Math 115 - Calculus I

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University of Michigan-Ann Arbor, Winter 2019

Introduction

The LaTeX coursenote template used is by Zev Chonoles (http://math.uchicago.edu/ chonoles/).

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Lecture 1 (2019-01-09)

Calculus is a mathematical framework for working with and extracting useful information from functions. Thus, in order to understand and appreciate the meat of calculus, we first need to make sure we know *a*) what functions are, *b*) why we care about functions/what functions are useful for, and *c*) many examples of common functions and their properties. These topics are roughly the content of "precalculus" courses, such as Math 105 here at UMich. Over the next few weeks, we are going to very rapidly review this material. Even if you feel comfortable with these concepts, don't check out! Use this as an opportunity to ensure facility and confidence with the basics. Try to notice new things you hadn't noticed before. Let the ideas marinate.

What is a function? Often in life and the world, we know, or at least have a strong intuition, that one quantity A depends on another quantity B in some fashion. In such a situation, we might say in a colloquial sense that A is a function of B. The formal definition of function is intended to capture this notion of "two quantities have some dependence relation" abstractly:

Definition. A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number.

Notation. We denote a function by the letter f (although other letters work just as well) and we denote a specific input to the function by the letter x (although other letters work just as well). We write $f(x)$ to denote the output obtained when you input the quantity x into the function f.

Examples.

- The functions you have seen already (and which we will often be working with) are those described by mathematical formulas. For example, $f(x) = 3x + 2$ is the function that takes as input a number x, then outputs the result of multiplying x by three and then adding two. Another example is $g(y) = y^2$, which takes as input a number y, then outputs the result of multiplying y with itself.
- Consider the function B which takes as input the number of drinks d that Homer has with dinner, and outputs the quantity $B(d)$ which is Homer's blood alcohol content 30 minutes later. This function illustrates some general features. We posit that there is a relationship between d and $B(d)$, but we don't necessarily know the precise nature of this relationship; in particular, the function B is not defined by an explicit formula. Furthermore, B only takes certain numbers as inputs (Homer can't have -7 drinks) and has a designated range of output values (Homer's blood alcohol content can't be 12, since BAC values are always between 0 and 1).

Definition. Suppose f is a function. The **domain** of f is defined to be the set of allowed input values; we often denote the domain of f by $\mathrm{Dom}(f)$. The **range** of f is defined to be the set of possible output values; we often denote the range of f by $\mathrm{Ran}(f)$.

Example. Consider the function

$$q(t) = \frac{3t+1}{t-2}$$

You can input any number t into this function *except for* $t = 2$ (since you can't divide by

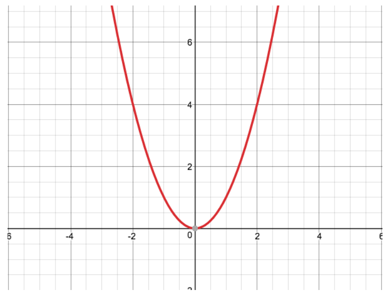


Figure 1: The graph of φ

zero). Thus, $\text{Dom}(q) = \{\text{all numbers except } 2\}$. If y is a number, then y is in the range of q if it is possible to solve the equation

$$y = \frac{3t+1}{t-2}$$

for t. Indeed, if we rearrange the equation, we get

$$t = \frac{2y+1}{y-3}$$

so it is possible to solve the equation for all y *except* $y = 3$. Thus, $\text{Ran}(q) = \{\text{all numbers except } 3\}$.

Exercise. For each of the following functions, determine (or estimate, if it's not possible to determine exactly) its domain and range.

(a) $h(y) = \sqrt{1-y^2}$

(b) $k(z) = (z^2+1)/z$ (*Hint:* to find the range of this function, use the quadratic formula)

(c) $\ell(t) =$ the number of people in Chipotle on State Street t hours after noon on January 1st, 2019

Definition. The **graph** of a function f is the set of all points in the coordinate plane with coordinates of the form $(a, f(a))$.

Example. Consider the function $\varphi(m) = m^2$. Its graph is the set of all points in the coordinate plane with coordinates of the form (m, m^2). For example, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 4)$ are all in the graph of φ. See the above picture.

Definition. Suppose g and h are functions such that $\text{Ran}(g)$ is contained in $\text{Dom}(h)$. This means that any output value from g can be used as an input into h. Thus, we can define the **composite** function $h \circ g$, defined by sending each input x to the value $h(g(x))$.

Examples.

- First, an example with mathematical formulas. If $k(p) = p^2 + 1$ and $\ell(t) = t + 7$, then we can explicitly calculate the composition $k \circ \ell$:

$$k(\ell(t)) = (t+7)^2 + 1 = t^2 + 14t + 50$$

- Let c be the function that takes as input the temperature f in degrees Fahrenheit and outputs the temperature $c(f)$ in degrees Celsius. Let q be the function that takes as input the temperature t in degrees Celsius and outputs the likelihood $q(t)$ that the temperature will dip below t degrees today. Then, the composition $q \circ c$ also outputs the likelihood that the temperature will dip below the input temperature, but accepts the input in degrees Fahrenheit rather than degrees Celsius.

Exercise. Express the function $f(t) = t^2 + 2t + 1$ as a composite in at least two different ways, i.e. find at least two different pairs of functions (g, h) such that $f(t) = g(h(t))$.

Definition. A function f is called **invertible** if there exists another function g such that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

for all input values x. In this situation, the function g is called the **inverse** of f, and denoted f^{-1}. Intuitively, f^{-1} "undoes" the effect of f.

A function f is invertible precisely when, for every number y in $\mathrm{Ran}(f)$, there is exactly one input a such that $f(a) = y$. This input a is the value $f^{-1}(y)$. We can think of $f^{-1}(y)$ as "the number which, when plugged into f, yields a result of y."

If you have available the graph of a function f, a quick way to check if f is invertible is the *horizontal line test*.

Exercise. Suppose f is an invertible function. Explain why $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$ and $\mathrm{Ran}(f^{-1}) = \mathrm{Dom}(f)$.

Important Remark: Even if a function is not invertible, it might become invertible when we *restrict its domain*. For example, consider the function $\varphi(m) = m^2$. We know that $\mathrm{Dom}(\varphi) = \{\text{all numbers}\}$ and $\mathrm{Ran}(\varphi) = \{\text{all nonnegative numbers}\}$.

The function φ is not invertible when thought of as a function on its entire domain: for any number z, there are two numbers m such that $\varphi(m) = z$ (the positive and negative square root of z). Thus, φ is not invertible.

However, suppose we think of φ as a function that is *only allowed to eat positive numbers.* In other words, we restrict the domain of φ to the interval $(0, \infty)$. Now, for every z, there is exactly one number m in the domain of φ such that $\varphi(m) = z$ (the positive square root of z). Thought of as a function with domain $(0, \infty)$, φ becomes invertible.

This trick is *very important/useful* to define inverses of functions that aren't invertible on their entire domains. We will use it especially when defining inverses of trigonometric functions.

Exercise. The function $q(t) = \sin(t)$ is not invertible on its entire domain. Give three different examples of domains on which it *is* invertible.

Tidbits I might mention in class: interval notation, set membership notation.

Lecture 2 (2019-01-11)

We now have a general sense of what a function is. Today, we're going to talk about the **growth** of functions, and review the properties of some common functions of interest (namely, **linear** and **exponential** functions).

Suppose f is a function. We are often interested in the problem of determining how the output $f(x)$ changes/grows/varies as the input x changes/grows/varies. A rough way to categorize growth behavior is via the notions of *increasing* and *decreasing* functions:

Definition. Suppose f is a function and S is a subset of the domain $\text{Dom}(f)$. We say f is **increasing on** S if whenever $x, y \in S$ satisfy $y > x$, then $f(y) > f(x)$. In words, *bigger inputs from S yield bigger outputs.*

Analogously, we say f is **decreasing on** S if whenever $x, y \in S$ satisfy $y > x$, then $f(y) < f(x)$. In words, *bigger inputs from S yield smaller outputs.*

If f is increasing on $\text{Dom}(f)$, we just call it **increasing**.

If f is decreasing on $\text{Dom}(f)$, we just call it **decreasing**.

Examples.

- Consider the function $\$(q)$, which is the rule that takes input a number of quarters q and outputs the value $\$(q)$ of q quarters, in cents. We all know that a formula for $\$(q)$ is given by $\$(q) = 25q$. Here, \$ is increasing on its entire domain, since bigger inputs numbers of quarters always correspond to larger monetary values.
- You're growing amoebas in a culture. At noon, you start with 1 amoeba in the culture. Suppose it takes any amoeba an hour to split into two amoebas, and they just keep splitting, because that's what amoebas do. At 1PM, you'll have 2 amoebas, at 2PM you'll have 4 amoebas, etc.

  Let $A(t)$ be the rule that outputs the number $A(t)$ of amoebas in the culture t hours after noon. We know $A(0) = 1$, $A(1) = 2$, $A(2) = 4$. In general, $A(t) = 2^t$. Note that A is also increasing on its entire domain, since later times always mean you'll have more amoebas in the culture.

Exercise. Think of another example of an increasing function and three examples of decreasing functions (either a mathematical formula, or describing some real-world relationship between quantities).

Important Observation: Both \$ and A as defined above are increasing functions. But intuitively, one of them is increasing *much* faster than the other. Even though the values of A are initially much smaller than the values of \$, the function A initially "overtakes" \$. Let's make careful sense of this notion of "increasing faster".

Definition. Suppose f is a function with the closed interval $[a, b]$ in its domain. The **average rate of change of f on the interval** $[a, b]$ is defined by the formula

$$\frac{f(b) - f(a)}{b - a} = \frac{\text{change in output}}{\text{change in input}}$$

The average rate of change of f on $[a, b]$ roughly answers the question "how much does the output of f change when going from a to b, relative to the length of the interval $[a, b]$?" We say "roughly" because the average rate of change ignores everything that's happening *inside* the interval $[a, b]$.

Let's apply this concept to our functions \$ and A. Suppose we restrict both functions to the domain $\{1, 2, 3, 4, 5\}$. We can calculate the following tables:

q	1	2	3	4	5
$\$(q)$	25	50	75	100	125
Average rate of change of \$ on the interval $[t, t+1]$	25	25	25	25	25

t	1	2	3	4	5
$A(t)$	2	4	8	16	32
Average rate of change of A on the interval $[q, q+1]$	2	4	8	16	32

The average rate of change of \$ is *constant* over every interval. Although \$ is always increasing, the rate at which it's increasing never changes.

The average rate of A is *itself increasing* as t gets bigger; in fact, it looks like you always get back $A(t)$ as the rate of change of A on the interval $[t, t+1]$. This is "compounded growth".

This is why we say A is "increasing faster" than \$: because the rate of change of \$ stays constant, but the rate of change of A itself increases as the input gets larger.

Definition. A function f is called **linear** if its average rate of change is constant on every interval. This constant average rate of change is called the **slope** of f.

Definition (can't quite make sense of this precisely yet)**.** A function f is called **exponential** if its "rate of change" is directly proportional to f.

Theorem. (*a*) *Every linear function is of the form* $g(y) = ay + b$ *for some numbers* a, b.

(*b*) *Every exponential function is of the form* $h(z) = ab^z$ *for some numbers* a, b *(here,* $b > 0$*).*

Exercise. For which pairs of numbers a and b is $g(y) = ay + b$ an increasing function? For which pairs is g decreasing?

For which pairs of numbers a and b is $h(z) = ab^z$ an increasing function (exponential growth)? For which pairs is h decreasing (exponential decay)?

Note: Exponential *growth* always eventually dominates linear growth.

Exercise. The table below depicts values sampled from two functions f and g. One of the functions is exponential and one is linear. Which one is which? Find explicit formulas for each one.

t	0	1	3	4
$f(t)$	64	96	216	324
$g(t)$	64	129	259	324

Lecture 3 (2019-01-15)

Quiz Today! -- Student Website + Grading Guidelines + Team HW. If you haven't already, please remember to fill out the Student Data Sheet by *midnight tonight.*

Before we start new material, some review problems:

- A certain region has a population of 10 million and an annual growth rate of 2%. Estimate the doubling time by guessing and checking. Then, calculate the doubling time exactly.
- Figure 1.29 (on the back of this page) is the graph of three exponential functions. What can you say about the values of the six constants a, b, c, d, p, q?

Our goal for today is to study some common transformations of functions. A **transformation** is an operation that you do to a function which yields a new function.

Some common transformations are the following:

- **Translation Of Input**: replace the function $x \mapsto f(x)$ with the function $x \mapsto f(x+a)$ for some number a
- **Scaling Of Input**: replace the function $x \mapsto f(x)$ with the function $x \mapsto f(cx)$ for some number c
- **Translation Of Output**: replace the function $x \mapsto f(x)$ with the function $x \mapsto f(x) + a$ for some number a
- **Scaling Of Output**: replace the function $x \mapsto f(x)$ wih the function $x \mapsto cf(x)$ for some number c
- **Any Combinations Of the Above**

One way to think about these transformations is *pre-processing* the input to the function or *post-processing* the output of the function.

To get a better handle on these transformations, we study carefully how they geometrically change the graph of a function. For example:

Example. Suppose $g(t) = t^2$ is the function that we start with. The graph of g is a standard parabola in the coordinate plane. What does the graph of $h(t) = g(t-2)$ look like?

Let's plot some points: on the graph of g we have $(0,0)$, $(1,1)$, $(2,4)$, $(3,9)$, etc. On the graph of h we have $(2,0)$, $(3,1)$, $(4,4)$, $(5,9)$. In general, the points on the graph of g all have the form (t,t^2), and the points on the graph of h all have the form $(t+2,t^2) = (t,t^2) + (2,0)$.

We get the graph of h by adding $(2,0)$ to the points of (t,t^2). This *shifts the whole graph two units to the right.*

Exercise. Without using a graphing calculator, sketch the graphs of $h(t+3)$, $h(2t)$, $3h(t-1)+1$, and $4h(t/2)$.

Explain in words what each of the transformations do geometrically.

Draw many more examples! We're going to spend a while today doodling.

Rough Slogans: "Inside transformations do the opposite of what you'd expect, outside transformations do what you'd expect"

"Scaling is like changing the dimensions of your picture. Translation is like picking it up and putting it somewhere else. These notions make the most sense when your graph goes through $(0,0)$." (??)

Some functions are especially symmetric: their graphs are unchanged under certain transformations.

The transformation $f(x) \rightsquigarrow f(-x)$ flips the graph of f over the y-axis. A function that is unchanged under this transformation, i.e. $f(x) = f(-x)$, is called an **even function**.

The transformation $f(x) \rightsquigarrow -f(-x)$ rotates the graph of f 180° about the origin. A function that is unchanged under this transformation, i.e. $f(x) = -f(-x)$, is called an **odd function**.

Exercise. Draw some pictures of graphs of even and odd functions. Give some examples of even and odd functions with formulas.

Exercise. In the table below, f is an even function and g is an odd function. Fill in the missing entries.

t	-2	-1	0	1	2
$f(t)$	2	3	1		
$g(t)$	2	3			
$f(g(t))$		0			

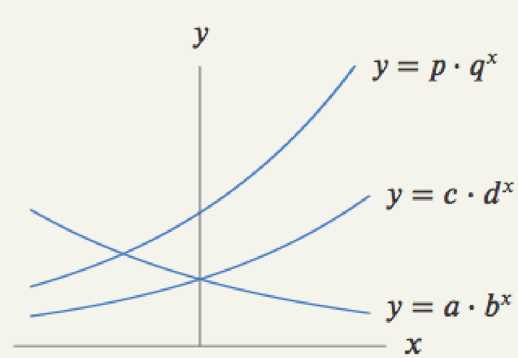


Figure 1.29: Three exponential functions

Lecture 4 (2019-01-16)

Exercise. Given the graph of some function f, how do you obtain the graph of f^{-1}? Describe some geometric transformation that turns the graph of f into the graph of f^{-1} and explain why this works.

To describe the growth of a function, we discussed the notions of *increasing* and *decreasing*. However, we also saw that these notions are insufficient to some extent: the linear function $f(s) = s$ and the exponential function $g(s) = 2^s$ are both increasing on their domains, but the exponential function g is "eventually increasing faster."

More precisely, we saw that the average rate of change of f is *constant* on every interval (what is this constant?) whereas the average rate of change of g is *increasing* as we look at "increasingly rightward" intervals.

There is an important definition that encapsulates the notion of "increasing rate of change":

Definition. Let f be a function and suppose S is a subset of $\text{Dom}(f)$. We say f is **concave up on** S if whenever a, b, $c \in S$ with $a < b < c$, the average rate of change of f on the interval $[b, c]$ is bigger than the average rate of change of f on the interval $[a, b]$. In words, the rate of change of f is increasing on S.

Similarly, we say f is **concave down on** S if whenever a, b, $c \in S$ with $a < b < c$, the average rate of change of f on the interval $[b, c]$ is smaller than the average rate of change on the interval $[a, b]$. In words, the rate of change of f is decreasing on S.

Important note: a function can be both concave up and decreasing, or concave down and increasing! Concavity says something about the *rate of change of the rate of change* of the function; it doesn't directly say something about the rate of change.

Exercise. Sketch some graphs of functions that are

- concave up and increasing
- concave up and decreasing
- concave down and increasing
- concave down and decreasing

Exercise. Suppose f is a function that is concave up on the open interval $(0, 6)$. If $f(1) = 2$, $f(3) = 5$, and $f(4) = 11$. What are the possible values of $f(2)$?

If g is a function satisfying $g(2) = 7$, $g(3) = 5$, and $g(4) = 6$, could g be concave down on the interval $(0, 6)$?

Let's remind ourselves of the algebraic properties of exponential and logarithmic functions. First of all, recall that $\log_a(x)$ is what we write to denote the inverse of the function a^x.

(Note that a^x always passes the horizontal line test, so it has an inverse defined on all of $(0, \infty)$).

We write $\ln(x)$ to denote $\log_e(x)$, where $e \approx 2.718..$ is that magic number that keeps showing up. We also often denote $\log_{10}(x)$ simply by $\log(x)$.

Recall that $a^{x+y} = a^x \cdot a^y$ for any numbers x and y. In particular, if we set $x = \log_a(m)$ and $y = \log_a(n)$, then

$$a^{\log_a(m)+\log_a(n)} = m \cdot n \implies \log_a(m) + \log_a(n) = m \cdot n$$

for any positive numbers m and n.

We also have the "change of base formula":

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

for any c. In particular, this formula implies that

$$\log_a(b^c) = \frac{\log_b(b^c)}{\log_b(a)} = \frac{c}{\log_b(a)} = \frac{c}{\log(b)/\log(a)} = \frac{c\log(a)}{\log(b)} = c\log_a(b)$$

so logarithms "turn powers into multiplications."

Exercise. A circle has a radius of $\log_{10}(a^2)$ and a circumference of $\log_{10}(b^4)$. What is $\log_a(b)$?

Exercise.
- Find all solutions x for the equation $7e^{x+3} + 5 = 13$.
- Find all solutions x for the equation $4^x = 2^{x+1} + 1$. (*Hint:* Use the quadratic formula)

A neat thing: Consider the powers of two, i.e. $1, 2, 4, 8, 16, 32$, etc. How often is the leftmost digit equal to 1? Explain/support your answer.

Lecture 5 (2019-01-18)

Today, our goal is to do a **rapid review of trigonometric functions**.

The convention is that angles input into trigonometric functions are always in *radians*. Recall that you can convert from degrees to radians by setting up a direct proportion using the fact that

$$\boxed{2\pi \text{ rad} = 360^\circ}$$

Constructing Sine and Cosine From Scratch

I'm going to construct the functions "sine" and "cosine" from scratch. Our goal in doing this is primarily to remind ourselves of some of the most important properties of these functions.

Suppose θ is an acute angle. Let $\triangle ABC$ be a right triangle with $\angle B = \pi/2$ and $\angle A = \theta$. We define

$$\sin(\theta) = \frac{BC}{AC}$$
$$\cos(\theta) = \frac{AB}{AC}$$

Note that if $\triangle A'B'C'$ is another right triangle with $\angle B' = \pi/2$ and $\angle A' = \theta$, then $\triangle ABC$ and $\triangle A'B'C'$ are similar. Thus, we have the equalities

$$\frac{BC}{AC} = \frac{B'C'}{A'C'} \ \ (= \sin(\theta))$$

$$\frac{AB}{AC} = \frac{A'B'}{A'C'} \ \ (= \cos(\theta))$$

Thus, the numbers $\sin(\theta)$ and $\cos(\theta)$ are independent of the right triangle you use to calculate them; they only depend on the angle θ.

This gives us a definition of sin and cos for acute angles θ. Note that by this definition, when θ is an acute angle, the numbers $\sin(\theta)$ and $\cos(\theta)$ are both positive. How do we extend make sense of $\sin(\theta)$ and $\cos(\theta)$ when θ is not an acute angle?

First, recall the *unit circle* is the set of points (x, y) in the coordinate plane with $x^2 + y^2 = 1$. If P is a point on the unit circle, let $\angle(P)$ be the (smallest nonnegative) angle at which P is located, measured counterclockwise from the positive x-axis. In particular, we insist $\angle((1, 0)) = 0$, and we have now a function

$$\angle : \{\text{points on the unit circle}\} \to [0, 2\pi)$$

(A function that eats points and spits out angles!) This $\angle$ is an invertible function, and we can write $\angle^{-1}(\theta)$ to denote the unique point P on the unit circle with $\angle(P) = \theta$.

Now, note that by our definition of sin and cos, if $0 \leq \theta < \pi/2$, then the coordinates of the point $\angle^{-1}(\theta)$ are exactly $(\cos(\theta), \sin(\theta))$. Turning this observation around on its head, we *define* $\sin(\theta)$ and $\cos(\theta)$ to be the y- and x-coordinates, respectively, of $\angle^{-1}(\theta)$ for $\pi/2 < \theta < 2\pi$ as well! Note that this means sin and cos now attain negative values too: for example $\sin(3\pi/2) = -1$ and $\cos(\pi) = -1$.

So, we have a definition of sin and cos for angles $0 \leq \theta < 2\pi$. Finally, we insist that sin and cos satisfy the rules

$$\sin(\theta + 2\pi) = \sin(\theta)$$
$$\cos(\theta + 2\pi) = \cos(\theta)$$

for all angles θ. In other words, sin and cos should be "2π-periodic". This gives a definition of sin and cos that works when you plug in *any* angle θ (it can be bigger than 2π, or even negative).

In particular, based on how we defined sin and cos via the coordinates of points on the unit circle, these functions satisfy the fundamental equation

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

for all angles θ.

Finally, we define a few additional trigonometric functions in terms of sin and cos:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\cot(\theta) = \frac{1}{\cot(\theta)}$$

Exercise. We defined the functions sin and cos each to have domain $(-\infty, \infty)$. What is the range of sin? What is the range of cos? What are the domain and the range of tan?

Exercise. Some values of sin and cos can be explicitly calculated, and it's important to know what they are. Use geometry to calculate the coordinates of $\angle^{-1}(\pi/6)$, $\angle^{-1}(\pi/4)$, and $\angle^{-1}(\pi/3)$.

Inverse Trig Functions

Since sin and cos are 2π-periodic, they fail the horizontal line test pretty badly. In fact, they already failed the horizontal line test even when we only had definitions for angles θ between 0 and 2π (why?).

However, the function $\sin(\theta)$ is invertible when restricted to the domain $[-\pi/2, \pi/2]$. Similarly, the function $\cos(\theta)$ is invertible when restricted to the domain $[0, \pi]$. We also see that the function $\tan(\theta)$ is invertible when restricted to the domain $(-\pi/2, \pi/2)$. Moreover, on all of these subdomains, every value in the range of each of the functions is in fact attained.

Accordingly, we define the *basic inverse trig functions* as follows:

$$\arcsin(x) = \text{ the unique angle } \theta \text{ in } [-\pi/2, \pi/2] \text{ such that } \sin(\theta) = x$$

$$\arccos(x) = \text{ the unique angle } \theta \text{ in } [0, \pi] \text{ such that } \cos(\theta) = x$$

$$\arctan(x) = \text{ the unique angle } \theta \text{ in } (-\pi/2, \pi/2) \text{ such that } \cos(\theta) = x$$

Keep in mind, however, that trigonometric equations have (many) more solutions than just the "canonical" solutions identified by the inverse trig functions. For example, $\arcsin(\sqrt{2}/2) = \pi/4$, but the equation $\sin(\theta) = \sqrt{2}/2$ has the (infinite!) solution set

$$\theta = \left\{ \cdots, -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \cdots \right\}$$

General Sinusoidal Functions

We now describe a notion of "general sinusoidal function". These functions are just the sin and cos functions scaled and translated via the transformations we discussed two classes ago. These functions are often used to (crudely) model periodic phenomena (real-world processes that repeat/recur).

Definition. A **general sinusoidal function** is any function of the form

$$f(t) = A\sin(Bt + C) + D$$

for numbers A, B, C, and D.

Example. Note that cos is indeed a general periodic function, since we always have the identity

$$\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$$

Definition. For a general sinusoidal function f of the form in the above definition, the quantity $|A|$ is called the **amplitude** of f. It is the amount by which the original sin curve has been vertically scaled. Note that the range of f is $[-|A| + D, |A| + D]$.

The quantity $2\pi/|B|$ is called the **period** of f. It is the smallest amount of time needed for the function to execute one complete cycle. This makes sense, since B measures the horizontal scaling of the original sin curve: large B (i.e. $|B| > 1$) means super squished curve means short period; small B (i.e. $|B| < 1$) means super stretched curve means long period.

Remark. For a general sinusoidal function f, the quantity C/B describes the amount of horizontal shift. More precisely, f is obtained from the function $g : t \mapsto A\sin(Bt) + D$ by shifting g by C/B units to the left. Indeed, the shifted function is then given by

$$g\left(t + \frac{C}{B}\right) = A\sin\left(B\left(t + \frac{C}{B}\right)\right) + D = A\sin(Bt + C) + D = f(t)$$

Lecture 6 (2019-01-23)

Today, we'll talk about **polynomials and rational functions**. These are an exceptionally important class of functions for a few reasons:

- They are easy to calculate, since they are built entirely out of the arithmetic operations of addition, subtraction, multiplication, and division.
- They are very rigid, in that they are (almost) entirely determined by their set of *roots* and *poles*.

Definition. A **polynomial** is a function that is entirely built out of the operations of repeated addition, subtraction, and multiplication of the input. More precisely, a polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where all the a_i's are constants. For example, $f(t) = t^2 + t + 1$, $g(s) = -s + 7$, and $h(q) = -2q^3 + q + 1$ are all polynomials.

Definition. The **degree** of a polynomial $p(x)$ is the largest power of x that appears in the expression of $p(x)$. For example, the degree of $f(t) = t^2+t+1$ is 2, the degree of $g(s) = -s+7$ is 1, and the degree of $h(q) = -2q^3 + q + 1$ is 3.

Remark. Here is a geometric interpretation of the degree of a polynomial. The graph of a polynomial of degree n always has at most $n-1$ *bends*. More precisely, if you imagine that the polynomial function $p(t)$ describes the location at time t of a particle moving on the number line, then if p has degree n, the particle changes its direction of movement at most $n-1$ times.

Definition. If f is a function, a **root** or **zero** of f is any number $r \in (-\infty, \infty)$ such that $f(r) = 0$.

Example. The polynomial $g(s) = -s + 7$ has a root at 7, since $g(7) = -7 + 7 = 0$. The polynomial $f(t) = t^2 + t + 1$ has *no* roots. This can be seen in (at least) two ways:

- Using the quadratic formula, the roots of f, if they exist, are of the form

  $$t = \frac{-1 \pm \sqrt{-3}}{2}$$

  But there is no square root of -3 in $(-\infty, \infty)$, so there are no real solutions.

- Completing the square, we see that

  $$f(t) = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4}$$

  which is the parabola with equation $m(t) = t^2$ shifted $1/2$ units to the left and $3/4$ units upwards. In particular, the graph of $f(t)$ does not intersect the x-axis, so f has no real roots.

Theorem. *Suppose $p(x)$ is a polynomial and r is a root of $p(x)$. Then $p(x) = (x - r)q(x)$ for some polynomial $q(x)$. In other words, roots can be "factored out" of polynomials.*

Example. Suppose $p(x)$ is a cubic polynomial with roots at 1, 2, and 3, such that $p(0) = 7$. Using the above theorem, $p(x)$ must be of the form

$$p(x) = c(x - 1)(x - 2)(x - 3)$$

Using the fact that $p(0) = 7$, we see that

$$7 = p(0) = -6c \implies c = -\frac{7}{6}$$

so that

$$p(x) = \frac{-7}{6}(x - 1)(x - 2)(x - 3)$$

Knowing the roots of p and one additional value allowed us to deduce the general form of the function $p(x)$.

Definition. A **rational function** is a function of the form $p(x)/q(x)$, where p and q are both polynomials. For example,

$$h(s) = \frac{3s + 2}{s - 1}$$

and

$$k(t) = \frac{-7t^2}{t^3 + t + 1}$$

are both rational functions.

Behavior of polynomials and rational functions as $x \to \infty$, $-\infty$: We are often interested in the "long run" behavior of a function, i.e. how it behaves when x gets very positive, or when x gets very negative. We say that a function f has a **horizontal asymptote** at $y = c$ if the graph of f approaches the line $y = c$ as $x \to \infty$ or as $x \to -\infty$.

Example. Consider the rational function

$$g(m) = \frac{3m^2 + 2}{m^2 + 7m}$$

Note that we can rewrite this as

$$g(m) = \frac{3 + (2/m^2)}{1 + (7/m)}$$

When $|m|$ is very large, the terms $2/m^2$ and $7/m$ are essentially zero, so $g(m)$ is essentially 3. In other words, g has a horizontal asymptote at $y = 3$, which it approaches as $m \to \infty$ and as $m \to -\infty$.

Definition. If $f(x) = p(x)/q(x)$ is a rational function, we say r is a **pole** of f if r is a root of $q(x)$. Basically, a pole is a place where f is undefined. If f has a pole at r, we say that f has a **vertical asymptote** at $x = r$. We can ask what the behavior of f is as x approaches a pole from the left or the right.

Domination: Given two functions $f(x)$ and $g(x)$, we say f **dominates** g **as** $x \to \infty$ if

$$\frac{f(x)}{g(x)} \to \infty$$

as $x \to \infty$. This is a way to compare the growth of f and g as x gets very large.

Example. If $f(x) = x$ and $g(x) = 2x + 7$, then neither function dominates the other, since $f/g \to 1/2$ as $x \to \infty$. On the other hand, if $q(x) = x^2$ and $r(x) = x$, then $q/r = x$, which goes to ∞ as $x \to \infty$. Thus, q dominates r.

In general, polynomial *growth* of higher degree dominates polynomial *growth* of lower degree. Moreover, exponential *growth* always dominates any polynomial *growth*.

Lecture 7 (2019-01-24)

Today, we're going to discuss the concept of limits. This concept is *the* technical heart of calculus, and it can take some doing to wrap your head around.

Limits Of Functions As We Approach A Number

Suppose f is a function, and the following table of values is sampled from f:

t	0.9	0.99	0.999	0.9999	0.99999
$f(t)$	2.11	2.04	2.005	2.00001	2.0000001

Based on this information, what would you guess the value of $f(1)$ is? The table strongly suggests that a prediction of $f(1) = 2$ is reasonable.

This is a very reasonable prediction, but it's not necessarily true. The value of a function at a point does not have to bear any resemblance to the values nearby. For all we know, it could be that $f(1) = 9$ billion. Moreover, even supposing that $f(1)$ is 9 billion, the function could actually reach this value in pretty different ways:

- One possibility is that, if we were to continue making the table above, with t values even closer to 1, we see something like this:

t	0.99999	0.999999	0.9999999	0.99999999	0.9999999999
$f(t)$	2.0000001	2.5	10	10^6	$8.7 \cdot 10^9$

  Upon zooming in to a sufficiently fine time-scale, the function actually stops decreasing to 2 and blows up towards 9 billion. The pattern of getting closer to 2, which we noticed in the earlier table, fails to continue.

- Another possibility is that, if we were to continue making the table above, with t values even closer to 1, we see something like this:

t	0.99999	0.999999	0.9999999	0.99999999
$f(t)$	2.0000001	2.000000001	2.0000000000001	2.0000000000000000001

  In this case, the values of the function continue to approach 2 as the inputs continue to approach 1. Keep in mind, this is still no guarantee that upon making inputs *even* closer to 1, the outputs don't start veering upwards towards 9 billion.

Very roughly speaking, we are suggesting two kinds of behaviors (these are not the only two things the function could do, it could also do even more complicated things!): either *eventually* as $t \to 1$, the function's outputs shoot up towards 9 billion, or they *always* get closer to 2 (and there is a discontinuous "jump" to 9 billion at $t = 1$ exactly).

In the latter case, we say that $f(t)$ has a **left limit** of 2 as t approaches 1 from below. This is written with the notation $\lim_{t\to 1^-} f(t) = 2$. More precisely, we have the following definition:

Definition. Suppose $f(t)$ is a function. We say f has a **left limit at** $t = a$ if, as t approaches a from below (i.e., through numbers slightly *smaller* than a), the values of $f(t)$ *always* get closer and closer to some number L. This number L is called the **left limit of** f **at** a, and we denote

$$\lim_{t \to a^-} f(t) = L$$

We similarly have a notion of **right limit** at $t = a$, denoted

$$\lim_{t \to a^+} f(t)$$

if it exists.

Remark. Note that these limits don't necessarily have anything to do with the *value* $f(a)$. In our above example with tables, we had a case where $\lim_{t \to 1^-} f(t) = 2$ but $f(1) = 9$ billion.

I'll say it again: **limits of** f **as the input approaches** a **are defined without reference to the value** $f(a)$!

Definition. Suppose $f(t)$ is a function and a is some number such that

$$\lim_{t \to a^-} f(t)$$

and

$$\lim_{t \to a^+} f(t)$$

both exist *and are equal*, say to some number L. Then, we say that the (nondirectional) **limit of** f **exists** as $t \to a$ (without a plus or minus sign!) and write

$$L = \lim_{t \to a} f(t)$$

In other words, when the left and right limits at a are the same, we just call it the limit.

Remark. One very useful way to conceptualize the concept of a limit as $t \to a$ is to ask yourself: if I knew nothing about the value of $f(a)$ but knew *everything* about the values $f(t)$ for t close to a, what would I predict $f(a)$ to be?

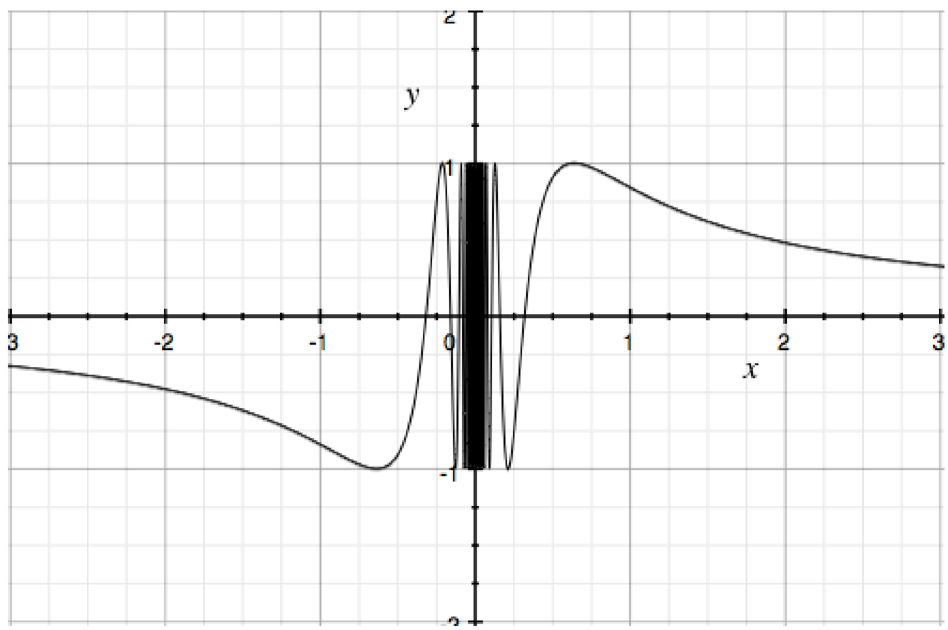
(Again, the prediction here is not necessarily *correct*, but if the function permits such a prediction, we call this prediction the limit of f as $t \to a$.)

Example. We have looked at examples where the values of $f(t)$ really do approach some number as t approaches a. This is **not** necessarily always going to happen! For example, consider the function $g(s) = 1/s$. As $s \to 0^+$, the values of $g(s)$ blow up to ∞ and don't approach any fixed number! In this case, we say the **limit does not exist**.

Limits can fail to exist even more extravagantly. In the case of g above, as $s \to 0^+$, the values $g(s)$ don't approach any fixed number but are always strictly increasing. A wilder example is the function

$$h(y) = \sin\left(\frac{1}{y}\right)$$

As $y \to 0^+$, note that $h(y)$ oscillates wildly between 1 and -1, and never settles on moving towards some fixed number. This is another case of the limit failing to exist.



Definition. Suppose $f(t)$ is a function defined in an open interval containing $t = a$ and $\lim_{t\to a} f(t)$ exists and equals $f(a)$. Then, we say f **is continuous at** $t = a$. In other words, we say f is continuous at $t = a$ if the prediction for $f(a)$ obtained by studying values $f(t)$ for t close to (but not equal to) a is *correct*.

Slogan: Continuity = correct prediction from nearby values.

Limits Of Functions As We Approach $+\infty$ Or $-\infty$

There is an analogous notion of limits as input values approach $+\infty$ or $-\infty$. Since ∞ is not actually a number, we do **not** have a corresponding notion of continuity at ∞!

Definition. Suppose $f(t)$ is a function. We say f has a **limit at infinity** if, as the input values t approach infinity (more precisely, get arbitrarily large), then the values of $f(t)$ *always* get closer and closer to some number L. This number L is called the **limit of f at infinity** and we denote

$$\lim_{t\to\infty} f(t) = L$$

Similarly, we have a notion of **limit at $-\infty$**, which is denoted

$$\lim_{t\to-\infty} f(t)$$

if it exists.

Remark. We can reconceptualize vertical and horizontal asymptotes in terms of limits. Horizontal asymptotes correspond to limits at ∞ and $-\infty$. Vertical asymptotes are a bit more complicated: they correspond to input values where the left/right limits do not exist, but only fail to exist in the "blow up to infinity or minus infinity" sense and not in the "oscillate wildly and never settle on any direction to go" sense.

Exercise: A Classic And Important Example

Always keep in mind that $\lim_{t\to a} f(t)$ does *not* depend on $f(a)$. In fact, f need not even be defined at a for a limit to exist. A classic example is the following:

Consider the function

$$h(\theta) = \frac{\sin(\theta)}{\theta}$$

Note that h is not even defined at $\theta = 0$. However, the limit $\lim_{\theta\to 0} h(\theta)$ still exists.

Can you guess what the limit is? *Make sure to justify your guess somehow.*

Can you explain/prove why this is the limit? (The proof is tricky, and counts for extra credit.)

Exercise: Limit Of A Composition

Consider the function

$$j(p) = \begin{cases} -p-3 & p < 0 \\ 0 & p = 0 \\ 3-p & p > 0 \end{cases}$$

What is $\lim_{p\to 0^-} j(p)$ What is $\lim_{p\to 0^+} j(p)$? Does $\lim_{p\to 0} j(p)$ exist?

Now, consider the composition $j(j(p))$. Does $\lim_{p\to 0} j(j(p))$ exist? Is $j(j(p))$ continuous at $p = 0$?

Lecture 8 (2019-01-29)

Imagine there is a magician, who performs the following two (incredibly breathtaking) tricks:

- She takes a grapefruit and tosses it into the air. The position of the grapefruit is given by a function $g(t)$, where t is measured in seconds since the grapefruit has been tossed, and $g(t)$ is measured in meters above the ground.
- She takes a dove in her hands and simply lets go of the dove. The dove falls momentarily, then catches itself and begins to fly upwards. The position of the dove is given by a function $d(t)$ where t is measured in seconds since the dove has been let go of, and $d(t)$ is measured in meters above the ground.

Suppose that $g(0) = h(0) = .5$ and $g(1) = h(1) = 2$. If this is the case, then both g and h will have the *same average rate of change on the interval* $[0, 1]$, namely $(2 - .5)/(1 - 0) = 1.5$.

However, the functions g and d are surely doing something different *near* or *at* $t = 0$: the function g is initially increasing, but the function d is initially decreasing. Moreover, this is actually reflected by calculating average rates of change on *short enough* time intervals near 0, perhaps like $[0, .2]$ or $[0, .02]$, etc.

This is an extremely important observation! If we calculate average rates of change on shorter time intervals $[0, \epsilon]$, we get numbers that are more reflective the behavior of the function *at* or *near* time $t = 0$.

Lecture 9 (2019-02-01)

Last time, we introduced the concept of *instantaneous rate of change*. There are three important perspectives to keep in mind on this concept:

The "kinematic" perspective: We have a function $f(x)$ and are interested in studying its behavior near some point $x = a$. The average rate of change of f on the interval $[a, a+h]$ is given by the formula

$$\mathrm{AvgRate}(a, h) := \frac{f(a+h) - f(a)}{h}$$

As h gets very small (equivalently, as the interval $[a, a+h]$ becomes very small in length), the average rates of change $\mathrm{AvgRate}(a, h)$ become more and more reflective of the behavior of f *near* or *at* a. Accordingly, we define the **instantaneous rate of change of f at a** to be

$$f'(a) := \lim_{h \to 0} \mathrm{AvgRate}(a, h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists!).

The "geometric" perspective: The point $(a, f(a))$ is on the graph of $f(x)$. If we choose h to be a small number, the slope of the line joining $(a, f(a))$ and $(a+h, f(a+h))$ is given by the formula

$$\frac{f(a+h) - f(a)}{h}$$

As h gets very small, the line joining $(a, f(a))$ and $(a+h, f(a+h))$ approaches the *tangent line* to the graph of f at a. Accordingly, if the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, it describes the **slope of the tangent line to f at** a.

The "approximation" perspective: We might be interested in calculating the value of $f(x)$ for some input x that is close to a. But f might be a complicated function for which it is not so clear how to calculate its values (for example, how would you calculate $\cos(1)$ without just plugging into a calculator? How is your calculator doing the computation?). A crucial observation is that for "nice" functions, when you zoom far enough into the picture of the function's graph, the picture *looks very much like a line*. In other words, we might be able to approximate f near a by a *linear function*. Here are the details:

First, translate f so that the point $(a, f(a))$ becomes the point $(0, 0)$. More precisely, replace f with $g(x) = f(x+a) - f(a)$. Now, using our knowledge of functional transformations, recall that "zooming in" to the graph of g near $(0, 0)$ with a "magnification factor" of c amounts to replacing g with

$$cg\left(\frac{x}{c}\right)$$

The "infinitely zoomed in" function is $h(x) = \lim_{c \to \infty} cg\left(\frac{x}{c}\right)$. If $h(x)$ is linear, then we can recover its slope by looking at $h(x)/x$. Indeed, we have

$$\frac{h(x)}{x} = \lim_{c \to \infty} \frac{c}{x} g\left(\frac{x}{c}\right) = \lim_{c \to \infty} \frac{c}{x}\left(f\left(\frac{x}{c} + a\right) - f(a)\right) = \lim_{\epsilon \to 0} \frac{f(\epsilon + a) - f(a)}{\epsilon}$$

which is exactly the definition from above. To summarize, at each input value a where the limit defining $f'(a)$ exists, we have a **linear approximation** at a, i.e. a linear function (whose equation coincides with the equation of the tangent line) that is a decent approximation for the values of $f(x)$ when the input x is *near* a. Another way of stating this is that for x close to a, we have

$$f(x) \approx f(a) + f'(a)(x - a)$$

The Derivative As a Function

We introduce some terminology/definitions. Suppose f is a function and the limit defining $f'(a)$ exists; then, we say f is **differentiable** at a. (Conversely, if the limit does not exist, we say f is **not differentiable** at a.)

Suppose f is a function that is differentiable everywhere. For each $a \in \text{Dom}(f)$, we have an assignment $a \mapsto f'(a)$. The rule that takes in an input a and outputs the instantaneous rate of change of f at a is *itself* a function. We call this function the **derivative** of f and often denote it by $f'(x)$.

Example. Consider $s(t) = t^2$. Rather than calculating the instantaneous rate of change at some fixed number a (like $a = 2$ as we did last time), we can carry out the same algebra for *arbitrary* a:

$$s'(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0}(2a + h) = 2a$$

Thus, as a *function*, the derivative of $s(t) = t^2$ is given by the function $s'(t) = 2t$.

Exercise. Some important special cases of the derivative as a function: if f is a *constant* function, what is f' (as a function)? If f is a *linear* function, what is f' (as a function)?

Exercise. Here's an example of a function that is *not* differentiable everywhere. Let $q(r) = |r|$. Graph the function $q(r)$ and explain why it is not differentiable at $r = 0$.

Exercise. See the image below.

In the graph of f in Figure 2.35, at which of the labeled x-values is

(a) $f(x)$ greatest? **(b)** $f(x)$ least?
(c) $f'(x)$ greatest? **(d)** $f'(x)$ least?

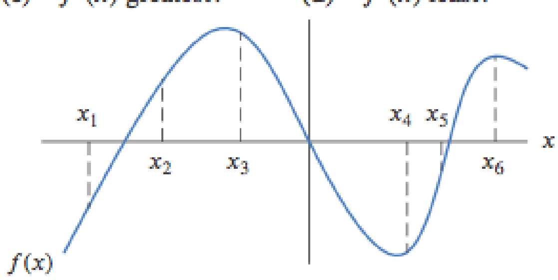


Figure 2.35

Lecture 10 (2019-02-05)

Some Important General Comments:

- In general, both on exams and often in life, getting the right answer counts for much less than being able to clearly communicate your ideas/efforts to other people. This is why it is so important to *organize* your work, make *clear and precise* statements, and to understand the *types* of various mathematical objects. (Examples: work through trig problem again from last quiz; "the function is symmetric"; "some men are doctors" and "some doctors are tall" does not imply "some men are tall"; if $f(t) = t^2$, the statement "at $t = 3$, the function is 9")
- Always ask yourself at the end: does this make sense? Be sure to check that your answer/solution is consistent with the information given in the problem. This is a great way to determine if you've worked out the problem correctly. If you discover that your answer is inconsistent with the information, go back to your work and see if you can figure out what went wrong. Make sure your work is clearly organized so that it's easy to review it if necessary.
- When you're stuck, you have to try to *actively* problem solve. Some potential strategies: draw a picture, make a table, remind yourself what the concepts mean, solve an easier problem, think wishfully and imaginatively, take a break and return. When faced with a novel mathematical problem/situation, we often find ourselves thinking "I Don't Know What To Do Here ☹" and giving up. But if we push through this and experiment, we will find that we often are able to prevail.

A Digression: Sums Of Powers

(a) Find a nice, simple formula for

$$1 + 2 + \cdots + n$$

where n is any positive whole number. Prove your answer.

(b) Can you find and prove a nice, simple formula for

$$1^3 + 2^3 + \cdots + n^3$$

where n is any positive whole number?

Lecture 11 (2019-02-06)

Interpreting The Derivative:

Exercise. Let $p(t)$ denote the price of gas in Ann Arbor, in cents per gallon, t days after January 1st, 2019. Suppose you are an applied mathematician who has come up with an accurate model for the fluctuating gas prices and has determined in your model that $p'(7) = 25$. How would you offer a practical interpretation of this fact to someone who does not know calculus?

If f is a differentiable function, the statement $f'(a) = b$ indicates that for inputs x *near* $x = a$, the function f is well approximated by the linear function with slope b passing through the point $(a, f(a))$.

Another way of restating this interpretation is to say: $f'(a) = b$ means that for all *sufficiently small* h, the difference $f(a+h) - f(a)$ is approximately directly proportional to h, with a proportionality factor of b.

Slogan: Differentiable functions are locally well-approximable by linear functions. You can use the values of these linear approximations to practically interpret derivatives.

An important point to keep in mind here is **how near is near enough**? In other words, if $f'(a) = b$, how close must x be to a in order for the linear approximation at a to be reasonable? This depends very much on the function itself, and in particular, on the rate of change of the rate of change (i.e., the derivative of f', as a function).

Example. Consider the function $f(x) = x + x^2$ and the function

$$g(x) = \begin{cases} 1, & x \in [-2\pi, 2\pi] \\ \cos(x), & x \in (-\infty, -2\pi) \cup (2\pi, \infty) \end{cases}$$

Note that $f'(0) = 1$ (why? Make sure you can carry out this calculation). In fact, looking at a graph of $f(x)$, the linear function x seems to be a decent approximation for $f(x)$ on a fairly big interval around 0, e.g. for all $x \in (-0.5, 0.5)$. This is because as $|x| \to 0$, the value of x^2 gets close to 0 much faster than the value of x does.

However, consider the functions $h_k(x) = f(x)g(kx)$ for large k (I will show a picture in class). When k is very big, the interval around 0 for which x is a decent approximation to $h_k(x)$ can be made as small as desired.

The point here is that the answer to the question *how near is near enough* depends strongly on the function we're studying. In particular, when the function comes from a real life context, it's important to take this context into account.

Example (Example/Exercise)**.** Let $g(v)$ denote the fuel efficiency, in miles per gallon, of a car going at speed v miles per hour. The units of $g'(v)$ are (miles per gallon) per (miles per hour), which can also be thought of as hours per gallon.

What is the practical meaning of $g'(55) = -0.54$? This example is from a multiple choice textbook problem, which indicates that more than one of the choices may be reasonable. Before we even look at the choices, let's think about what the equation means. It tells us that when the car is going 55 mph, the instantaneous rate of change of the fuel efficiency is -0.54 mpg/mph. This can be interpreted as suggesting that, if we were to wiggle the input 55 into $55 + \epsilon$ for some sufficently small number ϵ, then

$$g(55 + \epsilon) - g(5)) \approx -0.54\epsilon$$

So, for instance, assuming $\epsilon = .5$ is small enough for this approximation to be valid, then we may say

$$g(55.5) - g(55) \approx -0.27$$

In words, increasing speed from 55 mph to 55.5 mph reduces fuel efficiency by approximately 0.27 mpg. Similarly, if $\epsilon = 1$ is small enough for the approximation to be valid, then we may asay

$$g(56) - g(55) \approx -0.54$$

In words, increasing speed from 55 mph to 56 mph reduces fuel efficiency by approximately 0.54 mph.

Now, let's go through all the choices carefully:

(a) When the car is going 55 mph, the rate of change of the fuel efficiency decreases *to* approximately 0.54 miles/gal.

(b) When the car is going 55 mph, the rate of change of the fuel efficiency decreases *by* approximately 0.54 miles/gal.

(c) If the car speeds up from 55 to 56 mph, then the fuel efficiency is approximately -0.54 miles/gal.

(d) If the car speeds up from 55 to 56 mph, then the car becomes less fuel efficient by approximately 0.54 miles/gal.

Here, (d) is exactly the fact that we discovered above, so **statement** (d) **is a reasonable/correct interpretation** of the knowledge that $g'(55) = -.54$. How about statements (a), (b), and (c)? In fact, they are all **incorrect**. Why?

What Does $f' > 0$ Mean? $f' < 0$?

Theorem. *Suppose f is a differentiable function and $f'(x) > 0$ for all $x \in (a, b)$ (where (a, b) is some open interval). Then f is increasing on the interval (a, b). Similarly, if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).*

Lecture 12 (2019-02-13)

The Second Derivative

Given a function f, the function f' gives information about the rate of change of f. Oftentimes (if f' is again differentiable), we can carry out this process once more and study the derivative of f', denoted f''. This f'' called the **second derivative of** f. It gives information about the rate of change of f', or equivalently, the rate of change *of the rate of change* of f.

This is a concept we have already thought a bit about. Recall that a function f is **concave up** on an interval (a, b) if its rate of change is increasing on (a, b). In other words, the function f' should be increasing on (a, b), and this is the same as asking for $f'' > 0$ on (a, b). This is very useful: we have a criterion for checking the *geometric* property of concavity by studying derivatives.

Proposition. *Suppose f is defined on (a, b) and twice-differentiable (i.e., both f and f' are differentiable). Then if $f'' > 0$ on (a, b), the graph of f is concave up on (a, b). If $f'' < 0$ on (a, b), the graph of f is concave down on (a, b).*

Conversely, if f is concave up on (a, b), then $f'' \geq 0$ on (a, b). If f is concave down on (a, b), then $f'' \leq 0$ on (a, b).

Remark. Note that $f'' > 0$ implies concave up, but concave up only implies $f'' \geq 0$. An example of a concave up function for which $f'' \geq 0$ is $f(x) = x^4$. The key point is that f'' does not change sign at $x = 0$, so the concavity of f does not change from up to down as we move past 0.

Example. Suppose $s(t)$ is a function that describes the position of a moving particle at time t. Then $s'(t)$ outputs the *velocity* of the particle at time t, and $s''(t)$ outputs the *acceleration* of the particle at time t.

Exercise. A headline in the New York Times on December 14, 2014 read

"A Steep Slide In Law School Enrollment Accelerates"

What function is the author talking about? Draw a possible graph for the function. In terms of derivatives, what is the headline saying?

Exercise. Give an example of a function f for which $f'(0) = 0$ but $f''(0) \neq 0$.

Application of Calculus: Newton's Method

The quadratic formula lets us calculate the solution to a quadratic equation explicitly. There is a similar cubic formula (which is very complicated) and quartic formula (which I think is less complicated than the cubic formula?). But a famous theorem of Abel proves that there does not exist a quintic formula, or a sextic formula, or a n-tic formula for any $n \geq 5$.

If that's the case, how do we solve a degree 5 polynomial equation? For concreteness, consider the polynomial function $f(x) = x^5 - x + 1$. The polynomial $f(x)$ has a root in the interval

$(-2, -1)$. (Why? Hint: the intermediate value theorem.) In fact, this is the *only* root of f (Why? this is a little trickier and requires calculating a derivative). Call this unique root r.

How can we figure out what r is? (If your instinct is to say "use the equation solver on my calculator," ask yourself: how does a calculator figure out what r is? After all, there isn't a quintic formula to plug f into)

Newton's amazing idea for solving equations like this was via *iterated linear approximation*. Here's the idea: start with a guess for r. We know it's between -2 and -1, so let's just guess $r_0 = -1.5$. There is a linear approximation to f at r_0, call it ℓ_0. This ℓ_0 has a root, which we call r_1. Then, we find the linear approximation at r_1 and call it ℓ_1. Then, we find the root of ℓ_1 and call it r_2, and so on. We get a sequence $r_0, r_1, r_2, r_3 \cdots$ which converges to r! (In class, I will demonstrate this procedure with numbers and pictures.)

Lecture 13 (2019-02-15)

At this point, we are hopefully convinced that derivatives of functions are interesting/useful things to study. However, beyond very basic examples, we have not actually *calculated* many derivatives. Currently, the only method we really have for computing the derivative of a function is to plug the function into the limit definition and try to figure out the limit. This can get unwieldy very fast, as the functions we're interested in computing with increase in complexity. This lack of computational toolkit is a pretty glaring problem; after all, in order to obtain any useful information from the derivative of a function, you need to know what the derivative of the function is!

Our goal for the next few weeks (i.e., chapter 3) is to construct a basic toolkit that we can use to explicitly calculate derivatives of lots of functions. The toolkit has two main parts:

1. We will calculate the derivatives of some basic functions using the limit definition. By basic functions, I mean things like power functions, sin, cos, e^x, etc, which are used to build more complicated functions like rational functions, general sinusoidal functions, etc.

2. We will have theorems that allow us to *combine derivatives* of basic functions to obtain the *derivative of a combination* of basic functions. By combination of basic functions, I mean operations like compositions of functions, product/quotients of functions, etc.

To be especially clear and pedantic, I'll indicate when we're working on part 1 or part 2 of our project by writing the number and circling/bolding it. That said, let's get started.

Some Rules For Derivatives Of Combinations (2)

Notation. I want to introduce something called Leibniz notation, which is a very useful notation when working with derivatives of functions. Assume that, as is often the case, we are referring to the input variable of a function as x. We introduce an *operator* called "d/dx", which should be thought of as a machine that eats functions and spits out new functions (a function-valued function!). In particular, d/dx eats a function $f(x)$ and spits out its derivative $f'(x)$. This is written

$$\frac{d}{dx}[f(x)] = f'(x)$$

This is often further consolidated: the output of the operator d/dx after inputting any function f is written df/dx, so we have

$$\frac{df}{dx} = f'$$

In particular, df/dx is just another notation for the *function* f'.

Important remark: The Leibniz notation conceptually emphasizes the fact that the derivative measures the change in output "df" relative to an infinitesimal change in input "dx" near some particular input into the original function.

Proposition. *Throughout, assume f and g are differentiable functions.*

- *(Sum Rule)*
$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$
In words, the derivative of the sum of two functions equals the sum of the derivatives.

- *(Scaling Rule) Suppose c is any constant. Then*
$$\frac{d(cf)}{dx} = c\frac{df}{dx}$$
In words, the derivative of a scaled function is just the derivative of the original function, scaled.

- *(Product Rule)*
$$\frac{d(fg)}{dx} = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$
Another way to write this, using the usual notation f' for the derivative of f, is
$$\frac{d(fg)}{dx}(a) = f'(a)g(a) + f(a)g'(a)$$
for any input a.

Next time, I'll try to explain a bit why the product rule is true. For now, let's just apply it to **(1)** of our grand project.

The Power Rule For Positive Integer Powers (1)

Let's use the product rule to calculate the derivative of the function $f(x) = x^2$. We can certainly write f as a product of functions, namely, if we set $g(x) = x$, then $f(x) = g(x)^2$. Thus, the product rule tells us that

$$f'(x) = g'(x)g(x) + g(x)g'(x) = 1 \cdot x + x \cdot 1 = 2x$$

This is exactly the answer we obtained for $f'(x)$ using the limit definition of the derivative!

Now, let's go further: suppose we want to calculate the derivative of $h(x) = x^3$. We can write this as a product, i.e. $h(x) = f(x)g(x)$. Thus, the product rule says

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 2x \cdot x + x^2 \cdot 1 = 3x^2$$

Again! Let's calculate the derivative of $k(x) = x^4$. Again, this is a product, namely $k(x) = h(x)g(x)$, and then the product rule says

$$k'(x) = h'(x)g(x) + h(x)g'(x) = 3x^2 \cdot x + x^3 \cdot 1 = 4x^3$$

We seem to be obtaining a pattern:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

In fact, this is true, and it is called the **power rule**

Proposition (Power Rule)**.** *Let $n > 0$ be an integer. Then*

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof. We can actually prove this by extending the same calculation we did above; we used the derivative of x to calculate the derivative of x^2, we used the derivative of x^2 to calculate the derivative of x^3, we used the derivative of x^3 to calculate the derivative of x^4. In fact, you can keep going. Here are the details:

Suppose we already *know* that the derivative of x^n is nx^{n-1}. Then $x^{n+1} = x^n \cdot x$, so using the product rule, we have

$$\frac{d(x^{n+1})}{dx} = \frac{d(x^n)}{dx} \cdot x + x^n \cdot \frac{dx}{dx} = nx^{n-1} \cdot x + x^n \cdot 1 = (n+1)x^n$$

Thus, if the derivative of x^n is what the power rule predicts, then so is the derivative of x^{n+1}. Since we know the power rule is correct for the derivative of x^1, this argument shows that it's correct for x^2, x^3, x^4, x^5, and so on forever!

(This "knocking over all the dominoes argument" is an extremely important technique called **mathematical induction**.) □

Example. We now have enough technology to calculate the derivative of any polynomial function. This is because polynomials are just sums of scaled copies of the power functions x^n for $n \geq 0$, so we can use the Sum Rule, Scaling Rule, and Power Rule in tandem to calculate the derivative. For example,

$$\frac{d}{dx}[2x^3 - 7x + 2] = 2\frac{d}{dx}[x^3] - 7\frac{d}{dx}[x] + \frac{d}{dx}[2] = 2 \cdot 3x^2 - 7 \cdot 1 + 0 = 6x^2 - 7$$

For the rest of class, let's use this technology to study the graphs of polynomial functions.

Exercise. Consider the polynomial function $p(x) = x^3 - 3x + 1$.

(a) At what points on the graph of p is the slope of the tangent line equal to 45?

(b) On what intervals is p increasing? On what intervals is it decreasing?

(c) On what intervals is p concave up? On what intervals is it concave down?

Exercise. Consider the polynomial function $q(x) = x^2 - 2x + 4$. Find the equations of the lines through the origin that are tangent to the graph of q.

Exercise. Consider the polynomial function $r(x) = (x-1)(x+0.5)(x+1.5)($. There is a unique line that is tangent to the graph of r at *two* distinct points. Find the equation of this line.

(You may have to use a calculator/computer!)

Lecture 14 (2019-02-19)

Today, we start off by working on (**2**) of our differentiation toolkit project.

Why Is The Product Rule True?

Recall that if f and g are differentiable functions, then the *product rule* says that

$$\frac{d(fg)}{dx} = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

Why is this true? Recall that for any number a, we have linear approximations to f and g near a. These linear approximations say that for sufficiently small h, we have

$$f(a+h) \approx f(a) + f'(a)h$$

$$g(a+h) \approx g(a) + g'(a)h$$

Let $k(x) = f(x)g(x)$. Multiplying these, we get an approximation for k near $x = a$, namely

$$f(a+h)g(a+h) \approx f(a)g(a) + (f(a)g'(a) + f'(a)g(a))h + f'(a)g'(a)h^2$$

$$\implies k(a+h) \approx k(a) + (f(a)g'(a) + f'(a)g(a))h + f'(a)g'(a)h^2$$

for small h. (Look at that coefficient of h!) We can use this approximation to calculate $k'(a)$ using the limit definition of the derivative:

$$k'(a) = \lim_{h \to 0} \frac{k(a+h) - k(a)}{h} \approx \lim_{h \to 0} \frac{(f(a)g'(a) + f'(a)g(a))h + f'(a)g'(a)h^2}{h} = f(a)g'(a)+f'(a)g(a)$$

The Quotient Rule

The product rule tells us how to calculate the derivative of a product of functions (in terms of the original functions and their derivatives). There is a similar rule that tells us how to differentiate *quotients* of functions, which is aptly named the quotient rule:

Proposition (Quotient Rule)**.** *Suppose f and g are differentiable. Away from points where f/g is undefined, we have*

$$\frac{d(f/g)}{dx} = \frac{\frac{df}{dx} \cdot g - f \cdot \frac{dg}{dx}}{g^2}$$

Proof. We can deduce the quotient rule from the product rule. Let $h(x) = f(x)/g(x)$. We want to calculate $h'(x)$ in terms of f, g, f', and g'. Writing $f(x) = h(x)g(x)$, we can now apply the product rule to obtain

$$f'(x) = h'(x)g(x) + h(x)g'(x)$$

$$\implies h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)} = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

as desired. □

Example (Power Rule Holds For All Integer Exponents)**.** Last class, we showed that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

when n is a positive integer. In fact, this is true even when n is a negative integer. This is an immediate consequence of the quotient rule. We are interested in calculating the derivative of x^{-n} when $n > 0$. Rewriting $x^{-n} = 1/x^n$, note that we can represent x^{-n} as the quotient of two functions whose derivatives we already know, namely 1 and x^n. It follows that

$$\frac{d}{dx}[x^{-n}] = \frac{\frac{d(1)}{dx} \cdot x^n - 1 \cdot \frac{d(x^n)}{dx}}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

which is exactly what the power rule would predict!

Example. Now that we know the quotient rule, it is possible to calculate the derivative of any *rational function*, since rational functions are just quotients of polynomials. Here's a quick example:

$$\frac{d}{dx}\left[\frac{x+1}{2x-1}\right] = \frac{1 \cdot (2x-1) - 2(x+1)}{(2x-1)^2} = \frac{1}{(2x-1)^2}$$

The Chain Rule

The chain rule tells you how to calculate the derivative of the *composite* of two functions in terms of the original functions and their derivatives. Let's try to figure out what this rule should be, using linear approximation. Suppose f and g are differentiable functions, and we are interested in differentiating $k(x) = f(g(x))$ at $x = a$. Near a, we have the linear approximation

$$g(a+h) \approx g(a) + g'(a)h$$

Similarly, near $g(a)$, we have the linear approximation

$$f(g(a)+h) \approx f(g(a)) + f'(g(a))h$$

for sufficiently small h. It follows that

$$k'(a) = \lim_{h \to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

$$\approx \lim_{h \to 0} \frac{f(g(a) + g'(a)h) - f(g(a))}{h} \approx \lim_{h \to 0} \frac{f(g(a)) + f'(g(a))g'(a)h - f(g(a))}{h} = f'(g(a))g'(a)$$

We have therefore "proven" the:

Proposition (Chain Rule)**.** *If f and g are differentiable functions, then*

$$\frac{d(f \circ g)}{dx}(a) = \frac{df}{dx}(g(a)) \cdot \frac{dg}{dx}(a)$$

Another (less unwieldy) way of expressing this rule is to say that the derivative of $(f \circ g)(x)$ is $f'(g(x))g'(x)$.

Example (Power Rule Holds For All Rational Exponents)**.** At this point, we know the power rule

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

is true whenever n is an integer. In fact, this is true when n is any *rational* number, and we can deduce this quickly using the chain rule. Given a rational number n, write $n = p/q$ where p and q are integers. We want to differentiate $f(x) = x^n = x^{p/q}$. Taking the qth powers of both sides, we see that $f(x)^q = x^p$. Now, the left hand side can be expressed as $h(f(x))$ where $h(x) = x^q$, so the chain rule implies that its derivative is

$$\frac{d}{dx}\,[\text{left hand side}] = h'(f(x))f'(x) = q(f(x))^{q-1}f'(x)$$

Furthermore,

$$\frac{d}{dx}\,[\text{right hand side}] = px^{p-1}$$

It follows that

$$f'(x) = \frac{px^{p-1}}{q(f(x))^{q-1}} = \frac{px^{p-1}}{qx^{p(q-1)/q}} = \frac{p}{q}x^{p-1-p+(p/q)} = \frac{p}{q}x^{(p/q)-1} = nx^{n-1}$$

Example. We now know the power rule holds for any rational exponents. Using this fact in tandem with the chain rule, we can differentiate any function that involves only radicals and power functions. For example, consider the function $f(x) = \sqrt[3]{x^2+1}$. We can write this as a composition $f(x) = k(\ell(x))$ where $k(x) = \sqrt[3]{x}$ and $\ell(x) = x^2+1$. We know $k'(x) = (1/3)x^{-2/3}$ using the power rule for rational exponents. Thus, the chain rule implies

$$f'(x) = k'(\ell(x))\ell'(x) = \frac{1}{3}(x^2+1)^{-2/3} \cdot 2x$$

Let's finish off today with a bit of work on (**1**).

Derivatives of Exponential Functions

We want to differentiate the exponential function a^x. Recall that we can change bases and write $a^x = e^{\ln(a)x}$. Write $f(x) = e^x$, so that $a^x = f(\ln(a)x)$. Now, by the chain rule, it follows that

$$\frac{d}{dx}[a^x] = f'(\ln(a)x) \cdot \ln(a)$$

so to calculate the derivative of a^x for any exponential base a, it will suffice to know the derivative of e^x.

Let's try to calculate this derivative using the limit definition:

$$f'(x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} =: Le^x$$

Exercise: Estimate L and make a conjecture as to its exact value.

Lecture 15 (2019-02-20)

Expectations for Individual Meetings

I would like the individual meetings to be a way to discuss your performance on the first exam and your comfort with the material thus far, making a plan for moving forward if necessary. To prepare for the meeting, please review your first exam and be ready to explain what mistakes you made. Grading for these meetings will be solely participation-based (as in, show up and be ready to discuss the midterm).

I'll start by writing down some of the material we covered last class but that wasn't in the notes.

Derivatives of Logarithmic Functions

The *key fact* from which you can calculate the derivative of any exponential function is that

$$\boxed{\frac{d}{dx}[e^x] = e^x}$$

i.e. the function e^x is its own derivative. Similarly, recall that for any a, we have the change of base formula $\log_a(x) = \ln(x)/\ln(a)$. This implies that

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{\ln(a)}\frac{d}{dx}[\ln(x)]$$

so to calculate the derivative of any logarithmic function, it will suffice to figure out what the derivative of $\ln(x)$ is. To do this, recall that $\ln(x)$ is defined as the inverse of the exponential function e^x, so in particular satisfies the equation

$$e^{\ln(x)} = x$$

Differentiating both sides of this equation (and applying chain rule to do so) yields the equation

$$x \cdot \frac{d}{dx}[\ln(x)] = e^{\ln(x)} \cdot \frac{d}{dx}[\ln(x)] = \frac{d}{dx}[e^{\ln(x)}] = \frac{d}{dx}[x] = 1$$

It follows that

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

(Note that the derivative of $\ln(x)$, which is related to exponential/logarithmic functions, is a power function! This is a surprising and extremely important fact in math, and people have built careers off of exploiting this relationship.) Therefore, we have

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$$

Derivatives of Inverse Functions

To calculate the derivative of $\ln(x)$, we used the equation that defines it as the inverse of e^x, namely $e^{\ln(x)} = x$. More generally, if f is any invertible function, we have the defining equation $f(f^{-1}(x)) = x$. Just as above, we can differentiate this equation and obtain a general formula for the derivative of the inverse of a function:

$$f'(f^{-1}(x))(f^{-1})'(x) = \frac{d}{dx}[f(f^{-1}(x))] = \frac{d}{dx}[x] = 1$$

$$\implies (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

I would highly recommend that you do **not** memorize this formula, as it is easy to mix up the $'$ and $^{-1}$. Instead, understand that it comes from differentiating the defining relation of an inverse function.

Derivatives of Trigonometric Functions

Today, we will continue (**1**) by figuring out the derivatives of trigonometric functions. Since these are all built out of sin and cos, we need to calculate the derivatives of sin and cos and then use our rules to compute the derivatives of functions "built out of" sin and cos (e.g. tan, arcsin, etc). In fact, since cos is just a transformed copy of sin, it will suffice to calculate the derivative of sin by hand.

To do this, we use the geometry of the unit circle. Suppose θ is an angle and $\theta + h$ is a small perturbation of this angle. Let A be the point on the unit circle at an angle of θ (measured, as usual, counterclockwise from the positive x-axis) and let B be the point at an angle of $\theta + h$. Recall that $\sin(\theta)$ is the y-coordinate of A and $\sin(\theta + h)$ is the y-coordinate of B. Thus, $\sin(\theta + h) - \sin(\theta)$ is the length FB in the image below. Note that the length of the circular arc BA is exactly h, and as h gets smaller, this arc more and more closely resembles a straight line segment. Thus, when h is very small, we have

$$\frac{\sin(\theta + h) - \sin(\theta)}{h} \approx \sin(\angle BAF)$$

Moreover, when h is very small, the angle $\angle BAC$ gets very close to 90°, so

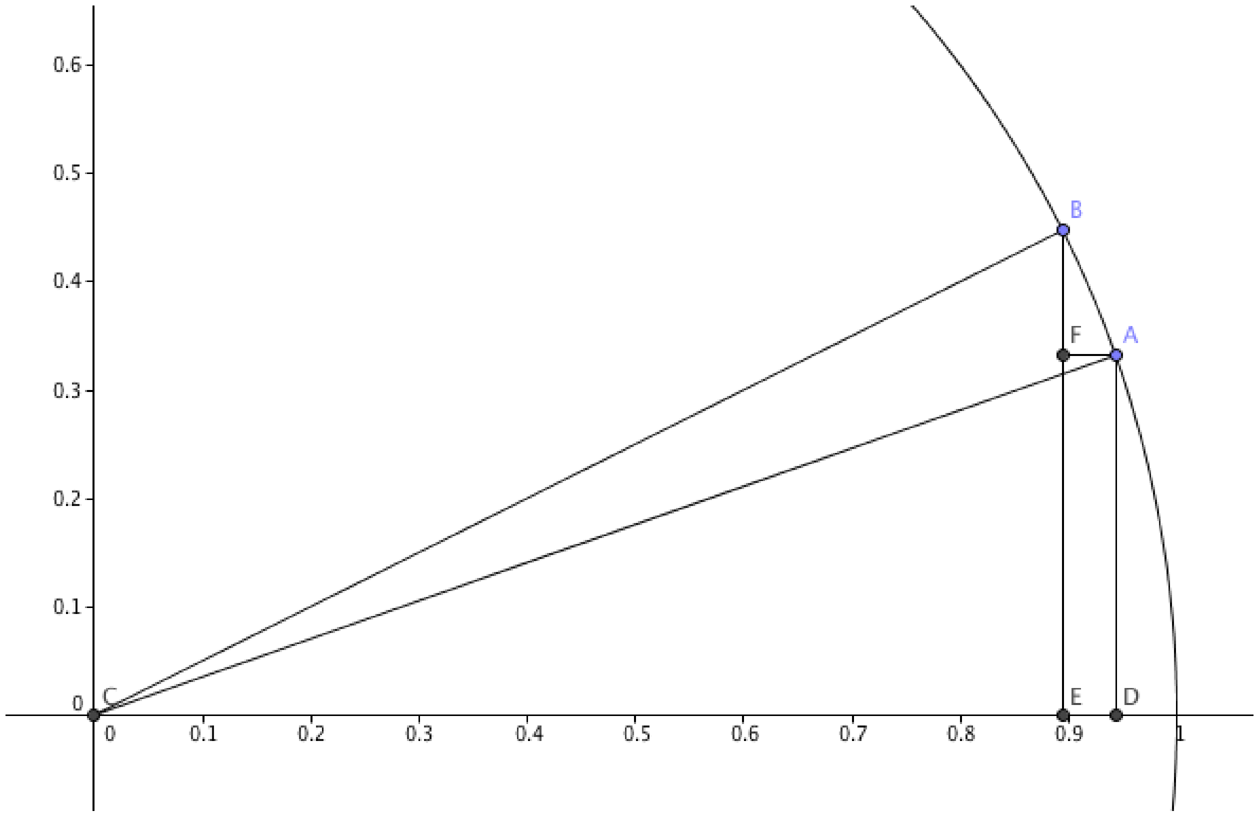
$$\angle BAF = \angle BAC - \angle FAC = \angle BAC - \angle ACD = \angle FAC - \theta$$

gets very close to $\frac{\pi}{2} - \theta$ (when measured in radians). Thus, we see that

$$\lim_{h \to 0} \frac{\sin(\theta + h) - \sin(\theta)}{h} = \sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

We conclude that

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$



Exercise (Important!)**.** Now that you know the derivative of $\sin(x)$, use the derivative rules to calculate the derivatives of the remaining trigonometric functions and their inverses:

- $\cos(x)$
- $\tan(x)$
- $\arcsin(x)$
- $\arccos(x)$
- $\arctan(x)$

Grab Bag Of Exercises

Exercise. How many solutions does the equation $\sin(x) = x/1000$ have on the interval $[0, 1000\pi]$?

Exercise. The function $g(x) = x^x$ is defined and differentiable on the interval $(0, \infty)$. Calculate $g'(x)$.

Exercise. Is the function $f(x) = x \sin\left(\frac{1}{x}\right)$ differentiable at $x = 0$? How about the function $g(x) = x^2 \sin\left(\frac{1}{x}\right)$?

Lecture 16 (2019-02-22)

A **plane curve** is the set of points in the coordinate plane determined by an equation of the form $f(x, y) = 0$. For example, if $f(x, y) = x^2 + y^2 - 1$, then the corresponding plane curve is the set of points whose coordinates satisfy $x^2 + y^2 = 1$, i.e. the unit circle.

We are interested in studying plane curves for two reasons:

- They are a vast class of examples with lots of interesting geometry.
- Oftentimes in real life, quantities of interest are constrained to satisfy particular equations. To visualize these quantities, one would then study the corresponding plane curves. For example, if a bug was moving on the unit circle, its coordinates (x, y) at any point in time would have to satisfy $x^2 + y^2 = 1$.

The key point is that by suitably restricting their domains (I won't get into this, but ask me if you're curious), the quantities x and y that are constrained to lie on the plane curve $f(x, y) = 0$ may be thought of as functions of one another, i.e. $y = y(x)$ or $x = x(y)$. More precisely, when the domain of x is suitably restricted, for each x in the restricted domain there is a unique $y = y(x)$ such that $f(x, y(x)) = 0$; we can then study this function $x \mapsto y(x)$.

Example. Suppose we are studying the plane curve $x^2 + y^2 = 4$. This curve includes the point $(\sqrt{2}, \sqrt{2})$. What is the tangent line to the curve at this point?

Thinking of y as a function of x (in some small neighborhood of $x = \sqrt{2}$), we have the equation

$$x^2 + y(x)^2 = 4$$

Differentiating both sides of this equation with respect to x, we obtain

$$2x + 2y(x)y'(x) = 0 \implies y'(x) = -\frac{x}{y(x)}$$

Thus, $y'(\sqrt{2}) = -\sqrt{2}/y(\sqrt{2}) = -\sqrt{2}/\sqrt{2} = -1$, so the slope of the tangent line is -1. Thus, its equation is

$$y - \sqrt{2} = -(x - \sqrt{2}) \implies y = -x + 2\sqrt{2}$$

This technique for *differentiating* a function that is *implicitly* defined by a constraint equation is called **implicit differentiation**.

Exercise. The curve defined by the equation $y^2 - x^3 + x = 0$ is called an **elliptic curve** and is extremely important in mathematics.

(*a*) There are two points on this curve with x-coordinate 2. What are they?

(*b*) Find the slope of the tangent line to the curve at the point $(2, a)$ where $(2, a)$ is the unique point on the curve with x-coordinate 2 and $a > 0$.

(*c*) At what points is the tangent line to the elliptic curve vertical? Horizontal?

(*d*) Sketch a graph of the elliptic curve.

Exercise. The curve defined by the equation $x^{2/3} + y^{2/3} = 2$ is called an **astroid**.

There is a unique point P on the astroid which is in the second quadrant and has an x-coordinate of $-1/8$. What is the slope of the tangent line to the astroid at P?

Exercise. Investigate the plane curve given by equation $x^3 + y^3 = 3xy - 1$. Calculate the slopes of the tangent lines at various points on this curve (use a calculator if necessary). Do you notice anything interesting? Can you explain your observations?

Lecture 17 (2019-02-26)

Activity/Exercise/Review: Identify the Plane Curves

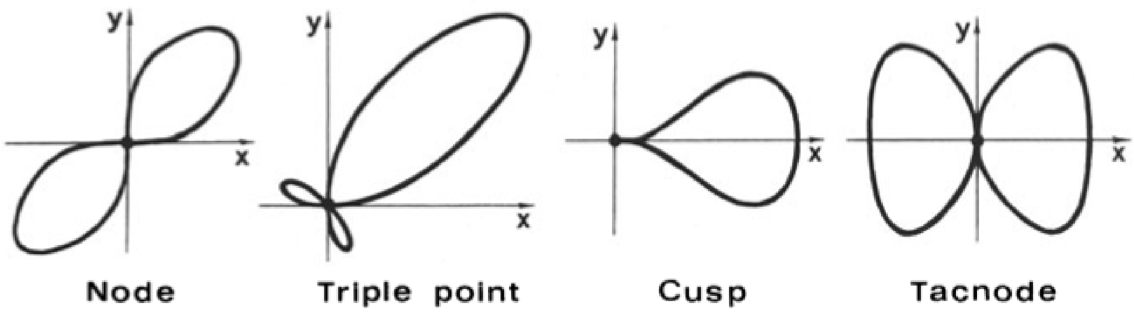


Figure 4. Singularities of plane curves.

The pictures shown above depict the curves defined by the equations below, in some order.

Curve A: $x^2 = x^4 + y^4$

Curve B: $xy = x^6 + y^6$

Curve C: $x^3 = y^2 + x^4 + y^4$

Curve D: $x^2y + xy^2 = x^4 + y^4$

(a) The point $P = (1, 1)$ is on exactly one of the four curves. Identify which curve passes through P and use implicit differentiation to calculate the tangent line to the curve at P.

(b) Match the four curves depicted to their equations.

Quadratic Approximation

This is a topic that is not covered in the textbook, but *will* be tested on exams. There are (or will be) official notes on this topic posted on the course website; make sure to read them!

Suppose f is a differentiable function. Recall that for every a, there is a linear function $\ell_a(x)$ that provides a good approximation to f for inputs close to a. We know by now that ℓ_a is the linear function whose graph is the tangent line to f at a.

A very important/useful way to think about ℓ_a is that it is *the linear function f would have to be if all you knew was $f(a)$ and $f'(a)$*. In other words, we know f is not (necessarily) linear, but let's pretend that it is. Then we would have

$$f(x) = Cx + D$$

If we knew the values of $f(a)$ and $f'(a)$, then it would have to be the case that $C = f'(a)$ and $f(a) = Ca + D$. Thus, pretending that f is linear, we would require that

$$f(x) = f'(a)x + f(a) - af'(a) = f'(a)(x - a) + f(a)$$

which is exactly the linear approximation ℓ_a to f at a!

Recap: if we pretend f is linear, then the "local" information $f(a)$ and $f'(a)$ would imply that $f = \ell_a$.

We can carry out an entirely analogous procedure where we pretend that f is quadratic and ask, based on information coming from $x = a$, what quadratic does f have to be?

Let's carry out this procedure in the case where $a = 0$. Pretend f is a quadratic function, i.e.

$$f(x) = Cx^2 + Dx + E$$

and suppose we know the values $f(0), f'(0)$, and $f''(0)$. It follows that $E = f(0)$, $D = f'(0)$, and $C = f''(0)/2$.

Thus, *if f was a quadratic function compatible with the given values of $f(0), f'(0)$, and $f''(0)$, it would have to equal* $(f''(0)/2)x^2 + f'(0)x + f(0)$. This is the "best quadratic function" that fits the "second order information" coming from f.

More generally, let a be any number. Near 0, the function $g(x) = f(x + a)$ looks exactly like f does near a; in particular, $g(0) = f(a)$, $g'(0) = f'(a)$, and $g''(0) = f''(a)$. As we saw above, the quadratic best approximating g near 0 is given by

$$(g''(0)/2)x^2 + g'(0)x + g(0) = (f''(a)/2)x^2 + f'(a)x + f(a)$$

Thus, the quadratic best approximating f near a should be the above function, but translated a units to the right.

Putting all this together, we have:

Fact. The quadratic approximation to $f(x)$ near $x = a$ is given by the formula

$$\boxed{q_a(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2}$$

In particular, note that

$$q_a(x) = \ell_a(x) + \frac{f''(a)}{2}(x - a)^2$$

i.e. it is the linear approximation to f at a, plus a "quadratic correction" term.

Exercise. Consider the function $f(x) = \ln(x)$.

(a) Is the linear approximation to $f(x)$ near $(1, 0)$ an overestimate or an underestimate (or neither) for the actual values of $f(x)$?

(b) Use the linear approximation for $f(x)$ near $(1, 0)$ to estimate $\ln(1.1)$. Then, use the quadratic approximation for $f(x)$ near $(1, 0)$ to estimate $\ln(1.1)$. Is the quadratic estimate better than the linear estimate?

Lecture 18 (2019-02-27)

Some Review on Linear and Quadratic Approximation

Exercise. 1. Let $g(s) = cos(s)$. Calculate the linear and quadratic approximations to g at $s = \pi/2$. Use these approximations to estimate $\cos(1.6)$.

2. Suppose z is a twice-differentiable function with $z(1) = 5$, $z'(1) = -2$, and $z''(1) = 3$. Estimate $z(0.9)$ using the linear approximation to z at 1. Do you expect this value will be an over-estimate or under-estimate? Explain your reasoning.

3. Suppose f is a differentiable function, and let q be the quadratic approximation to f at $x = 1$. Given that $q(1) = 1$, $q(2) = -2$, and $q(3) = 4$, find $f(1)$, $f'(1)$, and $f''(1)$.

The Mean Value Theorem

The mean value theorem is intuitively ‘‘obvious’’, but important on account of encapsulating the obvious intuition in a piece of rigorous mathematics.

Theorem (Mean value Theorem)**.** *Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, the average rate of change of f on $[a, b]$ is attained by the derivative f' at some point in (a, b). More precisely, there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example. Consider the function $f(x) = x^2 - 2x + 3$. On the interval $[0, 3]$, the average rate of change of f is

$$\frac{f(3) - f(0)}{3 - 0} = \frac{6 - 3}{3 - 0} = 1$$

so the mean value theorem predicts that there must exist some $c \in [0, 3]$ such that $f'(c) = 1$. Indeed, we can calculate $f'(x) = 2x - 2$; this equals 1 precisely when $x = 1.5$, which is certainly in the interval $[0, 3]$. Thus, the average rate of change of f on $[0, 3]$ is attained as an instantaneous rate of change at a ‘‘moment’’ in $[0, 3]$.

Example (A Fable)**.** The following story is definitely not true, but calculus teachers often like to claim that it is, because it makes them feel powerful.

A couple of years ago, I drove up to the Bay Area, which is 400 miles, and I drove fast, so it took me five hours. At the end of the trip, I slowed down, because I didn’t want to get a ticket, and when I got off the freeway, I was traveling at the speed limit. Then a police officer pulled me over, and he said, ”You were going a little fast there.” I said I was going the speed limit, but he responded, ”Maybe you were a little while ago, but earlier, you were speeding.” I asked how he knew that, and he said, ”Son, by the mean value theo rem of calculus, at some moment in the last five hours, you were going at *exactly* 80 m.p.h.”

I took the ticket to court, and when push came to shove, the officer was unable to prove the mean value theorem beyond a reasonable doubt.

Exercise (*based on the fable above*)**.** (*a*) Assuming the officer could prove the mean value theorem beyond a reasonable doubt, would his statement have been correct? Explain your reasoning.

(*b*) Suppose we change the ending of the story so that the officer said, "I can't prove the mean value theorem, your Honor, but I can prove the intermediate value theorem, and using this I can show that there was a time interval of exactly one minute during which the defendant drove at an average speed of 80 miles per hours." Explain the cop's reasoning.

Exercise. (*a*) Let $k(i) = i^2 + 7i + 5$. Find all values $c \in (-1, 2)$ at which h satisfies the conclusion of the mean value theorem for the interval $[-1, 2]$.

(*b*) Let $h(y) = \sin(y)$. Find all values $c \in (0, 5\pi/2)$ at which h satisfies the conclusion of the mean value theorem for the interval $[0, 5\pi/2]$.

Grab Bag Of Chapter 3 Review Exercises

Exercise. The function $g : \mathbb{R} \to \mathbb{R}$ is differentiable and has a differentiable inverse. If $(g^{-1})'(3) = 2$ and $g(2) = 3$, what is $g'(2)$?

Exercise. The *nth Chebyshev polynomial* is the unique polynomial T_n such that $T_n(\cos(\theta)) = \cos(n\theta)$ for all $\theta \in [0, 2\pi]$. What is $T_{10}{}'(1/2)$?

Exercise. Suppose a is the unique solution in $(-\infty, \infty)$ to the equation $x = \cos(x)$. *Without* using a calculator, estimate the value of a.

Exercise. The picture below depicts three surfaces in three-dimensional space, and the equations that cut them out. Match the equations to the pictures.

(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

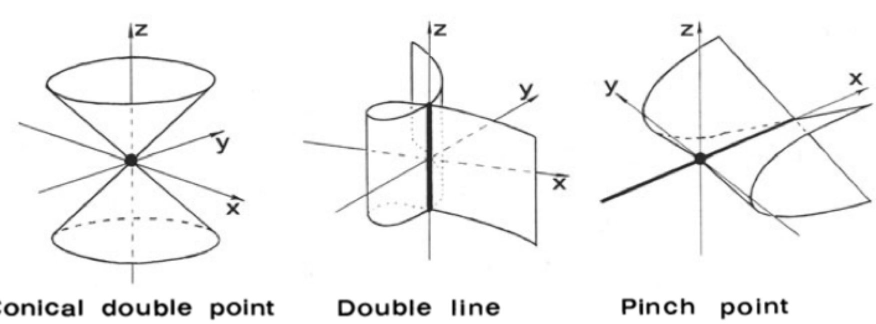


Figure 5. Surface singularities.

Exercise (Extra Credit -- tricky, but doable)**.** Calculate $f'''(0)$, where f is given by

$$f(x) = \underbrace{\sin(\sin(\cdots\sin(x)\cdots))}_{\text{2019 iterations of sine}}$$

Lecture 19 (2019-03-11)

Using Calculus For Optimization

In Chapter 4, our focus will be on **optimization** problems. From the perspective of many applications, optimization is essentially the raison d'être of calculus. The problem of optimization is, in a nutshell, to find the maximum/minimum values attained by a function of interest. For instance, a company interested in maximizing profits might develop a mathematical model for its activities then use the tools of calculus to find the parameters at which the company makes the most money (e.g., the optimal amount of labor to enlist, the optimal amount of initial capital to invest, etc.).

Definition. Let f be any function.

- We say f has a **local minimum** at $p \in \mathrm{Dom}(f)$ if $f(p) \leq f(p+h)$ for all sufficiently small h. In other words, the value of f at p is lower than all the nearby values.
- We say f has a **local maximum** at $p \in \mathrm{Dom}(f)$ if $f(p) \geq f(p+h)$ for all sufficiently small h. In other words, the value of f at p is higher than all the nearby values.
- We say f has a **local extremum** at $p \in \mathrm{Dom}(f)$ if f has either a local maximum or local minimum at p.

Definition. Let f be any function.

- We say f has a **global minimum** at $p \in \mathrm{Dom}(f)$ if $f(p) \leq f(x)$ for all $x \in \mathrm{Dom}(f)$. In other words, the value of f at p is lowest possible value attained by f.
- We say f has a **global maximum** at $p \in \mathrm{Dom}(f)$ if $f(p) \geq f(x)$ for all $x \in \mathrm{Dom}(f)$. In other words, the value of f at p is highest possible value attained by f.
- We say f has a **global extremum** at $p \in \mathrm{Dom}(f)$ if f has either a global maximum or global minimum at p.

Remark. Note that if f has a global minimum/maximum at p, then it certainly has a local minimum/maximum at p as well. However, a local minimum/maximum is *not* necessarily a global minimum/maximum.

Examples. (a) The function $s(t) = t^3 - t$ has a local maximum at $t = -1/\sqrt{3}$ but s does *not* have a global maximum, since as $t \to \infty$, we have $s(t) \to \infty$. Similarly, s has a local minimum at $t = 1/\sqrt{3}$ but has no global minimum. Note that the inputs $\pm 1/\sqrt{3}$ are precisely the *zeroes of the derivative* function $s'(t)$; studying the zeroes of the derivative is an important way to detect mins/maxes, as we'll see presently.

(b) The function $q(s) = 3 - |s-1|$ has a global maximum at $s = 1$. Note that this global maximum is attained at a point where q is *not differentiable*.

(c) Consider the function $\ell(q) = \lfloor q \rfloor$, where $\lfloor q \rfloor$ denotes the greatest integer less than or equal to q; in other words, ℓ is the "round down" function, with $\ell(3.4) = 3$, $\ell(5) = 5$, and $\ell(-2.3) = -3$. (We saw this function on a quiz a while ago!) Note that ℓ has no global extrema. However, by our definition of local min/max, the function ℓ has a local maximum at *every* input $q \in (-\infty, \infty)$! The local minima are precisely at the inputs that are not whole numbers.

Today, we will focus on how to use calculus to detect local maxima and minima. Here is the main theorem:

Theorem (Key Theorem On Local Extrema/First Derivative Test). *Suppose f is differentiable on some open interval I, and $p \in Dom(f)$.*

- *The function f has a local minimum at p if there is some $h > 0$ such that $f' < 0$ on the interval $(p-h, p)$ and $f' > 0$ on the interval $(p, p+h)$. In words, the sign of the derivative changes from negative to positive at p.*
- *The function f has a local maximum at p if there is some $h > 0$ such that $f' > 0$ on the interval $(p-h, p)$ and $f' < 0$ on the interval $(p, p+h)$. In words, the sign of the derivative changes from positive to negative at p.*

In particular, if f has a local extremum at p, then $f'(p) = 0$.

Remark (Warning). Even if $f'(p) = 0$, this does **not** imply that f must have a local extremum at p. For example, the function $g(s) = s^3$ has $g'(0) = 0$, but has neither a local maximum nor local minimum at 0. Indeed, the derivative does not *change sign* at 0.

Slogan: Local extrema occur where the derivative of a function changes sign.

Note that the key theorem tells us about the case where the function f is differentiable on the interval of interest. In this case, local extrema can only occur at points p with $f'(p) = 0$ (but again, every such point need not be a local extremum). On the other hand, we may be working with a function that is not differentiable at certain points; it is important to check these points as well to determine if they are local extrema.

Definition. Suppose f is a continuous function. A point $p \in \text{Dom}(f)$ is called a **critical point** if either:

- f is not differentiable at p
- f is differentiable at p and $f'(p) = 0$

Corollary (Corollary of Key Theorem). *If a function f has a local extremum at p, then p is a critical point of f.*

Exercise. Find all critical points of the following functions. Use this information to determine and classify their local extrema. Can you say anything about global extrema?

(*a*) $z(q) = q^3(q-1)^2$

(*b*) $\Theta(\beta) = \ln(\beta/\ln(\beta))$

(*c*) $t(x) = \sqrt{x + \frac{1}{\sqrt{x}}}$

(*d*) $x(h) = |h-1| - |h-2| + |h-3|$

Lecture 20 (2019-03-12)

Second Derivative Things

Yesterday, we talked about how to detect local extrema of functions using the first derivative test. If a function is in fact twice-differentiable, we can alternatively use the "second derivative test" to detect when a local extremum occurs and whether or not it is a local maximum or local minimum.

Theorem (Second Derivative Test)**.** *Suppose f is twice-differentiable on some interval I, and $p \in I$.*

- *If $f'(p) = 0$ and $f''(p) > 0$, then f has a local minimum at p.*
- *If $f'(p) = 0$ and $f''(p) < 0$, then f has a local maximum at p.*
- *If $f'(p) = 0$ and $f''(p) = 0$, then the test is inconclusive.*

Remark (Warning)**.** The second derivative test can be inconclusive, as in the example $f(x) = x^4$ at $x = 0$. Indeed, we then have $f'(0) = f''(0) = 0$. When the second derivative test is inconclusive, you should fall back on the first derivative test. In this case, $f'(x) = 4x^3$ so f' has a sign change at 0 from negative to positive. So the critical point at $x = 0$ is a local minimum.

Our interest in points where f' changes sign led us to study critical points. There is a similar notion of where f'' changes sign, called **inflection points**. Geometrically, this is a place where the graph of the function changes concavity (i.e. goes from concave up to concave down or concave down to concave up).

Definition. Let f be any function. A point p is an **inflection point** of f if the concavity of f changes at p.

If f is actually twice-differentiable, we can interpret this in terms of the second derivative:

- p is an inflection point if f'' changes sign at p.

In this situation, another useful rephrasing is to say that:

- p is an inflection point if f' has a local extremum at p.

Fact. If p is an inflection point of f, then $f''(p) = 0$. The converse is *not* necessarily true, e.g. $f(x) = x^4$ at $x = 0$.

Exercise. Study the function $f(x) = xe^{-x}$. Find its critical points and inflection points, determine where it is increasing/decreasing/concave up/concave down, and use all of this information to sketch an accurate graph of the function. (Don't use a graphing calculator!)

Global Optimization

The techniques we've discussed so far allow us to find the local extrema of a function. To find global extrema, we only need to work a little bit harder. The key point is that every global extremum of a function *is* a local extremum. Thus, to find the global extrema, we have the following heuristic procedure:

Heuristic Algorithm For Finding Global Extrema:

1. Find the critical points of your function. Use the first derivative/second derivative test to classify these critical points and determine the local extrema.
2. Compute the value of the function at the local extrema and pick out candidates for global maximum and global minimum (which, if they exist, must be among these local extrema).
3. Determine if the candidates for global max/min are actually global max/min by studying the end behavior of the function (i.e., the behavior as input values approach $\pm\infty$ or an endpoint of the domain).

Examples. (*a*) The global extrema of a function depend heavily on the domain of consideration. For example, consider the function $f(x) = x^2$. On the domain $[1, \infty)$ it attains a global minimum at $x = 1$ and has no global maximum. On the domain $[-0.5, 1]$ it attains a global maximum at $x = 1$ and global minimum at $x = 0$. On the domain $(-\infty, \infty)$ it attains a global minimum at $x = 0$ and has no global maximum.

(*b*) Let's work out an example in its entirety. We want to find the global extrema of the function

$$g(t) = \frac{e^{-t/2}}{1 + t^2}$$

Every global extremum has to be a local extremum, so let's start by finding the local extrema. We calculate

$$g'(t) = -\frac{e^{-t/2}(t^2 + 4t + 1)}{(1 + t^2)^2}$$

The zeroes of this function are at

$$t = -2 \pm \sqrt{3}$$

and these are the only critical points, since the function is everywhere differentiable. Call these zeroes $\alpha = -2 - \sqrt{3}$ and $\beta = -2 + \sqrt{3}$. Then we have

$$g'(t) = h(t)(t - \alpha)(t - \beta)$$

where $h(t)$ is some function that's always *negative*. Thus, we have that g' is

$$- \cdot - \cdot - = -$$

on $(-\infty, \beta)$, it's

$$- \cdot - \cdot + = +$$

on (β, α), and it's

$$- \cdot + \cdot + = -$$

on (α, ∞). So the critical point at β is a local minimum and the critical point at α is a local maximum (note that this is definitely a situation where we want to sue the first derivative test and not the second derivative test; calculating the derivative of g' looks like pain).

We now have to determine if the local extrema at α and β are global extrema. Since g was increasing on (β, α), we at least know $g(\beta) < g(\alpha)$. What's left is to figure out if the function ever takes on values lower than $g(\beta)$ or higher than $g(\alpha)$. Indeed, as $t \to \infty$, we have $g(t) \to 0$ (from above), so in particular it must eventually attain values lower than $g(\beta)$ (which is positive). Similarly, as $t \to -\infty$, we have $g(t) \to \infty$, so in particular it must attain values greater than $g(\alpha)$. So *neither* α nor β is a global extremum!

(c) Let's do another example. We want to find the global extrema of the function

$$h(x) = \begin{cases} -xe^x & x \leq 0 \\ -x & 0 \leq x \leq 4 \\ -\sqrt{x-1} - 4 + \sqrt{3} & 4 \leq x \leq 7 \end{cases}$$

Assume the domain of h is precisely $(-\infty, 7]$. First, we determine the critical points. On the intervals $(-\infty, 0)$, $(0, 4)$, and $(4, 7)$, the function is differentiable, so on those intervals we should just find where the derivative equals 0. On $(-\infty, 0)$ we find a critical point at $x = -1$, and this is a local max. On $(0, 4)$ and $(4, 7)$, the calculation reveals no critical points.

Now we should examine the points $x = 0$ and $x = 4$. At $x = 0$, the function is continuous and differentiable, so there is no critical point. At $x = 4$, the function is not differentiable, so there is a critical point, but a sign chart reveals that it is neither a local max or min; in fact, h is decreasing to the left and right of $x = 4$. Accordingly, on $[-1, 7]$ the function is decreasing, and on $(-\infty, -1)$ it is positive, so the global minimum occurs at the endpoint $x = 7$. The global maximum occurs at the critical point $x = -1$.

Lecture 21 (2019-03-12)

A Review Problem

Exercise. For this exercise, consider the function

$$g(x) = \begin{cases} e^x & x \leq 0 \\ -ex\ln(x) & 0 < x \leq 1 \\ C(\cos(\pi x) + 1) & x > 1 \end{cases}$$

where $C > 0$ is some positive constant.

(a) Where is g continuous? Where is g differentiable?

(b) Find the critical points of g, and use this information to find and classify the local extrema of g.

(c) Find the global maximum and minimum of g, if they exist. Your answer will depend on the value of C.

Optimization Problems Galore

Now that we have developed the techniques necessary to determine extrema of functions, the next step is to apply this technology to actual optimization problems. In this class, optimization problems are typically of the following form:

- You are given a function you wish to optimize. This function often depends on *more than one variable/quantity.*
- The input variables of the function of interest are required to satisfy certain *constraint* equations. This allows you to eliminate all but one of the variables in the function and treat the problem as a *single-variable optimization* problem.
- Once you have a single-variable function and a domain on which you are required to optimize it, you can go ahead and use single-variable calculus (i.e., the calculus we've been studying) to solve the problem.

In practice, this process can be very challenging! In particular, the tricky part is often figuring out to use your constraints to reduce to a single-variable optimization problem. Keep your wits about you.

Example. Alastair is trying to build a rectangular fence in his backyard in order to keep the neighbors' dogs from relieving themselves on his property. He has 500 meters of fencing material to use. What is the maximum area he can enclose with his fence?

Suppose x is the length of the fence and y is the width. We wish to maximize the quantity

$$\text{Area}(x, y) = x \cdot y$$

Since Alastair has 500 total meters of fencing material, assuming he uses all of the material, we have the constraint equation $2x + 2y = 500 \implies x + y = 250$. Thus, the area function can actually be thought of as a single-variable function

$$\text{Area}(x) = x \cdot (250 - x)$$

where x is allowed to vary in the domain $0 \leq x \leq 250$. We wish to maximize the function on this domain.

Note that the derivative of Area(x) is $\text{Area}'(x) = 250 - 2x$, so there is a critical point at $x = 125$ (and the derivative is defined everywhere in $(0, 250)$, so there are no other critical points in the interior). Furthermore, Area$'$ is negative for all $x > 125$ and positive for all $x < 125$, so the local maximum at $x = 125$ is actually a global maximum. We conclude that $\text{Area}(125) = 125^2$ meters^2 is the maximum area enclosed by a fence.

Example. Let's do a harder one. Suppose that the unit circle encloses a lake, in which Andy's friend is drowning. Andy's friend is located at the point $(-0.5, 0.5)$, and Andy is initially located at the point $(1, 0)$. Suppose that Andy swims half as fast as he is able to run on land. Where should he jump into the lake?

We assume that Andy runs along the circumference of the lake for some time, then jumps into the lake and swims in a straight line towards his friend. The question is, where should Andy jump in to the lake?

Let v be Andy's speed on land, so that $v/2$ is his swimming speed. Suppose he decides to jump into the lake at the point on the unit circle located at an angle θ counterclockwise from $(1, 0)$ (i.e., Andy's initial position). The coordinates of this point are $(\cos(\theta), \sin(\theta))$. The amount of time it takes Andy to run from $(1, 0)$ to $(\cos(\theta), \sin(\theta))$ is θ/v, and then the amount of time it takes Andy to swim to his friend is

$$\sqrt{(\cos(\theta) + .5)^2 + (\sin(\theta) - .5)^2}/(v/2)$$

so the function we want to minimize is

$$t(\theta) = \theta/v + \sqrt{(\cos(\theta) + .5)^2 + (\sin(\theta) - .5)^2}/(v/2)$$

We wish to minimize this function on the domain $0 \leq \theta \leq 2\pi$, on which the function is continuous and differentiable on the interior.

We calculate

$$t'(\theta) = \frac{1}{v}\left(\frac{-\sin(\theta) - \cos(\theta)}{\sqrt{(\sin(\theta) - .5)^2 + (\cos(\theta) + .5)^2}} + 1\right)$$

The critical points of t therefore occur at solutions to

$$-\sqrt{(\sin(\theta) - .5)^2 + (\cos(\theta) + .5)^2} = -\sin(\theta) - \cos(\theta)$$

Squaring this equation yields

$$((\sin(\theta) - .5)^2 + (\cos(\theta) + .5)^2) = (\sin(\theta) + \cos(\theta))^2$$

Expanding gives

$$(1.5 - \sin(\theta) + \cos(\theta)) = 1 + 2\sin(\theta)\cos(\theta)$$

At this point, we have the equation in a simple enough form that we may want to just put it in a calculator. The calculator will actually find that there are solutions in $[0, 2\pi]$ at $\theta = \pi/6, 2\pi/3, 5\pi/6, 4\pi/3$. Substituting these solutions back into the formula for $t'(\theta)$, we find that critical points only occur at $\pi/6$ and $2\pi/3$ (since we squared a lot to get the trig equation we ultimately solved, some vestigial solutions were introduced).

Finally, making a sign chart and classifying the critical points, we find a local max at $\pi/6$ and a local min at $2\pi/3$. Comparing the values $t(0) = t(2\pi)$ and $t(2\pi/3)$, we find that the local min is actually a global min on $[0, 2\pi]$. Thus, Andy's optimal strategy is to run around the lake for $2\pi/3$ radians, a.k.a. $120°$, and then begin swimming towards his friend!

Exercise. Suppose Andy is a faster swimmer than in the above scenario. For example, suppose he is only 1.75 times slower at swimming than running. Should he still run around the lake for a while or should he start swimming from the get-go? What if he's 1.5 times slower at swimming? For what threshold value of $v_{\text{land}}/v_{\text{water}}$ does his optimal rescue plan change from "run for a bit, then start swimming" to "start swimming immediately"?

Here is a collection of more optimization problems, of varying difficulty. For the exam, you should understand how to set up and solve problems like this. In particular, you'll need to refresh some geometry formulas/write them on your notecard; a list of fair game formulas is on the course website.

In general, these problems are *hard* because they involve a three-step process of thinking about a situation, converting it into math, and then doing calculus. For these word problems, and for optimization in general, **do not blindly apply formulas/algorithms. For basically every formula/algorithm, there is some situation where it doesn't apply, so make sure to understand what's going on and what the words mean!**

Exercise. You are building a cylindrical silo that holds 20 cubic meters of grain. The material with which the top and bottom are built costs \$10 per square meter, and the material with which the side is built costs \$8 per square meter. Find the radius and height of the most economical silo.

Exercise. A *Norman window* is a window constructed by adjoining a semicircle to the top of an ordinary rectangular window. What are the dimensions of the Norman window with perimeter 18 meters and maximum possible area?

Exercise. Let $P = (5, 2)$. What is the point on the parabola $y = x^2$ that is closest to P?

Exercise. Suppose three points are chosen on the unit circle. What is the maximum possible area of the triangle whose vertices are at these three points?

Lecture 22 (2019-03-29)

Families of Functions

In applications and in pure math, one is often interested in analyzing the behavior of *families of functions* as certain parameters that these families depend upon vary. For example, in statistics, one might be interested in how the shape of a statistical distribution changes as one varies parameters such as mean and standard deviation. Many questions of this type can be analyzed using tools from calculus, and we'll spend today working through some examples. A key tool to aid in visualizing/understanding here is Desmos, which has nice little sliders you can use to vary parameters of interest.

Example. Consider the family of functions of the form

$$f_{a,b}(x) = e^{ax+bx^2}$$

In other words, for every pair of numbers (a, b), we can plug them into the expression above to obtain a specific function. We are now interested in answering specific questions about this family of functions. For instance:

1. For which pairs (a, b) does $f_{a,b}$ have a global maximum? For which pairs does it have a global minimum? Neither?
2. For which pairs (a, b) does $f_{a,b}$ have no inflection point? For which pairs does it have two inflection points? Can $f_{a,b}$ have other numbers of inflection points?

Let's work through these questions. You'll see that the ideas/methods we're using are exactly the same as those we've been using to study extrema of functions for this whole chapter; the added difficulty comes from working with the "varying constants" a and b.

The First Part: We first calculate $f'_{a,b}(x)$ and find

$$f'_{a,b}(x) = e^{ax+bx^2}(a + 2bx)$$

Thus, the function always has exactly one critical point, at $x = -a/2b$. Let's now split into three cases: when $b > 0$, $b = 0$, and $b < 0$.

Case 1, $b > 0$: By making a sign chart, we see that $f'_{a,b}$ is negative for $x < -a/2b$ and positive for $x > -a/2b$. Since this is the only critical point of the function, it is a global minimum.

Case 2, $b = 0$: In this case, the function is just $f_{a,b}(x) = e^{ax}$, which is a normal exponential function; it has no global min/max.

Case 3, $b < 0$: By making a sign chart, we see that $f'_{a,b}$ is negative for $x > -a/2b$ and positive for $x < -a/2b$. Since this is the only critical point of the function, it is a global maximum.

The Second Part: We now calculate

$$f''_{a,b}(x) = e^{ax+bx^2}(a + 2bx)^2 + e^{ax+bx^2} \cdot 2b = e^{ax+bx^2}((a + 2bx)^2 + 2b)$$

The function $f_{a,b}$ will have no inflection points if $f''_{a,b}$ has no roots, which happens precisely when $b > 0$. Let's now look at the remaining two cases.

Case 2, $b = 0$: In this case, the function is a standard exponential, which has no inflection points.

Case 3, $b < 0$: In this case, we know from above that $f_{a,b}$ has a unique global maximum. We calculate that the critical points of $f'_{a,b}$ occur at $x = (\pm\sqrt{-2b} - a)/(2b)$; denote these values by r and s, where $r < s$. Then we can write

$$f''_{a,b}(x) = e^{ax+bx^2}\frac{1}{4b^2}(x-r)(x-s)$$

and now you can check that $f''_{a,b}$ does change sign at r and s, so they are both inflection points.

Exercise. Consider the family of functions $f_a(x) = \ln(x^2 + a)$, where a ranges over all positive numbers. For what value of a does the function $f_a(x)$ have inflection points at $x = \pm 2019$?

Exercise. Consider the family of functions

$$f_{a,b}(x) = e^{ax} + e^{bx}$$

where a, b range over all nonzero numbers. Explain why every function in this family is concave up. For which pairs (a, b) does $f_{a,b}$ have a global minimum?

Exercise. Consider the family of functions $f_{m,n}(x) = x^m(1-x)^n$ where m, n range over positive *integers*. For which pairs (m, n) does the function $f_{m,n}$ have a global maximum? Where does it occur and what is its value?

Exercise. Consider the family of plane curves

$$\mathcal{P}_a = \{(x, y) : y^2 = x^3 + ax + 1\}$$

There is a number c such that for all $a > c$, the graph of $\mathcal{P}_a$ consists of one connected piece, and for all $a < c$, the graph of $\mathcal{P}_a$ consists of two distinct connected pieces. What is c?

Exercise (Extra Credit)**.** Consider the family of functions

$$f_n(x) = 2^x + 3^x - n^x$$

For all $n > 3$, the function $f_n(x)$ has a global maximum; call this global maximum M_n. What is $\lim_{n\to\infty} M_n$? Justify your answer.

Lecture 23 (2019-04-02)

Review Problem: Families of Functions

Consider the family of functions

$$f_b(x) = \frac{x^3}{b + e^x}$$

where b ranges over all real numbers.

For which values of b does f_b have a global maximum? Global minimum? Local maximum? Local minimum? Justify your answers clearly using calculus.

(This problem, like many families of functions problems, is not so easy. Don't forget to keep sight of what you're trying to figure out, and make sure to draw/graph lots of pictures.)

Applications to Marginality a.k.a. the Econ Section

In this section, we will be doing exactly the same kinds of problems we've been doing so far, but in the context of some new economics vocabulary. Businesses (typically) want to maximize profit, which can be modeled as a function of the quantity q of goods produced. In other words, we have a **profit** function $\pi(q)$ which eats "quantity q of goods produced" and outputs "profit $\pi(q)$ made by the business at a production level of q."

We know that

$$\boxed{\text{profit} = \text{revenue} - \text{cost}}$$

so, after inventing the **revenue** function $R(q)$ (which eats quantity q of goods produced and outputs the revenue obtained at this production level) and the **cost** function $C(q)$ (which eats quantity of goods produced and outputs the cost to produce this many goods), we have the equation

$$\pi(q) = R(q) - C(q)$$

Here's a bit of vocabulary: the **fixed cost** of an enterprise is the amount of money it costs to produce no quantity of goods, i.e. $C(0)$. This is typically a positive number, and it represents the "startup" cost of the business, e.g., labor, rent, etc.

Here's some more vocabulary: **marginal cost** is a fancy way to say "derivative of the cost function." **Marginal revenue** is, similarly, a fancy way to say "derivative of the revenue function".

We know that the maxima of a function occur at its critical points, so the maxima of the profit function should occur at values of q such that $0 = \pi'(q) = R'(q) - C'(q) \implies R'(q) = C'(q)$. In words, the maxima of the profit function can occur at values of q where *marginal cost equals marginal revenue.* First off, keep in mind that this does not mean a maximum *must* occur when marginal cost equals marginal revenue (more generally, keep in mind that not every critical point has to be an extremum). Second, do **NOT** memorize this piece of vocabulary; instead, treat any of these "profit" problems exactly the way you would treat a standard optimization problem.

Exercise: Danny runs a company that used to make salt and vinegar chips. After realizing salt and vinegar chips are terrible, he changed the direction of the company, and it now produces microchips. Suppose that the revenue function is given by

$$R(q) = 5q - 0.003q^2$$

where q is the number of microchips produced and $R(q)$ is the corresponding revenue, in dollars. Similarly, suppose the cost function is given by

$$C(q) = 300 + 1.1q$$

where q is the number of microchips produced and $C(q)$ is the cost of production, in dollars. If the company cannot afford to produce more than 800 microchips in total, what number of microchips should it produce to maximize profit?

Lecture 24 (2019-04-03)

Related Rates

Today, we're going to discuss a new problem type called *related rates*. These problems are similar to optimization word problems, in the sense that the set-up is often the hardest part. Just as for optimization word problems, we reduced to a calculus "find the global optima" problem, here we will reduce related rates word problems to implicit differentiation problems. The premise of related rates problems is essentially the following:

- There are several quantities we are interested in, which are varying with respect to some variable, typically time. These quantities will satisfy constraint equations, which you must find.
- Once you find the constraint equations, implicitly differentiate the equations with respect to the (time) variable and use the resulting relations to calculate the desired values.

Let's see this premise in practice:

Example. Suppose $\triangle ABC$ is a right triangle, with $\angle ABC = 90°$. The legs AB and BC of $\triangle ABC$ are getting longer, at a rate of 3 m/s and 4 m/s, respectively. At what rate is the hypotenuse of $\triangle ABC$ growing when the legs are of length $AB = 5$ and $BC = 12$?

For simplicity, let's write $AB = a$, $BC = b$, and $AC = c$. These quantities depend on time, so we actually have functions $a(t)$, $b(t)$, and $c(t)$. We are given that a and b are linear functions, with $a'(t) = 3$ and $b'(t) = 4$. We are asked to compute $c'(s)$, where s is the time such that $a(s) = 5$ and $b(s) = 12$. Note that the Pythagorean theorem implies $c(s) = 13$.

We know that the functions a, b, and c satisfy the Pythagorean theorem:

$$a(t)^2 + b(t)^2 = c(t)^2$$

By differentiating both sides of this equation with respect to t, we find

$$a'(t)a(t) + b'(t)b(t) = c'(t)c(t)$$

We want to calculate $c'(s)$, so setting $t = s$ in the equation, we have

$$c'(s) = \frac{a'(s)a(s) + b'(s)b(s)}{c(s)} = \frac{3 \cdot 5 + 4 \cdot 12}{13} = \frac{63}{13}$$

(The units of $c'(s)$ are meters per second.)

Example. A spherical snowball is melting. Its radius is decreasing at a rate of 0.2 centimeters per hour, at the moment when its radius equals 15 centimeters. How fast is the volume decreasing at this moment?

We have two quantities of interest, the radius r of the snowball and its volume V. These are both functions $r(t)$ and $V(t)$ of time, which are related via the formula

$$V(t) = \frac{4\pi}{3} r(t)^3$$

We wish to find $V'(s)$, where s is the moment in time when $r(s) = 15$; we are given that $r'(s) = -0.2$. Differentiating the equation with respect to t, we find

$$V'(t) = 4\pi r(t)^2 r'(t) \implies V'(s) = 4\pi r(s)^2 r'(s) = 900\pi(-0.2) = -180\pi$$

(The units of $V'(s)$ are cubic centimeters per hour.)

Exercise. A ladder that is leaning against a wall starts slipping down. If the point where the ladder touches the ground is moving away from the wall at a constant rate, is the point where the ladder touches the wall falling at a constant rate? Explain your answer clearly.

Exercise. Suppose a street lamp is located 10 meters above the ground. You are walking away from the street lamp at a rate of 1 meter per second. When you are 5 meters away from the street lamp, what is the rate of change of the area of your shadow? For simplicity's sake, model yourself as a rectangle for this problem, say of height 1.5 meters and width 0.5 meters.

Exercise. When the growth of a spherical cell depends on the flow of nutrients through its surface, it is reasonable to assume that the volume growth rate dV/dt is proportional to the surface area S. Assume that for a particular cell, the volume growth rate satisfies the equation $dV/dt = S/3$ (as functions, not at a particular moment in time). At what rate is the radius of the cell increasing?

Exercise. Suppose the minutes hand of a clock is 15 millimeters long and the hours hand is 12 millimeters long. How fast is the distance between the hours and minutes hand changing at 2 P.M.?

(*Hint:* Use the Law of Cosines)

Exercise. Doppler radar measures the rate of change of the distance from an object to the observer. Suppose a police officer a meters from a straight road points a Doppler radar gun at a car travelling along the road, c meters away (this is the distance from the officer directly to the car), and measures a speed of v. What is the car's actual speed?

Lecture 25 (2019-04-03)

Two Questions

Consider the following two questions:

(a) Given the derivative $f'(x)$ of some function $f(x)$, how do we recover the original function $f(x)$?

(b) Given a function $f(x)$, how do we find the area of the region bounded by the graph of f and the x-axis?

Amazingly, it turns out that these questions are roughly "the same," in the sense that both are answered by the operation of **integration**. We will very soon make this more precise.

Areas Under Curves

We will begin by answering question (b): how to find the area under a curve. Suppose $f(x)$ is a function and we wish to find the area of the region bounded by the graph of f, the x-axis, and the two vertical delimiters $x = a$ and $x = b$; suppose $a < b$. The idea is to approximate the region by rectangles (whose areas are easy to calculate) and then take a limit as the number of rectangles goes to infinity.

Slogan: An infinitely accurate approximation ceases to be an approximation!

Definition. Imagine dividing the interval $[a, b]$ into n equal subintervals, each of which has length $(b - a)/n$: setting $x_0 = a$ and $x_n = b$, we suppose these subintervals are $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$. Now, on top of each interval $[x_i, x_{i+1}]$, we place a rectangle whose height is either $f(x_i)$ or $f(x_{i+1})$. In the former case, we are taking a **left-hand Riemann sum** and in the latter case, we are taking a **right-hand Riemann sum**.

We define the **left-hand Riemann sum** to be the sum of the areas of all the left-endpoint-heighted rectangles:

$$\mathcal{L}(f, [a, b], n) := \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k)$$

and similarly we define the **right-hand Riemann sum** to be the sum of the areas of all the right-endpoint-heighted rectangles:

$$\mathcal{R}(f, [a, b], n) = \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

(Note the different indexing on the two sums!)

Theorem. *If f is a piecewise continuous function, then*

$$\lim_{n\to\infty} \mathcal{L}(f, [a, b], n)$$

and

$$\lim_{n\to\infty} \mathcal{R}(f, [a, b], n)$$

both exist and are finite. Moreover, in this situation we have

$$\lim_{n\to\infty} \mathcal{L}(f,[a,b],n) = \lim_{n\to\infty} \mathcal{R}(f,[a,b],n)$$

and the common value is called the ***definite integral of*** f ***on*** $[a,b]$. *This number is denoted via the symbol*

$$\int_a^b f(x)\,dx$$

The definite integral of f on $[a,b]$ represents the area under the graph of f and over the interval $[a,b]$ on the x-axis. We have to be a little bit careful when we say "area"; for example, what if f is negative on the interval $[a,b]$? In fact, $\int_a^b f(x)\,dx$ gives the **signed area** under the curve: area sitting above the x-axis is counted positively and area sitting below the x-axis is counted negatively.

Example. Let

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Then $\int_{-7}^5 f(x)\,dx = -2$, because over the interval $[-7,5]$, we get a positively-counted rectangle sitting over $[0,5]$ with height 1, and a negatively-counted rectangle sitting over $[-7,0]$ with height 1, so the signed area is $5-7=-2$.

Proposition. *So far, we have defined the definite integral* $\int_a^b f(x)\,dx$ *for when* $a<b$*; we can also make sense of this notation when* $b<a$ *by defining*

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

With this definition, we have ***transitive bounds property of the definite integral*** *which says that for any numbers a, b, c, the equation*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

holds. At least in the setting where $a<b<c$ *and* f *is positive, this should be somewhat geometrically intuitive: it just says that areas add.*

Another important property is ***linearity of integration****; just like differentiation, we have that if* f *and* g *are functions, then*

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and if c is a constant, then

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Exercise. Suppose f is increasing, $f(2) = 7$ and $f(6) = 11$. Find a value of n such that the error between $\mathcal{R}(f, [2, 6], n)$ and $\int_2^6 f(x)\,dx$ is less than 10^{-9}.

Exercise. If a function f is increasing or decreasing, then the definite integral $\int_a^b f(x)\,dx$ is sandwiched between $\mathcal{R}(f, [a, b], n)$ and $\mathcal{L}(f, [a, b], n)$ for all n. This need not be the case if f is not monotonic: give an example of a function f and an interval $[a, b]$ such that $\mathcal{R}(f, [a, b], 2)$ and $\mathcal{L}(f, [a, b], 2)$ are both bigger than $\int_a^b f(x)\,dx$.

Recovering a Function From Its Derivative

Now, we turn to question (a): how do we recover a function $f(x)$ from its derivative $f'(x)$? The idea here is to use "iterated linear approximation." Suppose we know the value of $f(a)$, and we wish to find the value of $f(b)$, where $a < b$.

Divide the interval $[a, b]$ into n equal subintervals (just as before); again, each of these has length $(b - a)/n$ and setting $x_0 = a$ and $x_n = b$, we may suppose these subintervals are $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$. On each of these intervals, we will use linear approximation to approximate the value of $f(x_{i+1})$ from the value of $f(x_i)$.

Start with the value $f(a) = f(x_0)$. Using the linear approximation at a, we have

$$f(x_1) \approx f(x_0) + f'(x_0) \cdot \frac{b - a}{n}$$

and similarly, using the linear approximation at x_i, we have

$$f(x_{i+1}) =\approx f(x_i) + f'(x_i) \cdot \frac{b - a}{n} \iff f(x_{i+1}) - f(x_i) \approx f'(x_i) \cdot b - an$$

Adding these all up and "telescoping," we get

$$f(b) - f(a) \approx \frac{b - a}{n} \cdot \sum_{k=0}^{n-1} f'(x_k)$$

Note that this is exactly a left-hand Riemann sum for f' over the interval $[a, b]$! (If we had chosen to use the linear approximation at x_1 first, then we'd have gotten the right-hand Riemann sum.)

As we take $n \to \infty$, our approximation becomes exact, so we obtain the following (extremely important) theorem:

Theorem (Fundamental Theorem of Calculus (FTC))**.** *If f is differentiable, then for any interval $[a, b]$, we have*

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The Area Under a Parabola and Antiderivatives

We can use the FTC to calculate the area under the parabola $y = x^2$ over the interval $[0, 1]$. The idea is that x^2 is the derivative of a function we can explicitly write down, namely

$g(x) = x^3/3$. Thus,

$$\int_0^1 x^2\,dx = \int_0^1 g'(x)\,dx = g(1) - g(0) = \frac{1}{3}$$

(We also could have set up a Riemann sum and taken the limit as the number of subdivisions goes to infinity, but this would have been much harder; this is an example of the power of the FTC).

This example suggests making the following definition:

Definition. Given a function $f(x)$, an **antiderivative** of f is a function $F(x)$ such that $F'(x) = f(x)$.

Theorem. *Let f be a continuous function. Suppose F_1 and F_2 are antiderivatives of a function f. Then $F_1 - F_2$ is a constant function.*

Proof. We have $(F_1 - F_2)'(x) = f(x) - f(x) = 0$, so $F_1 - F_2$ is a constant function. $\square$

Thus, a function does not have a unique antiderivative; it has a *family* of antiderivatives. But any two antiderivatives are very close to each other: they differ only by a constant. The FTC gives us another way to describe the family of antiderivatives of a function $f(x)$:

Theorem (FTC: Second Form)**.** *If f is a continuous function, then the function*

$$F_a(x) = \int_a^x f(x)\,dx$$

is an antiderivative of f for each choice of a; these are precisely the family of antiderivatives of f. In particular, every function has an antiderivative.

Exercise. For each positive integer n, find an antiderivative of the function x^n. Use this antiderivative to calculate

$$\int_0^1 x^n\,dx$$

Exercise. Suppose $g(x)$ is a function such that $g(4) = 7$. Consider the function

$$G(x) := \int_{x^2}^{x^3} g(t)\,dt$$

Given that $G'(2) = 11$, find $g(8)$.

Exercise. In the picture, the horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant. For what value of c are the two shaded areas equal?