Orthogonal Sets of Young Symmetrizers

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Dedicated to my friend Dennis Stanton for his enthusiasm and encouragement

Abstract. Young symmetrizers are primitive idempotents in the group algebra of the symmetric group S_n that are indexed in a natural way by Young tableaux. Although the Young symmetrizers corresponding to standard tableaux may be used to decompose the group algebra into a direct sum of minimal left ideals, they are not pairwise orthogonal in general. We pose the problem of finding maximum sets of pairwise orthogonal (but not necessarily standard) Young symmetrizers, and show in particular that it is possible to find (nonstandard) complete orthogonal sets if and only if $n \leqslant 6$.

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1. Young Symmetrizers

Let $[n] = \{1, 2, \ldots, n\}$ and let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n; i.e., a weakly decreasing sequence of positive integers with sum n. The Young diagram associated to λ is the set of n lattice points $D_\lambda := \{(i,j) \in \mathbb{Z}^2 : 1 \leqslant j \leqslant \lambda_i\}$, and a Young tableau of shape λ is a bijection $T : D_\lambda \to [n]$. We use matrix-style coordinates so that for example,

$$\begin{matrix} 6 & 3 & 4 & 2 \\ 1 & 8 & & \\ 7 & 5 & & \end{matrix}$$

is a Young tableau of shape $(4, 2, 2)$, and the $(3, 1)$-entry of this tableau is 7.

Given a Young tableau T, let $R(T)$ denote the row-stabilizer of T; i.e., the subgroup of S_n that preserves the sets of entries in each row. We similarly let $C(T)$ denote the column stabilizer. In the group algebra $\mathbb{Q}S_n$, let

$$r_T := \sum_{w \in R(T)} w, \quad \bar{c}_T := \sum_{w \in C(T)} \mathrm{sgn}(w)w.$$

The product $e_T := \bar{c}_T r_T$ is a *Young symmetrizer*; it generates a minimal left ideal in the group algebra, and the irreducible representation of S_n this ideal affords depends only on the shape λ. Conversely, every irreducible representation of S_n arises this way.

Young symmetrizers are also known to be quasi-idempotent in the sense that

$$e_T^2 = \alpha_\lambda e_T \tag{1}$$

for some nonzero scalar α_λ. In fact, $\alpha_\lambda = n!/f^\lambda$, where f^λ denotes the dimension of the corresponding representation of S_n. (For example, see Theorem 3.1.10 of [**JK**].)

Given two Young tableaux S and T, we say that S is *rc-transverse* to T if every row of S intersects every column of T in at most one element. Dually, we say that T is *cr-transverse* to S. Note that T is rc-transverse and cr-transverse to itself.

Proposition 1. *If S and T are Young tableaux of shapes λ and μ (respectively), then $e_S e_T = 0$ if and only if $\lambda \neq \mu$ or S is not rc-transverse to T.*

Proof. If S is not rc-transverse to T, then there is a pair of elements i, j that appear in the same row of S and the same column of T. It follows that the transposition $t = (i, j)$ belongs to both $R(S)$ and $C(T)$, whence $r_S t = r_S$, $t\bar{c}_T = -\bar{c}_T$, and

$$r_S \bar{c}_T = r_S t \bar{c}_T = -r_S \bar{c}_T = 0.$$

Consequently, $e_S e_T = \bar{c}_S (r_S \bar{c}_T) r_T = 0$.

In the case $\lambda \neq \mu$, the ideal $\mathbb{Q}S_n e_S e_T$ is simultaneously a quotient of $\mathbb{Q}S_n e_S$ and a submodule of $\mathbb{Q}S_n e_T$. Since these $\mathbb{Q}S_n$-modules are irreducible and non-isomorphic, this is possible only if $e_S e_T = 0$.

For the converse, if S is rc-transverse to T and $\lambda = \mu$, then the unique permutation $w_{S,T} \in S_n$ that transforms $T \to S$ belongs to $R(S)C(T)$ (see Chapter 7, Lemma 1 in [**F**]). It follows that $r_S w_{S,T} \bar{c}_T = \pm r_S \bar{c}_T$, and hence (1) implies

$$\pm e_S e_T = e_S w_{S,T} e_T = w_{S,T} e_T^2 = \alpha_\lambda w_{S,T} e_T.$$

The fact that this expression is nonzero (i.e., $e_T \neq 0$) may be easily seen without taking for granted that e_T generates an irreducible $\mathbb{Q}S_n$-module. Indeed, note that $R(T)$ and $C(T)$ must intersect trivially, since any permutation common to both must stabilize the intersection of every row and column of T. From this observation, it follows that the coefficient of the identity element in $e_T = \bar{c}_T r_T$ is 1. $\square$

We define Young tableaux S and T to be *orthogonal* if $e_S e_T = e_T s_S = 0$. By the above proposition, this is equivalent to S and T either having different shapes, or S being neither rc-transverse nor cr-transverse to T.

As noted originally by Young, a set $\mathcal{Y}$ of pairwise orthogonal Young tableaux of shape λ may be used to give an explicit set of matrix units in $\mathbb{Q}S_n$. Indeed, if we set

$$e_{S,T} := \frac{1}{\alpha_\lambda} \bar{c}_S w_{S,T} r_T = \frac{1}{\alpha_\lambda} w_{S,T} e_T = \frac{1}{\alpha_\lambda} e_S w_{S,T} \quad (S, T \in \mathcal{Y}),$$

where (as in the proof of Proposition 1) $w_{S,T} \in S_n$ denotes the unique permutation that transforms T into S, then an easy calculation based on (1) shows that

$$e_{S,T} e_{P,Q} = \delta_{T,P} e_{S,Q}.$$

Since the minimal two-sided ideal of $\mathbb{Q}S_n$ corresponding to λ has dimension $(f^\lambda)^2$, one sees that f^λ is an upper bound for the cardinality of $\mathcal{Y}$. If this bound is achieved, then the matrix units $e_{S,T}$ form a basis for the corresponding two-sided ideal, and we refer to $\mathcal{Y}$ as a *complete* set of orthogonal Young tableaux. If we had such a set for each partition of n, we would have an explicit realization of $\mathbb{Q}S_n$ as a direct sum of matrix rings.

Recalling that f^λ is known to be the number of *standard* Young tableaux of shape λ (i.e., tableaux with increasing rows and columns), these particular tableaux are the most obvious candidates for a complete orthogonal set. However, it is easy to discover (and it is mentioned by Young in [**Y**]) that not all standard tableaux are orthogonal.[1][2] For example, the standard tableaux

$$S = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}, \qquad T = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix} \tag{2}$$

have the property that S is cr-transverse to T, so they are not orthogonal.

[1] Note that in Theorem 3.1.24 of [**JK**] it is incorrectly stated that the Young symmetrizers corresponding to standard tableaux are pairwise orthogonal.

[2] In [**Y**], Young also shows that standard tableaux are "half-orthogonal" in the sense that they may be totally ordered so that $e_S e_T = 0$ for $S > T$.

Proposition 2. *The standard Young tableaux of shape λ are pairwise orthogonal if and only if λ is a hook shape (i.e., $\lambda = (a, 1, ..., 1)$ for some $a \geqslant 1$) or $\lambda = (2, 2)$.*

Proof. The case $\lambda = (2, 2)$ is easy to check directly.

In the case $\lambda = (a, 1, ..., 1)$, consider any pair of distinct standard Young tableaux of shape λ, say S and T. Since S and T are uniquely determined by the entries > 1 in their first rows, it follows that there must be some element $i > 1$ in the first row of S that is in the first column of T (in addition to the element 1), and thus S is not rc-transverse to T.

In all other cases, the diagram of λ must contain the diagram of $(3, 2)$ or $(2, 2, 1)$. Transposing diagrams if necessary, we may assume the former. In that case, we may extend the two tableaux in (2) in an identical but arbitrary way to standard Young tableaux of shape λ, in which case the left tableau will remain cr-transverse to the right tableau. $\square$

Although the standard Young tableau of shape $(3, 2)$ are not pairwise orthogonal, it is surprising that one can still find a complete set of $f^{3,2} = 5$ (not necessarily standard) orthogonal Young tableaux of this shape. For example, the tableaux

$$\begin{matrix} 5 & 3 & 4 \\ 1 & 2 & \end{matrix} \qquad \begin{matrix} 1 & 4 & 5 \\ 2 & 3 & \end{matrix} \qquad \begin{matrix} 2 & 5 & 1 \\ 3 & 4 & \end{matrix} \qquad \begin{matrix} 3 & 1 & 2 \\ 4 & 5 & \end{matrix} \qquad \begin{matrix} 4 & 2 & 3 \\ 5 & 1 & \end{matrix} \tag{3}$$

provide one such set. It is even more surprising that, as we shall see below, there exist complete sets of orthogonal Young tableaux for every partition of size $\leqslant 6$.

Question 3. *For which partitions λ can one find a complete set of f^λ orthogonal Young tableaux of shape λ?*

Note that this may be formulated as a problem concerning the existence of a clique of size f^λ in a suitable graph whose size grows exponentially with n in the worst case. However, clique-existence is well known to be NP Complete, and is not known to be Co-NP (i.e., for a general graph, there is no known "short" witness to the fact that a clique of size f does not exist), so a naïve approach to Question 3 is unlikely to be successful.

While we are unable to answer this question in general, our main result is a structure theorem for orthogonal sets of Young tableaux of shape $(n-2, 2)$ that allows us to conclude in particular that complete orthogonal sets for this shape do not exist for $n \geqslant 7$. It would also be interesting to investigate the more general problem of determining, as a function of λ, the maximum size of an orthogonal set of Young tableaux of shape λ.

2. Orthogonal sets of shape $(n-2, 2)$.

Let us fix $\lambda = (n-2, 2)$ (and hence $n \geqslant 4$), and consider any set $\mathcal{Y}$ of Young tableaux of shape λ. We may associate to $\mathcal{Y}$ a loopless, undirected graph $\Gamma = \Gamma(\mathcal{Y})$ on the vertex set $[n]$ by creating an edge with endpoints a and b for each $T \in \mathcal{Y}$ with second row $\{a, b\}$.

For example, the graph associated to the set listed in (3) is a 5-cycle.

Note that in general, the graph Γ may have multiple edges with the same endpoints.

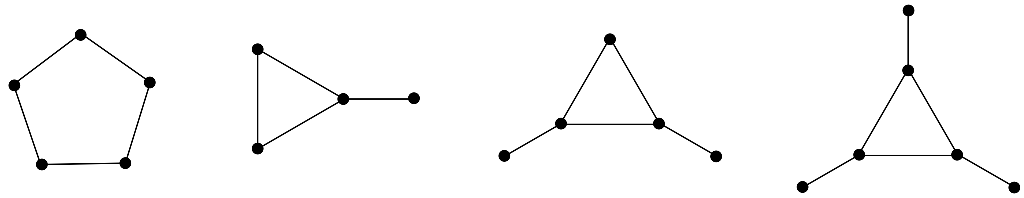

FIGURE 1: Graph complements for complete orthogonal sets.

THEOREM 4. *A loopless graph Γ with vertex set $[n]$ is the graph of an orthogonal set of Young tableaux of shape $(n-2,2)$ if and only if Γ is simple (no multiple edges), and for every adjacent pair of vertices i,j in Γ, there is a path of length 3 from i to j in the complementary graph Γ^c.*

Proof. If T is a Young tableau of shape $(n-2,2)$, say

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{n-2} \\ b_1 & b_2 & & & \end{array},$$

then T is cr-transverse to exactly those Young tableaux S such that a_1 and b_1, as well as a_2 and b_2, are in separate rows of S. Equivalently, these are the Young tableaux whose second row consists of a_1 or b_1, and a_2 or b_2, in some order.

Thus if $\mathcal{Y}$ is an orthogonal set that includes T, then T must be the unique member of $\mathcal{Y}$ whose set of second row entries is $\{b_1,b_2\}$, and there cannot exist tableaux in $\mathcal{Y}$ whose second rows are $\{b_1,a_2\}$, $\{a_1,a_2\}$, or $\{a_1,b_2\}$. It follows that in the corresponding graph Γ, there is only one edge whose endpoints are b_1 and b_2, and the vertex sequence b_1,a_2,a_1,b_2 is a path in Γ^c.

Conversely, if Γ is a graph satisfying the stated conditions, one may create a Young tableaux T of shape $(n-2,2)$ for each edge $\{i,j\}$ of Γ by selecting a 3-step path i,a,b,j in Γ^c and assigning

$$T \;=\; \begin{array}{ccccc} b & a & * & \cdots & * \\ i & j & & & \end{array},$$

with the $*$'s filled in any way without repetitions. Note that i,a,b,j are necessarily distinct (e.g., if $b=i$, then we contradict the fact that b and j are not adjacent in Γ), so T is a valid Young tableau. Furthermore, by construction, there will be no tableaux in $\mathcal{Y}$ with second row $\{a,i\}$, $\{a,b\}$, or $\{b,j\}$. As noted above, these are the only tableaux that are rc-transverse to T, so $\mathcal{Y}$ is an orthogonal set. $\square$

It is easy to check that $f^\lambda = \binom{n}{2} - n$ for $\lambda = (n-2,2)$, so in a complete orthogonal set $\mathcal{Y}$ of Young tableaux of shape λ, the complementary graph $\Gamma(\mathcal{Y})^c$ has n edges.

Theorem 5. *Let Γ be a simple graph such that there is a path of length 3 in Γ^c between every adjacent pair of vertices in Γ. If Γ has $n \geqslant 4$ vertices and $f^{(n-2,2)}$ edges, then Γ^c is isomorphic to one of the four graphs illustrated in Figure 1. In particular, there is a complete set of orthogonal Young tableaux of shape $(n-2,2)$ only if $n = 4, 5$ or 6.*

Proof. As noted above, the complementary graph Γ^c has n edges. It is also connected, since there is a path of length 1 or 3 between every pair of vertices. Thus Γ^c is a unicycle; i.e., a connected graph with a unique cycle. This forces the cycle C to have length 3 or 5, since in all other cases, there will exist pairs of non-adjacent vertices of C that cannot be connected by paths of length 3.

Suppose C is a 5-cycle. If there is a vertex v at distance 1 from C, then the second closest vertices to v on C can only be reached by paths of length 2 or $\geqslant 5$, a contradiction. Therefore in this case, C exhausts all of Γ^c.

Otherwise, C is a 3-cycle. If there is an off-cycle vertex v at distance 2 from the nearest vertex u of C, then there would be no Γ^c-path of length 3 connecting u and v, a contradiction. Therefore, all off-cycle vertices must be at distance 1 from C. If there were two such vertices u and v adjacent to the same cycle vertex, then the distance between u and v would be 2, again a contradiction. Thus each cycle vertex is adjacent to at most one off-cycle vertex, and the only graphs with this property (given that there are at least 4 vertices) are listed in Figure 1. Conversely, it is easy to check that each of the graphs listed in Figure 1 have the claimed properties. $\square$

Remark 6. (a) One sees that for the graphs in Figure 1, there is a *unique* path of length 3 between each pair of non-adjacent vertices, so from the proof of Theorem 4, it follows that there is a unique set of orthogonal Young tableaux of shape $(n-2,2)$ with these associated graphs, up to relabeling and permuting columns of equal length. Thus there is only one complete set of orthogonal Young tableaux for each of the shapes $(2,2)$ and $(4,2)$, and two of shape $(3,2)$, up to relabeling and column reordering.

(b) Using computer searches, we have found complete sets of orthogonal Young tableaux of shape $(3,2,1)$ and $(4,3)$. The results are listed in Table 1.

(c) Any orthogonal set of Young tableaux of shape $(n,n-1)$ may be extended to an orthogonal set of shape (n,n) by assigning the letter $2n$ to the southeast corner of each tableau in the set. It follows that there exist complete sets of orthogonal Young tableaux for the shapes $(3,3)$ and $(4,4)$ as well. Note also that the transpose of an orthogonal set is also orthogonal, so it follows from the above results that there is a complete set of orthogonal Young tableaux of shape λ for all λ of size $\leqslant 6$.

(d) By extracting the second row from each member of an orthogonal set of Young tableaux of shape $(n-k,k)$, one obtains a k-uniform hypergraph, and it is not hard to characterize the hypergraphs that arise in this fashion, generalizing Theorem 4. However, we lack a structure theorem analogous to Theorem 5 for this more general case.

1 2 3 4 5 6	1 2 3 4 6 5	1 2 4 3 5 6	1 2 4 3 6 5	1 2 5 3 4 6	1 2 5 3 6 4	1 2 6 3 4 5	1 2 6 3 5 4
1 4 3 5 6 2	1 4 5 3 6 2	1 4 6 3 5 2	1 5 3 4 6 2	4 1 3 5 2 6	4 2 5 3 1 6	4 2 6 3 1 5	4 5 6 1 3 2

1 2 3 4 5 6 7	1 2 3 5 4 6 7	1 2 3 6 4 5 7	1 2 3 7 4 5 6	1 2 5 4 3 6 7	1 2 6 4 3 5 7	1 2 6 5 3 4 7
1 3 5 4 2 7 6	1 3 6 7 4 2 5	1 4 7 3 2 5 6	2 4 6 5 1 3 7	2 7 5 6 1 3 4	3 2 6 7 1 4 5	3 6 5 7 1 2 4

TABLE 1: Complete sets of orthogonal Young tableaux of shape $(3, 2, 1)$ and $(4, 3)$.

(e) The partitions of 7 for which we do not know the answer to Question 3 are $(3, 3, 1)$ and $(4, 2, 1)$ (and their conjugates). In the former case, finding a complete orthogonal set amounts to finding a clique of size 21 in a vertex-transitive graph with 1260 vertices[3] in which each vertex-neighborhood has 888 vertices and an edge density within the neighborhood of roughly 69%. In the latter case, the corresponding numbers are 35 (desired clique size), 2520 (number of vertices), 2016 (vertex degree), and 79% (neighborhood edge density). In computer searches, we found cliques of size 20 and 32 in these two graphs.

3. Induced Matchings in the Partition Lattice

There is a natural reformulation of Question 3 as a problem involving the lattice of partitions of an n-element set.

A Young tableau T of shape λ provides two partitions of the set $[n]$: a partition $\rho(T)$ whose blocks are the rows of T, and a partition $\kappa(T)$ whose blocks are the columns of T. Note that $\rho(T)$ has "type" λ in the sense that the cardinalities of the blocks of $\rho(T)$ are the parts of λ. Similarly, $\kappa(T)$ has type λ' (the partition conjugate to λ).

To decide whether T is rc-transverse to S amounts to checking whether $\rho(T)$ and $\kappa(S)$ meet trivially in the lattice of partitions of $[n]$; i.e., the only common refinement of $\rho(T)$ and $\kappa(S)$ is the complete partition $\hat{0} := \{\{1\}, \ldots, \{n\}\}$.

By convention, the blocks of a set partition are unordered, so there is a loss of information in passing from T to the pair $(\rho(T), \kappa(T))$: if (say) rows i and $i + 1$ have the same length, then the tableau obtained by switching rows i and $i + 1$ has the same row and column partitions, and there is a similar degree of freedom available when there exist columns of the same length.

[3]The vertices correspond to normalized Young tableaux, as explained in Section 3.

We define a Young tableau T to be *normalized* if

(1) the smallest entry in row i is smaller than the smallest entry in row $i+1$ whenever rows i and $i+1$ have the same length, and
(2) the smallest entry in column j is smaller than the smallest entry in column $j+1$ whenever columns j and $j+1$ have the same length.

For any Young tableau T, there is a unique permutation of the equal-length rows and columns that normalizes T, so the total number of normalized Young tableaux of shape λ is $n!/(m_1!\,m'_1!\,m_2!\,m'_2!\cdots)$, where m_i (respectively, m'_i) denotes the multiplicity of i as a part of λ (respectively, λ').

Since tableau orthogonality depends only on row and column partitions, there is no loss of generality in restricting the domain of Question 3 to normalized Young tableaux. That is, rather than search for a clique of size f^λ in a graph with $n!$ vertices (one for each Young tableau of shape λ), it suffices instead to search for a clique of the same size in a (usually much smaller) graph whose vertices are the normalized Young tableaux of shape λ.

PROPOSITION 7. *The map $T \mapsto (\rho(T), \kappa(T))$ defines a bijection between the set of normalized Young tableaux of shape λ and the set of ordered pairs (ρ, κ) of partitions of $[n]$ with meet $\hat{0}$ such that ρ has type λ and κ has type λ'.*

Our proof of this proposition is based on the following well known fact.

LEMMA 8. *If λ is a partition, then there is a unique $0,1$-matrix $[a_{ij}]$ with row sum vector λ and column sum vector λ'. Moreover, this matrix is obtained by setting $a_{ij} = 1$ for $(i,j) \in D_\lambda$ and $a_{ij} = 0$ otherwise.*

Proof. Given that $A = [a_{ij}]$ has the stated properties, the number of 1's in the first j columns of row i of A is at most $\min(j, \lambda_i)$. However, by counting the number of nodes in the first j columns of D_λ in two ways, one obtains the identity

$$\min(j, \lambda_1) + \min(j, \lambda_2) + \cdots = \lambda'_1 + \cdots + \lambda'_j.$$

Hence, the number of 1's in the first j columns of row i of A must be exactly $\min(j, \lambda_i)$ for all i and j. In particular, taking $j = \lambda_i$, one sees that A is uniquely determined. $\square$

Proof of Proposition 7. Given that T is normalized, the i-th row R_i of T may be recovered as the i-th block of $\rho(T)$ after sorting the blocks first in order of decreasing cardinality, and then in order of increasing minimal elements among blocks of equal cardinality. Similar considerations allow one to recover the j-th column C_j of T from $\kappa(T)$. Thus the (i,j)-entry of T may be recovered from $R_i \cap C_j$, and the map is injective.

Conversely, given set partitions ρ and κ of the stated types, we may order the blocks as above and define a_{ij} to be the cardinality of the intersection of the i-th block of ρ with the j-th block of κ. If ρ and κ meet trivially, then $[a_{ij}]$ is a $0,1$-matrix with row sum vector λ and column sum vector λ'. By Lemma 8, it follows that if we define $T(i,j)$ to be the unique element in the intersection of the i-th block of ρ and the j-th block of κ (when the intersection is non-empty), we obtain a normalized Young tableau of shape λ. $\square$

This yields the following reformulation of Question 3.

QUESTION 9. *For which λ can one find set partitions $\rho_1, \dots, \rho_f$ of type λ and $\kappa_1, \dots, \kappa_f$ of type λ' (with $f = f^\lambda$) such that ρ_i and κ_j have meet $\hat{0}$ if and only if $i = j$?*

Define Π_λ to be the bipartite graph featuring the set partitions of type λ as black vertices, the set partitions of type λ' as white vertices, and edges connecting black and white vertices that meet trivially. The above question amounts to finding a matching in Π_λ of size f^λ that is *induced*; i.e., every edge of Π_λ whose endpoints belong to the matching must be an edge of the matching.

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