AFFINE DESCENTS AND THE STEINBERG TORUS

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Abstract. Let $W \ltimes L$ be an irreducible affine Weyl group with Coxeter complex Σ, where W denotes the associated finite Weyl group and L the translation subgroup. The Steinberg torus is the Boolean cell complex obtained by taking the quotient of Σ by the lattice L. We show that the ordinary and flag h-polynomials of the Steinberg torus (with the empty face deleted) are generating functions over W for a descent-like statistic first studied by Cellini. We also show that the ordinary h-polynomial has a nonnegative γ-vector, and hence, symmetric and unimodal coefficients. In the classical cases, we also provide expansions, identities, and generating functions for the h-polynomials of Steinberg tori.

1. Introduction

1.1. **Overview.** Let S_n denote the symmetric group of permutations of $[n] := \{1, \ldots, n\}$. For each $w \in S_n$, a *descent* is an index i $(1 \leqslant i < n)$ such that $w_i > w_{i+1}$. We let

$$d(w) := |\{i \in [n-1] : w_i > w_{i+1}\}|$$

denote the number of descents in w. The corresponding generating function

$$A_{n-1}(t) := \sum_{w \in S_n} t^{d(w)} \tag{1.1}$$

is known as an *Eulerian polynomial*, although this definition differs from the classical one by a power of t. Some interesting features of the Eulerian polynomials include the facts that they have symmetric and unimodal coefficients and are known to have all real roots.

More generally, if W is any finite Coxeter group with simple reflections $s_1, \ldots, s_n$ (such as the symmetric group S_{n+1} with simple transpositions $s_i = (i, i+1)$), then a descent in some $w \in W$ may be defined as an index i such that $\ell(ws_i) < \ell(w)$, where $\ell(w)$ denotes the minimum length of an expression for w as a product of simple reflections. Thus there is an analogous *W-Eulerian polynomial*

$$W(t) := \sum_{w \in W} t^{d(w)},$$

where $d(w)$ is defined to be the number of descents in w. Note that as a Coxeter group, S_n is often denoted A_{n-1}, so this notation is consistent with (1.1).

Like the classical Eulerian polynomials, the W-Eulerian polynomials are known to have symmetric and unimodal coefficients. An elegant explanation of this fact may be based on a topological interpretation of $W(t)$ as the h-polynomial of the Coxeter complex of W. Since

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every (finite) Coxeter complex is realizable as the boundary complex of a simplicial polytope, the symmetry and unimodality of the coefficients of $W(t)$ may thus be seen as a consequence of the g-theorem (e.g., see Section III.1 of [21]).

Recently, several authors (see for example [1, 12, 19, 25]) have identified interesting classes of simplicial complexes whose h-polynomials have expansions of the form

$$h(t) = \sum_{0 \leqslant i \leqslant n/2} \gamma_i t^i (1+t)^{n-2i},$$

where the coefficients γ_i are nonnegative. It is easy to see that each summand in this expansion has symmetric and unimodal coefficients centered at $n/2$, and thus any h-polynomial with a nonnegative "γ-vector" in this sense necessarily has symmetric and unimodal coefficients. In these terms, the h-polynomials of all finite Coxeter complexes (i.e., the W-Eulerian polynomials) are known to have nonnegative γ-vectors [25].

Another feature of γ-nonnegativity is that it is a necessary condition for a polynomial to have all real roots, given that the polynomial has nonnegative symmetric coefficients. In this direction, Brenti [2] has conjectured that the W-Eulerian polynomials have all real roots, a result that remains unproved only for the groups $W = D_n$.

In this paper, we study a family of Eulerian-like polynomials associated to irreducible affine Weyl groups. These "affine" Eulerian polynomials may be defined as generating functions for "affine descents" over the corresponding finite Weyl group. An affine descent is similar to an ordinary descent in a Weyl group, except that the reflection corresponding to the highest root may also contribute a descent, depending on its effect on length.

The affine Eulerian polynomials have a number of interesting properties similar to those of the ordinary W-Eulerian polynomials. In particular, we show that they have nonnegative γ-vectors (Theorem 4.2), and conjecture that all of their roots are real. Perhaps the most interesting similarity is that each affine Eulerian polynomial is the h-polynomial of a naturally associated relative cell complex (Theorem 3.1).

To describe this complex, one should start with an irreducible affine Coxeter arrangement. Such an arrangement induces a simplicial decomposition of the ambient space; by taking the quotient of this space by the translation subgroup of the associated affine Weyl group, one obtains a torus decomposed into simplicial cells. We refer to this cell complex as the *Steinberg torus* in recognition of the work of Steinberg, who gave a beautiful proof of Bott's formula for the Poincaré series of an affine Weyl group by analyzing the action of the finite Weyl group on the homology of this complex in two different ways (see Section 3 of [22]). In fact, Steinberg also allows the possibility of twisting the entire construction by an automorphism, but we will not consider this variation here.

It is important to note that the Steinberg torus is not a simplicial complex (distinct cells may share the same set of vertices), but it is at least a Boolean cell complex in the sense that all lower intervals in the partial ordering of cells are Boolean algebras.[1] For further information about Boolean complexes, see [20] and the references cited there.

[1]We thank V. Welker for bringing this to our attention.

For our purposes, it is essential to omit the empty cell of dimension -1 from the Steinberg torus; we refer to the resulting relative complex as the *reduced Steinberg torus.* It is this complex whose h-polynomial is the corresponding affine Eulerian polynomial; i.e., the generating function for affine descents.

It is noteworthy that affine descents in finite Weyl groups were first introduced by Cellini [3] in a construction of a variant of Solomon's descent algebra, and developed further for the groups of type A and C in several follow-up papers on "cyclic descents" by Cellini [4, 5], Fulman [10, 11], and Petersen [17]. In very recent work, Lam and Postnikov [15] study a weighted count of affine descents (the "circular descent number") that coincides with an ordinary count (only) in type A.

1.2. **Organization.** The paper is structured as follows. Section 2 introduces the necessary definitions, including details of the construction of the Steinberg torus. In Section 3 we show that the affine Eulerian polynomials are the h-polynomials of reduced Steinberg tori (Theorem 3.1). Although we do not know of any simple topological explanation for the nonnegativity of the h-vector,[2] we do show that reduced Steinberg tori are partitionable (Remark 3.6); this is a weak analogue of shellability that implies h-nonnegativity.

In Section 4, we present our second main result; namely, that the affine Eulerian polynomials have nonnegative γ-vectors (Theorem 4.2). As a corollary, it follows that the h-vectors of reduced Steinberg tori are symmetric and unimodal. In this section, we also present evidence supporting our conjecture that all roots of affine Eulerian polynomials are real. The proof of Theorem 4.2 is case-by-case, and relies on combinatorial expansions for the γ-vectors of affine Eulerian polynomials for the classical Weyl groups that we provide in Section 5. In this latter section, we also provide combinatorial expansions for the flag h-polynomials of reduced Steinberg tori, one of which suggests the possibility that a natural class of (reduced) polyhedral tori may have nonnegative cd-indices (see Question 5.5).

In Section 6, we present three unexpected identities relating ordinary and affine Eulerian polynomials (two new, one old), and use these to derive exponential generating functions for the affine Eulerian polynomials for each classical series of Weyl groups.

[2] However, Novik and Swartz have recently proved a lower bound for the h-vector of a Buchsbaum Boolean complex that, in the case of reduced tori, implies nonnegativity—see Theorem 6.4 of [16].

2. Preliminaries

2.1. **Finite and affine Weyl groups.** We assume the reader is familiar with the basic theory of reflection groups. We follow the notational conventions of [14].

Let Φ be a crystallographic root system embedded in a real Euclidean space V with inner product $\langle \cdot\,, \cdot\rangle$. For any root $\beta \in \Phi$, let $H_\beta := \{\lambda \in V : \langle \lambda, \beta\rangle = 0\}$ be the hyperplane orthogonal to β and let s_β denote the orthogonal reflection through H_β. Fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$, and let $S = \{s_1, \ldots, s_n\}$ denote the corresponding set of simple reflections. The latter generates a finite Coxeter group W (a Weyl group).

Unless stated otherwise, we always assume that Φ and W are irreducible.

For convenience, we assume that Δ spans V.

Having fixed a choice of simple roots, every root β either belongs to the nonnegative span of the simple roots and is designated *positive*, or else belongs to the nonpositive span of the simple roots and is designated *negative*. We write $\beta > 0$ or $\beta < 0$ accordingly.

The affine Weyl group $\widetilde{W}$ is generated by reflections $s_{\beta,k}$ through the affine hyperplanes

$$H_{\beta,k} := \{\lambda \in V : \langle \lambda, \beta\rangle = k\} \qquad (\beta \in \Phi,\ k \in \mathbb{Z}).$$

Alternatively, one may construct $\widetilde{W}$ as the semidirect product $W \ltimes \mathbb{Z}\Phi^\vee$, where $\mathbb{Z}\Phi^\vee$ denotes the lattice generated by all co-roots $\beta^\vee = 2\beta/\langle \beta, \beta\rangle$ $(\beta \in \Phi)$, acting on V via translations.

Given that Φ is irreducible, it has a unique highest root $\widetilde{\alpha}$, and it is well-known that $\widetilde{W}$ is generated by $\widetilde{S} := S \cup \{s_{\widetilde{\alpha},1}\}$ and that $(\widetilde{W}, \widetilde{S})$ is an irreducible Coxeter system.

Note that $\widetilde{W}$ depends on the underlying root system Φ (not merely W), so we are committing an abuse of notation. For example, B_n and C_n are isomorphic as Coxeter systems, but the affine groups $\widetilde{B}_n$ and $\widetilde{C}_n$ are not isomorphic as Coxeter systems for $n \geqslant 3$.

2.2. **Coxeter complexes.** The hyperplanes H_β $(\beta \in \Phi)$ induce a partition of V into a complete W-symmetric fan of simplicial cones. By intersecting this fan with the unit sphere in V, one obtains a topological realization of the Coxeter complex $\Sigma(W)$. The action of W on chambers (maximal cones) in the fan is simply transitive, and the choice of simple roots Δ is equivalent to designating a dominant chamber; namely,

$$C_\varnothing := \{\lambda \in V : \langle \lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Delta\}.$$

The closure of the dominant chamber is a fundamental domain for the action of W on V, and thus every cone in the fan has the form wC_J $(w \in W,\ J \subseteq [n])$, where

$$C_J := \{\lambda \in V : \langle \lambda, \alpha_j\rangle = 0 \text{ for } j \in J,\ \langle \lambda, \alpha_j\rangle > 0 \text{ for } j \in [n] \setminus J\}.$$

Notice that the rays (1-dimensional cones) have the form wC_J where $J = [n] \setminus \{j\}$ for some j. If we assign color j to all such rays, we obtain a *balanced* coloring of $\Sigma(W)$; i.e., every maximal face (chamber) has exactly one vertex (extreme ray) of each color.

Similarly, the affine hyperplanes $H_{\beta,k}$ $(\beta \in \Phi,\ k \in \mathbb{Z})$ may be used to partition V into a $\widetilde{W}$-symmetric simplicial complex that is isomorphic to the Coxeter complex $\Sigma(\widetilde{W})$. By abuse

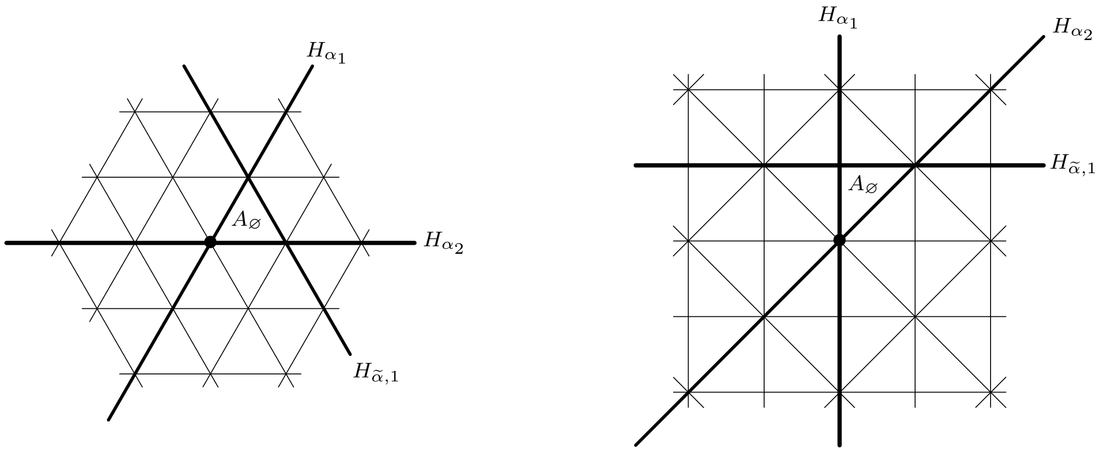


FIGURE 1. Portions of the Coxeter complexes for $\widetilde{A}_2$ and $\widetilde{C}_2$.

of notation, we will identify $\Sigma(\widetilde{W})$ with this particular geometric realization. The action of $\widetilde{W}$ on alcoves (maximal simplices) is simply transitive, and the fundamental alcove

$$A_\varnothing := C_\varnothing \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\}$$

is tied to the choice of $\widetilde{S}$ in the sense that the $\widetilde{W}$-stabilizer of every point in the closure of $A_\varnothing$ (a fundamental domain) is generated by a proper subset of $\widetilde{S}$. We index the faces of $A_\varnothing$ by subsets of $[0,n] := \{0,1,\ldots,n\}$ so that the J-th face is

$$A_J := \begin{cases} C_J \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle < 1\} & \text{if } 0 \notin J, \\ C_{J \setminus \{0\}} \cap \{\lambda \in V : \langle \lambda, \widetilde{\alpha} \rangle = 1\} & \text{if } 0 \in J. \end{cases}$$

Note that A_J is the empty face when $J = [0,n]$.

The Coxeter complexes for $\widetilde{A}_2$ and $\widetilde{C}_2$ are illustrated in Figure 1.

Since the closure of $A_\varnothing$ is a fundamental domain for the action of $\widetilde{W}$, each cell in this complex has the form $\mu + wA_J$ ($\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$, $J \subseteq [0,n]$). In particular, the vertices of $\Sigma(\widetilde{W})$ are of the form $\mu + wA_{\{j\}^c}$, where $J^c := [0,n] \setminus J$. If we assign color j to each of the vertices $\mu + wA_{\{j\}^c}$, then the vertices of the cell $\mu + wA_J$ are assigned color-set J^c (without repetitions), so this coloring is balanced.

Remark 2.1. If Φ and W are reducible, then the affine hyperplanes $H_{\beta,k}$ may still be used to partition V into a cell complex, but the result is not a geometric realization of the Coxeter complex of $\widetilde{W}$. Indeed, the cells of this complex are products of simplices, whereas the Coxeter complex of every Coxeter system is simplicial.

2.3. **Flag f-vectors and h-vectors.** Let Σ be a finite set of simplices (or abstractly, a hypergraph) that is properly colored; i.e., the vertices of Σ have been assigned colors from some index set, say $[0,n]$, so that no simplex has two vertices with the same color. The main examples we have in mind are balanced simplicial (or more generally, Boolean) complexes.

A basic combinatorial invariant of Σ that carries significant algebraic and topological information (e.g., see the discussion in Section III.4 of [21]) is the *flag h-vector*. The components of the flag h-vector are the quantities

$$h_J(\Sigma) := \sum_{I\subseteq J} (-1)^{|J\setminus I|} f_I(\Sigma) \qquad (J \subseteq [0,n]), \tag{2.1}$$

where $f_I(\Sigma)$ denotes the number of simplices in Σ whose vertices have color-set I.

The quantities $f_J(\Sigma)$ for $J \subseteq [0,n]$ are collectively referred to as the *flag f-vector* of Σ.

The corresponding generating functions

$$f(\Sigma; t_0, \ldots, t_n) := \sum_{J\subseteq[0,n]} f_J(\Sigma) \prod_{j\in J} t_j,$$
$$h(\Sigma; t_0, \ldots, t_n) := \sum_{J\subseteq[0,n]} h_J(\Sigma) \prod_{j\in J} t_j$$

are known as the *flag f-polynomial* and *flag h-polynomial* of Σ. The more familiar ordinary f-polynomial and h-polynomial may be obtained via the specializations

$$f(\Sigma; t) := f(\Sigma; t, \ldots, t) = \sum_{J\subseteq[0,n]} f_J(\Sigma) t^{|J|},$$
$$h(\Sigma; t) := h(\Sigma; t, \ldots, t) = \sum_{J\subseteq[0,n]} h_J(\Sigma) t^{|J|}.$$

The coefficients of these polynomials yield the (ordinary) f-vector and h-vector of Σ.

Note that (2.1) implies

$$h(\Sigma; t_0, \ldots, t_n) = (1-t_0)\cdots(1-t_n) f\Big(\Sigma; \frac{t_0}{1-t_0}, \ldots, \frac{t_n}{1-t_n}\Big), \tag{2.2}$$

and hence $h(\Sigma; t) = (1-t)^{n+1} f(\Sigma; t/(1-t))$.

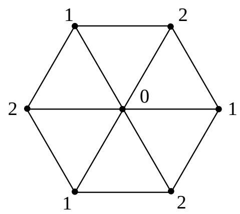
2.4. **The Steinberg torus.** As the translation subgroup of $\widetilde{W}$, the co-root lattice $\mathbb{Z}\Phi^\vee$ acts as a group of color-preserving automorphisms of the affine Coxeter complex $\Sigma(\widetilde{W})$. Letting T denote the n-torus $V/\mathbb{Z}\Phi^\vee$, it follows that the image of $\Sigma(\widetilde{W})$ under the natural map $V \to T$ is a balanced Boolean complex, denoted $\Sigma_T(\widetilde{W})$. That is,

$$\Sigma_T(\widetilde{W}) = \Sigma(\widetilde{W})/\mathbb{Z}\Phi^\vee.$$

As explained in the introduction, we refer to $\Sigma_T(\widetilde{W})$ as the *Steinberg torus*. We also define the *reduced Steinberg torus*, denoted $\Sigma'_T(\widetilde{W})$, to be the relative complex obtained by deleting the empty simplex of dimension -1 from $\Sigma_T(\widetilde{W})$.

Note that these are finite complexes; there is one maximal cell $wA_\varnothing + \mathbb{Z}\Phi^\vee$ for each $w \in W$.

There is an alternative way to construct the Steinberg torus that starts with the observation that the 0-colored vertices in $\Sigma(\widetilde{W})$ are the members of $\mathbb{Z}\Phi^\vee$. Since every alcove A has a unique 0-colored vertex, one may translate A via $\mathbb{Z}\Phi^\vee$ to a unique alcove that has the origin



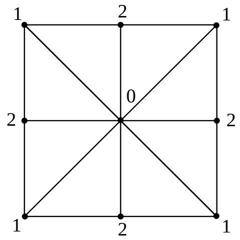


FIGURE 2. The Steinberg tori for $\widetilde{A}_2$ and $\widetilde{C}_2$.

as a vertex; i.e., to one of the alcoves in the W-orbit of $A_\varnothing$. The closure of this set of alcoves is the W-invariant convex polytope

$$P_\Phi = \{\lambda \in V : -1 \leqslant \langle \lambda, \beta \rangle \leqslant 1 \text{ for all } \beta \in \Phi\},$$

and the Steinberg torus is obtained by identifying the maximal opposite faces of P_Φ.

Example 2.2. The Steinberg torus for $\widetilde{A}_2$ is a hexagon with opposite sides identified, decomposed into six triangles, nine edges, and three vertices. See Figure 2. It has flag f-polynomial

$$f(\Sigma_T(\widetilde{A}_2); t_0, t_1, t_2) = 1 + t_0 + t_1 + t_2 + 3t_0t_1 + 3t_0t_2 + 3t_1t_2 + 6t_0t_1t_2.$$

Using (2.2) to compute the flag h-polynomial, we find

$$h(\Sigma_T(\widetilde{A}_2); t_0, t_1, t_2) = 1 + 2t_0t_1 + 2t_0t_2 + 2t_1t_2 - t_0t_1t_2.$$

On the other hand, the reduced Steinberg torus lacks the empty face, so its flag f-polynomial omits the constant term and we find

$$h(\Sigma'_T(\widetilde{A}_2); t_0, t_1, t_2) = t_0 + t_1 + t_2 + t_0t_1 + t_0t_2 + t_1t_2.$$

Specializing, we see that the reduced Steinberg torus has ordinary f-polynomial $3t+9t^2+6t^3$, and ordinary h-polynomial $3t + 3t^2$.

Example 2.3. The Steinberg torus for $\widetilde{C}_2$ (or the isomorphic $\widetilde{B}_2$) is a square with opposite sides identified, decomposed into eight triangles, twelve edges, and four vertices as in Figure 2. The reduced Steinberg torus has flag f-polynomial

$$f(\Sigma'_T(\widetilde{C}_2); t_0, t_1, t_2) = t_0 + t_1 + 2t_2 + 4t_0t_1 + 4t_0t_2 + 4t_1t_2 + 8t_0t_1t_2,$$

and (again via (2.2)) flag h-polynomial

$$h(\Sigma'_T(\widetilde{C}_2); t_0, t_1, t_2) = t_0 + t_1 + 2t_2 + 2t_0t_1 + t_0t_2 + t_1t_2.$$

As in the previous example, it is easy to check that the ordinary and flag h-polynomials of the unreduced Steinberg torus have (some) negative coefficients.

2.5. **Affine descents.** We define a root β to be negative with respect to $w \in W$ if $w\beta < 0$. The positive roots that are negative with respect to w are known as *inversions*. If $\ell(w)$ denotes the minimum length of an expression for w as a product of simple reflections, then β is negative with respect to w if $\ell(ws_\beta) < \ell(w)$ (for $\beta > 0$) or $\ell(ws_\beta) > \ell(w)$ (for $\beta < 0$).

A simple root that is negative with respect to w is said to be a (right) *descent*, and the descent set of w, denoted $D(w)$, records the corresponding set of indices. Thus,

$$D(w) = \{j \in [n] : w\alpha_j < 0\} = \{j \in [n] : \ell(ws_j) < \ell(w)\}.$$

We let $d(w) := |D(w)|$ denote the number of descents in w.

As noted in the introduction, the W-Eulerian polynomial is the h-polynomial of the Coxeter complex $\Sigma(W)$. That is,

$$W(t) = \sum_{w \in W} t^{d(w)} = h(\Sigma(W); t).$$

More generally, the generating function for descent *sets*; namely,

$$W(t_1, \ldots, t_n) := \sum_{w \in W} \prod_{j \in D(w)} t_j$$

is the flag h-polynomial of $\Sigma(W)$ (e.g., see the discussion at the end of Section III.4 in [21]).

Extending these concepts, set $\alpha_0 := -\widetilde{\alpha}$ (the lowest root), and let $s_0 = s_{\widetilde{\alpha}}$ denote the corresponding reflection in W. We define the *affine descent* set of w, denoted $\widetilde{D}(w)$, to be the set of indices of roots in $\Delta_0 := \Delta \cup \{\alpha_0\}$ that are negative with respect to w. Thus,

$$\widetilde{D}(w) = \{j \in [0, n] : w\alpha_j < 0\} = \begin{cases} D(w) \cup \{0\} & \text{if } \ell(ws_0) > \ell(w), \\ D(w) & \text{if } \ell(ws_0) < \ell(w). \end{cases}$$

We let $\widetilde{d}(w) := |\widetilde{D}(w)|$ denote the number of affine descents in w.

Note that only the identity element of W has an empty descent set (but has an affine descent at 0), and only the longest element w_0 has a full descent set (i.e., $D(w_0) = [n]$) but does not have an affine descent at 0. Thus $1 \leqslant \widetilde{d}(w) \leqslant n$ for all $w \in W$.

3. Affine Eulerian polynomials

We let $\widetilde{W}(t_0, \ldots, t_n)$ and $\widetilde{W}(t)$ denote the respective generating functions for affine descent sets and numbers of affine descent sets; i.e.,

$$\widetilde{W}(t_0, \ldots, t_n) := \sum_{w \in W} \prod_{j \in \widetilde{D}(w)} t_j, \tag{3.1}$$

$$\widetilde{W}(t) := \widetilde{W}(t, \ldots, t) = \sum_{w \in W} t^{\widetilde{d}(w)}. \tag{3.2}$$

We refer to these as *multivariate* and *univariate affine Eulerian polynomials*.

Theorem 3.1. *If $\widetilde{W}$ is an irreducible affine Weyl group, then the flag h-polynomial of the corresponding reduced Steinberg torus is the multivariate $\widetilde{W}$-Eulerian polynomial; i.e.,*

$$h(\Sigma'_T(\widetilde{W}); t_0, \ldots, t_n) = \widetilde{W}(t_0, \ldots, t_n). \tag{3.3}$$

In particular, for all $J \subseteq [0, n]$, we have

$$f_J(\Sigma'_T(\widetilde{W})) = |\{w \in W : \widetilde{D}(w) \subseteq J\}|, \tag{3.4}$$

$$h_J(\Sigma'_T(\widetilde{W})) = |\{w \in W : \widetilde{D}(w) = J\}|. \tag{3.5}$$

Furthermore,

$$\widetilde{W}(t_0, \ldots, t_n) = \sum_{J \subsetneq [0,n]} \frac{|W|}{|W_J|} \prod_{j \in J} (1 - t_j) \prod_{j \notin J} t_j, \tag{3.6}$$

where W_J denotes the (not necessarily parabolic) subgroup of W generated by $\{s_j : j \in J\}$.

Of course it follows immediately that the ordinary h-polynomial of the reduced Steinberg torus is the corresponding univariate affine Eulerian polynomial; i.e.,

$$h(\Sigma'_T(\widetilde{W}); t) = \widetilde{W}(t) = \sum_{w \in W} t^{\widetilde{d}(w)}.$$

Corollary 3.2. *The flag h-vector of the reduced Steinberg torus $\Sigma'_T(\widetilde{W})$ satisfies the generalized Dehn-Sommerville equations; that is, for all $J \subseteq [0, n]$, we have*

$$h_J(\Sigma'_T(\widetilde{W})) = h_{J^c}(\Sigma'_T(\widetilde{W})).$$

In particular, the $\widetilde{W}$-Eulerian polynomial is symmetric: $\widetilde{W}(t) = t^{n+1}\widetilde{W}(1/t)$.

Proof. Recall that the longest element $w_0 \in W$ is an involution that sends all positive roots to negative roots. It follows that a root β satisfies $w\beta < 0$ if and only if $w_0 w\beta > 0$, and hence

$$\widetilde{D}(w_0 w) = [0, n] \setminus \widetilde{D}(w),$$

for all $w \in W$. Now apply (3.5). □

Remark 3.3. The unreduced Steinberg torus $\Sigma_T(\widetilde{W})$ has nearly the same flag f-vector as its reduced counterpart, the only difference being $f_\varnothing(\Sigma_T(\widetilde{W})) = 1$ in place of $f_\varnothing(\Sigma'_T(\widetilde{W}))= 0$. However, as we noted in Example 2.2, the h-polynomial need not have symmetric or nonnegative coefficients in the unreduced case, and is therefore of less interest.

The following lemma is the key to our proof of Theorem 3.1.

Lemma 3.4. *If $\{\beta_i : i \in I\}$ is a set of simple roots for a reflection subgroup W' of W, then every coset in W/W' has a unique member w such that $w\beta_i > 0$ for all $i \in I$.*

Proof. Fix a dominant point $\lambda \in C_\varnothing$ (i.e., $\langle \lambda, \alpha \rangle > 0$ for all roots $\alpha \in \Delta$), so that the W-orbit of λ is generic and the map $w \mapsto w^{-1}\lambda$ is a bijection between W and the orbit $W\lambda$. Since

$$w\beta_i > 0 \Leftrightarrow \langle \lambda, w\beta_i \rangle > 0 \Leftrightarrow \langle w^{-1}\lambda, \beta_i \rangle > 0,$$

we see that w satisfies $w\beta_i > 0$ for all $i \in I$ if and only if $w^{-1}\lambda$ is dominant with respect to the simple roots of W'. However, every W'-orbit has a unique dominant member, and the image of the coset wW' under the bijection is the W'-orbit of $w^{-1}\lambda$, so the result follows. □

Remark 3.5. In the above lemma, it is interesting to note that by choosing the simple roots of W' so that they are positive relative to Φ, one may deduce that every coset of every reflection subgroup of W has a unique element of minimum length. This is a familiar fact for parabolic subgroups, but the less familiar general case also follows from work of Dyer (see Corollary 3.4 of [8]).

Proof of Theorem 3.1. For each nonempty $J \subseteq [0,n]$, the set of cells of the affine Coxeter complex with color-set J is the $\widetilde{W}$-orbit of A_{J^c}. These are the cells of the form $\mu + wA_{J^c}$ (for $\mu \in \mathbb{Z}\Phi^\vee$, $w \in W$), so

$$\{wA_{J^c} + \mathbb{Z}\Phi^\vee : w \in W\}$$

is the set of cells of the reduced Steinberg torus with color-set J. However, the $\widetilde{W}$-stabilizer of A_{J^c} (or indeed, any subset of the closure of the fundamental alcove) is generated by the subset of $\widetilde{S}$ that fixes A_{J^c}. The W-image of this subgroup (i.e., the W-stabilizer of $A_{J^c} + \mathbb{Z}\Phi^\vee$) is W_{J^c}, the reflection subgroup of W generated by $\{s_j : j \in [0,n] \setminus J\}$, and therefore

$$f_J(\Sigma'_T(\widetilde{W})) = |W|/|W_{J^c}|. \tag{3.7}$$

On the other hand, we have

$$\{w \in W : \widetilde{D}(w) \subseteq J\} = \{w \in W : w\alpha_j > 0 \text{ for } j \in [0,n] \setminus J\}$$

and every proper subset of Δ_0 is the set of simple roots of some root subsystem of Φ (this amounts to the fact that every proper subset of the extended Dynkin diagram, which records the geometry of Δ_0, is the Dynkin diagram of a finite root system), so Lemma 3.4 implies that $\{w \in W : \widetilde{D}(w) \subseteq J\}$ is a set of coset representatives for W/W_{J^c}. Hence,

$$f_J(\Sigma'_T(\widetilde{W})) = |W|/|W_{J^c}| = |\{w \in W : \widetilde{D}(w) \subseteq J\}|.$$

Noting that $f_J(\Sigma'_T(\widetilde{W})) = |\{w \in W : \widetilde{D}(w) \subseteq J\}| = 0$ when $J = \varnothing$, we obtain (3.4).

To complete the proof, note that (3.4) implies (3.5) and hence (3.3) via inclusion-exclusion. The latter allows one to deduce (3.6) as a corollary of (2.2) and (3.7). □

It is easy to compute the affine Eulerian polynomials for the groups of low rank via (3.6). Some examples, including all of the exceptional groups, are listed in Table 1.

Remark 3.6. Given $J \subsetneq [0,n]$, it follows from Lemma 3.4 that each coset in W/W_J has a unique representative w such that $\widetilde{D}(w) \cap J = \varnothing$. Thus each cell of the reduced Steinberg torus has the form $F(w,J) = wA_J + \mathbb{Z}\Phi^\vee$ for some unique pair (w,J) with $\widetilde{D}(w) \cap J = \varnothing$. Moreover, the cells of the form $F(w,*)$ are precisely the cells in the closure of $F(w,\varnothing)$ that have on their boundary the unique cell with color-set $\widetilde{D}(w)$; namely, $F(w,\widetilde{D}(w)^c)$. Thus the reduced Steinberg torus is "partitionable" in the sense defined in Section III.2 of [21].

W	$\widetilde{W}(t)$
B_3	$10t + 28t^2 + 10t^3$
B_4	$24t + 168t^2 + 168t^3 + 24t^4$
B_5	$54t + 904t^2 + 1924t^3 + 904t^4 + 54t^5$
B_6	$116t + 4452t^2 + 18472t^3 + 18472t^4 + 4452t^5 + 116t^6$
B_7	$242t + 20612t^2 + 157294t^3 + 288824t^4 + 157294t^5 + 20612t^6 + 242t^7$
D_4	$16t + 80t^2 + 80t^3 + 16t^4$
D_5	$44t + 464t^2 + 904t^3 + 464t^4 + 44t^5$
D_6	$104t + 2568t^2 + 8848t^3 + 8848t^4 + 2568t^5 + 104t^6$
D_7	$228t + 13192t^2 + 79580t^3 + 136560t^4 + 79580t^5 + 13192t^6 + 228t^7$
E_6	$351t + 5427t^2 + 20142t^3 + 20142t^4 + 5427t^5 + 351t^6$
E_7	$4064t + 115728t^2 + 710112t^3 + 1243232t^4 + 710112t^5 + 115728t^6 + 4064t^7$
E_8	$157200t + 9253680t^2 + 87417360t^3 + 251536560t^4$ $+ 251536560t^5 + 87417360t^6 + 9253680t^7 + 157200t^8$
F_4	$72t + 504t^2 + 504t^3 + 72t^4$
G_2	$6t + 6t^2$

TABLE 1. Some affine Eulerian polynomials.

Remark 3.7. If W is an irreducible but non-crystallographic finite reflection group, such as H_3 or H_4, then there is no corresponding affine Weyl group, and hence no Steinberg torus. The root system still has a unique dominant root, so one could define fake affine Eulerian polynomials $W^{\mathrm{fa}}(t_0, \ldots, t_n)$ and $W^{\mathrm{fa}}(t)$ analogous to (3.1) and (3.2), using the anti-dominant root in the role of α_0. For the groups H_3 and H_4, one obtains

$$H_3^{\mathrm{fa}}(t) = 26t + 68t^2 + 26t^3,$$

$$H_4^{\mathrm{fa}}(t) = 960t + 6240t^2 + 6240t^3 + 960t^4.$$

Although these polynomials have symmetric and unimodal coefficients (and real roots), we do not know if $W^{\mathrm{fa}}(t)$ is the h-polynomial of some naturally associated Boolean complex.

Remark 3.8. More generally, given any subset of roots $\Psi = \{\beta_i : i \in I\} \subset \Phi$, one could define a generalized descent set for $w \in W$ by setting

$$D_\Psi(w) := \{i \in I : w\beta_i < 0\},$$

whether or not Φ is crystallographic. Examining the proof of Theorem 3.1, one can see that the generating function for these generalized descent sets would satisfy a formula similar to (3.6) if for every $J \subseteq I$, either $\{\beta_j : j \in J\}$ is the set of simple roots of some finite root system (see Lemma 3.4), or $\{w \in W : w\beta_j > 0 \text{ for all } j \in J\}$ is empty. Applying this criterion to pairs $i, j \in I$, this forces the angle between β_i and β_j to be $(1 - 1/m)\pi$ for some integer $m \geqslant 2$, or $\beta_i = -\beta_j$ (i.e., $m = \infty$). Since the matrix $\langle \beta_i, \beta_j \rangle$ is necessarily positive semidefinite, it follows from the theory of reflection groups that (up to normalization) Ψ must be the simple roots of some root subsystem, or is an extension of the simple roots by the lowest root of some crystallographic root subsystem, or is an orthogonal disjoint union of such sets (e.g., see Section 2.7 of [14]). In particular, the identity in (3.6) is not valid for the fake affine Eulerian polynomials discussed in the previous remark.

4. Real roots, γ-vectors, and unimodality

The following is a companion to Brenti's conjecture [2] that the roots of all (ordinary) Eulerian polynomials $W(t)$ are real.

Conjecture 4.1. *The roots of all affine Eulerian polynomials $\widetilde{W}(t)$ are real.*

To complete a proof of this conjecture, we claim that it suffices to consider only the groups $\widetilde{B}_n$ and $\widetilde{D}_n$. Indeed, it follows from observations of Fulman [10, 11] and Petersen [17] that $\widetilde{A}_n(t)$ and $\widetilde{C}_n(t)$ are both multiples of $A_{n-1}(t)$ (see also the discussion in Section 5 below). Thus the conjecture for $\widetilde{A}_n$ and $\widetilde{C}_n$ follows from the fact that all roots of the classical Eulerian polynomials are known to be real [13]. Furthermore, using the data in Table 1, it is easy to check that the conjecture holds for the exceptional groups.

To collect supporting evidence for the remaining groups $\widetilde{B}_n$ and $\widetilde{D}_n$, we have determined explicit exponential generating functions for the corresponding affine Eulerian polynomials (see Proposition 6.4 below), and used these to verify the conjecture for $n \leqslant 100$. In a similar way, we have also confirmed that all roots of $D_n(t)$ are real (the only remaining open case of Brenti's conjecture) for $n \leqslant 100$.

A further supporting result involves γ-vectors in the sense of Brändén [1] and Gal [12]. To explain, consider a polynomial satisfying $h(t) = t^m h(1/t)$. It is clear that such a polynomial has a unique expansion of the form

$$h(t) = \sum_{0 \leqslant i \leqslant m/2} \gamma_i t^i (1+t)^{m-2i}.$$

We call $(\gamma_0, \gamma_1, \ldots)$ the *γ-vector* of $h(t)$.

It is elementary to show that if $h(t)$ has symmetric, nonnegative coefficients and all real roots, then it has a nonnegative γ-vector (see Lemma 4.1 of [1] or Section 1.4 of [25]).

Recall that $\widetilde{W}(t)$ is symmetric (Corollary 3.2), so it has a γ-vector.

Theorem 4.2. *The affine Eulerian polynomials $\widetilde{W}(t)$ have nonnegative γ-vectors.*

Proof. Given that we know Conjecture 4.1 holds for $\widetilde{A}_n$, $\widetilde{C}_n$, and the exceptional affine Weyl groups, it suffices to prove this result for $\widetilde{B}_n$ and $\widetilde{D}_n$. In these cases, we have explicit combinatorial expansions for $\widetilde{B}_n(t)$ and $\widetilde{D}_n(t)$ in Corollaries 5.9 and 5.11 below that transparently imply the nonnegativity of their γ-vectors. ∎

It would be interesting to have a conceptual (case-free) proof of this result.

Any polynomial with a nonnegative γ-vector has unimodal coefficients. Hence,

Corollary 4.3. *The affine Eulerian polynomials have unimodal coefficients.*

We remark that the γ-vectors of the Eulerian polynomials $W(t)$ are also known to be nonnegative, but the only existing proofs to date are case-by-case [6, 25].

5. Combinatorial expansions and γ-nonnegativity

In this section, we provide combinatorial expansions for the affine Eulerian polynomials (both multivariate and univariate) for the four infinite families of irreducible Weyl groups. As corollaries, we will deduce the nonnegativity of the γ-vectors for these polynomials.

5.1. **Type A.** Recall that the Weyl group A_{n-1} may be identified with S_n, the symmetric group of permutations of $[n]$, and the corresponding root system is

$$\{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n\},$$

where $\varepsilon_1, \dots, \varepsilon_n$ is the standard orthonormal basis of $\mathbb{R}^n$. For the simple roots, we choose $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ ($1 \leqslant i < n$). With respect to this choice, the simple reflection s_i transposes i and $i+1$ (as a permutation) and interchanges ε_i and ε_{i+1} (as a reflection acting on $\mathbb{R}^n$).

The positive roots are $\varepsilon_i - \varepsilon_j$ for $i > j$, and the lowest root $\alpha_0 = -\widetilde{\alpha}$ is $\varepsilon_1 - \varepsilon_n$.

We write permutations in one-line form $w = w_1 w_2 \cdots w_n$, where $w_i = w(i)$. In these terms, a root $\varepsilon_i - \varepsilon_j$ is negative with respect to a permutation w if and only if $w_j > w_i$. In particular, $D(w) = \{i \in [n-1] : w_i > w_{i+1}\}$ is the usual descent set of a permutation. Also, an extra "affine" descent occurs at 0 if and only if $w_n > w_1$, so $\widetilde{D}(w) = \{i \in [0, n-1] : w_i > w_{i+1}\}$, using the convention $w_0 = w_n$.

For example, $D(25413) = \{2, 3\}$ and $\widetilde{D}(25413) = \{0, 2, 3\}$.

Proposition 5.1. *For $n \geqslant 2$, we have*

$$\widetilde{A}_{n-1}(t_0, \dots, t_{n-1}) = \sum_{j=0}^{n-1} t_j A_{n-2}(t_{j+1}, \dots, t_{n-1}, t_0, \dots, t_{j-2}).$$

Proof. Let $c = 23 \cdots n1$ (an n-cycle in A_{n-1}), and note that one may obtain the affine descent set of $wc = w_2 \cdots w_n w_1$ by a cyclic shift of the affine descent set of $w \in A_{n-1}$; i.e.,

$$\widetilde{D}(wc) = \{i - 1 : i \in \widetilde{D}(w)\} \mod n.$$

Each coset of the cyclic subgroup $\langle c \rangle$ has a unique representative w such that $w_n = n$, and this set of representatives is in bijection with A_{n-2}. For each coset representative w, we have $0 \in \widetilde{D}(w)$, and the remaining affine descents coincide with the ordinary descents of the corresponding member of A_{n-2}. Thus, the generating function for the affine descent sets of these coset representatives is $t_0 A_{n-2}(t_1, \dots, t_{n-2})$, and the generating function corresponding to elements of the form wc^{-j} is obtained by substituting $t_i \to t_{i+j}$ (subscripts modulo n). $\square$

It follows that the univariate affine Eulerian polynomials of type A are multiples of classical Eulerian polynomials, as noted previously by Fulman [10] and Petersen [17].

Corollary 5.2. *For $n \geqslant 1$, we have $\widetilde{A}_n(t) = (n+1)tA_{n-1}(t)$.*

5.2. **Type C.** The root system of the Weyl group C_n has the form

$$\{\pm 2\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant j < i \leqslant n\},$$

and C_n acts as a group of permutations of $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_n\}$. More explicitly, if we identify $\pm i$ with $\pm\varepsilon_i$, then C_n may be viewed as the group of permutations of $\pm[n] = \{\pm 1, \ldots, \pm n\}$ such that $w(-i) = -w(i)$ for all i. The simple roots may be chosen so that $\alpha_1 = 2\varepsilon_1$ and $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 \leqslant i \leqslant n$. With respect to this choice, the positive roots are $2\varepsilon_i$ for all i and $\varepsilon_i \pm \varepsilon_j$ for all $i > j$, and the lowest root $\alpha_0 = -\widetilde{\alpha}$ is $-2\varepsilon_n$.

We write permutations $w \in C_n$ in one-line form $w = w_1 \cdots w_n$, where $w_i = w(i)$. In these terms, one can check that roots of the form $\varepsilon_i - \varepsilon_j$ with $i > j$ are negative with respect to w if and only if $w_j > w_i$, whereas roots of the form $2\varepsilon_i$ are negative with respect to w if and only if $w_i < 0$. In particular, the ordinary descent set is $D(w) = \{i \in [n] : w_{i-1} > w_i\}$, using the convention $w_0 = 0$, and 0 is in the affine descent set $\widetilde{D}(w)$ when $w_n > 0$.

For example, if $w = 23\bar{5}\bar{1}4$ (bars indicate negative values), then $\widetilde{D}(w) = \{0, 3\}$.

In the following formula for the multivariate $\widetilde{C}_n$-Eulerian polynomial, it is more convenient to use $n+1$ in place of 0 to mark the extra affine descent, or equivalently, set $t_0 = t_{n+1}$. Note that by specializing this extra variable (i.e., setting $t_0 = t_{n+1} = 1$), we recover Stembridge's formula for the flag h-polynomial of the Coxeter complex $\Sigma(C_n)$ (Proposition A.1 in [25]).

Below, we use $\chi(\cdot)$ as an indicator function: $\chi(S) = 1$ if S is true; 0 if S is false.

Proposition 5.3. *For $n \geqslant 1$, we have*

$$\widetilde{C}_n(t_{n+1}, t_1, \ldots, t_n) = \sum_{u \in S_n} \prod_{i=1}^{n} \left(t_i^{\chi(u_{i-1} < u_i)} + t_{i+1}^{\chi(u_i > u_{i+1})} \right),$$

using the convention $u_0 = u_{n+1} = 0$.

Proof. Following the proof of Proposition A.1 in [25], each member of C_n has the form $w = \sigma u$, where $u \in S_n$ and $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}_2^n$ (meaning that $w_i = \sigma_i u_i$). Given that $u_0 = u_{n+1} = 0$ and that $n+1$ replaces 0 in $\widetilde{D}(w)$, we see that for $i = 1, \ldots, n+1$,

- if $u_{i-1} < u_i$, then $i \in \widetilde{D}(w) \Leftrightarrow \sigma_i = -1$,
- if $u_{i-1} > u_i$, then $i \in \widetilde{D}(w) \Leftrightarrow \sigma_{i-1} = +1$.

Thus for each $i \in [n+1]$, there is a unique j (depending on u) such that the presence or absence of i in $\widetilde{D}(w)$ is controlled by the value of σ_j. More specifically, σ_j controls the presence of j (if $u_{j-1} < u_j$) and $j+1$ (if $u_j > u_{j+1}$), and nothing else. Furthermore, if we define

$$c_j(u) := t_j^{\chi(u_{j-1} < u_j)} + t_{j+1}^{\chi(u_j > u_{j+1})}, \tag{5.1}$$

then $c_j(u)$ records the sum of the weights of the effects of $\sigma_j = -1$ and $\sigma_j = +1$ on the affine descent set of σu. Since the effects of $\sigma_1, \dots, \sigma_n$ are mutually independent, we conclude that

$$\sum_{\sigma \in \mathbb{Z}_2^n} \prod_{i \in \widetilde{D}(\sigma u)} t_i \;=\; c_1(u) \cdots c_n(u),$$

and the result follows by summing over $u \in S_n$. □

Remark 5.4. It is well-known that the Coxeter complex $\Sigma(C_n)$ is isomorphic to the barycentric subdivision of an n-dimensional cube, and as explained in Remark A.3 of [25], one may recognize Stembridge's formula for the flag h-polynomial of $\Sigma(C_n)$ as a disguised formula for the *cd*-index of the n-cube. Similarly, the Steinberg torus $\Sigma_T(\widetilde{C}_n)$ may be constructed from the barycentric subdivision of an n-cube by identifying the opposite maximal faces of the cube (recall Example 2.3), and the above formula may be reinterpreted as a nonnegative *cd*-index for the reduced complex.

The above remark suggests the possibility of a more general result. Given a tiling of $\mathbb{R}^n$ by lattice translates of a convex polytope P, the quotient of $\mathbb{R}^n$ by the lattice may be viewed as an n-torus decomposed into polyhedral cells. Lattice translates of the cells in the barycentric subdivision of P may be identified, thereby yielding a "polytorus" with a well-defined flag h-vector, and our findings here suggest that one should study the "reduced polytorus" obtained by deleting the empty face.

Question 5.5. *Does every reduced polytorus have a nonnegative cd-index?*

It will be convenient for what follows to introduce three conventions for counting peaks in a permutation $u \in S_n$; namely,

$$\begin{aligned}
\mathrm{pk}(u) &:= |\{i \in [2, n-1] : u_{i-1} < u_i > u_{i+1}\}|,\\
\mathrm{lpk}(u) &:= |\{i \in [1, n-1] : u_{i-1} < u_i > u_{i+1}\}|,\\
\mathrm{epk}(u) &:= |\{i \in [1, n] : u_{i-1} < u_i > u_{i+1}\}|,
\end{aligned}$$

again using the convention $u_0 = u_{n+1} = 0$. We refer to these quantities as the number of *ordinary*, *left*, and *extended peaks* in u, respectively.

The following expansions show that $\widetilde{C}_n(t)$, $C_n(t)$, and $A_{n-1}(t)$ have nonnegative γ-vectors. Part (b) is due to Petersen (Proposition 4.15 in [17]), and part (c) is equivalent to an identity due to Foata and Schützenberger (Théorème 5.6 of [9]; see also Remark 4.8 of [24]).

Corollary 5.6. *For $n \geqslant 1$, we have*

(a) $\widetilde{C}_n(t) = (1/2) \sum_{u \in S_n} (4t)^{\mathrm{epk}(u)} (1+t)^{n+1-2\,\mathrm{epk}(u)}$,

(b) $C_n(t) = \sum_{u \in S_n} (4t)^{\mathrm{lpk}(u)} (1+t)^{n-2\,\mathrm{lpk}(u)}$,

(c) $A_{n-1}(t) = 2^{-(n-1)} \sum_{u \in S_n} (4t)^{\mathrm{pk}(u)} (1+t)^{n-1-2\,\mathrm{pk}(u)}$.

Proof. (a) Proposition 5.3 implies that $\widetilde{C}_n(t_{n+1}, t_1, \dots, t_n) = \sum_{u \in S_n} c_1(u) \cdots c_n(u)$, where $c_i(u)$ is defined as in (5.1). Specializing the variables so that $t_i \to t$ for all i, one sees that

$$c_i(u) \to \begin{cases} 2t & \text{if } u_{i-1} < u_i > u_{i+1} \text{ (a peak)}, \\ 2 & \text{if } u_{i-1} > u_i < u_{i+1} \text{ (a valley)}, \\ 1{+}t & \text{otherwise}. \end{cases} \tag{5.2}$$

However, any sequence $(0, u_1, \dots, u_n, 0)$ that begins with an increase and ends with a decrease must have exactly one more peak than it has valleys, so the first possibility occurs $\operatorname{epk}(u)$ times, the second $\operatorname{epk}(u) - 1$ times, and the last $n + 1 - 2\operatorname{epk}(u)$ times.

(b) We have $C_n(t) = \widetilde{C}_n(1, t, \dots, t)$. The analysis is similar to (a), the only change being that $c_n(u)$ now specializes to $1{+}t$ or 2 according to whether $u_{n-1} < u_n$. An equivalent way to obtain the same result would be to use the rules in (5.2) but with $u_{n+1} = \infty$. In these terms, the sequence $(0, u_1, \dots, u_n, \infty)$ has $\operatorname{lpk}(u)$ peaks each contributing factors of $2t$, along with $\operatorname{lpk}(u)$ valleys each contributing factors of 2, and the remaining $n - 2\operatorname{lpk}(u)$ contributions are factors of $1 + t$.

(c) A sum over $w \in C_n$ may be viewed as 2^n sums over permutations of n distinct objects (first choose which subset of letters in $[n]$ to negate). In this way, it is not hard to see that $\widetilde{C}_n(1, t \dots, t, 1) = 2^n A_{n-1}(t)$. The analysis of this case is similar to (b), but now using the convention that $u_0 = u_{n+1} = \infty$. □

The following result was first obtained by Fulman (using the combinatorics of shuffling [11]) and later by Petersen (using a variation of the theory of P-partitions [17]).

Corollary 5.7. *For $n \geqslant 1$, we have*

$$\widetilde{C}_n(t) = 2^n t A_{n-1}(t).$$

Proof #1. Comparing parts (a) and (c) of Corollary 5.6, it suffices to show that $\operatorname{epk}(u) - 1$ and $\operatorname{pk}(u)$ have the same distribution as u varies over S_n. To see this, recall from the proof of Corollary 5.6(a) that every $u \in S_n$ has exactly $\operatorname{epk}(u) - 1$ valleys, and that these valleys occur in internal positions. Thus $\operatorname{epk}(u) - 1 = \operatorname{pk}(v)$, where $v_i = n + 1 - u_i$. □

Proof #2. It suffices to show that $\widetilde{d}(w)$ and $d(u) + 1$ have the same probability distributions as w and u vary uniformly over C_n and S_n, respectively. To see this, first consider

$$f_{ij}(t) := \sum_{u \in S_n : u_i = j} t^{\widetilde{d}(u)},$$

so that $\widetilde{A}_{n-1}(t) = f_{1j}(t) + \cdots + f_{nj}(t)$. Since $\widetilde{d}(u)$ is invariant under cyclic shifts (recall the proof of Proposition 5.1), it follows that $f_{1j}(t) = \cdots = f_{nj}(t)$, and hence $f_{ij}(t) = tA_{n-2}(t)$ (Corollary 5.2). Thus, the distribution of $\widetilde{d}(u)$ as $u \in S_n$ varies over all $(n-1)!$ permutations with a fixed value in one position is the same as the distribution of $d(v) + 1$ over $v \in S_{n-1}$.

Now consider $w \in C_n$. If we fix in advance the set $\{w_1, \dots, w_n\}$ (one of 2^n equally likely possibilities), one may view the word $\hat{w} := w_1 \cdots w_n 0$ as a permutation of $n + 1$ objects, and thus identify $\hat{w}$ as one of the $n!$ members of $A_n = S_{n+1}$ with a fixed value in its last position. However, it is not hard to see that $\hat{w}$ (as a member of A_n) and w (as a member of C_n) have the same number of affine descents. $\square$

5.3. **Type B.** The Weyl group B_n is identical to C_n, but has a root system that is a rescaling of the C_n root system; namely,

$$\{\pm\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant j < i \leqslant n\}.$$

We can likewise rescale the choice of simple roots; the only change is that α_1 is now ε_1. In this way, the positive roots are (positive) rescalings of the positive roots for C_n, so ordinary descents in B_n and C_n are the same. On the other hand, the lowest root $\alpha_0 = -\widetilde{\alpha}$ is now $-\varepsilon_{n-1} - \varepsilon_n$, so 0 is in the affine descent set of $w \in B_n$ if and only if $w_{n-1} + w_n > 0$.

For example, if $w = 23\bar{4}5\bar{1}$, then $\widetilde{D}(w) = \{0, 3, 5\}$.

Proposition 5.8. *For $n \geqslant 2$, we have*

$$\widetilde{B}_n(t_0, \ldots, t_n) = \sum_{u \in S_n} b_1(u) \cdots b_n(u),$$

where $b_i(u) = c_i(u)$ for $i < n-1$ as defined in (5.1)*, and*

$$b_{n-1}(u) := t_{n-1}^{\chi(u_{n-2}<u_{n-1})} + (t_0 t_n)^{\chi(u_{n-1}>u_n)},$$
$$b_n(u) := t_n^{\chi(u_{n-1}<u_n)} + t_0^{\chi(u_{n-1}<u_n)}.$$

Proof. By factoring elements of B_n in the form $w = \sigma u$ ($\sigma \in \mathbb{Z}_2^n$, $u \in S_n$), the analysis proceeds as in the proof of Proposition 5.3, except that

- if $u_{n-1} < u_n$, then $0 \in \widetilde{D}(w) \Leftrightarrow \sigma_n = +1$,
- if $u_{n-1} > u_n$, then $0 \in \widetilde{D}(w) \Leftrightarrow \sigma_{n-1} = +1$.

Thus for each $i \in [0, n]$, there is still a unique j (depending on u) such that the presence or absence of i in $\widetilde{D}(w)$ is controlled by the value of σ_j. Hence,

$$\sum_{\sigma \in \mathbb{Z}_2^n} \prod_{i \in \widetilde{D}(\sigma u)} t_i \;=\; \big(b_1^-(u) + b_1^+(u)\big) \cdots \big(b_n^-(u) + b_n^+(u)\big),$$

where $b_j^-(u)$ denotes the product of all t_i such that $\sigma_j = -1$ forces $i \in \widetilde{D}(\sigma u)$, and $b_j^+(u)$ denotes the analogous product when $\sigma_j = +1$. These products are the same as their counterparts for C_n in Proposition 5.3 except for those involving t_0; namely, $b_{n-1}^+(u)$ and $b_n^+(u)$. In the latter, there should be an extra factor of t_0 only when $u_{n-1} < u_n$, and in the former there should be an extra factor of t_0 when $u_{n-1} > u_n$. $\square$

Corollary 5.9. *For $n \geqslant 2$, we have*

$$\widetilde{B}_n(t) = \sum_{u \in S_n} \phi(u)(4t)^{\operatorname{epk}(u)}(1+t)^{n+1-2\operatorname{epk}(u)},$$

where

$$\phi(u) := \begin{cases} 1 & \text{if } u_{n-2} > u_{n-1} > u_n, \\ 0 & \text{if } u_{n-2} > u_n > u_{n-1}, \\ 1/2 & \text{otherwise.} \end{cases}$$

In particular, $\widetilde{B}_n(t)$ has a nonnegative γ-vector.

Given our convention that $u_0 = 0$, one should understand that $\phi(u) = 1/2$ for $u \in S_2$.

Proof. Recall from the proof of Corollary 5.6(a) that if we specialize the variables so that $t_i \to t$ for all i, we obtain

$$c_1(u) \cdots c_n(u) \to (1/2)(4t)^{\operatorname{epk}(u)}(1+t)^{n+1-2\operatorname{epk}(u)}. \tag{5.3}$$

Comparing the definitions of $b_i(u)$ and $c_i(u)$, we see that

$$\frac{b_1(u) \cdots b_n(u)}{c_1(u) \cdots c_n(u)} = \frac{b_{n-1}(u)b_n(u)}{c_{n-1}(u)c_n(u)} \to \begin{cases} 2(1+t^2)/(1+t)^2 & \text{if } u_{n-2} > u_{n-1} > u_n, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, $b_1(u) \cdots b_n(u)$ usually specializes in the same way as in (5.3).

Now consider that transposing u_{n-1} and u_n yields a bijection $u \leftrightarrow u'$ between permutations in S_n that satisfy $u_{n-2} > u_{n-1} > u_n$ and $u'_{n-2} > u'_n > u'_{n-1}$. Furthermore, we have

$$\frac{b_1(u') \cdots b_n(u')}{c_1(u) \cdots c_n(u)} = \frac{b_{n-1}(u')b_n(u')}{c_{n-1}(u)c_n(u)} \to \frac{4t}{(1+t)^2}.$$

Therefore, if we combine the terms indexed by u' and u in these cases (and eliminate u' from the sum), the net contribution of u is $(2(1+t^2)+4t)/(1+t)^2 = 2$ times (5.3). ☐

5.4. **Type D.** The Weyl group D_n is the subgroup of B_n consisting of signed permutations $w = w_1 \cdots w_n$ with an even number of negative entries. It has a root system of the form

$$\{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant j < i \leqslant n\},$$

and one can choose simple roots so that $\alpha_1 = \varepsilon_2 + \varepsilon_1$ and $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 \leqslant i \leqslant n$. This choice is compatible with our previous choices for B_n and C_n in the sense that a D_n root is positive if and only if it is positive as a B_n or C_n root.

It is important to note that D_n is irreducible only for $n \geqslant 3$. In such cases, the lowest root $\alpha_0 = -\widetilde{\alpha}$ is $-\varepsilon_{n-1} - \varepsilon_n$ (the same as in B_n), and thus the affine descent set of $w \in D_n$ consists of all $i \in [2, n]$ such that $w_{i-1} > w_i$, together with 1 (if $w_1 + w_2 < 0$) and 0 (if $w_{n-1} + w_n > 0$).

For example, if $w = 3\bar{4}2\bar{1}5$, then $\widetilde{D}(w) = \{0, 1, 2, 4\}$.

Note that by specializing $t_0 = 1$ in the following, we recover Stembridge's formula for the flag h-polynomial of the Coxeter complex $\Sigma(D_n)$ (Proposition A.4 in [25]).

Proposition 5.10. *For $n \geqslant 4$, we have*

$$\widetilde{D}_n(t_0, \ldots, t_n) = \frac{1}{2} \sum_{u \in S_n} d_1(u) \cdots d_n(u),$$

where $d_i(u) = b_i(u)$ for $i > 2$ as defined in Proposition 5.8, and

$$d_1(u) := t_1^{\chi(u_1 > u_2)} + t_2^{\chi(u_1 > u_2)},$$
$$d_2(u) := (t_1 t_2)^{\chi(u_1 < u_2)} + t_3^{\chi(u_2 > u_3)}.$$

Proof. Following the proof of Proposition A.4 in [25], note that the definition of an affine descent set in D_n makes sense for any signed permutation $w \in B_n$. Since replacing 1 with -1 or vice-versa in $w_1 \cdots w_n$ does not change this set, it follows that

$$\widetilde{D}_n(t_0, \ldots, t_n) = \frac{1}{2} \sum_{u \in S_n} \sum_{\sigma \in \mathbb{Z}_2^n} \prod_{i \in \widetilde{D}(w)} t_i.$$

The analysis of $w = \sigma u$ now proceeds as in the proof of Proposition 5.8, except that

- if $u_1 > u_2$, then $1 \in \widetilde{D}(w) \Leftrightarrow \sigma_1 = -1$,
- if $u_1 < u_2$, then $1 \in \widetilde{D}(w) \Leftrightarrow \sigma_2 = -1$.

Again it follows that for each $i \in [0, n]$, there is a unique j (depending on u) such that the presence or absence of i in $\widetilde{D}(w)$ is controlled by the value of σ_j. Hence,

$$\sum_{\sigma \in \mathbb{Z}_2^n} \prod_{i \in \widetilde{D}(w)} t_i \;=\; \left(d_1^-(u) + d_1^+(u)\right) \cdots \left(d_n^-(u) + d_n^+(u)\right),$$

where $d_j^-(u)$ denotes the product of all t_i such that $\sigma_j = -1$ forces $i \in \widetilde{D}(\sigma u)$, and $d_j^+(u)$ denotes the analogous product when $\sigma_j = +1$. These products are the same as their counterparts for B_n in Proposition 5.8 except for those that involve t_1; namely, $d_1^-(u)$ and $d_2^-(u)$. In the former, there should be a factor of t_1 only when $u_1 > u_2$, and in the latter, there should be an extra factor of t_1 when $u_1 < u_2$. □

Specializing, we obtain nonnegative γ-expansions for both $\widetilde{D}_n(t)$ and $D_n(t)$, the latter of which is due to Stembridge (Corollary A.5 in [25]; compare also Theorem 6.9 in [6]).

Corollary 5.11. *For $n \geqslant 4$, we have*

(a) $\widetilde{D}_n(t) = \sum_{u \in S_n} \phi(u)\phi(u^*)(4t)^{\mathrm{epk}(u)}(1+t)^{n+1-2\,\mathrm{epk}(u)}$.

(b) $D_n(t) = \sum_{u \in S_n} \phi(u^*)(4t)^{\mathrm{lpk}(u)}(1+t)^{n-2\,\mathrm{lpk}(u)}$.

where $\phi(u)$ is defined as in Corollary 5.9 and $u^* := u_n \cdots u_2 u_1$.

Proof. (a) Specializing the variables so that $t_i \to t$ for all i, we obtain

$$\frac{d_1(u) \cdots d_n(u)}{b_1(u) \cdots b_n(u)} = \frac{d_1(u) d_2(u)}{b_1(u) b_2(u)} \to \begin{cases} 2(1+t^2)/(1+t)^2 & \text{if } u_1 < u_2 < u_3, \\ 1 & \text{otherwise.} \end{cases}$$

Now pair each permutation $u \in S_n$ such that $u_1 < u_2 < u_3$ with the permutation u'' obtained by switching u_1 and u_2. In such cases, we have

$$\frac{d_1(u'')\cdots d_n(u'')}{b_1(u)\cdots b_n(u)} = \frac{d_1(u'')d_2(u'')}{b_1(u)b_2(u)} \to \frac{4t}{(1+t)^2},$$

so when the expansion in Proposition 5.10 is specialized, the terms indexed by u and u'' such that $u_1 < u_2 < u_3$ may be combined into a single term with twice the b-weight of u, yielding

$$\sum_{u\in S_n} \phi(u^*)b_1(u)\cdots b_n(u) \to \widetilde{D}_n(t).$$

Now proceed as in the proof of Corollary 5.9, combining the terms indexed by $u \in S_n$ such that $u_{n-2} > u_{n-1} > u_n$ with the terms indexed by the permutations u' obtained by switching u_{n-1} and u_n, and note that $\phi(u^*) = \phi((u')^*)$, even when $n = 4$.

(b) Similarly, we have $D_n(t) = \widetilde{D}_n(1, t, \dots, t)$. Under this specialization, the effects on $b_i(u)$ and $d_i(u)$ are similar to the previous case; the only differences occur in the terms that involve t_0; namely $b_i(u)$ and $d_i(u)$ for $i = n-1$ and $i = n$. However, we have $b_i(u) = d_i(u)$ in these cases (even without specialization), so the same reasoning as above implies

$$\sum_{u\in S_n} \phi(u^*)b_1(u)\cdots b_n(u)\Big|_{t_0=1} \to\ D_n(t).$$

Now observe that $b_i(u) = c_i(u)$ for all i when $t_0 = 1$, so the result follows by the reasoning in the proof of Corollary 5.6(b). □

6. Identities and generating functions

6.1. **Strange Identities.** Here we provide several unexpected identities (two new, one old) relating the ordinary and affine Eulerian polynomials.

Proposition 6.1. *For $n \geqslant 2$, we have*

$$2\widetilde{C}_n(t) = \widetilde{B}_n(t) + 2ntC_{n-1}(t).$$

Proof. Given $u \in S_n$, let $u{\downarrow} = u_1 \cdots u_{n-1}$, a permutation of $n-1$ distinct positive integers. Noting that the definitions of peak numbers make sense for any sequence of positive integers, we see that the distribution of $\mathrm{lpk}(u{\downarrow})$ as u varies over S_n is the same as n copies of the distribution of $\mathrm{lpk}(v)$ as v varies over S_{n-1}. Thus Corollary 5.6(b) implies

$$2ntC_{n-1}(t) = 2t\sum_{u\in S_n} (4t)^{\mathrm{lpk}(u\downarrow)}(1+t)^{n-1-2\,\mathrm{lpk}(u\downarrow)}.$$

Now recall that swapping u_{n-1} and u_n provides a bijection between the permutations $u \in S_n$ satisfying $\phi(u) = 1$ (i.e., $u_{n-2} > u_{n-1} > u_n$) with the permutations u' satisfying $\phi(u') = 0$ (i.e., $u'_{n-2} > u'_n > u'_{n-1}$). Noting that $\mathrm{lpk}(u{\downarrow}) = \mathrm{lpk}(u'{\downarrow})$ for such pairs, we can achieve an equivalent result by doubling the contribution of u' and eliminating u, or simply modify the contribution of every permutation u by the factor $2(1-\phi(u))$. Thus,

$$2ntC_{n-1}(t) = \sum_{u \in S_n} (1-\phi(u))(4t)^{\operatorname{lpk}(u\downarrow)+1}(1+t)^{n-1-2\operatorname{lpk}(u\downarrow)}. \tag{6.1}$$

On the other hand, Corollaries 5.6(a) and 5.9 imply

$$2\widetilde{C}_n(t) - \widetilde{B}_n(t) = \sum_{u \in S_n} (1-\phi(u))(4t)^{\operatorname{epk}(u)}(1+t)^{n+1-2\operatorname{epk}(u)}.$$

Noting that $\operatorname{epk}(u) = \operatorname{lpk}(u\downarrow)+1$ whenever $\phi(u) \neq 1$, the result follows. ☐

Proposition 6.2. *For $n \geqslant 3$, we have*

$$\widetilde{B}_n(t) = \widetilde{D}_n(t) + 2ntD_{n-1}(t).$$

Proof. It is easy to check the case $n = 3$ (note that $D_2(t) = (1+t)^2$), so we assume $n \geqslant 4$.

Following the argument we used to deduce (6.1) from Corollary 5.6(b), one may similarly use Corollary 5.11(b) to show that

$$\begin{aligned} 2ntD_{n-1}(t) &= \sum_{u \in S_n} (1-\phi(u))\phi(u^*)(4t)^{\operatorname{lpk}(u\downarrow)+1}(1+t)^{n-1-2\operatorname{lpk}(u\downarrow)} \\ &= \sum_{u \in S_n} (1-\phi(u))\phi(u^*)(4t)^{\operatorname{epk}(u)}(1+t)^{n+1-2\operatorname{epk}(u)}, \end{aligned}$$

again using the fact that $\operatorname{epk}(u) = \operatorname{lpk}(u\downarrow)+1$ when $\phi(u) \neq 1$. The only caveats are that one needs to check that $\phi((u\downarrow)^*) = \phi(u^*)$ for all $u \in S_n$, and $\phi(u^*) = \phi((u')^*)$ when $\phi(u) = 1$. One should also check that the formula provided in Corollary 5.11(b) is valid for D_3, since the argument given there is not.

On the other hand, Corollaries 5.9 and 5.11(a) imply

$$\widetilde{B}_n(t) - \widetilde{D}_n(t) = \sum_{u \in S_n} (1-\phi(u^*))\phi(u)(4t)^{\operatorname{epk}(u)}(1+t)^{n+1-2\operatorname{epk}(u)}.$$

Comparing the two expansions and noting that $\operatorname{epk}(u^*) = \operatorname{epk}(u)$, the result follows. ☐

The following identity is due to Stembridge (set $l = 0$ in [23, Lemma 9.1]).

Proposition 6.3. *For $n \geqslant 2$, we have*

$$B_n(t) = C_n(t) = D_n(t) + n2^{n-1}tA_{n-2}(t).$$

Proof. It is easy to check the cases $n = 2$ and $n = 3$, so assume $n \geqslant 4$.

By the same reasoning we used in the proof of Proposition 6.1, Corollary 5.6(c) implies

$$n2^{n-1}tA_{n-2}(t) = 2t \sum_{u \in S_n} (4t)^{\operatorname{pk}(\downarrow u)}(1+t)^{n-2-2\operatorname{pk}(\downarrow u)},$$

where $\downarrow u := u_2 \cdots u_n$. Now consider that if $\phi(u^*) = 1$ (i.e., $u_1 < u_2 < u_3$) and u'' is obtained from u by switching u_1 and u_2 (hence $\phi((u'')^*) = 0$) then $\operatorname{pk}(\downarrow u) = \operatorname{pk}(\downarrow u'')$. It follows that we can create an equivalent sum by modifying the contribution of every permutation u by the factor $2(1-\phi(u^*))$, yielding

$$\begin{aligned} n2^{n-1}tA_{n-2}(t) &= \sum_{u\in S_n}(1-\phi(u^*))(4t)^{\mathrm{pk}(\downarrow u)+1}(1+t)^{n-2-2\,\mathrm{pk}(\downarrow u)} \\ &= \sum_{u\in S_n}(1-\phi(u^*))(4t)^{\mathrm{lpk}(u)}(1+t)^{n-2\,\mathrm{lpk}(u)}, \end{aligned}$$

using the fact that $\mathrm{lpk}(u)=\mathrm{pk}(\downarrow u)+1$ whenever $\phi(u^*)\neq 1$. On the other hand, it is clear from Corollaries 5.6(b) and 5.11(b) that this sum is $C_n(t)-D_n(t)$. □

6.2. **Generating functions.** First let us review the (known) generating functions for the Eulerian polynomials corresponding to the Weyl groups A_n, $B_n=C_n$, and D_n:

$$A(t,z) := 1+\sum_{n\geqslant 1} tA_{n-1}(t)\frac{z^n}{n!} = \frac{(1-t)}{1-te^{z(1-t)}}, \tag{6.2}$$

$$B(t,z)=C(t,z) := 1+(1+t)z+\sum_{n\geqslant 2} B_n(t)\frac{z^n}{n!} = \frac{(1-t)e^{z(1-t)}}{1-te^{2z(1-t)}}, \tag{6.3}$$

$$D(t,z) := 1+tz+\sum_{n\geqslant 2} D_n(t)\frac{z^n}{n!} = \frac{(1-t)(e^{z(1-t)}-z)}{1-te^{2z(1-t)}}. \tag{6.4}$$

The first of these is classical (e.g., see [7, p. 244]), and for proofs of the latter two see Theorem 3.4 and Corollary 4.9 in [2], but note that our initial terms for $D(t,z)$, and hence the resulting closed form, are slightly different from those in [2].

For small n, the values for $A_n(t)$, $B_n(t)$, $D_n(t)$ implicit in the above expansions are

$$A_{-1}(t)=1/t,\ B_0(t)=1,\ B_1(t)=1+t,\ D_0(t)=1,\ D_1(t)=t.$$

In this way, Proposition 6.3 is valid even for $n=0$ or 1, and immediately implies

$$B(t,z)=D(t,z)+zA(t,2z).$$

Thus (6.4) may be viewed as a corollary of (6.3) and Proposition 6.3.

Turning to the affine Eulerian polynomials, let

$$\begin{aligned} \widetilde{A}(t,z) &:= z+\sum_{n\geqslant 2}\widetilde{A}_{n-1}(t)\frac{z^n}{n!}, \\ \widetilde{C}(t,z) &:= 1+\sum_{n\geqslant 1}\widetilde{C}_n(t)\frac{z^n}{n!}, \\ \widetilde{B}(t,z) &:= 2+2tz+\sum_{n\geqslant 2}\widetilde{B}_n(t)\frac{z^n}{n!}, \\ \widetilde{D}(t,z) &:= 2+4t\frac{z^2}{2}+\sum_{n\geqslant 3}\widetilde{D}_n(t)\frac{z^n}{n!}. \end{aligned}$$

Proposition 6.4. *We have*

$$\widetilde{A}(t,z) = \frac{z(1-t)}{1-te^{z(1-t)}}, \tag{6.5}$$

$$\widetilde{C}(t,z) = \frac{1-t}{1-te^{2z(1-t)}}, \tag{6.6}$$

$$\widetilde{B}(t,z) = \frac{2(1-t)(1-tze^{z(1-t)})}{1-te^{2z(1-t)}}, \tag{6.7}$$

$$\widetilde{D}(t,z) = \frac{2(1-t)(1+tz^2-2tze^{z(1-t)})}{1-te^{2z(1-t)}}. \tag{6.8}$$

Proof. Corollaries 5.2 and 5.7 immediately imply $\widetilde{A}(t,z) = zA(t,z)$ and $\widetilde{C}(t,z) = A(t,2z)$, so these generating functions are consequences of (6.2). In the remaining two cases, the values implicit for small n in the series defined above (namely, $\widetilde{B}_0(t) = \widetilde{D}_0(t) = 2$, $\widetilde{C}_0(t) = 1$, $\widetilde{B}_1(t) = \widetilde{C}_1(t) = 2t$, $\widetilde{D}_1(t) = 0$, and $\widetilde{D}_2(t) = 4t$) have been deliberately chosen so that Propositions 6.1 and 6.2 remain valid for all $n \geqslant 0$, and thus respectively imply

$$2\widetilde{C}(t,z) = \widetilde{B}(t,z) + 2tzC(t,z),$$
$$\widetilde{B}(t,z) = \widetilde{D}(t,z) + 2tzD(t,z).$$

The first of these, together with (6.3) and (6.6), yields the formula claimed for $\widetilde{B}(t,z)$, and then the second (with (6.4)) yields the formula claimed for $\widetilde{D}(t,z)$. □

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