Appendix A. Depth-zero supercuspidal L-packets on general inner forms

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Let F be a p-adic field with ring of integers O_F and residue field k_F of cardinality q and characteristic p. We fix an algebraic closure $\bar{F}$ of F and let Γ be the absolute Galois group of $\bar{F}/F$ and W the absolute Weil group. Let G be a connected reductive algebraic group defined over F and unramified, i.e. quasi-split and split over the maximal unramified extension F^u of F. Let (ξ, G') be an inner twist of G: That is, G' is a connected reductive algebraic group defined over F, and $\xi : G \times \bar{F} \to G' \times \bar{F}$ is an isomorphism with the property that for all $\sigma \in \Gamma$, the automorphism $\xi^{-1\sigma}\xi$ of $G \times \bar{F}$ is inner.

Let ${}^LG$ be the Weil-form of the Langlands dual group of G. Since we are identifying the dual groups of G and G' via ξ, ${}^LG$ is also the Langlands dual group of G'. Let $\phi : W \to {}^LG$ be a TRSELP parameter [DR09, §4.1]. If the inner twist (G', ξ) comes from a pure inner twist (G', ξ, z) (see [Kal11, §2] for the terminology), an L-packet Π_ϕ^z consisting of irreducible supercuspidal depth-zero representations of $G'(F)$ was defined in [DR09]. This was later generalized to the cases when (G', ξ) comes from an extended pure inner twist in [Kal11b]. Moreover, the packets constructed in [DR09] and [Kal11b] were shown to have a natural parameterization in terms of the centralizer of ϕ in the dual group $\hat{G}$ of G, and satisfy stability and endoscopic transfer.

In this appendix, we are going to repeat some of the arguments of [Kal11b] in order to obtain a special case of the results of that paper in a more general setting. Namely, we will first construct the L-packet $\Pi_\phi^{G'}$ on G' corresponding to ϕ without assuming any condition on (G', ξ). In the cases where an L-packet on G' has already been constructed, we will obtain the same set of representations. What we lose in this more general setting is the ability to provide a natural parameterization of the set $\Pi_\phi^{G'}$ in terms of $S_\phi = \mathrm{Cent}(\phi, \hat{G})$. Instead, the following much weaker statement is all we can show: Let $\zeta_{G'}$ be any character of $Z(\hat{G}_{\mathrm{sc}})$ whose restriction to $Z(\hat{G}_{\mathrm{sc}})^\Gamma$ corresponds to the class of (ξ, G) in $H^1(F, G_{\mathrm{ad}})$ under the Kottwitz isomorphism. Let $\mathcal{A}_\phi$ be the inverse image in $\hat{G}_{\mathrm{sc}}$ of the image in G_{ad} of S_ϕ. Then we can show that there exists a *non-canonical* bijection between $\Pi_\phi^{G'}$ and the set $\mathrm{Irr}(\mathcal{A}_\phi, \zeta_{G'})$ of irreducible representations of $\mathcal{A}_\phi$ which transform under $Z(\hat{G}_{\mathrm{sc}})$ by $\zeta_{G'}$.

Second, we will prove the following character identity: Let Π_ϕ^G be the L-packet on G, and $\Pi_\phi^{G'}$ that on G'. Then for any strongly-regular semi-simple elements $\gamma \in G(F)$ and $\gamma' \in G'(F)$ whose stable classes correspond to each other, we have

$$\sum_{\pi \in \Pi_\phi^G} \Theta_\pi(\gamma) = e(G') \sum_{\pi' \in \Pi_\phi^{G'}} \Theta_{\pi'}(\gamma').$$

Recall that the stable classes of γ and γ' correspond to each other if there exists $g \in G(\bar{F})$ such that $\xi \circ Ad(g)\gamma = \gamma'$.

This character identity is a special case of the endoscopic character identities for depth-zero supercuspidal L-packets established in [Kal11] and [Kal11b], and the necessary argument – a slight generalization of the stability calculations of [DR09] – is very similar to the proof of [Kal11b, Thm. 4.3.1].

A.1. **Construction of L-packets.** Given a TRSELP parameter $\phi : W \to {}^LG$ (see [DR09, §4.1] or [Kal11b, §3.2]), we apply "Step 1" in [Kal11b, §3.3] and obtain a triple $(S_0, [a], [{}^Lj])$ where S_0 is an elliptic unramified maximal torus of G whose unique fixed point in $\mathcal{B}^{\mathrm{red}}(G, F)$ is a vertex, $[a]$ is an equivalence class of Langlands parameters for S, and $[{}^Lj]$ is a $\hat{G}$-conjugacy class of L-embeddings ${}^LS \to {}^LG$, with the property that ϕ belongs to the $\hat{G}$-conjugacy class $[{}^Lj \circ a]$. The construction of this triple involves choices of χ-data and hyperspecial vertex $o \in \mathcal{B}^{\mathrm{red}}(G, F)$. However, using [Kal11b, Lemma 3.4.1] we see that the $G_{\mathrm{ad}}(F)$-conjugacy class of this triple is independent of all choices. We let $\theta_0 : S_0(F) \to \mathbb{C}^\times$ be the character corresponding to $[a]$.

Next we consider admissible embeddings of S_0 into G'. We recall that an embedding $j : S_0 \to G'$ is called admissible if it is defined over F and is of the form $\xi \circ \mathrm{Ad}(g)$ for some $g \in G(\bar{F})$. The set of admissible embeddings forms a unique stable conjugacy class by definition. Given such an embedding j, we obtain the maximal torus $S_j := j(S_0)$ of G' and the character $\theta_j : S_j(F) \to \mathbb{C}^\times$ given by $\theta_j = j_*\theta_0 = \theta_0 \circ j^{-1}$. Thus, j gives rise to the representation π_{G', S_j, θ_j} discussed in [Kal11b, §3.1]. We set

$$\Pi_\phi^{G'} = \{\pi_j\},$$

where j runs over the set of $G'(F)$-conjugacy classes inside the stable class of admissible embeddings $j : S_0 \to G'$.

Lemma A.1. *The group $\mathcal{A}_\phi$ is abelian, and there exists a bijection between $\Pi_\phi^{G'}$ and $\mathrm{Irr}(\mathcal{A}_\phi, \zeta_{G'})$.*

Proof. Choose arbitrarily one admissible embedding $j_a : S_0 \to G'$. Then set of $G'(F)$-conjugacy classes of admissible embeddings $S_0 \to G'$ is in canonical bijection with the kernel of the map

$$H^1(j_a) : H^1(F, S_0) \to H^1(F, G').$$

Using [Kot86, Thm 1.2], this kernel is in bijection with the kernel of

$$\pi_0(S_0^\Gamma)^D \to \pi_0(Z(\hat{G})^\Gamma)^D.$$

Since S_0 is elliptic, we know that $[\hat{S}_0^\Gamma]^\circ = [Z(\hat{G})^\Gamma]^\circ$, which implies that the kernel of the last displayed map is equal to $[\hat{S}_0^\Gamma / Z(\hat{G})^\Gamma]^D$.

On the other hand, the map ${}^Lj$ provides an isomorphism $\hat{S}_0^\Gamma \to \mathrm{Cent}(\phi, \hat{G})$, from which we obtain an isomorphism between $\hat{S}_0^\Gamma / Z(\hat{G})^\Gamma$ and the image of $\mathrm{Cent}(\phi, \hat{G})$ in $\hat{G}_{\mathrm{ad}}$. The pre-image $\mathcal{A}_\phi$ of $\hat{S}_0^\Gamma / Z(\hat{G})^\Gamma$ in $\hat{G}_{\mathrm{sc}}$ is certainly contained in $[\hat{S}_0]_{\mathrm{sc}}$, and is thus abelian. It follows that $\mathrm{Irr}(\mathcal{A}_\phi, \zeta_{G'})$ is a torsor under the abelian group $\mathrm{Irr}(\mathcal{A}_\phi, 1)$. The latter is by definition is equal to $[\hat{S}_0^\Gamma / Z(\hat{G})^\Gamma]^D$. □

A.2. **Stability and transfer to inner forms.** Given that $\mathcal{A}_\phi$ is abelian, we define the stable character of the packet $\Pi_\phi^{G'}$ to be

$$S\Theta_\phi^{G'} = e(G') \sum_{\pi' \in \Pi_\phi^{G'}} \Theta_\pi,$$

where $e(G')$ is the Kottwitz sign of G' [Kot83]. We recall that

$$e(G') = (-1)^{\mathrm{rk}_F(G) - \mathrm{rk}_F(G')}.$$

More generally, we define for any two reductive groups G_1, G_2 the sign

$$e(G_1, G_2) = (-1)^{\mathrm{rk}_F(G_1) - \mathrm{rk}_F(G_2)}.$$

Clearly $e(G') = e(G, G')$.

Before discussing stability, we need to give a formula for $S\Theta_\phi^{G'}$. For this, we will closely follow the proof of [Kal11b, Prop 4.2.2]. In order to obtain the result, we need to assume that the residual characteristic of F is large enough. More precisely, we assume that the cardinality of the residue field k_F of F at least as large as the number of positive roots of G, and that both G and G' have faithful rational representations of dimension not exceeding $p/(2+e)$, where $p = \mathrm{char}(k_F)$ and e is the ramification index of $F/\mathbb{Q}_p$.

Under these conditions, DeBacker and Reeder have shown [DR09, Lemma 12.4.1] that the following statements hold for any group H which is the connected component of the centralizer of a topologically semi-simple element of $G(F)$ or $G'(F)$:

- Let $x \in \mathcal{B}^{\mathrm{red}}(H, F)$ be a point and H_x^{red} be the reductive quotient of the special fiber of the corresponding parahoric group scheme. Then the Lie-algebra of any maximal torus of H_x^{red} defined over k_F contains a strongly-regular semi-simple element defined over k_F.
- There exists an $H(F)$-equivariant bijection $\log : H(F)_{0+} \to \mathrm{Lie}(H)(F)_{0+}$ which restricts, for each $x \in \mathcal{B}^{\mathrm{red}}(H, F)$, to an $H_x^{\mathrm{red}}(k_F)$-equivariant bijection from the set of unipotent elements of $H_x^{\mathrm{red}}(k_F)$ to the set of nilpotent elements of $\mathrm{Lie}(H_x^{\mathrm{red}})(k_F)$.

We fix a character $\psi : F \to \mathbb{C}^\times$ whose restriction to O_F factors through a non-trivial character of k_F. Furthermore, we fix a good bilinear form B [DR09, §A] on the Lie-algebra $\mathfrak{h}$ of H. Reductive groups and their Lie-algebras will be endowed with the canonical Haar measures described in [DR09, §5.1]. Then, for any regular semi-simple element $X \in \mathrm{Lie}(H)(F)$, the distribution on $\mathfrak{h}(F)$ defined by

$$f \mapsto \int_{H(F)/H_X(F)} \int_{\mathfrak{h}(F)} f(Y)\psi(B(\mathrm{Ad}(g)X, Y))dY dg$$

is represented by a function, which we call $\hat{\mu}_X^H$. We also define

$$S\hat{\mu}_X^H = \sum_{X'} \hat{\mu}_{X'}^H,$$

where the sum runs over a set of representatives for the $H(F)$-conjugacy classes inside the stable class of X.

We now let $Q_0 \in \mathrm{Lie}(S_0)(O_F)$ be an element with strongly-regular reduction. This element exists due to the work of DeBacker and Reeder recalled above. Of course, Q_0 is automatically a regular semi-simple element of $\mathrm{Lie}(G)(F)$, and hence

$$j \leftrightarrow dj(Q_0)$$

is a bijection between the set of $G'(F)$-conjugacy classes of admissible embeddings $j : S_0 \to G'$ and the set of $G'(F)$-conjugacy classes inside the set of elements of $\mathrm{Lie}(G')(F)$ which are stably-conjugate to Q_0. We will denote the inverse of $j \mapsto dj(Q_0)$ by $P \mapsto \phi_{Q_0,P}$.

The inner twist $\xi : G \to G'$ restricts to an isomorphism $Z_G \to Z_{G'}$ defined over F. We will identify these two groups from now on.

Lemma A.2. *The function $S\Theta_\phi^{G'}$ is supported on the subset $G'_{\mathrm{sr}}(F)_0 Z_G(F)$. Moreover, for any $\gamma' \in G'_{\mathrm{sr}}(F)_0$ and $z \in Z_G(F)$, the value of $S\Theta_\phi^{G'}(z\gamma')$ is given by*

$$e(G,H)\theta_0(z)\sum_{P'}[\phi_{Q_0,P'}]_*\theta_0(\gamma'_s)S\hat{\mu}^H_{P'}(\log(\gamma'_u)),$$

where $\gamma' = \gamma'_s\gamma'_u$ is the topological Jordan decomposition of γ', the sum over P' runs over a set of representatives for the $G'_{\gamma'_s}$-stable classes of elements of $\mathrm{Lie}(G'_{\gamma_s})(F)$ *which are G'-stably conjugate to Q_0, and H is equal to $[G'_{\gamma'_s}]^\circ$.*

Proof. The vanishing result follows at once from [DR09, Lemma 9.3.1]. We turn to the formula. Let $j : S_0 \to G'$ be an admissible embedding, and $\pi_j \in \Pi_\phi^{G'}$ the corresponding representation. According to [Kal11b, Lemma 3.1.2] and [DR09, Lemma 12.4.3], we have

$$\Theta_{\pi_j} = e(G',A_G)\theta_j(z)\sum_Q[\phi_{dj(Q_0),Q}]_*\theta_j(\gamma'_s)e(H,A_H)\hat{\mu}^H_Q(\log(\gamma'_u))$$

where Q runs over a set of representatives for the $H(F)$-conjugacy classes inside the intersection of the $G'(F)$-conjugacy class of $dj(Q_0)$ with $\mathrm{Lie}(H)(F)$. We recall that A_G (resp A_H) is the maximal split torus in the center of G (resp. H).

Under the identification of $Z(G)$ with $Z(G')$ via ξ, the characters θ_0 and θ_j have the same restriction to $Z(G)(F)$. Moreover, $\phi_{dj(Q_0),Q}\circ j = \phi_{Q_0,Q}$. Finally, we have $A_H = A_G$, since the centralizer of Q in H is a maximal torus of H which is also an elliptic maximal torus of G' (being stably-conjugate to the elliptic torus S_0). We conclude

$$\Theta_{\pi_j} = e(G',H)\theta_0(z)\sum_Q[\phi_{Q_0,Q}]_*\theta_0(\gamma'_s)\hat{\mu}^H_Q(\log(\gamma'_u)).$$

To obtain $S\Theta_\phi^{G'}$, we sum these formulas over the set of $G'(F)$-conjugacy classes of admissible embeddings $j : S_0 \to G'$. Using the bijection $j \leftrightarrow dj(Q_0)$ and its inverse $\phi_{Q_0,P} \leftrightarrow P$, we obtain

$$S\Theta_\phi^{G'} = e(G')e(G',H)\theta_0(z)\sum_P\sum_Q[\phi_{Q_0,Q}]_*\theta_0(\gamma'_s)\hat{\mu}^H_Q(\log(\gamma'_u)),$$

where now the sum over P runs over the set of $G'(F)$-conjugacy classes inside the set of elements of $\mathrm{Lie}(G')(F)$ which are stably conjugate to Q_0, while Q runs over a set of representatives for the $H(F)$-conjugacy classes inside the intersection of the $G'(F)$-conjugacy class of P with $\mathrm{Lie}(H)(F)$. The double sum over P,Q can be replaced by a single sum running over the $H(F)$-conjugacy classes inside the set of elements of $\mathrm{Lie}(G')(F)$ which are stably-conjugate to Q_0 and belong to $\mathrm{Lie}(H)(F)$. This single sum can then be written again as a double sum, where we first sum over a set of representatives P' for the sets of H-stable classes in $\mathrm{Lie}(H)(F)$ of elements which are G'-stably-conjugate to Q_0, and then sum over a set of representatives Q' for the sets of $H(F)$-conjugacy classes inside the H-stable class of P'. With this re-indexing, we obtain

$$S\Theta_\phi^{G'} = e(G')e(G',H)\theta_0(z)\sum_{P'}\sum_{Q'}[\phi_{Q_0,Q'}]_*\theta_0(\gamma'_s)\hat{\mu}^H_{Q'}(\log(\gamma'_u)).$$

Since P' and Q' are elements of $\mathrm{Lie}(H)(F)$ which are H-stably-conjugate, we have

$$[\phi_{Q_0,Q'}]_*\theta_0(\gamma'_s) = [\phi_{Q_0,P'}]_*\theta_0(\gamma'_s)$$

Combining this with $e(G')e(G',H) = e(G,H)$ we obtain

$$S\Theta_\phi^{G'} = e(G,H)\theta_0(z)\sum_{P'}[\phi_{Q_0,P'}]_*\theta_0(\gamma'_s)\sum_{Q'}\hat{\mu}_{Q'}^H(\log(\gamma'_u)).$$

and the statement follows. □

Proposition A.3. *Let $\gamma \in G(F)$ and $\gamma' \in G(F)$ be strongly-regular semi-simple elements whose stable classes correspond. Then*

$$S\Theta_\phi^G(\gamma) = S\Theta_\phi^{G'}(\gamma').$$

Proof. We apply Lemma A.2 to both groups G and G'. Since $\gamma \in G_{\mathrm{sr}}(F)_0 Z_G(F)$ if and only if $\gamma' \in G'_{\mathrm{sr}}(F)_0 Z_G(F)$, we see that we have to show

$$e(G,H)\sum_P[\phi_{Q_0,P}]_*\theta_0(\gamma_s)S\hat{\mu}_P^H(\log(\gamma_u)) = e(G,H')\sum_{P'}[\phi_{Q_0,P'}]_*\theta_0(\gamma'_s)S\hat{\mu}_{P'}^{H'}(\log(\gamma'_u)),$$

when $\gamma \in G_{\mathrm{sr}}(F)_0$ and $\gamma' \in G'_{\mathrm{sr}}(F)_0$. Applying [Kal11b, Lemma 4.2.3] to the trivial ep twist $(G,1,1)$ and $\gamma \in G(F)$ we may assume that $\gamma_s \in S_0(F)$. According to [Kot86, Prop 7.1], the group H is then unramified. Choose $g \in G(\bar{F})$ such that $\xi\mathrm{Ad}(g)\gamma = \gamma'$. Then

$$\xi \circ \mathrm{Ad}(g) : H \to H'$$

is an inner twist. On the one hand, this implies via [Kal11b, Lemma 4.1.1] that

$$e(G,H)e(G,H') = e(H,H') = \gamma_\psi(B)\gamma_\psi(B')^{-1}$$

where B and B' are compatible good bilinear forms on $\mathfrak{h}(F)$ and $\mathfrak{h}'(F)$, and γ are the corresponding Weil constants [Wal95, VIII]. On the other hand, the inner twist $\xi\circ\mathrm{Ad}(g)$ provides a bijection between the stable classes of regular semi-simple elliptic elements in $\mathfrak{h}(F)$ and $\mathfrak{h}'(F)$. This bijection restricts to a bijection $P \leftrightarrow P'$ between the summation sets $\{P\}$ and $\{P'\}$ which has the property that whenever $P \leftrightarrow P'$, we have

$$[\phi_{Q_0,P}]_*\theta_0(\gamma_s) = [\phi_{Q_0,P'}]_*\theta_0(\gamma'_s).$$

In order to complete the proof, it will be enough to show

$$\gamma_\psi(B)S\hat{\mu}_P^H(\log(\gamma_u)) = \gamma_\psi(B')S\hat{\mu}_{P'}^{H'}(\log(\gamma'_u))$$

whenever $P \leftrightarrow P'$. This is the main result of [Wal97], which is now unconditional thanks to [Ngo10] and [Wal06]. □

Corollary A.4. *The function $S\Theta_\phi^{G'}$ is stable.*

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