h)$ for all $\alpha \in \Phi$. For a given x, the co-weight λ is well-defined modulo $2\Lambda^\vee$, so the Ad-involutions in H are parameterized by $\Lambda^\vee / 2\Lambda^\vee$. In the case of the special involution $t_0 = \mathrm{Exp}(i\pi h_0)$, one can take $\lambda = \rho^\vee$ as the corresponding co-weight.

In any case, it follows that x acts as the scalar $(-1)^{\langle \lambda, \alpha \rangle}$ on $\mathfrak{g}_\alpha$, and hence

$$\mathrm{tr}_{\mathfrak{g}}(x) = n + \sum_{\alpha \in \Phi} (-1)^{\langle \lambda, \alpha \rangle} = n - |\Phi| + 2|\Phi_\lambda^\vee|.$$

By Lemma 5.4 (applied to $\Phi^\vee$ and $\Lambda^\vee$), this quantity is minimized by taking $\lambda = \rho^\vee$; i.e., special involutions minimize $\mathrm{tr}_{\mathfrak{g}}(x)$. Furthermore, any Ad-involution in H achieving this minimum trace must have the property that the corresponding co-weight λ belongs to the W-orbit of $\rho^\vee$ mod $2\Lambda^\vee$, again by Lemma 5.4. However, the conjugation action of $N_G(H)$ on H is obtained by exponentiating the action of W on $\mathfrak{h}$, so any such Ad-involution must be conjugate to t_0. $\square$

Proof of Lemma 5.4. By adding suitable multiples of 2ρ, we can assume that λ is of the form $\mu + \rho$, where $\mu \in \Lambda$ is dominant. Since $\varphi_\mu(\rho^\vee; q)$ is by definition a Laurent polynomial in $q^{1/2}$, it cannot have a pole at $q = -1$. Thus in the product formula for $\varphi_\mu(\rho^\vee; q)$ (Lemma 2.5), the multiplicity of -1 as a zero of the denominator (namely, $|\Phi_\rho^+|$) cannot exceed the multiplicity of -1 as a zero of the numerator (namely, $|\Phi_{\mu+\rho}^+|$).

Having proved $|\Phi_\lambda| \geq |\Phi_\rho|$, we now examine the situations in which equality occurs. For this we may assume that Φ is irreducible and resort to a case-by-case argument. In each of these cases, it will be convenient to work with explicit coordinates for λ with respect to a particular basis $\varepsilon_1, \ldots, \varepsilon_n$ described in the Appendix. Thus we suppose $\lambda = a_1\varepsilon_1 + \cdots + a_n\varepsilon_n$. We will also need the fundamental weight coordinates for λ; say $\lambda = b_1\omega_1 + \cdots + b_n\omega_n$, where $b_j = \langle \lambda, \alpha_j^\vee \rangle$. Note that the 2^n choices for λ (mod 2Λ) can be obtained by choosing $b_j \in \{0, 1\}$ arbitrarily.

Case 1: $\Phi = A_{n-1}$. In this case, the standard coordinates for Φ are in an n-dimensional space; thus there are n standard coordinates a_j and $n-1$ weight coordinates $b_j = a_{j+1} - a_j$. The 2^{n-1} possible choices for λ can be obtained (twice each) by choosing $a_j \in \{0, 1\}$ arbitrarily. Since $\langle \lambda, \alpha^\vee \rangle = a_i - a_j$ for $\alpha = \varepsilon_i - \varepsilon_j$, it follows that $|\Phi_\lambda^+| = \binom{k}{2} + \binom{n-k}{2}$, where k denotes the number of even a_j's. If n is even, this is minimized by taking $k = n/2$. Since the Weyl group acts by permuting the ε_j's, it follows that all such choices are conjugate. If n is odd, the minimum is obtained by taking $k = (n \pm 1)/2$. Since the substitution

$a_j \mapsto 1 - a_j$ (for all j) does not change $\lambda \pmod{2\Lambda}$, we can assume $k = (n-1)/2$. Again, all such choices are conjugate under the Weyl group.

Case 2: $\Phi = D_n$. Using the coordinates in the Appendix, we have $b_j = a_j - a_{j-1}$ (for $j > 1$) and $b_1 = a_1 + a_2$. Therefore, the 2^{n-1} choices for λ with $b_1 + b_2$ even (resp., odd) can be obtained by choosing $a_j \in \{0, 1\}$ (resp., $a_j \in \{\pm 1/2\}$) arbitrarily. Since $\langle \lambda, \alpha^\vee \rangle = a_i \pm a_j$ for $\alpha = \varepsilon_i \pm \varepsilon_j$, it follows that $|\Phi_\lambda^+| = 2\binom{k}{2} + 2\binom{n-k}{2}$ if there are k even a_j's (assuming $a_j \in \{0, 1\}$), or $|\Phi_\lambda^+| = \binom{n}{2}$ (assuming $a_j \in \{\pm 1/2\}$). It is easy to show that $|\Phi_\lambda|$ is minimized only in the former case. Since the Weyl group includes all permutations of the ε_j's (and since $\varepsilon_1 + \cdots + \varepsilon_n = 0 \bmod 2\Lambda$), we can complete the argument for this case in the same manner as Case 1.

Case 3: $\Phi = C_n$. Using the coordinates in the Appendix, we have $b_j = a_j - a_{j-1}$ (for $j > 1$) and $b_1 = a_1$. It follows that the 2^n choices for λ can be obtained by choosing $a_j \in \{0, 1\}$ arbitrarily. Since we have $\langle \lambda, \alpha^\vee \rangle = a_i \pm a_j$ for $\alpha = \varepsilon_i \pm \varepsilon_j$ and $\langle \lambda, \alpha^\vee \rangle = a_j$ for $\alpha = 2\varepsilon_j$, it follows that $|\Phi_\lambda^+| = k + 2\binom{k}{2} + 2\binom{n-k}{2}$, where k denotes the number of even a_j's. This quantity is minimized at $k = \lfloor n/2 \rfloor$. Since the Weyl group includes all permutations of the ε_j's, it follows that all choices for λ with $k = \lfloor n/2 \rfloor$ belong to the same W-orbit.

Case 4: $\Phi = B_n$. Using the coordinates in the Appendix, we have $b_j = a_j - a_{j-1}$ (for $j > 1$) and $b_1 = 2a_1$. It follows that the 2^{n-1} choices for λ with b_1 even (resp., odd) can be obtained by choosing the a_j's independently from $\{0, 1\}$ (resp., $\{\pm 1/2\}$). Since we have $\langle \lambda, \alpha^\vee \rangle = a_i \pm a_j$ for $\alpha = \varepsilon_i \pm \varepsilon_j$ and $\langle \lambda, \alpha^\vee \rangle = 2a_j$ for $\alpha = \varepsilon_j$, it follows that $|\Phi_\lambda^+| = n + 2\binom{k}{2} + 2\binom{n-k}{2}$ if there are k even a_j's (assuming $a_j \in \{0, 1\}$), or $|\Phi_\lambda^+| = \binom{n}{2}$ (assuming $a_j \in \{\pm 1/2\}$). It is easy to show that $|\Phi_\lambda|$ is minimized only in the latter case. Since the Weyl group includes arbitrary changes of sign among the ε_j's, it follows that all choices for λ with $a_j = \pm 1/2$ belong to the same W-orbit.

Finally, the exceptional cases were analyzed using the author's Maple package for root systems and finite Coxeter groups [**Ste2**]. (The programs are freely available upon request.) We checked that for the root systems G_2, F_4, E_6, E_7, and E_8, there are 3, 12, 36, 36, and 135 distinct choices for $\lambda \pmod{2\Lambda}$ that minimize $|\Phi_\lambda|$. We then verified that the W orbit of $\rho \pmod{2\Lambda}$ did indeed have sizes 3, 12, 36, 36, and 135, respectively. $\square$

Appendix: A Minuscule Atlas

In this appendix we provide a description of every minuscule poset P, together with other related data. In the following, $\varepsilon_1, \ldots, \varepsilon_n$ denotes a standard choice of basis for $\mathfrak{h}^*$, except for $\Phi = A_{n-1}$, where these coordinates are subject to $\varepsilon_1 + \cdots + \varepsilon_n = 0$. In each case, we provide either an explicit choice of simple roots $\alpha_1, \ldots, \alpha_n$ in terms of the coordinates ε_j, or else a labeled Dynkin diagram indicating our preferred ordering of the simple roots.

We also indicate which fundamental weights ω_j (relative to our ordering of the simple roots) are minuscule. In such cases, the associated minuscule poset is denoted P_j.

$\Phi = A_{n-1}$, $\mathfrak{g} = sl(n)$.
Simple roots: $\alpha_j = \varepsilon_{j+1} - \varepsilon_j$ $(1 \leq j < n)$.
Minuscule weights: $\omega_j = \varepsilon_{j+1} + \cdots + \varepsilon_n$ $(1 \leq j < n)$.
Minuscule reps.: $\bigwedge^j(\mathbf{C}^n)$ $(1 \leq j < n)$.
Minuscule posets: $P_j \cong [j] \times [n-j]$ $(1 \leq j < n)$.

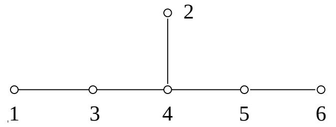
$\Phi = B_n$, $\mathfrak{g} = so(2n+1)$.
Simple roots: $\alpha_1 = \varepsilon_1$, $\alpha_j = \varepsilon_j - \varepsilon_{j-1}$ $(2 \leq j \leq n)$.
Minuscule weights: $\omega_1 = (1/2)(\varepsilon_1 + \cdots + \varepsilon_n)$.
Minuscule reps.: spin representation.
Minuscule posets: $P_1 \cong ([n] \times [n])/S_2 \cong J([n-1] \times [2])$.

$\Phi = C_n$, $\mathfrak{g} = sp(2n)$.
Simple roots: $\alpha_1 = 2\varepsilon_1$, $\alpha_j = \varepsilon_j - \varepsilon_{j-1}$ $(2 \leq j \leq n)$.
Minuscule weights: $\omega_n = \varepsilon_n$.
Minuscule reps.: defining representation.
Minuscule posets: $P_n \cong [2n-1]$.

$\Phi = D_n$, $\mathfrak{g} = so(2n)$.
Simple roots: $\alpha_1 = \varepsilon_1 + \varepsilon_2$, $\alpha_j = \varepsilon_j - \varepsilon_{j-1}$ $(2 \leq j \leq n)$.
Minuscule weights: $\omega_{1,2} = (1/2)(\pm\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$, $\omega_n = \varepsilon_n$.
Minuscule reps.: two spin representations, defining representation.
Minuscule posets: $P_{1,2} \cong ([n-1] \times [n-1])/S_2$, $P_n \cong J^{n-3}([2] \times [2])$.

In Figure 1 is an illustration of P_n for the case $n = 6$. One sees that in general, P_n has two order-reversing involutions. For even n, the involution induced by the action of w_0 has two fixed points; for odd n it has no fixed points.

$\Phi = E_6$.
Dynkin diagram:



Minuscule weights: ω_1, ω_6.
Minuscule posets: $P_{1,6} \cong J(([4] \times [4]/S_2) \cong J^2([2] \times [3])$. See Figure 2.

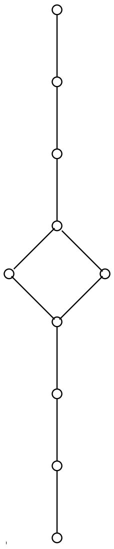


FIGURE 1.

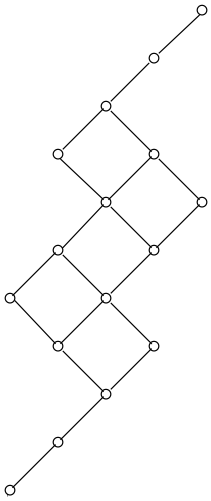


FIGURE 2.

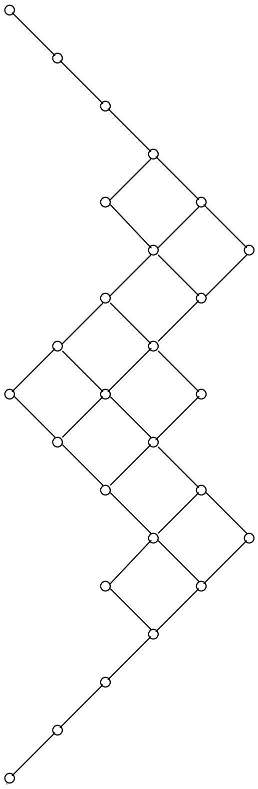
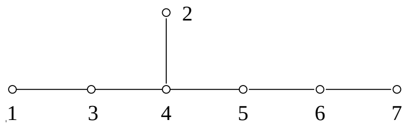


FIGURE 3.

$\Phi = E_7$.

Dynkin diagram:



Minuscule weights: ω_7.

Minuscule posets: $P_7 \cong J^2(([4] \times [4]/S_2) \cong J^3([2] \times [3])$. See Figure 3.

The exceptional root systems G_2, F_4 and E_8 have no minuscule weights.

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