Regular supercuspidal representations

Tasho Kaletha

Abstract

We show that, in good residual characteristic, most supercuspidal representations of a tamely ramified reductive p-adic group G arise from pairs (S, θ), where S is a tame elliptic maximal torus of G, and θ is a character of S satisfying a simple root-theoretic property. We then give a new expression for the roots of unity that appear in the Adler-DeBacker-Spice character formula for these supercuspidal representations and use it to show that this formula bears a striking resemblance to the character formula for discrete series representations of real reductive groups. Led by this, we explicitly construct the local Langlands correspondence for these supercuspidal representations and prove stability and endoscopic transfer in the case of toral representations. In large residual characteristic this gives a construction of the local Langlands correspondence for almost all supercuspidal representations of reductive p-adic groups.

Contents

AMS 2010 Mathematics subject classification: 22E50; 11S37; 11F70.
This research is supported in part by NSF grant DMS-1161489 and a Sloan Fellowship.

1 Introduction

This paper pursues multiple interconnected goals, all of which are related to Yu's construction of supercuspidal representations of reductive p-adic groups [Yu01], which generalizes Adler's earlier construction [Adl98]. Recall briefly that if G is a connected reductive group over a p-adic field F that splits over a tamely ramified extension of F, a supercuspidal representation of $G(F)$ can be constructed by giving the following data: a tower $G^0 \subsetneq \cdots \subsetneq G^d = G$ of connected reductive subgroups that become Levi subgroups of G over some tame Galois extension of F, a sequence of characters $\phi_i : G^i(F) \to \mathbb{C}^\times$ for all $i \geq 0$ satisfying a certain genericity condition, and a depth-zero supercuspidal representation π_{-1} of $G^0(F)$, which we may call the socle of the Yu-datum. Representations obtained from this construction are customarily called tame, even though they can have arbitrary depth (in the case of $G = \mathrm{GL}_N$, these representations are called essentially tame in the work of Bushnell and Henniart; when $p \nmid N$ all supercuspidal representations are essentially tame). Different Yu-data can give rise to the same representation and Hakim and Murnaghan [HM08] have made a precise study of when this happens. This leads to the natural question of whether one can use simpler data to parameterize the supercuspidal representations resulting from Yu's construction. Ideally, such data would consist simply of a maximal torus $S \subset G$ and a character $\theta : S(F) \to \mathbb{C}^\times$, in analogy with the classification of discrete series representations of real reductive groups, as well as that of supercuspidal representations of GL_N when $p \nmid N$. There is an immediate obstruction to this: Many reductive groups over finite fields (but not GL_N) have cuspidal representations that are not immediately parameterizable by such pairs (for example cuspidal unipotent representations), and this obstruction propagates to depth-zero supercuspidal representations of reductive groups over F. We therefore restrict our attention to Yu-data that satisfy a slight regularity condition, which is automatically satisfied for $G = \mathrm{GL}_N$, and whose main part is that the socle π_{-1} (when it is non-trivial) corresponds to a Deligne-Lusztig representation (of the reductive quotient of a parahoric subgroup of G) that is associated to a character in general position. Let us call supercuspidal representations arising from such Yu-data regular. The first main goal of this paper is to give an explicit parameterization of regular supercuspidal representations in terms of $G(F)$-conjugacy classes of pairs (S,θ). Partial results towards this were obtained earlier by Murnaghan in [Mur11], where a further technical restriction is imposed on π_{-1} and an injective map is constructed from the set of equivalence classes of regular supercuspidal representations satisfying this additional technical restriction to the set of $G(F)$-conjugacy classes of pairs (S,θ) consisting of an elliptic maximal torus and a character of it. No effective description of the image of this map was known. For many purposes it is important to have a map in the opposite direction – from pairs (S,θ) to representations. In the current paper we introduce the notion of a tame regular elliptic pair (S,θ). This notion is defined in simple and explicit root-theoretic terms. We show that in the case of GL_N it specializes to the classical notion of an admissible character. We give an explicit algorithm that, starting from a tame regular elliptic pair, produces a Yu-datum for a regular supercuspidal representation. This algorithm can be seen as a generalization to arbitrary reductive groups of the Howe factorization lemma ([How77, Lemma 11 and Corollary]) that plays an important role in the construction of supercuspidal representations of GL_N. Just as in the case of GL_N, the factorization we obtain is not unique, but we show that two possible factorizations are related to each other by a process already introduced by Hakim and Murnaghan, called refactorization. Their work implies that the resulting supercuspidal representation is unaffected by this ambiguity, and may thus be called $\pi_{(S,\theta)}$. We then show that two such representations are isomorphic if and only if the pairs giving rise to them are $G(F)$-conjugate. It is then straightforward to check that the map $(S,\theta) \mapsto \pi_{(S,\theta)}$ is the inverse to Murnaghan's injection (after removing the additional technical restriction on π_{-1} imposed in [Mur11]). This implies that the image of Murnaghan's map is precisely the set of $G(F)$-conjugacy classes of tame regular elliptic pairs. In this way, we obtain explicit mutually inverse bijections between the set of $G(F)$-conjugacy classes of tame regular elliptic pairs and the set of isomorphism classes of regular supercuspidal representations. This result includes as a special basic case the classification of regular supercuspidal representations of depth zero. In fact, this special case is needed as the basis of our argument. When G splits over an unramified extension, regular depth-zero supercuspidal representations were studied by DeBacker and Reeder [DR09]. As a preparation for the study of regular supercuspidal representations of general depth, we extend their classification results to the case of tamely ramified groups G.

When the residual characteristic of F is not too small for G the work of Kim [Kim07] shows that all supercuspidal representations of $G(F)$ arise from Yu's construction. Most of these are regular and thus of the form $\pi_{(S,\theta)}$. For $G = \mathrm{GL}_N$ with $p \nmid N$ all supercuspidal representations are regular, but for other groups non-regular supercuspidal representations do exist, as the example of the four exceptional supercuspidal representations of SL_2 shows. We believe that our work can be used to reduce the description of general supercuspidal representations in terms of elliptic (but not necessarily regular) pairs (S,θ) to the description of cuspidal representations of finite groups of Lie type in terms of Deligne-Lusztig virtual characters. It would be interesting to pursue this question.

In the second part of the paper we study the Harish-Chandra character of supercuspidal representations, and in particular of the representations $\pi_{(S,\theta)}$. A formula for the character of a supercuspidal representation π arising from Yu's construction has been given by Adler and Spice [AS09] and subsequently refined by DeBacker and Spice [DS18]. At the moment this formula is only valid under the assumption that $G^{d-1}(F)/Z(G)(F)$ is compact, but in private communication the authors have assured me that this assumption will soon be removed. In the mean time we have proved in this paper a technical result which removes this condition in a certain special case that still allows us to draw conclusions from it. The character formula [DS18, Theorem 4.6.2] has the following form. Recall first that Yu's construction produces not just a supercuspidal representation π of $G(F)$, but in fact a supercuspidal representation π_i of $G^i(F)$ for each i. Let r be the depth of π_{d-1}. Given a regular semi-simple element $\gamma \in G(F)$ admitting a decomposition (or approximation) $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ in the sense of [AS08], the value at γ of the normalized Harish-Chandra character function of $\pi = \pi_d$ is

$$\Phi_{\pi_d}(\gamma) = \sum_g \epsilon_{\text{sym,ram}}(\gamma_{<r}^g)\epsilon^{\text{ram}}(\gamma_{<r}^g)\tilde{e}(\gamma_{<r}^g)\Phi_{\pi_{d-1}}(\gamma_{<r}^g)\widehat{\mu}_{{}^g X^*}(\log(\gamma_{\geq r})), \quad (1.0.1)$$

where the sum is over certain elements $g \in G(F)$. The term $\widehat{\mu}$ is the Fourier-transform of an orbital integral (on the Lie-algebra of the connected centralizer of $\gamma_{<r}$), and $\epsilon_{\text{sym,ram}}$, ϵ^{ram}, and $\tilde{e}$ are roots of unity. This formula mirrors the inductiveness of Yu's construction by expressing the character of $\pi = \pi_d$ in terms

of that of π_{d-1}. The function ϵ^{ram} is a character of $S(F)$ and this allows it to be handled easily. In fact, as is pointed out in [DS18], this function might be an artifact resulting from certain choices inherent in Yu's construction of supercuspidal representations, and a modification of this construction suppresses it. On the other hand, the two functions $\epsilon_{\text{sym,ram}}$ and $\tilde{e}$ are not characters of $S(F)$. Their definition is quite subtle and involves the fine structure of the p-adic group $G(F)$, in particular its Bruhat-Tits building and associated Moy-Prasad filtrations. This makes the analysis of these two functions with respect to stable conjugacy and related questions difficult. The second main result of this paper gives a new expression for the product $\epsilon_{\text{sym,ram}}(\gamma_{<r}) \cdot \tilde{e}(\gamma_{<r})$. This expression is a quotient of two terms of the form

$$e(G)e(J)\epsilon_L(X^*(T_G)_{\mathbb{C}} - X^*(T_J)_{\mathbb{C}}, \Lambda)\Delta_{II}^{\text{abs}}[a,\chi](\gamma_{<r}). \tag{1.0.2}$$

Here J is the connected centralizer of $\gamma_{<r}$ and the terms $e(G)$ and $e(J)$ are the Kottwitz signs [Kot83] of the connected reductive groups G and J. The tori T_G and T_J are the minimal Levi subgroups in the quasi-split inner forms of G and J, and ϵ_L is the ϵ-factor at $s = 1/2$ of the given virtual Galois representation. Finally, the term Δ_{II}^{abs} is an absolute version of the corresponding term of the Langlands-Shelstad endoscopic transfer factor [LS87, §3.3]. What we mean here by the word "absolute" is that while the term Δ_{II} of Langlands-Shelstad is associated to a group G and an endoscopic group $G^{\mathfrak{e}}$, the term Δ_{II}^{abs} depends only on the group G, and moreover one obtains the Langlands-Shelstad term Δ_{II} as a quotient of the terms Δ_{II}^{abs}, with the one for G in the numerator and the one for $G^{\mathfrak{e}}$ in the denominator.

If we apply the formula (1.0.1) to a regular supercuspidal representation $\pi_{(S,\theta)}$ and a regular semi-simple element $\gamma \in S(F)$ that is very far from the identity (this is the special case in which we have been able to remove the compactness hypothesis), we obtain as a consequence of (1.0.2) the following formula for the un-normalized Harish-Chandra character $\Theta_{\pi_{(S,\theta)}}(\gamma)$

$$\frac{e(G)\epsilon_L(X^*(T_G)_{\mathbb{C}} - X^*(S)_{\mathbb{C}})}{|D(\gamma)|^{\frac{1}{2}}} \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\text{abs}}[a,\chi](\gamma^w)\theta'(\gamma^w). \tag{1.0.3}$$

Here we have set $\theta' = \epsilon^{\text{ram}} \cdot \theta$ using the fact that ϵ^{ram} is a character of $S(F)$.

Before we discuss the main implication of (1.0.2) and (1.0.3), let us consider some of its features. First, none of the terms in (1.0.2) involve Bruhat-Tits theory in their construction. Rather, they come from Lie-theory and basic p-adic arithmetic and are thus more elementary (the reader might argue that ϵ-factors of non-abelian local Galois representations are not elementary, but a result of Kottwitz computes the particular ϵ-factor we are dealing with in elementary terms, see [Kal15, §5.5]). Note also that the first three terms in (1.0.2) depend only on the stable conjugacy class of $\gamma_{<r}$, and it is just the term Δ_{II}^{abs} that depends on the full triple $(S, \theta, \gamma_{<r})$.

Another interesting feature of the character formula (1.0.3) is that it provides an interpretation of most of the Langlands-Shelstad endoscopic transfer factor in terms of the characters of supercuspidal representations. Recall that the Langlands-Shelstad transfer factor is given as a product

$$\epsilon_L \cdot \Delta_I \cdot \Delta_{II} \cdot \Delta_{III_1} \cdot \Delta_{III_2} \cdot \Delta_{IV}.$$

In view of (1.0.3), each of the factors ϵ_L, Δ_{II}, Δ_{III_2}, and Δ_{IV} has an interpretation as the quotient of a piece of the character formula for G by the corresponding piece for $G^{\mathfrak{e}}$. The factors ϵ_L, Δ_{II}, and Δ_{IV}, are directly visible in

(1.0.3), and so is also the factor Δ_{III_2}, being the quotient of the character θ by the corresponding character on the side of $G^{\mathfrak{e}}$. The factor Δ_I also appears, albeit in a more subtle way: It measures which representation in the L-packet is generic, as we will discuss in Subsection 6.2. We believe that in this way the character formula (1.0.3) sheds a different light on the Langlands-Shelstad transfer factor and shows that almost all parts of it are actually not of strictly endoscopic nature (except for the term Δ_{III_1}, which is indeed purely endoscopic and not a relative term in the sense discussed here). We hope that this point of view will be fruitful in the study of more general functoriality beyond the endoscopic case, and might in particular help with the study of the transfer factors occurring there [Lan13].

The most striking feature of (1.0.3) is however the following: Each term in it has an interpretation for groups G defined over an arbitrary local field, not just a p-adic field, and when $F = \mathbb{R}$ then (1.0.3) becomes the formula for the character of the discrete series representation of the real group $G(\mathbb{R})$ associated to the elliptic maximal torus S and the character $\theta' : S(\mathbb{R}) \to \mathbb{C}^\times$, i.e. the well-known formula

$$(-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\theta'(w^{-1}\gamma)}{\prod_{\alpha>0}(1-\alpha(w^{-1}\gamma)^{-1})}.$$

This can be seen as an instance of Harish-Chandra's Lefschetz principle, which suggests a mysterious analogy between the behaviors of real and p-adic reductive groups. In fact, if we consider the full character formula (1.0.1), we see that it combines two extreme behaviors – the behavior at elements near the identity ($\gamma = \gamma_{\geq r}$), which is controlled by $\widehat{\mu}$, and the behaviour at elements far from the identity ($\gamma = \gamma_{<r}$), to which all the roots of unity contribute. The Fourier-transform of the orbital integral $\widehat{\mu}$ appears to belong to the world of finite groups of Lie type. For example, when π has depth zero, the term $\widehat{\mu}$ is a lift [DR09, Lemma 12.4.3] of a Green function, expressing the character of a cuspidal representation of a finite group of Lie type at a unipotent element. The roots of unity on the other hand seem to belong to the world of real reductive groups. This suggests that the behavior of p-adic groups is an interpolation between the behavior of finite groups of Lie type and the behavior of real reductive groups.

The close parallel between the characters of regular supercuspidal representations at shallow elements and the characters of real discrete series, besides being alluring in its own right, also has practical value, which brings us to the third main goal of this paper – the construction and study of L-packets of regular supercuspidal representations and their matching with Langlands parameters. The original approach [Lan89] of Langlands to the construction of L-packets of discrete series representations for real reductive groups was to first extract from the Langlands parameter a character of the elliptic maximal torus and to then use this character to write down the Harish-Chandra characters of the constituents of the L-packet. The recent explicit constructions of L-packets for p-adic groups [DR09], [Ree08], [Kal15] have followed this procedure to the extent that they extract from the Langlands parameter a character θ of an elliptic maximal torus, but then they determine the constituents of the L-packet not via their Harish-Chandra characters, but by plugging in some modification of the character θ into Adler's construction in the case of $r > 0$, or into the construction of [DR09, §4.4] in the case of $r = 0$. The works of Adler-Spice [AS09], DeBacker-Reeder [DR09], and DeBacker-Spice [DS18] on the character formula for supercuspidal representations and our reinterpretation of it from the first part of this paper allow us to implement a much closer analog of Langlands' construction and use it to construct the L-packets that consist of regular supercuspidal representations and associate them to Langlands parameters.

The class of parameters we consider in this paper contains as special cases those considered in the above mentioned papers, but is much larger. More precisely, it consists of those discrete Langlands parameters $\varphi : W_F \to {}^LG$ for which $\varphi(P_F)$ is contained in a maximal torus of $\widehat{G}$ and $\mathrm{Cent}(\varphi(I_F), \widehat{G})$ is abelian (as well as a small amount of slightly more complicated parameters that we need in order to obtain a balanced theory). Guided by the character formula (1.0.3) we assemble the L-packet corresponding to a given parameter in the same way as Langlands constructs the packets of real discrete series representations – by writing down (a piece) of the Harish-Chandra character of each constituent of the L-packet. Our construction is nonetheless completely explicit: Given a parameter, we explicitly give the inducing data for each constituent of the L-packet. Conversely, one can also explicitly recover the L-parameter from this inducing data. Important for this is the fact that the notion of a tame regular elliptic pair (S, θ) has a direct interpretation in terms of ${}^LG$.

Let us now describe the construction of L-packets in more detail. Initially it follows the framework laid out in [Kal15]. A parameter φ satisfying the above conditions determines an algebraic torus S. We use χ-data to produce an embedding of ${}^Lj_\chi : {}^LS \to {}^LG$ whose image contains the image of φ, and hence leads to a factorization $\varphi = {}^Lj_\chi \circ \varphi_{S,\chi}$, after which the parameter $\varphi_{S,\chi} : W_F \to {}^LS$ leads to a character $\theta_\chi : S(F) \to \mathbb{C}^\times$ via the Langlands correspondence for tori. The torus S comes equipped with a stable class of embeddings into G (and in fact into any inner form of G). For any embedding $j : S \to G$ belonging to this stable class, we obtain an elliptic maximal torus $jS \subset G$ with a character $j\theta_\chi$ of it. It is at this point that the construction of the current paper diverges from the previous constructions. We write down the formula

$$e(G)\epsilon_L(X^*(T_G)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(G,jS)(F)/jS(F)} \Delta_{II}^{\mathrm{abs}}[a,\chi](\gamma^w) j\theta_\chi(\gamma^w),$$

and demand that π_j be the regular supercuspidal representation whose normalized character at shallow elements $\gamma \in jS(F)$ is given by this formula. In practice we ensure that this demand is met by explicitly providing the pair (S, θ) that parameterizes the regular supercuspidal representation, but we feel that the difference in point of view is essential. At this point, a remark is in order about the choice of χ-data involved. In [Kal15, §5.2] we spent a lot of effort to choose the correct χ-data so that the character θ_χ of S we obtain would be the right one for Adler's construction. From the current point of view, the choice of χ-data is irrelevant. This is because both θ_χ and $\Delta_{II}^{\mathrm{abs}}[a,\chi]$ depend on this choice in a parallel way and the dependence cancels in the product. However, $\Delta_{II}^{\mathrm{abs}}[a,\chi]$ also depends on a-data, and there is no other object in the character formula with this dependence. This means that the burden is now on choosing the a-data correctly. It turns out that this choice is given by a simple formula (4.10.1) that is uniform for real and p-adic groups. The only difference in the p-adic case is that one needs to pay attention to the first upper numbering filtration subgroup of inertia whose image under φ is detected by a given root of $\widehat{G}$. This is reminiscent of the study of the jumps of an admissible character in the work of Bushnell and Henniart [BH05a], [BH05b]. In fact, our work here might be seen as a generalization to arbitrary tamely ramified p-adic groups of the work of Bushnell-Henniart, insofar as both have the goal of giving an explicit realization of the local Langlands correspondence.

Once the representations π_j are determined, the L-packet is defined to be the set $\{\pi_j\}$ where j runs over all rational classes of embeddings of S into G. The internal parameterization of this L-packet is again done as in [Kal15, §5.3], the only difference being that now we are using the cohomology functor $H^1(u \to W, -)$ introduced in [Kal16] instead of the set $B(G)_{\text{bas}}$ used in [Kal15]. This allows us to uniformly treat all connected reductive groups, without conditions on the center. A reader interested in having a parameterization in terms of $B(G)_{\text{bas}}$, say for the purpose of studying Rapoport-Zink spaces, can either replace in the construction all occurrences of $H^1(u \to W, -)$ with $B(-)_{\text{bas}}$, or appeal to the general results of [Kal18b].

We now give a brief overview of the contents of this paper. Section 3 contains the study of regular supercuspidal representations. In Subsection 3.1 we collect some basic facts about p-adic tori, and in particular extend Yu's theorem [Yu09, Theorem 7.10] that the local Langlands correspondence for tamely ramified tori preserves depth from the case of positive depth to the case of characters vanishing on the Iwahori subgroup and on the maximal bounded subgroup of a torus. In Subsection 3.4 we classify the regular depth-zero supercuspidal representations of tamely ramified groups. This is based on the notion of a maximally unramified maximal torus of a tamely ramified group, that generalizes the notion of an unramified maximal torus of a group that splits over an unramified extension. This notion, suggested by Dick Gross, already appears in [Roe11], where the regular depth-zero supercuspidal representations of ramified unitary groups are studied. We then review and extend results of DeBacker [DeB06] on the parameterization of such tori, focusing on the case of elliptic tori that will be needed later. Using these results, we classify the regular depth-zero supercuspidal representations of tamely ramified groups, extending results of DeBacker-Reeder [DR09]. The main hurdle in the construction of regular depth-zero supercuspidal representations is that if S is a maximally unramified maximal torus of the connected reductive group G, then the equality $S(F) = S(F)_0 \cdot Z(G)(F)$ is not always true, as was pointed out to us by Cheng-Chiang Tsai. This equality holds in the unramified case, as well as in the case of ramified unitary groups, and makes the passage from a cuspidal representation of a parahoric subgroup of $G(F)$ to a supercuspidal representation of $G(F)$ straightforward. This equality is also equivalent to the technical condition imposed on π_{-1} in [Mur11]: the $G(F)$-conjugacy class of pairs (S, θ) corresponding to π_{-1} satisfies [Mur11, Definition 10.1(3)] precisely when $S(F) = S(F)_0 Z(G)(F)$. We deal with the additional difficulty in the general ramified case in Subsubsection 3.4.4 by exploiting the fact that the Deligne-Lusztig variety $\tilde{X}(\dot{w})$ associated to $\dot{w} \in N(T^*)$ (notation as in [DL76, §1.8]) admits an action of $\mathbf{T}_{\text{ad}}(w)^F$ by conjugation.

The rest of Section 3 is devoted to the study of the positive-depth case. In Subsection 3.5 we review the study of Hakim-Murnaghan on when different Yu-data produce the same representation and remove a hypothesis from their results, namely Hypothesis $C(\vec{G})$ of [HM08, §2.6]. In Subsection 3.6 we introduce the notion of a *Howe factorization* and prove that any pair (S, θ) consisting of a tame maximal torus and a character possesses a Howe factorization, generalizing to arbitrary reductive groups the Howe factorization lemma. In Subsection 3.7 we apply these results to the classification of regular supercuspidal representations of positive depth. In Subsubsection 3.7.1 we define the notion of *regular Yu-data*. In Subsubsection 3.7.2 we define the notion of a *tame regular elliptic pair* and show that it specializes in the case of GL_N to the classical notion of an admissible character. In Subsubsection 3.7.3 we use Howe factorization and the results of Hakim-Murnaghan to show that G-equivalence classes of regular Yu-data are in a natural bijection with $G(F)$-conjugacy classes of tame regular elliptic pairs. This is done in under the assumption that p does not divide the order of the fundamental group of the derived subgroup of G, which is also imposed in Subsections 3.5 and 3.6. We remove this assumption in Subsubsection 3.7.4, where we only require that p is odd and not a bad prime for G.

Before moving on to Section 4 we mention here a recent draft [Hak17] that was sent to us after this paper was written, in which Hakim reinterprets Yu's construction and gives a parameterization of the resulting representations in terms of a different kind of data. His interpretation has the advantage that the refactorization process studied in [HM08] becomes unnecessary. The goal and results of this draft are quite disjoint from ours. It would be interesting to see if the two approaches can be combined.

Section 4 is devoted to our reinterpretation of the Adler-DeBacker-Spice character formula. The technical heart of this section is Subsection 4.5, in which we give a formula for a certain subset $\mathrm{ord}_x(\alpha) \subset \mathbb{R}$ associated by [DS18, Definition 3.1.3] to a tame maximal torus T of G, a symmetric root α of T, and a point x in the Bruhat-Tits building of T seen as embedded into the building of G. This set plays a fundamental role in the character formula, because all roots of unity occurring in the formula are defined based on it. According to [DS18, Corollary 3.1.9] there are only two possibilities for this set and Proposition 4.5.1 shows that these possibilities are distinguished by the toral invariant introduced in [Kal15, §4]. After giving the definition of the term $\Delta_{II}^{\mathrm{abs}}$ in Subsection 4.6 we are in a position to rewrite the character formula. We need however to pay attention to the technical assumption that $G^{d-1}(F)/Z(G)(F)$ is compact, under which the character formula of [AS09] and [DS18] is valid. For toral supercuspidal representations this assumption is automatically satisfied and we can write the full character formula in this case, which is done in Subsection 4.8. For general regular supercuspidal representations $\pi_{(S,\theta)}$ we are able to show in Subsection 4.4 that this assumption can be dropped provided we consider sufficiently shallow elements belonging to the torus S. We use this fact, together with a computation in the depth-zero case done in Subsection 4.9, to prove (1.0.3) in Subsection 4.10. We conclude with Subsection 4.11, where we compare (1.0.3) with the character formula for real discrete series representations.

Section 5 contains the construction of regular supercuspidal L-packets. We also give a description of the internal structure of each L-packet Π_φ by showing that it has a simply transitive action of the abelian group $\pi_0(S_\varphi^+)^D$. In order to convert this into a bijection, we need to know that the choice of a Whittaker datum for the quasi-split group G determines a base point in the L-packet Π_φ, in accordance with the strong form of Shahidi's tempered L-packet conjecture [Sha90, §9]. Due to the technical compactness assumption necessary for the current form of the Adler-DeBacker-Spice character formula for elements close to the identity, we are not in a position to do so for general regular supercuspidal L-packets. For the same reason, we can only prove stability or endoscopic transfer for these packets for shallow elements, but not for general regular semi-simple elements. Both of these points will be addressed in forthcoming joint work with DeBacker and Spice, based on ongoing work of Spice on removing the compactness assumption from [DS18].

We are however able to prove these statements for toral L-packets, which are the topic of Section 6, where we specialize the construction of L-packets to the case of toral supercuspidal representations. These are the representations obtained from a Yu-datum for which the twisted Levi sequence is of the form $S = G^0 \subsetneq G^1 = G$, where S is an elliptic maximal torus of G. For these representations the compactness assumption is satisfied and thus the Adler-DeBacker-Spice character formula is valid for general elements, rather than just for shallow elements. Using it, we are able to prove the existence and uniqueness of a generic constituent in each compound L-packet as well as the stability and endoscopic transfer of these L-packets (the stability of toral L-packets under the additional assumption that S is unramified was already shown in [DS18]). We expect the same arguments to apply to the case of the general regular supercuspidal L-packets of Section 5, once the compactness assumption on the Adler-DeBacker-Spice character formula has been removed.

Acknowledgements: The initial spark for this paper came from a remark of Robert Kottwitz that the pieces ϵ_L and Δ_{II} of the Langlands-Shelstad transfer factor can be directly observed in the character formulas for supercuspidal representations of SL_2 due to Sally and Shalika. We are grateful to Kottwitz for drawing our attention to this. We were also influenced by Moshe Adrian's thesis [Adr10] and subsequent paper [Adr13], which carries out in the case of GL_n (for prime n) the main idea we employ here – the description of the local Langlands correspondence in terms of Harish-Chandra characters. We further thank Cheng-Chiang Tsai for multiple helpful discussions about Bruhat-Tits theory and in particular for a useful counterexample that he provided, and Jeffrey Hakim for his careful reading and feedback.

2 Notation and assumptions

2.1 Assumptions on the ground field and the group

Throughout most of the paper, F denotes a non-archimedean local field of zero or positive characteristic. The only exceptions to this are §3.2 and §5.1, where F can be any field, and §4.11, where $F = \mathbb{R}$.

For convenience, we collect here the assumptions on F placed in different parts of the paper. In §3.1-§3.4 there are no further assumptions on F or G. Starting with §3.5 we assume that the residual characteristic of F is odd and that G splits over a tame extension of F; these assumptions are kept throughout. In §3.6 and §3.7, we assume further that the residual characteristic is not a bad prime for G and does not divide the order of $\pi_1(G_{\mathrm{der}})$. The last of these assumptions is only for technical convenience and is removed in §3.7.4.

We recall from [SS70, I,§4] the list of bad primes for each irreducible root system: For type A_n there are no bad primes, for types B_n, C_n, or D_n the only bad prime is 2, for types E_6, E_7, F_4, or G_2 the bad primes are 2 and 3, and for type E_8 the bad primes are 2, 3, and 5. A prime is bad for G if it is bad for some irreducible component of its absolute root system.

In §4.5, §4.6 and §4.7 the only assumption on the local field F is that its residual characteristic is odd. For the rest of §4 we assume further that the residual characteristic is not a bad prime for G.

In §5 and §6 we assume that the residual characteristic is odd, is not a bad prime for G, and does not divide $|\pi_0(Z(G))|$. All prime divisors of $|\pi_0(Z(G))|$ are bad primes unless G has components of type A_n. If G has a component of type A_n, a sufficient condition would be $p \nmid (n+1)$. We are moreover forced to assume that the characteristic of F is zero due to the usage of [Kal16], where this assumption is made. We believe that the results of [Kal16] are valid without this assumption, but have not checked this carefully. In the meantime, if [Kal16] is replaced by [Kot], the characteristic zero assumption can be dropped at the expense of possibly not reaching all inner forms.

Finally, in §6.3 we must assume that F has characteristic zero and large residual characteristic. More precisely, we require $p \geq (2+e)n$, where e is the ramification degree of $F/\mathbb{Q}_p$ and n is the dimension of a faithful rational representation of G. A result [DR09, App. B] of DeBacker-Reeder ensures that then the exponential map converges for all topologically nilpotent elements of the Lie algebra of G.

In some parts of the paper we appeal to papers such as [LS87] or [KS99], where a blanket assumption is made that the ground field is of characteristic zero. It is however easy to check that for the results we use this assumption is not needed.

The following assumptions are cruder than necessary but easier to remember: If F has characteristic zero, $p > 5$ and $p \nmid n+1$ for any component of G of type A_n, then all results of this paper hold except for §6.3, for which the added assumption $p \geq (2+e)n$ is sufficient.

2.2 Further notation

We denote the ring of integers of F by O_F, its maximal ideal by $\mathfrak{p}_F$, and its residue field by k_F, of cardinality q. We fix a separable closure F^s of F and let $\Gamma = \Gamma_F$ be the Galois group of F^s/F, $W = W_F$ the Weil-group, $I = I_F$ the inertia group, and $P = P_F$ the wild inertia group. If E/F is a finite separable extension, which we will assume to be contained in F^s, we will use the subscript E to denote the analogous objects relative to E instead of F. Moreover, we will denote the relative Galois group of E/F by $\Gamma_{E/F}$ and the relative Weil group by $W_{E/F}$. We will write F^u for the maximal unramified extension of F within F^s and by Fr the element of $\Gamma_{F^u/F}$ that induces the automorphism $x \mapsto x^q$ on the residue field $\overline{k_F}$.

Given a connected reductive group G defined over F, we denote by G_{der} its derived subgroup, by G_{sc} and G_{ad} the simply connected cover and adjoint quotient of G_{der}, and by $\mathfrak{g}$ the Lie-algebra of G. For an element $g \in G$ we will write $\text{Ad}(g)$ for the conjugation action of g on G (i.e. $x \mapsto gxg^{-1}$) as well as for the adjoint action of g on $\mathfrak{g}$. We will write $\mathfrak{g}^*$ for the dual space of $\mathfrak{g}$ and $\text{Ad}^*(g)$ for the coadjoint action of g.

Whenever we refer to a maximal torus $S \subset G$, we will always assume that it is defined over F, unless explicitly stated otherwise. We will write $N(S,G)$ for the normalizer of S in G and $\Omega(S,G) = N(S,G)/S$ for the absolute Weyl group, a finite algebraic group defined over F. We write $R(S,G)$ for the corresponding set of roots. This set has an action of Γ and for any $\alpha \in R(S,G)$ we will write Γ_α and $\Gamma_{\pm\alpha}$ for the stabilizers of the subsets $\{\alpha\}$ and $\{\alpha, -\alpha\}$ respectively, and F_α and $F_{\pm\alpha}$ for the corresponding fixed subfields of F^s. Then $F_\alpha/F_{\pm\alpha}$ is an extension of degree at most 2. Following [LS87] we will call α symmetric if the degree of this extension is 2, and asymmetric if the degree is 1. Moreover, following [AS09] we will call α ramified or unramified if the extension $F_\alpha/F_{\pm\alpha}$ is such. Note that α is symmetric and ramified if and only if it is inertially symmetric in the sense if [Kal15]. For each $\alpha \in R(S,G)$ we have the 1-dimensional root subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$, which is defined over F_α.

We will write $\mathcal{B}^{\text{red}}(G,F)$ for the reduced Bruhat-Tits building of $G(F)$ and $\mathcal{A}^{\text{red}}(T,F)$ for the apartment associated to any maximal torus of G which is maximally split (this notation is slightly different than the one used by other authors, who prefer to write $\mathcal{A}^{\text{red}}(A_T,F)$, where A_T is the maximal split subtorus of T). For any $x \in \mathcal{B}^{\text{red}}(G,F)$ we shall write $G(F)_x$ for the stabilizer of x in $G(F)$, $G(F)_{x,0}$ for the parahoric subgroup associated to x, and $G(F)_{x,r}$ for the Moy-Prasad filtration subgroup [MP94, MP96] at depth $r \in \mathbb{R}_{\geq 0}$. On the Lie-algebra we have the analogous filtration lattices $\mathfrak{g}(F)_{x,r}$ for any $r \in \mathbb{R}$. It is sometimes convenient to use the notation $G(F)_{x,r:s} = G(F)_{x,r}/G(F)_{x,s}$ for $r < s$, as well as $G(F)_{x,r+} = \bigcup_{s>r} G(F)_{x,s}$.

In the special case $G = \text{Res}_{E/F}\mathbb{G}_m$ for a finite separable extension E/F the reduced Bruhat-Tits building is a singleton and the Moy-Prasad filtration can be described simply as $E_0^\times = O_E^\times$ and $E_r^\times = 1 + \mathfrak{p}_E^{\lceil er \rceil}$ for $r > 0$, where e is the ramification degree of E/F. The corresponding Lie-algebra filtration on $\mathfrak{g}(F) = E$ is given by $E_0 = O_E$ and $E_r = \mathfrak{p}_E^{\lceil er \rceil}$ for $r \in \mathbb{R}$.

3 Regular supercuspidal representations

3.1 Basics on p-adic tori

Let S be a torus defined over F. The topological group $S(F)$ has a unique maximal bounded subgroup $S(F)_b$ (which is also the unique maximal compact subgroup, as F is locally compact) and this subgroup is equipped with a decreasing filtration $S(F)_r$ indexed by the non-negative real numbers, namely the Moy-Prasad filtration corresponding to the unique point in the reduced Bruhat-Tits building of S. When the splitting field of S is wildly ramified over F it is known that this filtration exhibits some pathologies, which are not present when for some tamely ramified extension E/F the torus $S \times E$ becomes induced, see [Yu15, §4]. In particular, the pathologies are not present when the splitting field of S is tamely ramified over F. We will call such S *tame* for short.

We recall the definition of $S(F)_r$. For $r = 0$ there are two ways to define the subgroup $S(F)_0$. The torus S possesses an lft-Neron model $\mathfrak{S}^{\text{lft}}$ by [BLR90, §10]. This is a smooth group scheme over O_F satisfying a certain universal property. It is locally of finite type and the maximal subgroup-scheme of finite type is called the ft-Neron model $\mathfrak{S}^{\text{ft}}$. Both models share the same neutral connected component, called the connected Neron model $\mathfrak{S}^\circ$. Then $S(F)_0 = \mathfrak{S}^\circ(O_F)$. One also has

$$\mathfrak{S}^{\text{ft}}(O_F) = S(F)_b = \{s \in S(F) | \forall \chi \in X^*(S),\ \text{ord}(\chi(s)) \geq 0\}.$$

Note that $S(F)/S(F)_b$ is a finitely generated free abelian group.

A second way to define $S(F)_0$ is via the Kottwitz homomorphism. This is a functorial surjective homomorphism $S(F) \to X_*(S)_I^{\text{Fr}}$ introduced in [Kot97, §7], see in particular [Kot97, §7.2,§7.6]. The kernel of this homomorphism is $S(F)_0$, and the preimage of the torsion subgroup $[X_*(S)_I^{\text{Fr}}]_{\text{tor}}$ is $S(F)_b$. See the first note at the end of [Rap05].

For $r > 0$, the definition of $S(F)_r$ is

$$S(F)_r = \{s \in S(F)_0 | \forall \chi \in X^*(S),\ \text{ord}(\chi(s) - 1) \geq r\},$$

see [MP96, §3.2] and [Yu15, §4.2]. Denoting by $S(F)_{r+}$ the union of $S(F)_s$ over $s > r$ we see that $S(F)_{0+}$ is precisely the pro-p-Sylow subgroup of $S(F)_0$.

These descriptions make it clear that when F is unramified $S(F)_0 = S(F)_b$ and $S(F)_{0+}$ is the pro-p-Sylow subgroup of $S(F)_b$ and hence of $S(F)$. In order to generalize these statements we introduce the following notions.

Definition 3.1.1. We say that a torus S is *inertially induced* if the following equivalent conditions hold:

1. $X^*(S)$ has a basis invariant under the action of I_F.
2. $X_*(S)$ has a basis invariant under the action of I_F.
3. $S \times F^u$ is an induced torus, where F^u is the maximal unramified extension of F.

We say that S is *wildly induced* if the following equivalent conditions hold:

1. $X^*(S)$ has a basis invariant under the action of P_F.
2. $X_*(S)$ has a basis invariant under the action of P_F.
3. $S \times F^{\mathrm{tr}}$ is an induced torus, where F^{tr} is the maximal tamely ramified extension of F.

□

It is obvious that an unramified torus is inertially induced, and that a tame torus is wildly induced. The notion of wildly induced is the same as "Condition (T)" in [Yu15, §4.7.1].

Fact 3.1.2.

1. If S is inertially induced, then $S(F)_0 = S(F)_b$.
2. If S is wildly induced, then
$$S(F)_r = \{s \in S(F) | \forall \chi \in X^*(S),\ \mathrm{ord}(\chi(s) - 1) \geq r\}. \tag{3.1.1}$$
In particular, $S(F)_{0+}$ is precisely the pro-p-Sylow subgroup of $S(F)_b$ and hence of $S(F)$.

□

Proof. When S is inertially induced, $X_*(S)_I$ is torsion-free, hence the first point. When S is wildly induced, $X_*(S)_I = [X_*(S)_P]_{I/P}$ has no p-torsion and hence $S(F)_b/S(F)_0$ is a finite group of order prime to p. The right-hand side of (3.1.1) is contained in $S(F)_b$ and is a pro-p group, hence lies in $S(F)_0$ and thus equals $S(F)_r$. ■

The second part of this Fact was also proved in [Yu15, 4.7.2] by a different method.

Lemma 3.1.3. If $1 \to A \to B \to C \to 1$ is an exact sequence of tame tori and $r > 0$, then
$$1 \to A(F)_r \to B(F)_r \to C(F)_r \to 1$$
is also exact. If $A \to B$ is an isogeny of tame tori whose kernel has order prime to p and $r > 0$, then $A(F)_r \to B(F)_r$ is a bijection. □

Proof. Let E/F be a tame finite Galois extension splitting the tori. We have $A(E)_r = X_*(A) \otimes E_r^\times$ and $A(F)_r = A(E)_r^{\Gamma_{E/F}}$, according to (3.1.1). Applying the functor X_* to the exact sequence of tori produces an exact sequence of finite rank free $\mathbb{Z}$-modules with $\Gamma_{E/F}$ action, which remains exact after $\otimes_{\mathbb{Z}} E_r^\times$, leading to

$$1 \to A(E)_r \to B(E)_r \to C(E)_r \to 1.$$

Taking $\Gamma_{E/F}$-invariants and applying [Yu01, Proposition 2.2] finishes the proof.

For the second point we recall that for any integer n and abelian group T the groups $\mathrm{Tor}_1^{\mathbb{Z}}(T, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Tor}_0^{\mathbb{Z}}(T, \mathbb{Z}/n\mathbb{Z})$ are the kernel and cokernel of the multiplication-by-n-map on T, and thus both vanish if T is pro-finite with pro-order prime to n. Since $X_*(A) \to X_*(B)$ is an injection with finite cokernel of order prime to p and $E_r^\times$ is pro-p, the functor $X_*(-) \otimes_{\mathbb{Z}} E_r^\times$ turns the isogeny $A \to B$ into a $\Gamma_{E/F}$-equivariant bijection, which remains bijective after taking $\Gamma_{E/F}$-fixed points. ■

Lemma 3.1.4. Let $1 \to A \to B \to C \to 1$ be an exact sequence of tori and assume that A is inertially induced.

1. The sequence $0 \to X_*(A)_I \to X_*(B)_I \to X_*(C)_I \to 0$ is exact, where I denotes the inertia subgroup of Γ.
2. For $r = 0$ and $r = 0+$ the sequence $1 \to A(F)_r \to B(F)_r \to C(F)_r \to 1$ is exact.

□

Proof. For the first point we apply again X_* to the exact sequence of tori to obtain an exact sequence of finite-rank free $\mathbb{Z}$-modules with Γ-action. We claim that after taking inertial co-invariants the sequence remains exact. The only issue would be the injectivity of $X_*(A)_I \to X_*(B)_I$. We may of course replace I by a suitable finite quotient through which it acts. The kernel of this map is the image of the connecting homomorphism $H_1(I, X_*(C)) \to X_*(A)_I$. But $H_1(I, X_*(C))$ is finite, while by assumption $X_*(A)_I$ is torsion-free, so this connecting homomorphism is zero.

For the second point, we see that the case of $r = 0+$ follows from the case of $r = 0$ because $(-)_{0+}$ is the pro-p-Sylow subgroup of $(-)_0$ as remarked above. For the case $r = 0$ we consider the commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & A(F^u) \arrow[r] \arrow[d] & B(F^u) \arrow[r] \arrow[d] & C(F^u) \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & X_*(A)_I \arrow[r] & X_*(B)_I \arrow[r] & X_*(C)_I \arrow[r] & 0 \end{tikzcd}$$

The exactness of the top row on the right follows from $H^1(I, A(F^s)) = 0$ due to Steinberg's theorem [Ste65, Theorem 1.9]. The vertical maps are surjective. The kernel-cokernel lemma implies that the sequence

$$1 \to A(F^u)_0 \to B(F^u)_0 \to C(F^u)_0 \to 1$$

is exact. It is well known that $H^1(\mathrm{Fr}, A(F^u)_0) = 0$, see e.g. [DR09, Lemma 2.3.1]. Taking Frobenius-invariants finishes the proof. ■

Example 3.1.5. It is tempting to hope that the above lemma might hold when $(-)_0$ is replaced by $(-)_b$. This is however not the case. Let E/F be a ramified quadratic extension of residual characteristic not 2, $B = \mathrm{Res}_{E/F}\mathbb{G}_m$, $A = \mathbb{G}_m$, and $A \to B$ the usual embedding. Then the exact sequence of F-points is

$$1 \to F^\times \to E^\times \to E^1 \to 1,$$

where E^1 is the subgroup of $E^\times$ of elements whose E/F-norm is 1, the first map is the natural embedding, and the second map sends $x \in E^\times$ to $x/\sigma(x)$, where σ is the non-trivial F-automorphism of E.

We have $A(F)_0 = A(F)_b = O_F^\times$, $B(F)_0 = B(F)_b = O_E^\times$, and $C(F)_b = E^1 \neq C(F)_0 = (1+\mathfrak{p}_E)$. The map $x/\sigma(x)$ maps $O_E^\times$ surjectively onto $(1+\mathfrak{p}_E)$. □

Every torus S defined over F has a maximal unramified subtorus $S' \to S$, characterized by $X_*(S') = X_*(S)^{I_F}$, as well as a maximal unramified quotient $S \to S''$, characterized by $X^*(S'') = X^*(S)^{I_F}$. One has $X^*(S') = X^*(S)_{I_F,\mathrm{free}}$ and $X_*(S'') = X_*(S)_{I_F,\mathrm{free}}$, i.e. the torsion-free quotient of the inertial coinvariants of $X^*(S)$ or $X_*(S)$ respectively.

Lemma 3.1.6. Let S be a torus defined over F and let $S' \subset S$ be the maximal unramified subtorus. The natural map

$$S'(F)_0/S'(F)_{0+} \to S(F)_0/S(F)_{0+}$$

is an isomorphism. □

Proof. The injectivity of this map is equivalent to $S'(F)_0 \cap S(F)_{0+} = S'(F)_{0+}$, which follows from the description of $S(F)_{0+}$ and $S'(F)_{0+}$ as the pro-p-Sylow subgroups of $S(F)_0$ and $S'(F)_0$. The surjectivity is equivalent to $S(F)_0 = S'(F)_0 \cdot S(F)_{0+}$. This follows by applying Lemma 3.1.4 with $r = 0$ and $r = 0+$ to the exact sequence $1 \to S' \to S \to S/S' \to 1$, provided we can show that $[S/S'](F)_{0+} = [S/S'](F)_0$. This is equivalent to saying that the special fiber of the connected Neron model of S/S', which is a smooth connected commutative algebraic group defined over k_F and hence a product of a torus and a unipotent abelian group, is purely unipotent. This statement is proved in [NX91, Theorem 1.3] (there is a blanket assumption that F has characteristic zero, which is however not used in that proof), or alternatively in [HN10, Proposition 3.12]. ■

Lemma 3.1.7. Let S be a torus defined over F. Then we have

$$H^2(\Gamma_F/I_F, S(F^u)_b) = H^2(\Gamma_F/I_F, S(F^u)_0) = H^2(\Gamma_F/I_F, S(F^u)_{0+}) = 0.$$

□

Proof. We shall use [Ser79, Ch. XIII, §1,Prop. 2], according to which for any torsion Γ_F/I_F-module A we have $H^2(\Gamma_F/I_F, A) = 0$. This applies in particular when A is the set of $\overline{k_F}$-rational points of a commutative linear algebraic group defined over k_F. Kottwitz's homomorphism leads to the exact sequence of Γ_F/I_F-modules

$$1 \to S(F^u)_0 \to S(F^u)_b \to [X_*(S)_I]_{\mathrm{tor}} \to 0.$$

From $H^2(\Gamma_F/I_F, [X_*(S)_I]_{\mathrm{tor}}) = 0$ we see that $H^2(\Gamma_F/I_F, S(F^u)_b) = 0$ would follow from $H^2(\Gamma_F/I_F, S(F^u)_0) = 0$. In the same way, $H^2(\Gamma_F/I_F, S(F^u)_0) =$

0 would follow from $H^2(\Gamma_F/I_F, S(F^u)_{0+}) = 0$, because $S(F^u)_0/S(F^u)_{0+} = \mathfrak{S}^\circ(\overline{k_F})$ implies $H^2(\Gamma_F/I_F, S(F^u)_0/S(F^u)_{0+}) = 0$.

We have $H^2(\Gamma_F/I_F, S(F^u)_{0+}) = \varinjlim H^2(\Gamma_{F'/F}, S(F')_{0+})$, where the colimit runs over the finite unramified extensions of F. The group $S(F')_{0+}$ is equal to $\varprojlim_r S(F')_{0+}/S(F')_r$, where $S(F')_r$ is the r-th filtration subgroup with respect to any admissible schematic connected filtration in the sense of [Yu15, §4]. One could take for example the connected Moy-Prasad filtration of [Yu15, §4.6.3] or the minimal congruent filtration of [Yu15, §5]. The steps of the filtration are discrete and the quotients are the $k_{F'}$-points of an abelian connected unipotent algebraic group U_r defined over k_F. From the inflation restriction sequence

$$H^1(\Gamma_{k_{F'}}, \mathsf{U}(\overline{k_F})) \to H^2(\Gamma_{k_{F'}/k_F}, \mathsf{U}(k_{F'})) \to H^2(\Gamma_{k_F}, \mathsf{U}(\overline{k_F}))$$

and the vanishing of the two outer terms we see that the middle term vanishes. From [Ser79, Chap. XII, §3, Lem. 3] we see that $H^2(\Gamma_{F'/F}, S(F')_{0+})$ vanishes, and hence that $H^2(\Gamma_F/I_F, S(F^u)_{0+})$ vanishes. ■

Consider now a tame torus S defined over F and its complex dual torus $\widehat{S}$. The local Langlands correspondence provides an isomorphism of abelian groups $H^1_{\text{cts}}(W_F, \widehat{S}) \to \mathrm{Hom}_{\text{cts}}(S(F), \mathbb{C}^\times)$. This bijection is functorial in S and is characterized uniquely by a short list of properties [Yu09, Theorem 7.5]. According to [Yu09, Theorem 7.10], if $\varphi \in H^1(W_F, \widehat{S})$ corresponds to $\theta : S(F) \to \mathbb{C}^\times$, then for any $r > 0$ the restriction $\theta|_{S(F)_r}$ is zero if and only if the restriction $\varphi|_{I^r}$ is zero. The following two lemmas extend this result to the restrictions of θ to $S(F)_0$ and $S(F)_b$.

Lemma 3.1.8. The restriction $\theta|_{S(F)_0}$ is trivial if and only if φ lies in the kernel of the restriction map $H^1(W_F, \widehat{S}) \to H^1(I_F, \widehat{S})$, or, equivalently, belongs to the image of the inflation map $H^1(W_F/I_F, \widehat{S}^{I_F}) \to H^1(W_F, \widehat{S})$. □

Proof. Assume first that S is split. The claim reduces immediately to the case $S = \mathbb{G}_m$, where it follows from the fact that the Artin reciprocity map $W_F \to F^\times$ carries I_F surjectively onto $O_F^\times$. Assume next that S is unramified. Let E/F be the splitting field of S and $R = \mathrm{Res}_{E/F}(S \times E)$. The kernel of the norm map $R \to S$ is an unramified torus. Applying Lemma 3.1.4 to the resulting sequence of unramified tori we obtain a surjection $S(E)_0 = R(F)_0 \to S(F)_0$. Thus $\theta|_{S(F)_0}$ is trivial if and only if $[\theta \circ N]|_{S(E)_0}$ is trivial. But $\theta \circ N$ is a character of the split torus $S(E)$ whose parameter is equal to $\varphi|_{W_E}$. According to the split case, $\theta \circ N$ is trivial on $S(E)_0$ if and only if $\varphi|_{W_E}$ has trivial restriction to I_E. But $I_E = I_F$ and the unramified case is complete.

Assume now that S is tamely ramified. Let $S' \subset S$ be the maximal unramified subtorus. According to Lemma 3.1.6, $\theta|_{S(F)_0}$ is trivial if and only if $\theta|_{S(F)_{0+}}$ and $\theta|_{S'(F)_0}$ are trivial. The parameter of $\theta|_{S'(F)}$ is the image of φ in $H^1(W_F, \widehat{S}_{I_F})$. If φ has trivial image in $H^1(I_F, \widehat{S})$, then it has trivial images in $H^1(I_F^{0+}, \widehat{S})$ and $H^1(I_F, \widehat{S}_{I_F})$, so we conclude that $\theta|_{S(F)_{0+}}$ and $\theta|_{S'(F)_0}$ are trivial. Conversely, if $\theta|_{S(F)_{0+}}$ is trivial, then the image of φ in $H^1(I_F^{0+}, \widehat{S})$ is trivial, so φ is inflated from $H^1(W_F/I_F^{0+}, \widehat{S})$ and its restriction to I_F is inflated from $H^1(I_F/I_F^{0+}, \widehat{S})$. The group I_F/I_F^{0+} is pro-cyclic, let x be a pro-generator and let $\bar{x}$ be the finite-order automorphism of $\widehat{S}$ through which x acts. We have $\widehat{S}_{I_F} = \widehat{S}/(1-\bar{x})\widehat{S}$. If $\theta|_{S'(F)_0}$ is also trivial, then the image of $\varphi|_{I_F}$ in $H^1(I_F/I_F^{0+}, \widehat{S}_{I_F})$ is zero and hence $\varphi|_{I_F}$ comes from an element of $H^1(I_F/I_F^{0+},(1-\bar{x})\widehat{S})$. But we claim that this cohomology group is zero. Indeed, let $N_{\bar{x}}:\widehat{S}\to\widehat{S}$ be the norm map for the action of $\bar{x}$. Evaluating 1-cocycles at the pro-generator x provides an isomorphism from $H^1(I_F/I_F^{0+},(1-\bar{x})\widehat{S})$ to the quotient of $\ker(N_{\bar{x}}|_{(1-\bar{x})\widehat{S}})$ by $(1-\bar{x})(1-\bar{x})\widehat{S}$. But $N_{\bar{x}}$ is zero on $(1-\bar{x})\widehat{S}$, so the numerator of this quotient is equal to $(1-\bar{x})\widehat{S}$. We claim that the denominator is also equal to that. This follows from the fact that the map $(1-\bar{x}):(1-\bar{x})\widehat{S}\to(1-\bar{x})\widehat{S}$ is an isogeny. Indeed, its kernel consists of those elements of $(1-\bar{x})\widehat{S}$ that are fixed by $\bar{x}$ and is thus equal to the intersection $(1-\bar{x})\widehat{S}\cap\widehat{S}^{\bar{x}}$. This intersection is contained in the kernel of the restriction of $N_{\bar{x}}$ to $\widehat{S}^{\bar{x}}$. But that restriction is just the $\mathrm{ord}(\bar{x})$-power map and its kernel is finite. ■

Note that the abelian group $\widehat{S}^{I_F}$ might be disconnected. In fact,

$$X^*(\widehat{S}^{I_F}/\widehat{S}^{I_F,\circ})=X_*(S)_{I_F,\mathrm{tor}}$$

which means that the disconnectedness of $\widehat{S}^{I_F}$ mirrors exactly the disconnectedness of the ft-Neron model of S. This motivates the following.

Lemma 3.1.9. The restriction $\theta|_{S(F)_b}$ is trivial if and only if φ belongs to the image of the inflation map $H^1(W_F/I_F,\widehat{S}^{I_F,\circ})\to H^1(W_F,\widehat{S})$. □

Proof. Let $S\to S''$ be the maximal unramified quotient of S and let $S_1\subset S$ be the kernel of this quotient. Thus $X_*(S_1)$ is the kernel of the norm map $X_*(S)\to X_*(S)$ for the action of inertia. Note that $X_*(S_1)^{I_F}=\{0\}$, which means that S_1 is inertially anisotropic and in particular $S_1(F)$ is compact.

We claim that $S(F)_b$ is the preimage of $S''(F)_0$. Indeed, the image of $S(F)_b$ in $S''(F)$ is compact and hence belongs to $S''(F)_b=S''(F)_0$. Thus the preimage $\Theta\subset S(F)$ of $S''(F)_0$ contains $S(F)_b$ so it is enough to show that it is compact. Now Θ is an open subgroup of $S(F)$, hence locally compact and σ-compact. The open mapping theorem [HR79, Theorem 5.29] implies that the surjection $\Theta\to S''(F)_0$ is open, and hence a quotient map. We thus have the exact sequence

$$1\to S_1(F)\to\Theta\to S''(F)_0\to 1$$

of Hausdorff topological groups, the outer terms of which are compact. Then [HR79, Theorem 5.25] implies that Θ is compact.

We conclude that the natural map $S(F)/S(F)_b\to S''(F)/S''(F)_0$ is injective. Since its cokernel is finite, the characters of $S(F)$ that are trivial on $S(F)_b$ are precisely those obtained from characters of $S''(F)$ that are trivial on $S''(F)_0$ by composing them with the natural map $S(F)\to S''(F)$. But the dual of the natural map $S(F)\to S''(F)$ is the map $\widehat{S}^{I_F,\circ}\to\widehat{S}$ and the statement follows from Lemma 3.1.8. ■

3.2 Review of stable conjugacy of tori

In this subsection we review the standard notions and results concerning stable conjugacy and transfer of tori between inner forms, mainly in order to have a convenient reference that does not impose conditions on the ground field. We work over an arbitrary ground field F with a fixed separable closure F^s and let $\Gamma = \mathrm{Gal}(F^s/F)$.

Let G and G' be connected reductive groups defined over F. Recall that an inner twist $\psi : G \to G'$ is an isomorphism $\psi : G \times F^s \to G' \times F^s$ such that $\psi^{-1}\sigma(\psi)$ is an inner automorphism of G for all $\sigma \in \Gamma$. Let $T \subset G$ be a maximal torus. Recall that T is said to transfer to G' if there exists $g \in G(F^s)$ such that $\psi \circ \mathrm{Ad}(g)|_T : T \to G'$ is defined over F, i.e. invariant under Γ. The image $T' \subset G'$ of $\psi \circ \mathrm{Ad}(g)|_T$ is a maximal torus of G' and one says that T and T' are stably conjugate. In the special case where $G = G'$ and $\xi = \mathrm{id}$ this recovers the usual notion of stable conjugacy of maximal tori of G. Note that since every torus splits over F^s, for any two tori $T, T' \subset G$ there exists $g \in G(F^s)$ such that $\mathrm{Ad}(g)T = T'$. However, usually the homomorphism $\mathrm{Ad}(g) : T \to T'$ will not be defined over F.

Fix a maximal torus $T \subset G$. Given any other maximal torus $T' \subset G$ choose $g \in G(F^s)$ such that $\mathrm{Ad}(g)T = T'$. Then $\sigma \mapsto g^{-1}\sigma(g)$ is an element of $Z^1(\Gamma_F, N(T,G))$ whose cohomology class $\mathrm{cls}(T')$ is independent of the choice of g. Two tori T' and T'' are conjugate in $G(F)$ if and only if $\mathrm{cls}(T') = \mathrm{cls}(T'')$, and are stably conjugate if and only if $\mathrm{cls}(T')$ and $\mathrm{cls}(T'')$ have the same image in $H^1(\Gamma_F, \Omega(T,G))$.

This criterion can be extended across inner forms, at least when the groups in question are adjoint, which we assume for the rest of this paragraph. Let $\psi_i : G \to G_i$ for $i = 1, 2$ be inner twists and let $T_i \subset G_i$ be maximal tori. Replace each ψ_i by $\psi_i \circ \mathrm{Ad}(g_i)$ for $g_i \in G(F^s)$ to achieve that $\psi_i(T) = T_i$. Then the class $\mathrm{cls}(T_i)$ of $\psi_i^{-1}\sigma(\psi_i) \in Z^1(\Gamma, N(T,G))$ is independent of the choice of g_i. The image of $\mathrm{cls}(T_i)$ in $H^1(\Gamma, G)$ is the class of ψ_i. The tori T_1 and T_2 are called rationally conjugate if there exists $g \in G(F^s)$ s.t. $\psi_2 \circ \mathrm{Ad}(g) \circ \psi_1^{-1} : G_1 \to G_2$ is defined over F and restricts to an isomorphism $T_1 \to T_2$. This is the case if and only if $\mathrm{cls}(T_1) = \mathrm{cls}(T_2)$. This implies in particular that the classes of ψ_1 and ψ_2 in $H^1(F, G)$ are equal. Furthermore, T_1 and T_2 are stably conjugate if and only if the images of $\mathrm{cls}(T_i)$ in $H^1(\Gamma, \Omega(T,G))$ are equal.

Note that for the purpose of checking stable conjugacy of tori we can always replace G by its adjoint group. This is not true for the purposes of checking rational conjugacy. The above discussion can be extended to rational conjugacy of not necessarily adjoint groups by replacing $H^1(\Gamma, -)$ with the cohomology sets $H^1(u \to W, Z \to G)$ or $H^1(P \to \mathcal{E}, Z \to G)$ of [Kal16] or [Kal18a], in the case of a local and global fields of characteristic zero.

The following result is well-known, e.g. [Kot86, §10], but we have not been able to find a reference that allows positive characteristic.

Lemma 3.2.1. Assume that F is local. If T is elliptic then it transfers to G'. $\square$

Proof. We assume that F is non-archimedean and refer to [Kot86, §10] for the proof in the archimedean case. As above we may assume without loss of generality that G is adjoint. One checks that T transfers to G' if and only if the class of ψ in $H^1(F, G)$ lies in the image of $H^1(F, T)$. Let G_{sc} be the simply connected cover of G and $Z \subset G_{\mathrm{sc}}$ its center. Let T_{sc} be the preimage of T in G_{sc}. The exact sequences of algebraic groups

$$1 \to Z \to G_{\mathrm{sc}} \to G \to 1 \qquad \text{and} \qquad 1 \to Z \to T_{\mathrm{sc}} \to T \to 1$$

give exact sequences of sheaves on $\mathrm{Spec}(F)$ for the fpqc topology. All groups above are smooth except possibly Z. For them, the first fpqc-cohomology group coincides with the first etale cohomology group by [ABD$^+$64, exp XXIV, Proposition 8.1]. Since T_{sc} is anisotropic, Tate-Nakayama duality implies that $H^2(F, T_{\text{sc}}) = 0$. On the other hand, Kneser's theorem [BT87, §4.7] implies that all inner twists of G_{sc} have vanishing first cohomology. This leads to the commutative diagram of pointed sets

$$\begin{tikzcd} H^1(F,T) \arrow[r] \arrow[d] & H^1(F,G) \arrow[d, hook] \\ H^2_{\text{fpqc}}(F,Z) \arrow[r, equal] & H^2_{\text{fpqc}}(F,Z) \end{tikzcd}$$

from which we conclude that the top map must be surjective. ■

Lemma 3.2.2. If G is quasi-split then any maximal torus $T' \subset G'$ transfers to G. □

Proof. This has been proved by Raghunathan [Rag04] and independently by Gille [Gil04]. The proofs are based on Steinberg's work [Ste65] on rational elements in conjugacy classes, which assumes that F is perfect. We learned from Jessica Fintzen that this assumption was later removed by Borel and Springer [BS68, §8.6]. Since neither of the papers [Rag04] and [Gil04] cites [BS68], which can cause confusion about the validity of their results, we have taken the opportunity to point this out here. ■

3.3 Short remarks about parahoric subgroups

Let G be a connected reductive group defined over F. Recall that Borovoi has defined in [Bor98, §1.3,§1.4] the algebraic fundamental group $\pi_1(G)$ of G. The assignment $G \mapsto \pi_1(G)$ is a functor from the category of connected reductive groups defined over F to the category of finitely generated abelian groups with Γ-action. Let L denote the completion of the maximal unramified extension of F. In [Kot97, §7] Kottwitz has constructed a surjective homomorphism $\kappa_G : G(L) \to \pi_1(G)_I$. It is a natural transformation from the identity functor to the functor $\pi_1(-)_I$. Note that in loc. cit. Kottwitz uses $X^*(Z(\widehat{G})^I)$ instead of $\pi_1(G)_I$. These two abelian groups are equal, and just as in [RR96] we prefer to use $\pi_1(G)_I$ because it is obviously a functor. In [PR08, Appendix], Haines and Rapoport prove that for any $x \in \mathcal{B}^{\text{red}}(G,F)$ one has

$$G(F)_{x,0} = G(F)_x \cap \ker(\kappa_G).$$

Corollary 3.3.1. Let $f : H \to G$ be a homomorphism of connected reductive groups defined over F, $x \in \mathcal{B}^{\text{ed}}(H,F)$ and $y \in \mathcal{B}^{\text{ed}}(G,F)$. Then

$$f(H(F)_{x,0}) \cap G(F)_y \subset G(F)_{y,0}.$$

In particular, if $T \subset G$ is a maximal torus, then

$$T(F)_0 \cap G(F)_y \subset G(F)_{y,0}.$$

□

Lemma 3.3.2. Let $x \in \mathcal{B}^{\text{red}}(G,F)$ and $r \geq 0$.

1. Let $K \subset G$ be a central torus and $\bar{G} = G/K$. The sequence
$$1 \to K(F)_r \to G(F)_{x,r} \to \bar{G}(F)_{x,r} \to 1$$
is exact if $r = 0$ or $r = 0+$ and K is inertially induced.

Assume now that G splits over a tame extension and $r > 0$.

2. The above sequence is exact without assuming that K is inertially induced.
3. Let $G' \subset G$ be a connected subgroup containing G_{der} and let $D = G/G'$. The following sequence is exact
$$1 \to G'(F)_{x,r} \to G(F)_{x,r} \to D(F)_r \to 1.$$
4. Let $G' \to G$ is an isogeny whose kernel has order prime to p. Then $G'(F)_{x,r} \to G(F)_{x,r}$ is a bijection.

□

Proof. The argument for the first point and $r = 0$ is essentially the same as for Lemma 3.1.4, where now one uses $\pi_1(-)_I$ instead of $X_*(-)_I$.

For all the other points, we use [BT72, Lemma 6.4.48] which shows that $G(F)_{x,r}$ is the direct product (as topological spaces) of $T(F)_r$ and the appropriate affine root subgroups, where T is a maximally unramified maximally split maximal torus, whose existence is guaranteed by [BT84, Corollary 5.1.12]. Since the maps $G \to \bar{G}$ and $G' \to G$ induces isomorphisms on the affine root subgroups, the claims reduce to the corresponding claims for the maps $T \to \bar{T}$ and $T' \to T$, where $T' = T \cap G'$ and $\bar{T} = T/K$. These follow from Lemmas 3.1.3 and 3.1.4. ■

3.4 Regular supercuspidal representations of depth zero

Let G be a connected reductive group defined over F. In this subsection we will define and classify regular depth-zero supercuspidal representations of G. This extends results of DeBacker-Reeder [DR09], where G was assumed to split over the maximal unramified extension of F.

In §3.4.1 we review results of DeBacker on the classification of $G(F)$-conjugacy classes of maximally unramified maximal tori, focusing on the elliptic case needed later on, and prove some additional results, particularly involving comparisons of various Weyl groups. In §3.4.2 we review the concepts of regular and non-singular characters due to Deligne-Lusztig and prove some supplementary results. The framework of the construction of regular depth-zero supercuspidal representations is provided by the results of Moy and Prasad [MP96] and is reviewed in §3.4.3. In order to apply this framework in the ramified case, we need a technical result that is not needed in the unramified case treated in [DR09]. Its necessity is explained in Example 3.4.21. This technical result, which we view as the heart of the construction of regular supercuspidal representations, is proved in §3.4.4, together with an extension of the Deligne-Lusztig character formula to our more general setting. In §3.4.5 we prove the classification of regular depth-zero supercuspidal representations. This is again a generalization of results of [DR09], but requires additional arguments due to the complication in the ramified setting exposed in Example 3.4.21.

3.4.1 Maximally unramified elliptic maximal tori

Fact 3.4.1. Let $S \subset G$ be a maximal torus and $S' \subset S$ be the maximal unramified subtorus. The following statements are equivalent.

1. S' is of maximal dimension among the unramified subtori of G.
2. S' is not properly contained in an unramified subtorus of G.
3. S is the centralizer of S' in G.
4. $S \times F^u$ is a minimal Levi subgroup of $G \times F^u$.
5. The action of I_F on $R(S, G)$ preserves a set of positive roots.

□

Proof. This follows from the fact that $G \times F^u$ is quasi-split. ■

Definition 3.4.2. A maximal torus $S \subset G$ will be called *maximally unramified* if it satisfies the above equivalent conditions. □

When G splits over F^u, i.e. when it is an inner form of an unramified group, then S is unramified. Therefore, this notion generalizes the notion of an unramified maximal torus to the case of ramified groups (in [Roe11, Definition 3.1.1], such tori were called "unramified", but we prefer the term maximally unramified because it emphasizes that the splitting field of S need not be an unramified extension of F).

The assignments $S \mapsto S'$ and $S' \mapsto \mathrm{Cent}(S', G)$ are mutually inverse bijections between the set of maximally unramified maximal tori of G and the set of maximal unramified tori of G. The $G(F)$-conjugacy classes of the latter were classified by DeBacker in [DeB06]. We shall review here some of DeBacker's work and prove some additional results that will be needed later.

Let $S \subset G$ be a maximally unramified elliptic maximal torus. We can associate to S a point $x \in \mathcal{B}^{\mathrm{red}}(G, F)$ as follows: Since $S' \subset G$ becomes a maximal split torus over F^u, we have the apartment $\mathcal{A}^{\mathrm{red}}(S, F^u) \subset \mathcal{B}^{\mathrm{red}}(G, F^u)$. This apartment is Frobenius-invariant, since S is defined over F, and the action of Frobenius on $\mathcal{A}^{\mathrm{red}}(S, F^u)$ has a unique fixed point, since S is elliptic.

Lemma 3.4.3. The point x is a vertex of $\mathcal{B}^{\mathrm{red}}(G, F)$. □

Proof. This follows from [DeB06, Lemma 2.2.1(1)] applied to $\mathbf{T} = S'$. ■

As shown in [BT84, §5], the vertex x specifies a smooth connected O_F-group scheme $\mathfrak{G}_x^\circ$ with $\mathfrak{G}_x^\circ(F) = G(F)$ and $\mathfrak{G}_x^\circ(O_F) = G(F)_{x,0}$. We shall write G_x° for the reductive quotient of the special fiber of $\mathfrak{G}_x^\circ$. Then $\mathsf{G}_x^\circ(k_F) = G(F)_{x,0:0+}$. We further have the O_F-group scheme $\widehat{\mathfrak{G}}_x$ with $\widehat{\mathfrak{G}}_x(F) = G(F)$ and $\widehat{\mathfrak{G}}_x(O_F) = \mathrm{Stab}(x, G(F)^1)$, where $G(F)^1$ denotes the intersection of the kernels of the group homomorphisms $\mathrm{ord} \circ \chi : G(F) \to \mathbb{Z}$ for all F-rational characters $\chi : G \to \mathbb{G}_m$. We shall write G_x for the quotient of the special fiber of this group scheme by its maximal connected normal unipotent subgroup. Then G_x° coincides with the neutral connected component of G_x. In particular, G_x is a usually disconnected algebraic group over k_F with reductive neutral connected component.

Lemma 3.4.4.
1. The special fiber of the (automatically connected) ft-Neron model of S' embeds canonically as an elliptic maximal torus S' of the reductive group G_x°. Explicitly, $\mathsf{S}'(k_{F'}) \subset \mathsf{G}_x^\circ(k_{F'})$ is the image in $G(F')_{x,0:0+}$ of $S(F') \cap G(F')_{x,0}$, or equivalently of $S'(F') \cap G(F')_{x,0}$ for every unramified extension F'.
2. Every elliptic maximal torus of G_x° arises in this way.

□

Proof. This is [DeB06, Lemma 2.2.1(3) and Lemma 2.3.1] applied to $\mathbf{T} = S'$. ■

Lemma 3.4.5. Let $S_1, S_2 \subset G$ be two maximally unramified elliptic maximal tori. Assume that their points in $\mathcal{B}^{\mathrm{red}}(G, F)$ coincide, call them x. Assume furthermore that $S_1(F^u) \cap G(F^u)_{x,0}$ and $S_2(F^u) \cap G(F^u)_{x,0}$ have the same projection to $\mathsf{G}_x^\circ(\overline{k_F})$. Then S_1 and S_2 are $G(F)_{x,0+}$-conjugate. □

Proof. This is [DeB06, Lemma 2.2.2] applied to $\mathbf{T}_i = S_i'$. ■

Lemma 3.4.6. Let $S \subset G$ be a maximally unramified elliptic maximal torus with associated point $x \in \mathcal{B}^{\mathrm{red}}(G, F)$. Then

$$S(F) \cap G(F)_{x,0} = S(F)_0.$$

□

Proof. This follows immediately from [PR08, Lemma 5 in Appendix] by taking Frobenius-fixed points. ■

Lemma 3.4.7. Assume that G is either simply connected or adjoint. If $S \subset G$ is a maximally unramified elliptic maximal torus, then $S(F) = S(F)_0$. □

Proof. The set of fundamental weights, respectively simple roots, corresponding to a set of positive roots preserved by inertia, forms a basis of $X^*(S)$ preserved by inertia. This shows that S is inertially induced in the sense of Definition 3.1.1. Thus $S(F)_b = S(F)_0$ by Fact 3.1.2. On the other hand, S is anisotropic, so $S(F) = S(F)_b$. ■

Definition 3.4.8. We shall call a vertex $x \in \mathcal{B}^{\mathrm{red}}(G, F)$ *superspecial* if it is a (necessarily special) vertex that is special in $\mathcal{B}^{\mathrm{red}}(G, F')$ for every finite unramified extension F' of F. □

Remark 3.4.9. If G is quasi-split then superspecial vertices exist: The Chevalley valuation [BT84, §4.2.1] associated to any F-pinning of G is such a vertex.

On the other hand, a superspecial vertex need not be a Chevalley valuation. All vertices in the building of a ramified unitary group in 3 variables are superspecial, but not all of them are Chevalley valuations. $\square$

Lemma 3.4.10. Let $S \subset G$ be a maximally unramified elliptic maximal torus with associated point $x \in \mathcal{B}^{\mathrm{red}}(G, F)$.

1. We have the exact sequence

   $$1 \to N(S, G(F)_{x,0})/S(F)_0 \to N(S, G)(F)/S(F) \to G(F)_x/[G(F)_{x,0} \cdot S(F)].$$

2. The natural map

   $$N(S, G(F)_{x,0})/S(F)_0 \to N(\mathsf{S}', \mathsf{G}_x^\circ)(k_F)/\mathsf{S}'(k_F)$$

   is bijective.

3. If x is superspecial, the natural inclusions

   $$N(S, G(F)_{x,0})/S(F)_0 \to N(S, G)(F)/S(F) \to \Omega(S, G)(F)$$

   are both bijective.

$\square$

Proof. At the first spot exactness is equivalent to the equality $S(F) \cap G(F)_{x,0} = S(F)_0$ of Lemma 3.4.6, while at the second spot exactness is obvious. We only observe that the inclusion $N(S, G)(F) \to G(F)$ takes image in $G(F)_x$, because the action of $N(S, G)(F)$ on $\mathcal{B}^{\mathrm{red}}(G, F^u)$ preserves the apartment $\mathcal{A}^{\mathrm{red}}(S, F^u)$ and commutes with the action of $\mathrm{Gal}(F^u/F)$.

Consider now the second map. If we replace F by F^u then its bijectivity is the content of [BT84, 4.6.12]. Moreover, this bijective map is Frobenius-equivariant. The claim now follows from the fact that both $S(F^u)_0$ and $\mathsf{S}'(\overline{k_F})$ have trivial $H^1(\mathrm{Fr}, -)$.

For the third point, Steinberg's theorem [Ste65, Theorem 1.9] implies the equality $\Omega(S, G)(F^u) = N(S, G)(F^u)/S(F^u)$. The point x remains special over F^u and according to [BT72, 6.2.19] the natural map $N(S, G(F^u)_{x,0})/S(F^u)_0 \to N(S, G)(F^u)/S(F^u)$ is an isomorphism . Applying again $H^1(\mathrm{Fr}, S(F^u)_0) = \{0\}$ we obtain the lemma. $\blacksquare$

Example 3.4.11. If the point x is not superspecial, the inclusion

$$N(S, G(F)_{x,0})/S(F)_0 \to N(S, G)(F)/S(F)$$

may be proper. Consider the adjoint group $G = \mathrm{PSp}_4$ and let x be a non-special vertex in the standard apartment. The connected reductive group G_x° is of type $A_1 \times A_1$. Let S' be the unique (up to $\mathsf{G}_x^\circ(k_F)$-conjugacy) anisotropic maximal torus in G_x° and let $S \subset G$ be an anisotropic unramified maximal torus corresponding to S' by Lemma 3.4.4. Then using Lemma 3.4.10 we have

$$N(S, G(F)_{x,0})/S(F)_0 = N(\mathsf{S}', \mathsf{G}_x^\circ)(k_F)/\mathsf{S}'(k_F) = \Omega(\mathsf{S}', \mathsf{G}_x^\circ)(k_F) = (\mathbb{Z}/2\mathbb{Z})^2.$$

On the other hand, the extended affine Weyl group of the split diagonal torus in G has an element that fixes x and switches the two hyperplanes passing through it. This element is represented in $G(F)_x$ and generates the group $G(F)_x/G(F)_{x,0} \cong \mathbb{Z}/2\mathbb{Z}$. Its action on G_x° is outer and switches the two irreducible factors of the root system $A_1 \times A_1$. Since the $\mathsf{G}_x^\circ(k_F)$-conjugacy class of S' is unique, it is preserved by this action. Thus there exists an element of $G(F)_x \smallsetminus G(F)_{x,0}$ that normalizes S'. By Lemma 3.4.5 we may assume that this element normalizes S. It is thus an element of $N(S,G)(F)/S(F)$ that does not lie in $N(S,G(F)_{x,0})/S(F)_0$. In fact, we have the exact sequence

$$1 \to N(S,G(F)_{x,0})/S(F)_0 \to N(S,G)(F)/S(F) \to G(F)_x/G(F)_{x,0} \to 1.$$

$\square$

Lemma 3.4.12. Let G be quasi-split, $x \in \mathcal{B}^{\text{red}}(G,F)$ superspecial, and $S \subset G$ a maximally unramified elliptic maximal torus. There exists a maximal torus S_1 stably conjugate to S with associated point x. Moreover, if T is a minimal Levi subgroup whose apartment contains x, then S_1 can be chosen to be conjugate to T under $G(F^u)_{x,0}$. $\square$

Proof. Fix a minimal Levi subgroup $T \subset G$ whose apartment contains x. Since both S and T become minimal Levi subgroups over F^u we see that the class $\text{cls}(S)$ defined in §3.2 is inflated from $H^1(\text{Fr}, N(T,G)(F^u))$. Projecting to the Weyl group we obtain an element of $H^1(\text{Fr}, \Omega(T,G)(F^u))$. We have $\Omega(T,G)(F^u) = N(T,G)(F^u)/T(F^u)$ by Steinberg's theorem [Ste65, Theorem 1.9]. Since the point x remains special over F^u [BT72, 6.2.19] implies that the natural map $N(T,G(F^u)_{x,0})/T(F^u)_0 \to N(T,G)(F^u)/T(F^u)$ is an isomorphism. We map $\text{cls}(S)$ to $H^1(\text{Fr}, N(T,G(F^u)_{x,0})/T(F^u)_0)$ via this isomorphism. Lemma 3.1.7 implies that it lifts to an element of $H^1(\text{Fr}, N(T,G(F^u)_{x,0}))$. This element can be represented by $\sigma \mapsto g^{-1}\sigma(g)$ for some $g \in G(F^u)_{x,0}$, because $H^1(\text{Fr}, G(F^u)_{x,0})$ is trivial by [DR09, Lemma 2.3.1]. By construction the images in $\Omega(T,F)(F^u)$ of $g^{-1}\sigma(g)$ and $\text{cls}(S)$ coincide. We conclude that $\text{Ad}(g)T$ is stably conjugate to S. The point $x = gx$ belongs to $\mathcal{A}^{\text{red}}(\text{Ad}(g)T, F^u)$ and being Frobenius-fixed it must be the unique Frobenius-fixed element of $\mathcal{A}^{\text{red}}(\text{Ad}(g)T, F^u)$ and hence the point associated to $\text{Ad}(g)T$. ■

3.4.2 Depth zero characters

Let G be a connected reductive group defined over a finite field k, let $\mathsf{S}' \subset \mathsf{G}$ be a maximal torus, and let $\bar\theta : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$ be a character. In [DL76, Definition 5.15], Deligne and Lusztig define two regularity conditions for a character $\bar\theta : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$, which we shall now recall. They say that $\bar\theta$ is *in general position*, if its stabilizer in $\Omega(\mathsf{S}',\mathsf{G})(k_F)$ is trivial. They say that $\bar\theta$ is *non-singular*, if it is *not orthogonal to any coroot*, which means that the composition of $\bar\theta$ with the map $X_*(\mathsf{S}') \to \mathsf{S}'(k)$ given by (3.4.1) below is non-trivial on each coroot $\alpha^\vee \in R^\vee(\mathsf{S}',\mathsf{G}) \subset X_*(\mathsf{S}')$. To recall the map $X_*(\mathsf{S}') \to \mathsf{S}'(k_F)$, we choose an embedding $\bar{k}^\times \to \mathbb{Q}/\mathbb{Z}$. Its image is $(\mathbb{Q}/\mathbb{Z})_{p'}$, the subgroup of elements whose order is prime to p, and this embedding allows us to identify $\mathsf{S}'(\bar k)$ with $X_*(\mathsf{S}') \otimes (\mathbb{Q}/\mathbb{Z})_{p'}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_*(\mathsf{S}') \arrow[r] \arrow[d, "\text{Fr}-1"] & X_*(\mathsf{S}') \otimes \mathbb{Q} \arrow[r] \arrow[d, "\text{Fr}-1"] & X_*(\mathsf{S}') \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\text{Fr}-1"] & 0 \\
0 \arrow[r] & X_*(\mathsf{S}') \arrow[r] & X_*(\mathsf{S}') \otimes \mathbb{Q} \arrow[r] & X_*(\mathsf{S}') \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where Fr is the endomorphism of $X_*(\mathsf{S}')$ obtained functorially from the Frobenius endomorphism of S'. We alert the reader that this convention, which is in use in [DL76, §5] and [Car93], implies that Fr is not of finite order, but rather $\mathrm{Fr}^n = q^n$ for some natural number n. Applying the kernel-cokernel lemma to this diagram and noting the middle vertical arrow is an isomorphism we obtain the exact sequence

$$0 \to X_*(\mathsf{S}') \to X_*(\mathsf{S}') \to \mathsf{S}'(k) \to 1. \tag{3.4.1}$$

We will now reinterpret the notion of non-singular in a way that does not involve the choice of an isomorphism $\bar{k}^\times \to (\mathbb{Q}/\mathbb{Z})_{p'}$ and is closer to the p-adic torus S.

Fact 3.4.13. Let k' be a finite extension of k. The exact sequence (3.4.1) fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_*(\mathsf{S}') \arrow[r, "\mathrm{Fr}^n-1"] & X_*(\mathsf{S}') \arrow[r] \arrow[d, "\mathrm{id}"] & \mathsf{S}'(k') \arrow[r] \arrow[d, "N"] & 1 \\
0 \arrow[r] & X_*(\mathsf{S}') \arrow[r, "\mathrm{Fr}-1"] & X_*(\mathsf{S}') \arrow[r] & \mathsf{S}'(k) \arrow[r] & 1
\end{tikzcd}$$

where $n = [k' : k]$ and $N : \mathsf{S}'(k') \to \mathsf{S}'(k)$ is the norm map. □

Proof. This is a direct computation. ■

Lemma 3.4.14. Let k' be a finite extension of k splitting S', $\bar{\theta} : \mathsf{S}'(k) \to \bar{\mathbb{Q}}_l^\times$ a character, and $\alpha^\vee \in R^\vee(\mathsf{S}', \mathsf{G})$. Then $\bar{\theta}$ is orthogonal to $\alpha^\vee$ if and only if the character $\bar{\theta} \circ N \circ \alpha^\vee : k'^\times \to \bar{\mathbb{Q}}_l^\times$ is trivial. In particular, $\bar{\theta}$ is non-singular if and only if for each $\alpha^\vee \in R^\vee(\mathsf{S}', \mathsf{G})$ the character $\bar{\theta} \circ N \circ \alpha^\vee : k'^\times \to \bar{\mathbb{Q}}_l^\times$ is non-trivial. □

Proof. According to Fact 3.4.13 we may reduce the proof to the case where S' is split. In that case the Frobenius endomorphism Fr of $X_*(\mathsf{S}')$ is simply given by multiplication by q (again, we are using here the conventions of [DL76, §5]) and the map $X_*(\mathsf{S}') \to \mathsf{S}'(k)$ sends $\lambda \in X_*(\mathsf{S}')$ to $\lambda(\zeta)$, where $\zeta \in k^\times$ is the generator whose image under the chosen isomorphism $\bar{k}^\times \to (\mathbb{Q}/\mathbb{Z})_{p'}$ is $1/(q-1) \in (\mathbb{Q}/\mathbb{Z})_{p'}$. By definition, $\bar{\theta}$ is orthogonal to $\alpha^\vee \in R^\vee(\mathsf{S}', \mathsf{G})$ if and only if $\bar{\theta}(\alpha^\vee(\zeta)) = 1$. Since ζ is a generator of $k^\times$ this is equivalent to requiring that the character $\bar{\theta} \circ \alpha^\vee$ be trivial. ■

We will now define a third regularity condition on $\bar{\theta}$. We say that $\bar{\theta}$ is absolutely regular, if for some (hence any) finite extension k' of k splitting S' the character $\bar{\theta} \circ N$ has trivial stabilizer in $\Omega(\mathsf{S}', \mathsf{G})$. It is clear that absolutely regular implies general position. By [DL76, Corollary 5.18] general position implies non-singular.

Lemma 3.4.15. If the center of G is connected, then the notions of non-singular, general position, and absolutely regular, are equivalent. □

Proof. According to [DL76, Proposition 5.16], the notions of non-singular and general position are equivalent. By Fact 3.4.13 $\bar{\theta}$ is non-singular if and only if $\bar{\theta} \circ N$ is non-singular. But for $\bar{\theta} \circ N$ the notions of general position and absolute regularity coincide. ■

We now return to the p-adic group G defined over F. Let $x \in \mathcal{B}^{\text{red}}(G,F)$ be a vertex. We take $k = k_F$ and $\mathsf{G} = \mathsf{G}_x^\circ$. Assume that S' is elliptic. By Lemma 3.4.4 there exists a maximally unramified elliptic maximal torus $S \subset G$ such that the reductive quotient of the special fiber of the connected Neron model of S is S'. We have $S'(F)_0/S'(F)_{0+} = S(F)_0/S(F)_{0+} = \mathsf{S}'(k_F)$ by Lemma 3.1.6.

Definition 3.4.16. We shall call $\bar\theta : \mathsf{S}'(k_F) \to \bar{\mathbb{Q}}_l^\times$ (or $\bar\theta : \mathsf{S}'(k_F) \to \mathbb{C}^\times$) *regular* if its stabilizer in $N(S(F),G(F))/S(F)$ is trivial. We shall call $\bar\theta$ *extra regular* if its stabilizer in $\Omega(S,G)(F)$ is trivial. If $\theta : S(F) \to \bar{\mathbb{C}}^\times$ is a depth-zero character such that $\theta|_{S(F)_0}$ equals the inflation of $\bar\theta$, we shall call θ (extra) regular if $\bar\theta$ is such. □

Remark 3.4.17. Whether $\bar\theta$ is (extra) regular does not depend on the choice of S: this follows from Lemma 3.4.5. □

Fact 3.4.18. We have

$$\bar\theta \text{ extra regular } \Rightarrow \bar\theta \text{ regular } \Rightarrow \bar\theta \text{ in general position .}$$

If the point of $\mathcal{B}^{\text{red}}(G,F)$ associated to S is superspecial, then the converse implications also hold. □

Proof. This follows from Lemma 3.4.10. ■

3.4.3 Definition and construction

We now come to the definition and construction of regular depth-zero supercuspidal representations. Let π be an irreducible supercuspidal representation of $G(F)$ of depth zero. According to [MP96, Proposition 6.8] there exists a vertex $x \in \mathcal{B}^{\text{red}}(G,F)$ such that the restriction $\pi|_{G(F)_{x,0}}$ contains the inflation to $G(F)_{x,0}$ of an irreducible cuspidal representation κ of $G(F)_{x,0:0+}$.

Definition 3.4.19. We shall call π *regular* (resp. *extra regular*) if κ is a Deligne-Lusztig cuspidal representation $\pm R_{\mathsf{S}',\bar\theta}$ associated to an elliptic maximal torus S' of G_x° and a character $\bar\theta : \mathsf{S}'(k_F) \to \mathbb{C}^\times$ that is regular (resp. extra regular) in the sense of Definition 3.4.16. □

Note that, since $\mathsf{S}'(k_F)$ is a finite group, $\bar\theta$ takes values in $\bar{\mathbb{Q}}^\times$, so replacing $\bar{\mathbb{Q}}_l$ with $\mathbb{C}$ is inconsequential. By Fact 3.4.18 a regular π is automatically extra regular if the vertex of S is superspecial.

Regular depth-zero supercuspidal representations of $G(F)$ are constructed as follows. Let S be a maximally unramified elliptic maximal torus of G and let $\theta : S(F) \to \mathbb{C}^\times$ be a regular depth-zero character. The restriction $\theta|_{S(F)_0}$ factors through a character $\bar\theta$ of $S(F)_{0:0+}$ that is in general position according to Fact 3.4.18. Let $x \in \mathcal{B}^{\text{red}}(G,F)$ be the vertex (by Lemma 3.4.3) associated to S. Let $\kappa_{(S,\bar\theta)} = \pm R_{\mathsf{S}',\bar\theta}$ be the irreducible cuspidal representation of $\mathsf{G}_x^\circ(k_F)$ arising from the Deligne-Lusztig construction applied to the reductive quotient S' of the special fiber of the connected Neron model of S and the character $\bar\theta$. Identify $\kappa_{(S,\bar\theta)}$ with its inflation to $G(F)_{x,0}$.

Note that $S(F)$ normalizes $G(F)_{x,0}$. It is easy to check, and we shall do so soon, that the normalizer in $G(F)_x$ of $\kappa_{(S,\bar\theta)}$ is equal to $S(F) \cdot G(F)_{x,0}$. This means that, in order to obtain an irreducible representation of $G(F)$, we need to extend $\kappa_{(S,\bar\theta)}$ to $S(F)\cdot G(F)_{x,0}$ before inducing it. In [DR09, §4.4] this is done using the fact that, when S is unramified, $S(F) = Z(F)\cdot S(F)_0$ (see e.g. [Kal11, Lemma 7.1.1]), which implies $S(F)\cdot G(F)_{x,0} = Z(F)\cdot G(F)_{x,0}$. Since $Z(F)\cap G(F)_{x,0}$ acts on $\kappa_{(S,\bar\theta)}$ via $\bar\theta|_{Z(F)\cap S(F)_0}$ an extension of $\kappa_{(S,\bar\theta)}$ to $Z(F)\cdot G(F)_{x,0}$ is given by letting $Z(F)$ act by the character $\theta|_{Z(F)}$. The same is also true for ramified unitary groups [Roe11, Theorem 3.4.1, Proposition 3.4.2, Proposition 5.2.3]. However, for general tamely ramified groups the equality $S(F) = Z(F)\cdot S(F)_0$ is generally false and a counterexample was shown to us by Cheng-Chiang Tsai, which we have included at the end of this subsubsection.

The extension of $\kappa_{(S,\bar\theta)}$ to $S(F)\cdot G(F)_{x,0}$ must thus be obtained differently. We shall construct this extension and study its character in the next subsubsection. For now we just assume that an extension $\kappa_{(S,\theta)}$ of $\kappa_{(S,\bar\theta)}$ to $S(F)\cdot G(F)_{x,0}$ is given.

Lemma 3.4.20. The representation $\pi_{(S,\theta)} = \text{c-Ind}_{S(F)G(F)_{x,0}}^{G(F)}\kappa_{(S,\theta)}$ is irreducible (and hence supercuspidal). □

Proof. The proof is the same as for [DR09, Lemma 4.5.1]. By [MP96, Proposition 6.6] it is enough to show that $\kappa_{(S,\theta)}$ induces irreducibly to the normalizer of $G(F)_{x,0}$ in $G(F)$. Note here that the Levi subgroup $\mathcal{M}$ in loc. cit. is equal to G in our case since x is a vertex. The normalizer of $G(F)_{x,0}$ is equal to $G(F)_x$, the stabilizer of the vertex x for the action of $G(F)$ on $\mathcal{B}^{\text{red}}(G,F)$. It is enough to show that the normalizer of $\kappa_{(S,\theta)}$ in $G(F)_x$ is equal to $S(F)\cdot G(F)_{x,0}$. For this, let $h\in G(F)_x$ normalize $\kappa_{(S,\theta)}$. Then it in particular normalizes $\kappa_{(S,\bar\theta)}$, so by [DL76, Theorem 6.8] there is $g\in G(F)_{x,0}$ so that $\mathrm{Ad}(gh)(\mathsf{S}',\bar\theta) = (\mathsf{S}',\bar\theta)$. By Lemma 3.4.5 there is $l\in G(F)_{x,0+}$ so that $\mathrm{Ad}(lgh)(S,\theta|_{S(F)_0}) = (S,\theta|_{S(F)_0})$. Thus $lgh\in N(S,G)(F)$ and then the regularity of θ implies that $lgh\in S(F)$. ■

It is clear that $\pi_{(S,\theta)}$ is regular, as its restriction to $G(F)_{x,0}$ contains $\kappa_{(S,\bar\theta)}$. We shall see in Proposition 3.4.27 that every regular depth-zero supercuspidal representation is of the form $\pi_{(S,\theta)}$ for some pair (S,θ) consisting of a maximally unramified elliptic maximal torus S and regular character $\theta : S(F)\to\mathbb{C}^\times$.

Example 3.4.21. We come to example of the failure of $Z(F)\cdot S(F)_0 = S(F)$ due to Cheng-Chiang Tsai. Consider F of odd residual characteristic and not containing a fourth root of unity, for example $F = \mathbb{Q}_p$ with $4|(p-3)$, and a ramified quadratic extension E/F. Let $\bar G = \mathrm{Res}_{E/F}\mathrm{PGL}_4$ and $\bar T = \mathrm{Res}_{E/F}\bar T_0$, where $\bar T_0 = \mathbb{G}_m^4/\mathbb{G}_m$ is the standard maximal torus of PGL_4. We have $X_*(\bar T) = \mathbb{Z}^4/\mathbb{Z}\oplus\mathbb{Z}^4/\mathbb{Z}$. Let L be the kernel of the addition map $\mathbb{Z}^4/\mathbb{Z}\oplus\mathbb{Z}^4/\mathbb{Z}\to\mathbb{Z}/4\mathbb{Z}$. Let $T\to\bar T$ be the isogeny of tori specified by $X_*(T) = L$ and let $G\to\bar G$ be the corresponding isogeny of connected reductive groups. Then G is semi-simple, quasi-split, and its center is μ_4. A direct computation shows $X_*(T)_I = \mathbb{Z}^3\oplus\mathbb{Z}/2\mathbb{Z}$. Let S be an anisotropic maximal torus of G that is conjugate to T over F^u (such an S does exist). Then $X_*(S)_I = \mathbb{Z}^3\oplus\mathbb{Z}/2\mathbb{Z}$ with an action of Frobenius. This action must be trivial on the $\mathbb{Z}/2\mathbb{Z}$ factor, which is the torsion subgroup of $X_*(S)_I$. Since S is anisotropic, we conclude that $[X_*(S)_I]^{\mathrm{Fr}} = \mathbb{Z}/2\mathbb{Z}$. Since this is the quotient $S(F)/S(F)_0$, in order for $Z(F)\cdot S(F)_0 = S(F)$ to hold the composed map $Z(F)\to S(F)\to[X_*(S)_I]^{\mathrm{Fr}}$ must be surjective. However, $Z(F^u) = \mu_4(F^u)$ has order 4, while $Z(F) = \mu_4(F)$ has order 2, by our assumption on F. We see that whatever the map $Z(F^u)\to S(F^u)\to X_*(S)_I$ might be, its restriction to $Z(F)\to S(F)\to[X_*(S)_I]^{\mathrm{Fr}}$ is the zero map. □

3.4.4 An extension and its character

We shall now construct the representation $\kappa_{(S,\theta)}$ of $S(F)\cdot G(F)_{x,0}$ that extends $\kappa_{(S,\bar\theta)}$. For this we must first recall the construction of $\kappa_{(S,\bar\theta)}$. Let $\mathsf{U}\subset\mathsf{G}_x^\circ$ be the unipotent radical of a Borel subgroup defined over $\overline{k_F}$ and containing the maximal torus S'. Let

$$X=\{g\in\mathsf{G}_x^\circ(\overline{k_F})|g^{-1}\mathrm{Fr}(g)\in\mathsf{U}\}$$

be the corresponding Deligne-Lusztig variety, where Fr stands for the Frobenius automorphism of G_x°. By construction, $\mathsf{G}_x^\circ(k_F)$ acts on this variety by multiplication on the left, and $\mathsf{S}'(k_F)$ acts by multiplication on the right. The l-adic cohomology (for some fixed auxiliary prime l different from p) with compact support $H_c^i(X,\bar{\mathbb{Q}}_l)$ is thus a $(\mathsf{G}_x^\circ(k_F),\mathsf{S}'(k_F))$-bimodule. It is shown in [DL76, Corollary 9.9] that if X is affine and $\bar\theta:\mathsf{S}'(k_F)\to\bar{\mathbb{Q}}_l^\times$ is a non-singular character, then the $\bar\theta$-isotypic subspace $H_c^i(X,\bar{\mathbb{Q}}_l)_{\bar\theta}$ is non-zero for exactly one value of i, namely $i=l(w)$, where w is the Weyl element determined by the maximal torus S'. In fact, according to [DL76, Remark 9.15.1] the affineness assumption on X can be relaxed to the assumption that some $X(w')$ is affine, where w' is an element of the Weyl group that is Frobenius-conjugate to w. The latter assumption has been proved to always hold [He08, Theorem 1.3]. We let $V=H_c^{l(w)}(X,\bar{\mathbb{Q}}_l)$ and

$$V_{\bar\theta}=\{v\in V|vt=\bar\theta(t)v,\quad\forall t\in\mathsf{S}'(k_F)\}.$$

The $\mathsf{G}_x^\circ(k_F)$-module $V_{\bar\theta}$, inflated to $G(F)_{x,0}$, is the representation $\kappa_{(S,\bar\theta)}$. It is irreducible if $\bar\theta$ is in general position [DL76, Theorem 6.8].

We now assume that $\bar\theta:\mathsf{S}'(k_F)\to\mathbb{C}^\times$ is obtained by restricting to $S(F)_0$ a regular depth-zero character $\theta:S(F)\to\mathbb{C}^\times$. The extension $\kappa_{(S,\theta)}$ of $\kappa_{(S,\bar\theta)}$ to $S(F)\cdot G(F)_{x,0}$ begins with the observation that besides a left $\mathsf{G}_x^\circ(k_F)$-action and a right $\mathsf{S}'(k_F)$-action, the variety X also carries an $\mathsf{S}'_{\mathrm{ad}}(k_F)$-action by conjugation, where $\mathsf{S}'_{\mathrm{ad}}$ is the image of S' in the adjoint group of G_x°. This action – a special case of [DL76, 1.21] – is simply the restriction to $\mathsf{S}'_{\mathrm{ad}}(k_F)$ of the action of $\mathsf{S}'_{\mathrm{ad}}$ on G_x° by conjugation, and is readily seen to preserve X. We shall write $\mathrm{Ad}(\bar s)$ for the action of $\bar s\in\mathsf{S}'_{\mathrm{ad}}(k_F)$ by conjugation on G_x° as well as on the subvariety X of G_x°. The three actions on X have the following compatibility relation:

$$\mathrm{Ad}(\bar s)[g\cdot x\cdot s']=[\mathrm{Ad}(\bar s)g]\cdot[\mathrm{Ad}(\bar s)x]\cdot s',\bar s\in\mathsf{S}'_{\mathrm{ad}}(k_F),g\in\mathsf{G}_x^\circ(k_F),s'\in\mathsf{S}'(k_F),x\in X.$$

In particular, the action of $\mathsf{S}'_{\mathrm{ad}}(k_F)$ by conjugation commutes with the action of $\mathsf{S}'(k_F)$ on the right. Furthermore, the action of $s'\in\mathsf{S}'(k_F)$ on the right can be recovered as $vs'=\mathrm{Ad}(\bar s'^{-1})s'v$, where $\bar s'\in\mathsf{S}'_{\mathrm{ad}}(k_F)$ is the image of s'.

There is a natural map $S(F)\to\mathsf{S}'_{\mathrm{ad}}(k_F)$ given as follows. To avoid confusion, let H be the adjoint group of G. Then the natural map $\mathsf{G}_x^\circ\to[\mathsf{G}_x^\circ]_{\mathrm{ad}}$ factors as the composition $\mathsf{G}_x^\circ\to\mathsf{H}_x^\circ\to[\mathsf{G}_x^\circ]_{\mathrm{ad}}$. In particular, we have the map $\mathsf{S}'_H\to\mathsf{S}'_{\mathrm{ad}}$, where S_H is the image of S in H. Since S is maximally unramified, so is S_H and Lemma 3.4.7 implies $S_H(F)_0=S_H(F)$. The map $S(F)\to\mathsf{S}'_{\mathrm{ad}}(k_F)$ is then obtained as the composition $S(F)\to S_H(F)=S_H(F)_0\to S_H(F)_{0:0+}=\mathsf{S}'_H(k_F)\to\mathsf{S}'_{\mathrm{ad}}(k_F)$.

Via the map $S(F)\to\mathsf{S}'_{\mathrm{ad}}(k_F)$ we can let $S(F)$ act on G_x° and X by conjugation, and the action on X gives an action on V, which preserves the subspace $V_{\bar\theta}$. This action factors through $S(F)/S(F)_{0+}$. The subgroup $\mathsf{S}'(k_F)$ of $\mathsf{S}(k_F)=$

$S(F)/S(F)_{0+}$ embeds via $s \mapsto (s, s^{-1})$ into the center of $\mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}(k_F)$ and we let $\tilde{\mathsf{G}}(k_F)$ be the quotient. We extend the representation of $\mathsf{G}_x^\circ(k_F)$ on $V_{\bar{\theta}}$ to $\tilde{\mathsf{G}}(k_F)$ via the formula

$$(g, s) \cdot v = \theta(s) g[\mathrm{Ad}(s)v]. \tag{3.4.2}$$

It is immediate to check that this formula gives an action of $\mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}(k_F)$ which descends to $\tilde{\mathsf{G}}(k_F)$. The maps $G(F)_{x,0} \to \mathsf{G}_x^\circ(k_F)$ and $S(F) \to \mathsf{S}(k_F)$ splice together to a map $G(F)_{x,0} \cdot S(F) \to \tilde{\mathsf{G}}(k_F)$ via which we obtain an action of $G(F)_{x,0} \cdot S(F)$ on $V_{\bar{\theta}}$. This is the extension $\kappa_{(S,\theta)}$ of $\kappa_{(S,\bar{\theta})}$ that we were seeking.

We will now compute the character of the representation $\kappa_{(S,\theta)}$ of the group $G(F)_{x,0} \cdot S(F)$. The resulting formula will be used in the classification of regular depth-zero supercuspidal representations in the next subsubsection. In fact, we will give two formulations of the character formula: one more technical, but valid in complete generality, and one more palatable, but only valid when G splits over a tamely ramified extension and for elements of $G(F)_{x,0} \cdot S(F)$ that are semi-simple.

To state the more palatable version, we begin with a discussion of a variant of the topological Jordan decomposition. When S is unramified, the equation $G(F)_{x,0} \cdot S(F) = G(F)_{x,0} \cdot Z(F)$ allows one to use the usual topological Jordan decomposition of elements of $G(F)_{x,0}$ for the computation of the character of $\kappa_{(S,\theta)}$. The failure of this equation when S is ramified precludes this, because elements of $G(F)_{x,0} \cdot S(F)$ are in general not compact and hence do not have a usual topological Jordan decomposition. However, if we let A_G be the maximal split central torus of G, and $\bar{G} = G/A_G$, then the image of $G(F)_{x,0} \cdot S(F)$ in $\bar{G}(F)$ is contained in $\bar{G}(F)_x$, whose elements are compact and have a topological Jordan decomposition. We recall from [Spi08] that an element is called *topologically unipotent* if has pro-p order, and *topologically semi-simple* if it has finite order prime to p. An element is called topologically unipotent/semi-simple *modulo* A_G if its image in $\bar{G}(F)$ has the corresponding property.

Lemma 3.4.22. Assume that G splits over a tame extension. Let $\gamma \in G(F)_{x,0} \cdot S(F)$ be a semi-simple element. Then $\gamma = \gamma_s \gamma_u$, where $\gamma_s \in G(F)_{x,0} \cdot S(F)$ is topologically semi-simple modulo A_G and $\gamma_u \in G(F)_{x,0}$ is topologically unipotent. Both γ_s and γ_u are unique up to multiplication by elements of $A_G(F)_{0+}$. The image of the decomposition $\gamma = \gamma_s\gamma_u$ in $\bar{\mathsf{G}}_x(k_F)$ is the usual Jordan decomposition in the (possibly disconnected) finite group of Lie type $\bar{\mathsf{G}}_x$. If T is a maximal torus containing γ, then $\gamma_s, \gamma_u \in T(F)$. In particular, γ_s and γ_u commute. □

Proof. Let $\bar{\gamma} \in \bar{G}(F)$ be the image of γ. Since $\bar{G}$ has anisotropic center, the group $\bar{G}(F)_{x,0}\bar{S}(F) \subset \bar{G}(F)_x$ is compact. By [Spi08, Proposition 1.8] there exist commuting elements $\bar{\gamma}_s, \bar{\gamma}_u \in \bar{G}(F)_{x,0}\bar{S}(F)$ with $\bar{\gamma} = \bar{\gamma}_s \cdot \bar{\gamma}_u$ such that $\bar{\gamma}_s$ is of finite order prime to p, and $\bar{\gamma}_u$ is of pro-p order. This decomposition is unique by [Spi08, Proposition 1.7]. The orders of $\bar{\gamma}_s$ and $\bar{\gamma}_u$ show that their images in $\bar{\mathsf{G}}_x$ form the usual Jordan decomposition of the image of $\bar{\gamma}$ there.

We claim that $\bar{\gamma}_u \in \bar{G}(F)_{x,0}$. Indeed, the image of $\bar{\gamma}_u$ in $\bar{G}(F)_{x,0}\bar{S}(F)/\bar{G}(F)_{x,0}$ still has pro-p order. But $\bar{G}(F)_{x,0}\bar{S}(F)/\bar{G}(F)_{x,0} \cong \bar{S}(F)/\bar{S}(F)_0 \cong X_*(\bar{S})_I^{\mathrm{Fr}}$. Since S becomes a minimal Levi subgroup of G over F^u, S splits over the same extension of F^u as G does. In particular, S is tame, and therefore the abelian group $X_*(\bar{S})_I$ has no p-torsion. Thus the image of $\bar{\gamma}_u$ in $\bar{G}(F)_{x,0}\bar{S}(F)/\bar{G}(F)_{x,0}$ is trivial.

Using the surjectivity of $G(F)_{x,0} \to \bar{G}(F)_{x,0}$ guaranteed by Lemma 3.3.2 we lift $\bar{\gamma}_u$ to an element of $G(F)_{x,0}$. This lift may not have pro-p-order. Apply [Spi08, Proposition 1.8] to this lift and the group $G(F)_{x,0}$ to write this lift as a commuting product $\delta \cdot \gamma_u$ with $\delta \in G(F)_{x,0}$ having finite order prime to p and $\gamma_u \in G(F)_{x,0}$ having pro-p-order. The image of $\delta \cdot \gamma_u$ in $\bar{G}(F)$ equals $\bar{\gamma}_u$ and thus has pro-p order. This implies that the image of δ in $\bar{G}(F)$ is trivial, and hence that γ_u lifts $\bar{\gamma}_u$.

We claim that if T is a maximal torus and $\gamma \in T(F)$ then also $\gamma_u \in T(F)$. Indeed, $\bar{\gamma} \in \bar{T}(F) \cap \bar{G}(F)_x \subset \bar{T}(F)_b$. Thus $\bar{\gamma}$ has a decomposition according to [Spi08, Proposition 1.8] relative to $\bar{T}(F)_b$, but then [Spi08, Proposition 1.7] implies that this decomposition coincides with the one relative to $\bar{G}(F)_x$, thus $\bar{\gamma}_s, \bar{\gamma}_u \in \bar{T}(F)$. Since $\gamma_u \in G(F)$ is a lift of $\bar{\gamma}_u$ we must have $\gamma_u \in T(F)$.

Set now $\gamma_s = \gamma\gamma_u^{-1}$. Then γ_s is a lift of $\bar{\gamma}_s$ and hence topologically semi-simple modulo A_G. Moreover, if T is a maximal torus with $\gamma \in T(F)$, then also $\gamma_s \in T(F)$.

Finally, since $\bar{\gamma}_s, \bar{\gamma}_u$ are uniquely determined by γ and γ_u is a lift of $\bar{\gamma}_u$ of pro-p-order, γ_u is uniquely determined up to multiplication by $A_G(F)_{0+}$, and the same is then true for γ_s. ■

Proposition 3.4.23. Assume that G splits over a tamely ramified extension. The character of $\kappa_{(S,\theta)}$ at a semi-simple element $\gamma \in G(F)_{x,0} \cdot S(F)$ is given by the formula

$$(-1)^{r_G - r_S} |C(\gamma_s)^\circ(k_F)|^{-1} \sum_{\substack{h \in \mathsf{G}_x^\circ(k_F) \\ h^{-1}\gamma_s h \in \mathsf{S}(k_F)}} \theta(h^{-1}\gamma_s h) Q_{h\mathsf{S}'h^{-1}}^{C(\gamma_s)^\circ}(\gamma_u),$$

where $C(\gamma_s) \subset \mathsf{G}_x^\circ$ is the subgroup whose action on $[G(F)_{x,0} \cdot S(F)]/G(F)_{x,0+}$ fixes the image of γ_s, and r_G and r_S are the split ranks of G and S, respectively. □

We now state the more technical version of this formula, valid without assumptions on G and γ. Let $\gamma \in G(F)_{x,0} \cdot S(F)$. Write $\gamma = gr$, where $r \in S(F)$ and $g \in G(F)_{x,0}$. We map r to $\bar{r} \in \mathsf{S}'_{\mathrm{ad}}(k_F)$ via the natural map $S(F) \to \mathsf{S}'_{\mathrm{ad}}(k_F)$ described above and then lift $\bar{r}$ arbitrarily to $\dot{r} \in \mathsf{S}'(\overline{k_F})$. We also map g to $\mathsf{G}_x^\circ(k_F)$. Let $g\dot{r} = su$ be the Jordan decomposition of $g\dot{r} \in \mathsf{G}_x^\circ$. Let $z \in Z(\mathsf{G}_x^\circ)$ be such that $\mathrm{Fr}(\dot{r}) = z\dot{r}$. Then $\mathrm{Fr}(g\dot{r}) = zg\dot{r}$ and the uniqueness of the Jordan decomposition implies $u \in \mathsf{G}_x^\circ(k_F)$ and $\mathrm{Fr}(s) = zs$. We see that $s\dot{r}^{-1} \in \mathsf{G}_x^\circ(k_F)$, and moreover the centralizer $C(s)$ of s in G_x° has a k_F-structure. Since the action of $S(F^u)/S(F^u)_{0+}$ on $G(F^u)_{x,0}/G(F^u)_{x,0+} = \mathsf{G}_x^\circ(\overline{k_F})$ by conjugation factors through the natural map $S(F^u) \to \mathsf{S}'_{\mathrm{ad}}(\overline{k_F})$ we see that $\dot{r}^{-1}r$ acts trivially on $\mathsf{G}_x^\circ(\overline{k_F})$.

Proposition 3.4.24. The character of $\kappa_{(S,\theta)}$ at an element $\gamma \in G(F)_{x,0} \cdot S(F)$ is given by the formula

$$(-1)^{r_G - r_S} |C(s)^\circ(k_F)|^{-1} \sum_{\substack{h \in \mathsf{G}_x^\circ(k_F) \\ h^{-1}sh \in \mathsf{S}'}} \bar{\theta}(h^{-1}sh\dot{r}^{-1})\theta(r) Q_{h\mathsf{S}'h^{-1}}^{C(s)^\circ}(u).$$

where $C(s) \subset \mathsf{G}_x^\circ$ is the centralizer of s in G_x°, r_G and r_S are the split ranks of G and S, respectively, and $\bar{\theta} : \mathsf{S}'(k_F) \to \mathbb{C}^\times$ is obtained by restricting θ to $S(F)_0$. □

Proof of Propositions 3.4.23 and 3.4.24. We first show that Proposition 3.4.23 follows from Proposition 3.4.24. For this we claim that in $[G(F)_{x,0} \cdot S(F)]/G(F)_{x,0+}$ we have the identities $\gamma_s = s\dot{r}^{-1}r$ and $\gamma_u = u$. Indeed, recalling the notation $H = G_{\text{ad}}$, we map the decomposition $\gamma = \gamma_s\gamma_u$ to $H(F)$. The image of γ in $H(F)$ belongs to $H(F)_{x,0}S_H(F)$ which equals $H(F)_{x,0}$ by Lemma 3.4.7. The image of γ in $\mathsf{H}_x^\circ(k_F)$ has Jordan decomposition given by the images of γ_s and γ_u. Since the images of γ and $g\dot{r}$ in the adjoint group of G_x° agree, the images of γ_u and u there also agree. Both of these elements being unipotent elements of G_x° we conclude that they are equal in G_x°. Now $\gamma_s = s\dot{r}^{-1}r$ follows from $s\dot{r}^{-1}ru = su\dot{r}^{-1}r = gr = \gamma$, which uses the fact that $\dot{r}^{-1}r$ commutes with G_x°.

Having established the claimed identities, the fact that $\dot{r}^{-1}r$ commutes with every element of G_x° implies that $C(\gamma_s) = C(s)$, that $h^{-1}sh\dot{r}^{-1}r = h^{-1}(s\dot{r}^{-1}r)h^{-1} = h^{-1}\gamma_s h$, and finally that $h^{-1}sh \in \mathsf{S}'$ is equivalent to $h^{-1}\gamma_s h \in \mathsf{S}$.

We come to the proof of Proposition 3.4.24 and will compute the character of $\kappa_{(S,\theta)}$ at $\gamma = gr$ using the Jordan decomposition $g\dot{r} = su$ following the arguments in [Car93, §7]. The virtual $(\mathsf{G}_x^\circ(k_F), \mathsf{S}'(k_F))$-bimodule $\sum_i (-1)^i H_c^i(X, \bar{\mathbb{Q}}_l)$ will be denoted by W and its $\bar{\theta}$-isotypic component for the right action of $\mathsf{S}'(k_F)$ will be denoted by $W_{\bar{\theta}}$. Thus $W_{\bar{\theta}}$ is a virtual $\mathsf{G}_x^\circ(k_F)$-module. By the above mentioned vanishing result we have $W_{\bar{\theta}} = (-1)^{l(w)}V_{\bar{\theta}}$, so it will be enough to compute the character of the action of $S(F) \cdot G(F)_{x,0}$ on $W_{\bar{\theta}}$, noting that $(-1)^{l(w)} = (-1)^{r_{\mathsf{G}_x^\circ} - r_{\mathsf{S}'}}$, $r_S = r_{\mathsf{S}'}$, and $r_G = r_{\mathsf{G}_x^\circ}$, the latter according to [BT84, Corollary 5.1.11].

We now use all three actions we have on W, i.e. the fact that it is a $([\mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}(k_F)], \mathsf{S}'(k_F))$-bimodule. According to (3.4.2), the action of gr on $W_{\bar{\theta}}$ is given by $\theta(r)$ times the action of $g \rtimes \bar{r} \in \mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}'_{\text{ad}}(k_F)$ on $W_{\bar{\theta}}$. The element in the group algebra of $\mathsf{S}'(k_F)$ given by $e = |\mathsf{S}'(k_F)|^{-1} \sum_{t \in \mathsf{S}'(k_F)} \bar{\theta}(t^{-1})t$ projects W to $W_{\bar{\theta}}$. Then the trace of $g \rtimes \bar{r} \in \mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}'_{\text{ad}}(k_F)$ on $W_{\bar{\theta}}$ is equal to the trace of $(g \rtimes \bar{r}, e)$ on W. The element $(g \rtimes \bar{r}, e)$ of the group algebra of $[\mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}'_{\text{ad}}(k_F)] \times \mathsf{S}'(k_F)$ has the same action on W as the element

$$e' = |\mathsf{S}'(k_F)|^{-1} \sum_{t \in \mathsf{S}'(k_F)} \bar{\theta}(t^{-1})[gt \rtimes \bar{t}^{-1}\bar{r}]$$

of the group algebra of $\mathsf{G}_x^\circ(k_F) \rtimes \mathsf{S}'_{\text{ad}}(k_F)$. The trace of gr on $W_{\bar{\theta}}$ is thus equal to $\theta(r)$ times the trace of e', i.e. to the expression

$$|\mathsf{S}'(k_F)|^{-1} \sum_{t \in \mathsf{S}'(k_F)} \theta(r)\bar{\theta}(t^{-1})\mathcal{L}(gt \rtimes \bar{t}^{-1}\bar{r}, X),$$

where $\mathcal{L}$ denotes the Lefschetz number. For the computation of the Lefschetz number, we use [Car93, Property 7.1.10], which involves the Jordan decomposition in the algebraic group $\mathsf{G}_x^\circ \rtimes \mathsf{S}'_{\text{ad}}$. This decomposition is computed as follows: Given $g' \rtimes \bar{t}'$, lift $\bar{t}'$ to $t' \in \mathsf{S}'$ and let $g't' = s'u'$ be the Jordan decomposition of $g't'$ in the algebraic group G_x°. Then $[g' \rtimes \bar{t}'] = [s't'^{-1} \rtimes \bar{t}'] \cdot [u' \rtimes 1]$ is the Jordan decomposition of $g' \rtimes \bar{t}'$ in the algebraic group $\mathsf{G}_x^\circ \rtimes \mathsf{S}'_{\text{ad}}$. Note that s' depends on the choice of lift t' of $\bar{t}'$ and that $s't'^{-1}$ is independent of this choice. Note moreover that if $\bar{t}'$ is a k_F-point, then so is $s't'^{-1}$, even though neither s' nor t' has to be a k_F-point in general.

Applying this to the element $gt \rtimes \bar{t}^{-1}\bar{r}$, we decompose $g\dot{r} = su$ in G_x° and then obtain $[gt \rtimes \bar{t}^{-1}\bar{r}] = [st\dot{r}^{-1} \rtimes \bar{t}^{-1}\bar{r}] \cdot [u \rtimes 1]$ as the Jordan decomposition in $\mathsf{G}_x^\circ \rtimes \mathsf{S}'_{\text{ad}}$. The subvariety of X fixed by the action of the semi-simple part $[st\dot{r}^{-1} \rtimes \bar{t}^{-1}\bar{r}]$ is $X^{s,t\dot{r}^{-1}} = \{x \in X | x^{-1}sx = (t\dot{r}^{-1})^{-1}\}$. We are following here the notation of [Car93, Proposition 7.2.5], but need to keep in mind that s and $\dot{r}$ are not Frobenius-fixed, but rather satisfy the relation $F(\dot{r}) = \dot{r}z$ and $F(s) = sz$, for some z in the center of G_x°. Nonetheless, the conclusions of Propositions 7.2.6 and Propositions 7.2.7 remain valid with the same proofs, and the arguments in the proof of Theorem 7.2.8 carry over as well. We give a brief sketch.

The trace of gr on $W_{\bar{\theta}}$ is now seen to equal

$$|\mathsf{S}'(k_F)|^{-1} \sum_{t \in \mathsf{S}'(k_F)} \theta(r)\bar{\theta}(t^{-1})\mathcal{L}(u, X^{s,t\dot{r}^{-1}}).$$

One checks that the centralizer $C(t\dot{r}^{-1})$ of $t\dot{r}^{-1}$ in G_x° is defined over k_F, even though $t\dot{r}^{-1}$ is not. Let $[\mathsf{G}_x^\circ(k_F)]^{s,t\dot{r}^{-1}}$ denote the subset $\{g \in \mathsf{G}_x^\circ(k_F) | g^{-1}sg = (t\dot{r}^{-1})^{-1}\}$ and let $Y_{t\dot{r}^{-1}} = X \cap C(t\dot{r}^{-1})^\circ$. Just as in the proof of Proposition 7.2.6 we see that the morphism

$$[\mathsf{G}_x^\circ(k_F)]^{s,t\dot{r}^{-1}} \times Y_{t\dot{r}^{-1}} \to X^{s,t\dot{r}^{-1}}, \quad (g, y) \mapsto gy$$

is surjective and its fibers are the orbits for the action of the group $C(t\dot{r}^{-1})^\circ(k_F)$ on the variety $[\mathsf{G}_x^\circ(k_F)]^{s,t\dot{r}^{-1}} \times Y_{t\dot{r}^{-1}}$ given by $c(g, y) = (gc^{-1}, cy)$. This implies that $X^{s,t\dot{r}^{-1}}$ is the disjoint union of closed subsets

$$X^{s,t\dot{r}^{-1}} = \bigsqcup_{h \in [\mathsf{G}_x^\circ(k_F)]^{s,t\dot{r}^{-1}}/C(t\dot{r}^{-1})^\circ(k_F)} hY_{t\dot{r}^{-1}},$$

each of which is invariant under left multiplication by u. Plugging this into the Lefschetz number we obtain the trace of gr on $W_{\bar{\theta}}$ as

$$|\mathsf{S}'(k_F)|^{-1} \sum_{t \in \mathsf{S}'(k_F)} |C(t\dot{r}^{-1})^\circ(k_F)|^{-1} \sum_{h \in [\mathsf{G}_x^\circ(k_F)]^{s,t\dot{r}^{-1}}} \theta(r)\bar{\theta}(t^{-1})\mathcal{L}(u, hY_{t\dot{r}^{-1}}).$$

Define $Z_s = hXh^{-1} \cap C(s)^\circ$ and note that $Z_s = Z_{s^{-1}} = hY_{t\dot{r}^{-1}}h^{-1}$. Then $\mathcal{L}(u, hY_{t\dot{r}^{-1}}) = \mathcal{L}(h^{-1}uh, Y_{t\dot{r}^{-1}}) = \mathcal{L}(u, Z_s)$. Since Z_s is the Deligne-Lusztig variety associate to the group $C(s)^\circ$, its maximal torus $h\mathsf{S}'h^{-1}$, and its maximal unipotent subgroup $h\mathsf{U}h^{-1} \cap C(s)^\circ$, we have $\mathcal{L}(u, Z_s) = |\mathsf{S}'(k_F)|Q_{h\mathsf{S}'h^{-1}}^{C(s)^\circ}(u)$. Combining the two sums and re-indexing completes the proof. ∎

Corollary 3.4.25. Let $\phi : G(F)_x \to \mathbb{C}^\times$ be a character that is trivial on $G(F)_{x,0} \cap G(F)_{y,0+}$ for all $y \in \mathcal{B}^{\text{red}}(G, F)$. Then $\kappa_{(S,\phi\cdot\theta)} = \phi|_{S(F)\cdot G(F)_{x,0}} \otimes \kappa_{(S,\theta)}$. □

Proof. It is enough to show that the characters of both sides are equal. For this we use Proposition 3.4.24 and the notation of its statement. Given $\gamma \in [G(F)_{x,0} \cdot S(F)]/G(F)_{x,0+}$ we write it again as $\gamma = gr = g\dot{r}\dot{r}^{-1}r = su\dot{r}^{-1}r = s\dot{r}^{-1}ru$. For the character of $\phi \otimes \kappa_{(S,\theta)}$ at γ we note that $\phi(\gamma) = \phi(s\dot{r}^{-1}r)$, since u, being unipotent in $G(F)_{x,0}/G(F)_{x,0+}$, is contained in $G(F)_{y,0+}$ for some $y \in \mathcal{B}^{\text{red}}(G, F)$. To compare this with the character of $\kappa_{(S,\phi\cdot\theta)}$ at γ we compute for $h \in \mathsf{G}_x^\circ(k_F)$

$$\phi(h^{-1}sh\dot{r}^{-1}r) = \phi(h^{-1}s\dot{r}^{-1}rh^{-1}) = \phi(s\dot{r}^{-1}r).$$

∎

Corollary 3.4.26. If $\gamma \in G(F)_{x,0} \cdot S(F)$ is regular semi-simple and its image in $G_{\text{ad}}(F)$ is topologically semi-simple, then the character of $\kappa_{(S,\theta)}$ at γ is zero unless γ is ($G(F)_{x,0}$-conjugate to) an element of $S(F)$, in which case it is given by the formula

$$(-1)^{r_G - r_S} \sum_{w \in N(S, G(F)_{x,0})/S(F)_0} \theta(\gamma^w),$$

where again r_G and r_S are the split ranks of G and S, respectively. $\square$

Proof. We apply Proposition 3.4.24. The character of $\kappa_{(S,\theta)}$ is zero unless s is conjugate in $\mathsf{G}_x^\circ(k_F)$ to an element of S'. This is equivalent to γ being $G(F)_{x,0}$-conjugate to an element of $S(F)$, because the images of s and γ in $\mathsf{H}_x^\circ(k_F)$ coincide.

We now assume that $\gamma \in S(F)$ and use that in the decomposition $\gamma = su\dot{r}^{-1}r$ we have $u = 1$ and $s = \dot{r}$. Since $\gamma = r$ is regular in $G(F)$, $\dot{r}$ is regular in G_x°, so the summation index k runs over $N(\mathsf{S}', \mathsf{G}_x^\circ)(k_F)$. Taking into account the normalizing factor $|C(s)^\circ(k_F)|^{-1}$ and the fact that every $h \in \mathsf{S}'(k_F)$ commutes with s, we see that the formula becomes

$$(-1)^{r_G - r_S} \sum_{w \in N(\mathsf{S}', \mathsf{G}_x^\circ)(k_F)/\mathsf{S}'(k_F)} \theta(\gamma^w).$$

According to Lemma 3.4.10 the indexing set of this sum is $N(S, G(F)_{x,0})/S(F)_0$ and the proof is complete. $\blacksquare$

3.4.5 Classification

Proposition 3.4.27. Every regular depth-zero supercuspidal representation of $G(F)$ is of the form $\pi_{(S,\theta)}$ for some maximally unramified elliptic maximal torus S and regular depth-zero character $\theta : S(F) \to \mathbb{C}^\times$. Two representations $\pi_{(S_1,\theta_1)}$ and $\pi_{(S_2,\theta_2)}$ are isomorphic if and only if the pairs (S_1, θ_1) and (S_2, θ_2) are $G(F)$-conjugate. $\square$

Proof. For the first statement, let π be regular depth-zero supercuspidal representation. By definition (which involves [MP96, Proposition 6.8]), there exists a vertex $x \in \mathcal{B}^{\mathrm{red}}(G, F)$ such that the restriction of π to $G(F)_{x,0}$ contains the representation $\kappa_{(S,\bar{\theta}')}$ for some maximally unramified elliptic maximal torus S whose vertex is x and some regular character $\theta' : S(F) \to \mathbb{C}^\times$. By [MP96, Proposition 6.6], we have

$$\pi = \text{c-Ind}_{G(F)_x}^{G(F)} \tau,$$

for an irreducible representation τ of $G(F)_x$ which upon restriction to $G(F)_{x,0}$ contains $\kappa_{(S,\bar{\theta}')}$. Now

$$\tau' = \mathrm{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x} \kappa_{(S,\theta')}$$

is one such representation, so $\tau = \tau' \otimes \chi$ for a character $\chi : G(F)_x/G(F)_{x,0} \to \mathbb{C}^\times$. But Corollary 3.4.25 shows

$$\tau' \otimes \chi = \mathrm{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x}(\kappa_{(S,\theta')} \otimes \chi|_{S(F)G(F)_{x,0}}) = \mathrm{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x} \kappa_{(S,\theta' \cdot \chi|_{S(F)})}.$$

The character $\theta = \theta' \cdot \chi|_{S(F)}$ has the same restriction to $S(F)_0$ as θ' and is thus regular. We conclude $\pi = \pi_{(S,\theta)}$.

We come to the second statement. The first half of the proof is the same as for [Kal14, Lemma 3.1.1]. It is clear that conjugate pairs lead to isomorphic representations, so we need to prove the opposite implication. Let $x_i \in \mathcal{B}^{\mathrm{red}}(G, F)$ be the point for S_i and κ_i and $\bar\kappa_i$ the representations of $S_i(F)G(F)_{x_i,0}$ and $G(F)_{x_i,0}$ respectively. By [MP96, Theorem 3.5] the unrefined minimal K-types of depth zero $(G(F)_{x_1,0},\bar\kappa_1)$ and $(G(F)_{x_2,0},\bar\kappa_2)$ are associate. This implies that there exists $g\in G(F)$ with $gx_1=x_2$ and $\mathrm{Ad}(g)\bar\kappa_1=\bar\kappa_2$: Indeed, assume $gx_1\neq x_2$. The group $G(F)_{x_2,0}\cap G(F)_{gx_1,0}$ fixes the unique geodesic in $\mathcal{B}^{\mathrm{red}}(G,F)$ connecting gx_1 and x_2. This geodesic meets a facet of positive dimension containing x_2 in its closure. If y is a point in the intersection of the geodesic and the facet, then $G(F)_{x_2,0}\cap G(F)_{gx_1,0}$ is contained in $G(F)_y$. It is moreover contained in the kernel of the Kottwitz map, and hence in $G(F)_{y,0}$, see §3.3. But the image of $G(F)_{y,0}$ in $G(F)_{x_2,0:0+}$ is a parabolic subgroup of $G(F)_{x_2,0:0+}$ and this precludes $G(F)_{x_2,0}\cap G(F)_{gx_1,0}$ mapping surjectively onto $G(F)_{x_2,0:0+}$.

Conjugating (S_1,θ_1) by g we may assume $g=1$, so $x_1=x_2=:x$ and $\bar\kappa_1=\bar\kappa_2$. By [DL76, Theorem 6.8] we can find $g\in G(F)_{x,0}$ such that $\mathrm{Ad}(g)(\mathsf{S}_1,\bar\theta_1)=(\mathsf{S}_2,\bar\theta_2)$. By Lemma 3.4.5 there is $l\in G(F)_{x,0+}$ such that $\mathrm{Ad}(lg)(S_1,\bar\theta_1)=(S_2,\bar\theta_2)$. We again conjugate (S_1,θ_1) by lg and assume that $S_1=S_2$ and $\bar\theta_1=\bar\theta_2$. Let's write $S=S_1=S_2$ and $\bar\theta_1=\bar\theta_2=\bar\theta$.

In the unramified case the proof is now complete, because $S(F)=S(F)_0\cdot Z(F)$, which implies $\theta_1=\theta_2$, because the central character of $\pi_{(S,\theta_i)}$ is $\theta_i|_{Z(F)}$. Since this fails in the ramified case, we need an additional argument. It is similar to the proof of [Mor89, Proposition 4.2]. The argument given there, combined with [Mor89, Proposition 5.2], shows that for any $g\notin G(F)_x$ the group $\mathrm{Hom}_{G(F)_x\cap {}^gG(F)_x}(\dot\kappa_1,\dot\kappa_2)$ vanishes, where $\dot\kappa_i=\mathrm{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x}\kappa_i$. We conclude

$$\mathrm{Hom}_{G(F)}(\pi_{(S,\theta_1)},\pi_{(S,\theta_2)})=\mathrm{Hom}_{G(F)_x}(\dot\kappa_1,\dot\kappa_2),$$

using Kutzko's Mackey formula [Kut77]. We use again the (this time ordinary) Mackey formula to the subgroup $S(F)\cdot G(F)_{x,0}$ of finite index of $G(F)_x$. Then we get

$$\mathrm{Hom}_{G(F)_x}(\dot\kappa_1,\dot\kappa_2)=\bigoplus_{g\in G(F)_x/S(F)G(F)_{x,0}}\mathrm{Hom}_{S(F)G(F)_{x,0}}(\kappa_1,{}^g\kappa_2).$$

For any coset g the corresponding summand on the right is a subgroup of $\mathrm{Hom}_{G(F)_{x,0}}(\bar\kappa_1,{}^g\bar\kappa_2)$. But we already know $\bar\kappa_1\cong\bar\kappa_2$, so by the same argument as above there exist $h\in G(F)_{x,0}$ and $l\in G(F)_{x,0+}$ such that $\mathrm{Ad}(g)(S,\bar\theta)=\mathrm{Ad}(lh)(S,\bar\theta)$. The regularity of $\bar\theta$ implies that $g^{-1}lh\in S(F)$, which means that g must represent the trivial coset in $G(F)_x/S(F)G(F)_{x,0}$. This implies

$$\mathrm{Hom}_{G(F)}(\pi_{(S,\theta_1)},\pi_{(S,\theta_2)})=\mathrm{Hom}_{S(F)G(F)_{x,0}}(\kappa_1,\kappa_2).$$

From (3.4.2) we see that both κ_1 and κ_2 act on the same vector space $V_{\bar\theta}$ and $\kappa_2=\theta_2\theta_1^{-1}\otimes\kappa_1$, because $\theta_2\theta_1^{-1}$ is a character of $S(F)/S(F)_0=S(F)G(F)_{x,0}/G(F)_{x,0}$. Since $\bar\kappa_1=\bar\kappa_2$ is already an irreducible representation of $G(F)_{x,0}$, any element of $\mathrm{Hom}_{S(F)G(F)_{x,0}}(\kappa_1,\kappa_2)$ is a scalar multiple of the identity, which forces $\theta_2=\theta_1$. ■

The following lemma will be needed for the classification of positive depth regular supercuspidal representations.

Lemma 3.4.28. Let $S\subset G$ be maximally unramified elliptic, $\theta:S(F)\to\mathbb{C}^\times$ a regular character, and $\phi:G(F)\to\mathbb{C}^\times$ a character. If the depth of $\phi\otimes\pi_{(S,\theta)}$ is zero, then the depth of ϕ is zero. If the depth of ϕ is zero, then $\phi\otimes\pi_{(S,\theta)}=\pi_{(S,\phi\cdot\theta)}$. □

Proof. The representation $\phi \otimes \pi_{(S,\theta)}$ is supercuspidal and if its depth is zero, then by [MP96, Proposition 6.8] it is of the form $\text{c-Ind}_{G(F)_y}^{G(F)} \tau$ for an irreducible representation τ of the stabilizer $G(F)_y$ of some vertex $y \in \mathcal{B}^{\text{red}}(G, F)$, whose restriction to $G(F)_{y,0}$ contains a cuspidal representation σ. At the same time, writing $\dot{\kappa}_{(S,\theta)} = \text{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x} \kappa_{(S,\theta)}$ we see $\phi \otimes \pi_{(S,\theta)} = \text{c-Ind}_{G(F)_x}^{G(F)}(\phi \otimes \dot{\kappa}_{(S,\theta)})$. Applying Kutzko's Mackey formula [Kut77] we see that

$$\begin{aligned} \text{End}_{G(F)}(\phi \otimes \pi_{(S,\theta)}) &= \text{Hom}_{G(F)}(\text{c-Ind}_{G(F)_y}^{G(F)} \tau, \text{c-Ind}_{G(F)_x}^{G(F)}(\phi \otimes \dot{\kappa}_{(S,\theta)})) \\ &= \bigoplus_g \text{Hom}_{G(F)_y \cap G(F)_{gx}}(\tau, {}^g[\phi \otimes \dot{\kappa}_{(S,\theta)}]), \end{aligned}$$

where g runs over $G(F)_y \setminus G(F)/G(F)_x$. Since the left hand side is non-zero there must exist g for which the corresponding summand on the right is non-zero. This summand is a subgroup of $\text{Hom}_{G(F)_{y,0+} \cap G(F)_{gx,0+}}(\tau, {}^g[\phi \otimes \dot{\kappa}_{(S,\theta)}])$. Since both τ and ${}^g\dot{\kappa}_{(S,\theta)}$ are 1-isotypic upon restriction to $G(F)_{y,0+} \cap G(F)_{gx,0+}$ we see that ϕ must be trivial upon restriction to this group. By [HM08, Lemma 2.45, Definition 2.46] this implies that ϕ has depth zero.

The equality $\phi \otimes \pi_{(S,\theta)} = \pi_{(S,\phi\cdot\theta)}$ now follows from Corollary 3.4.25 and the obvious equality $\phi \otimes \pi_{(S,\theta)} = \text{c-Ind}_{S(F)G(F)_{x,0}}^{G(F)}(\phi \otimes \kappa_{(S,\theta)})$. ∎

3.5 Review of the work of Hakim and Murnaghan

In accordance with [Yu01] and [HM08] we now assume that the residual characteristic of F is not 2 and that G splits over a tame extension of F. Let $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ be a reduced generic cuspidal G-datum, in the sense of [HM08, Definition 3.11]. We recall that each G^i is a tame twisted Levi subgroup of G, i.e. a connected reductive subgroup of G that is defined over F and becomes a Levi subgroup of G over a tame extension of F, $G^d = G$, π_{-1} is a depth-zero supercuspidal representation of $G^0(F)$, and $\phi_i : G^i(F) \to \mathbb{C}^\times$ is a smooth character of depth $r_i > 0$, which is G^{i+1}-generic when $i \neq d$. We refer the reader to [HM08, §3.1] for the notion of a generic character, as well as for the precise list of conditions imposed on this data.

From a reduced generic cuspidal G-datum, the construction of [Yu01] produces an irreducible supercuspidal representation of $G(F)$. We can think of Yu's construction as a map from the set of reduced generic cuspidal G-data to the set of isomorphism classes of irreducible supercuspidal representations of $G(F)$. One of the main results of [HM08] is the description of the fibers of this map. Hakim and Murnaghan introduce three operations on the set of reduced generic cuspidal G-data: elementary transformation, G-conjugation, and refactorization. According to [HM08, Theorem 6.6], the equivalence relation generated by these operations, called G-equivalence in [HM08], places two reduced generic cuspidal G-data in the same equivalence class precisely when they lead to isomorphic supercuspidal representations. This theorem is valid under a certain technical hypothesis, called $C(\vec{G})$.

Our goal in this subsection is to recall the notion of G-equivalence and Hypothesis $C(\vec{G})$, and then show that [HM08, Theorem 6.6] is valid even without assuming $C(\vec{G})$. For this, we will first prove that Hypothesis $C(\vec{G})$ holds for all G for which the fundamental group of G_{der} has order prime to p. In particular, it holds when G_{der} is simply connected. We will then use this to treat the case of general G.

We now recall Hypothesis $C(\vec{G})$ from [HM08, §2.6]. Given a tower $\vec{G} = (G^0 \subsetneq G^1 \cdots \subsetneq G^d)$ of twisted Levi subgroups of G, Hypothesis $C(\vec{G})$ is the concatenation of hypotheses $C(G^i)$. In turn, Hypothesis $C(G)$ stipulates that whenever $\phi : G(F) \to \mathbb{C}^\times$ is a character of positive depth $r > 0$ and $x \in \mathcal{B}^{\text{red}}(G, F)$, the restriction $\phi|_{G(F)_{x,(r/2)+}}$ is realized by an element of $\text{Lie}^*(Z(G)^\circ)(F)_{-r}$. This means the following. Let $\mathfrak{g} = \text{Lie}(G)$ and let $\Lambda : F \to \mathbb{C}^\times$ be an additive character of depth zero. For any $r > s$ the Pontryagin dual of the abelian group $\mathfrak{g}(F)_{x,s+}/\mathfrak{g}(F)_{x,r+}$ is identified with the abelian group $\mathfrak{g}^*(F)_{x,-r}/\mathfrak{g}^*(F)_{x,-s}$, via the pairing

$$(Y, X^*) \mapsto \Lambda\langle X^*, Y\rangle, \qquad Y \in \mathfrak{g}(F)_{x,s+}, X^* \in \mathfrak{g}^*(F)_{x,-r}.$$

Whenever $r > s \geq r/2$ we have the Moy-Prasad isomorphism (see [Yu01, Corollary 2.4] and the discussion following [HM08, Definition 2.46])

$$\text{MP}_x : \mathfrak{g}(F)_{x,s+}/\mathfrak{g}(F)_{x,r+} \to G(F)_{x,s+}/G(F)_{x,r+},$$

via which $\mathfrak{g}^*(F)_{x,-r}/\mathfrak{g}^*(F)_{x,-s}$ is identified with the Pontryagin dual of the abelian group $G(F)_{x,s+}/G(F)_{x,r+}$. An element $X^* \in \mathfrak{g}^*(F)_{x,-r}/\mathfrak{g}^*(F)_{x,-s}$ is said to realize the character of $G(F)_{x,s+}/G(F)_{x,r+}$ that it corresponds to under this identification. Now let $\mathfrak{z} = \text{Lie}(Z(G)^\circ)$. As discussed in [Yu01, §8], there is a natural way to view $\mathfrak{z}^*$ as a subspace of $\mathfrak{g}^*$. Namely, the natural projection $\mathfrak{g}^* \to \mathfrak{z}^*$ that is dual to the inclusion $\mathfrak{z} \to \mathfrak{g}$ becomes an isomorphism upon restriction to the subspace of $\mathfrak{g}^*$ that is 1-isotypic for the adjoint action of G.

Lemma 3.5.1. If the fundamental group of G_{der} has order prime to p, then every character of $G_{\text{der}}(F)$ has depth at most zero. □

Proof. Assume first that G_{der} is simply connected and write $G_{\text{der}} = G_{\text{sc}}$ as the product $G_{\text{sc}} = G_{\text{sc},1} \times \cdots \times G_{\text{sc},n}$, with each $G_{\text{sc},i}$ simple over F. Then $G_{\text{sc},i}$ is either isotropic, or else by [BT87, §4.5,§4.6] isomorphic to $\text{Res}_{E/F}\text{SL}_1(D)$, where E/F is a finite extension and D/E is a central division algebra. In the first case, $G_{\text{sc},i}$ satisfies the Kneser-Tits conjecture [Tit78, §1.2] and hence $G_{\text{sc},i}(F)$ has no characters. In the second case $G_{\text{sc},i}(F)$ is isomorphic to the group $D^{(1)}$ of elements of D whose reduced norm is equal to 1. According to [Rie70, §5 Corollary], the derived subgroup of $D^{(1)}$ is equal to $(1 + \mathfrak{p}_D) \cap D^{(1)}$, where $\mathfrak{p}_D$ is the maximal ideal of D. In terms of Moy-Prasad filtrations this means that the derived subgroup of $G_{\text{sc},i}(F)$ is $G_{\text{sc},i}(F)_{x,0+}$, where x is the unique point in the reduced building of $G_{\text{sc},i}(F)$. We conclude that every character of $G_{\text{sc}}(F)$ is trivial on $G_{\text{sc}}(F)_{x,0+}$ for any $x \in \mathcal{B}^{\text{red}}(G, F)$.

For the general case, let $x \in \mathcal{B}^{\text{red}}(G, F)$. Let $A \subset G$ be a maximal split torus such that x belongs to the apartment of A. According to [BT84, Corollaire 5.1.12] there exists a maximal torus $T \subset G$ containing A and maximally split over F^u. Since G is tame, so is T. Let T_{der} and T_{sc} be the corresponding maximal tori of G_{der} and G_{sc}. Lemma 3.3.2 implies that the natural map $G_{\text{sc}}(F)_{x,0+} \to G_{\text{der}}(F)_{x,0+}$ is bijective. ■

Lemma 3.5.2. Assume the fundamental group of G_{der} has order prime to p. Then Hypothesis $C(G)$ holds. More generally, Hypothesis $C(\vec{G})$ holds for any tower of twisted Levi subgroups of G. □

Proof. The fundamental group of the derived subgroup of any twisted Levi subgroup of G is a subgroup of the fundamental group of the derived subgroup of G. It is therefore enough to establish Hypothesis $C(G)$. When G is a torus the statement is clear, because then $\mathfrak{g}^* = \mathfrak{z}^*$. For the general case, let $D = G/G_{\text{der}}$ and let $\phi : G(F) \to \mathbb{C}^\times$ be a character of depth $r > 0$. The restriction of ϕ to $G_{\text{der}}(F)_{x,(r/2)+}$ is trivial by Lemma 3.5.1, hence its restriction to $G(F)_{x,(r/2)+}$ factors through a character of $D(F)_{(r/2)+}$ by Lemma 3.3.2. This character is represented by an element $X^* \in \text{Lie}^*(D)(F)_{-r}$. Under the exact sequence of dual Lie algebras

$$1 \to \text{Lie}^*(D) \to \text{Lie}^*(G) \to \text{Lie}^*(G_{\text{der}}) \to 1$$

the image of $\text{Lie}^*(D)$ in $\text{Lie}^*(G)$ is precisely the subspace that Yu identifies with $\text{Lie}^*(Z(G)^\circ)$ in [Yu01, §8]. Thus the image of X^* in $\text{Lie}^*(G)_{x,-r}$ realizes $\phi|_{G(F)_{x,(r/2)+}}$. ■

At the moment we do not know if Hypothesis $C(G)$ holds without the assumption on the fundamental group of G_{der}. However, we can still prove that [HM08, Theorem 6.6] is valid without this assumption, by reducing to the case where G_{der} is simply connected. The main tool that we exploit for that is z-extensions, introduced by Langlands and Kottwitz. Recall from [Kot82, §1] that a z-extension of G is an exact sequence $1 \to K \to \tilde{G} \to G \to 1$ of connected reductive groups defined over F, where the derived subgroup of $\tilde{G}$ is simply connected, and K is an induced torus (automatically central). In particular, the map $\tilde{G}(F) \to G(F)$ is surjective. Such a z-extension always exists.

Lemma 3.5.3. Let $1 \to K \to \tilde{G} \to G \to 1$ be a z-extension and let $x \in \mathcal{B}^{\text{red}}(G, F)$. For any $r \geq 0$ the sequence

$$1 \to K(F)_r \to \tilde{G}(F)_{x,r} \to G(F)_{x,r} \to 1$$

is exact. For any $r > 0$ the sequence

$$1 \to G_{\text{sc}}(F)_{x,r} \to \tilde{G}(F)_{x,r} \to D(F)_r \to 1$$

is exact, where $D = \tilde{G}/G_{\text{sc}}$. □

Proof. This is a special case of Lemma 3.3.2. ■

Given such a z-extension, we can pull-back the reduced generic cuspidal G-datum to $\tilde{G}$ to obtain $((\tilde{G}^0 \subsetneq \tilde{G}^1 \cdots \subsetneq \tilde{G}^d), \tilde{\pi}_{-1}, (\tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_d))$. Here $\tilde{G}^i$ is the preimage of G^i in $\tilde{G}$ and is a twisted Levi subgroup of $\tilde{G}$, $\tilde{\pi}_{-1}$ is the composition of π_{-1} with the surjective homomorphism $\tilde{G}^0(F) \to G^0(F)$ and is an irreducible supercuspidal representation of depth-zero, and $\tilde{\phi}_i$ is the composition of ϕ_i with the surjection $\tilde{G}^i(F) \to G^i(F)$ and is a character of the same depth as ϕ_i, generic when $i \neq d$. The result of this procedure is a reduced cuspidal generic datum for $\tilde{G}$. The irreducible supercuspidal representation of $\tilde{G}(F)$ associated to this datum by Yu's construction is the pull-back of the irreducible supercuspidal representation of $G(F)$ associated to the reduced cuspidal G-datum we started with. Note here that $K(F)$ is contained in the compact open subgroup of $\tilde{G}(F)$ from which the supecuspidal representation of $\tilde{G}(F)$ is compactly induced.

We now recall the notion of G-equivalence of reduced generic cuspidal G-data. It is the equivalence relation generated by three operations: G-conjugation, elementary transformation, and refactorization. The operation of G-conjugation is obvious from its name – one replaces each member of the G-datum by its conjugate under a given $g \in G(F)$. An elementary transformation consists of replacing π_{-1} by an isomorphic representation. If we are already thinking of π_{-1} as an isomorphism class of representations, then this operation is vacuous. Finally, a datum $((G'^0 \subsetneq G'^1 \cdots \subsetneq G'^d), \pi'_{-1}, (\phi'_0, \phi'_1, \ldots, \phi'_d))$ is a refactorization of $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ if $G'^i = G^i$ for all i and the following conditions involving

$$\chi_i : G^i(F) \to \mathbb{C}^\times, \qquad \chi_i(g) := \prod_{j=i}^{d} \phi_j(g)\phi'_j(g)^{-1},$$

are satisfied:

F0. If $\phi_d = 1$ then $\phi'_d = 1$;

F1. χ_i is of depth at most r_{i-1} for all $i = 0, \ldots, d$, where $r_{-1} = 0$;

F2. $\pi'_{-1} = \pi_{-1} \otimes \chi_0$.

Note that the three operations of G-conjugation, elementary transformation, and refactorization, commute.

Lemma 3.5.4. Let $1 \to K \to \tilde{G} \to G \to 1$ be a z-extension. Two reduced generic cuspidal G-data are G-equivalent if and only if their pull-backs to $\tilde{G}$ are $\tilde{G}$-equivalent. □

Proof. Let $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ and $((G'^0 \subsetneq G'^1 \cdots \subsetneq G'^d), \pi'_{-1}, (\phi'_0, \phi'_1, \ldots, \phi'_d))$ be the two reduced data for G. It is enough to check the three relations that generate G-equivalence: G-conjugacy, elementary transformation, and refactorization. For these, the statement follows immediately from the surjectivity of the maps $\tilde{G}^i(F) \to G^i(F)$ and $\tilde{G}^i(F)_{x,r} \to G^i(F)_{x,r}$ for any $x \in \mathcal{B}^{\text{red}}(G^i, F)$ and $r \geq 0$. ■

Corollary 3.5.5. Let Ψ and Ψ' be two reduced generic cuspidal G-data, and let $\pi(\Psi)$ and $\pi(\Psi')$ be the corresponding irreducible supercuspidal representations of $G(F)$. Then $\pi(\Psi)$ and $\pi(\Psi')$ are isomorphic if and only if Ψ and Ψ' are G-equivalent (without assuming Hypotheses $C(\vec{G})$ and $C(\vec{G}')$). □

Proof. This follows immediately from Lemma 3.5.4 and [HM08, Theorem 6.6]. ■

3.6 Howe factorization

In this subsection we assume that p is not a bad prime (in particular also not a torsion prime) for G. We further assume that p does not divide the order of the fundamental group of G_{der}.

Imagine we are given a reduced generic cuspidal G-datum $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ such that the depth-zero supercuspidal representation π_{-1} of $G^0(F)$ is regular in the sense of Definition 3.4.19. Proposition

3.4.27 produces from π_{-1} a $G^0(F)$-conjugacy class of pairs (S, ϕ_{-1}). We can let $\theta : S(F) \to \mathbb{C}^\times$ be the product $\prod_{i=-1}^{d} \phi_i|_{S(F)}$. It was observed by Murnaghan [Mur11], in a more technically restricted setting, that the $G(F)$-conjugacy class of the pair (S, θ) obtained in this way does not change if we replace the G-datum by a G-equivalent one. Thus, the $G(F)$-conjugacy class of the pair (S, θ) is an invariant of the representation π obtained from the G-datum via Yu's construction. In order to turn this observation into an effective classification of representations we need a reverse process – one that takes a pair (S, θ) and "unfolds" the information contained in it into a reduced generic cuspidal G-datum. Motivated by this we shall introduce in this subsection a factorization algorithm that generalizes to arbitrary connected reductive groups (split over a tame extension) the Howe factorization lemma ([How77, Lemma 11 and Corollary]).

Let (S, θ) be a pair consisting of a tame maximal torus $S \subset G$ and a character $\theta : S(F) \to \mathbb{C}^\times$. Let E be the splitting field of S. For each positive real number r consider the set of roots

$$R_r = \{\alpha \in R(S, G) | \theta(N_{E/F}(\alpha^\vee(E_r^\times))) = 1\}. \tag{3.6.1}$$

Then $r \mapsto R_r$ is a Γ-invariant filtration of $R(S, G)$. We have $R_s \subset R_r$ for $s < r$ and define $R_{r+} = \bigcap_{s>r} R_s$. Let $r_{d-1} > r_{d-2} > \cdots > r_0 > 0$ be the breaks of this filtration, that is, the positive real numbers r with $R_{r+} \neq R_r$. We allow here $d = 0$, which signifies that there are no breaks, i.e. $R_{0+} = R(S, G)$. We set in addition $r_{-1} = 0$ and $r_d = \text{depth}(\theta)$, so that $r_d \geq r_{d-1} > \cdots > r_0 > r_{-1} = 0$ if $d > 0$ and $r_0 \geq r_{-1} = 0$ if $d = 0$. For each $d \geq i \geq 0$ let G^i be the connected reductive subgroup of G with maximal torus S and root system $R_{r_{i-1}+}$. By definition the root system of G^d is $R(S, G)$, so $G^d = G$. Moreover, the root system of G^0 is R_{0+}, which may or may not be empty. If it is empty, then $G^0 = S$. We set $G^{-1} = S$. Since S splits over a tamely ramified extension of F, so does each G^i. The following Lemma shows that each G^i is a tame twisted Levi.

Lemma 3.6.1. The subset $R_r \subset R(S, G)$ is a Levi subsystem of $R(S, G)$. □

Proof. Let $\varphi \in Z^1(W_F, \widehat{S})$ be the Langlands parameter of θ. The Langlands parameter of the character $\theta \circ N_{E/F} \circ \alpha^\vee : E^\times \to \mathbb{C}^\times$ is $\widehat{\alpha} \circ \varphi|_{W_E}$. By local class field theory, $\theta \circ N_{E/F} \circ \alpha^\vee$ has non-trivial restriction to $E_r^\times$ if and only if $\widehat{\alpha} \circ \varphi|_{W_E}$ has non-trivial restriction to $I_E^r = I_F^r$. Thus

$$R_r = \{\alpha \in R(S, G) | \widehat{\alpha}(\varphi(I^r)) = 1\}.$$

But $\varphi(I^r)$ is a finite subgroup of $\widehat{S}$ and then $R_r^\vee = \{\widehat{\alpha} \in R(\widehat{S}, \widehat{G}) | \widehat{\alpha}(\varphi(I^r)) = 1\}$ is the root system of the connected centralizer in $\widehat{G}$ of this finite group. This connected centralizer is a Levi subgroup, as one sees by applying [AS08, Proposition A.7] repeatedly to the elements in the image of $\varphi(I^r)$. Thus $R_r^\vee$ is a Levi subsystem of $R(S, G)^\vee$. ■

Definition 3.6.2. A *Howe factorization* of (S, θ) is a sequence of characters $\phi_i : G^i(F) \to \mathbb{C}^\times$ for $i = -1, \ldots, d$ with the following properties.

1.
$$\theta = \prod_{i=-1}^{d} \phi_i|_{S(F)}. \tag{3.6.2}$$

2. For all $0 \leq i \leq d$ the character ϕ_i is trivial on $G^i_{\text{sc}}(F)$.

3. For all $0 \leq i < d$, ϕ_i has depth r_i and is G^{i+1}-generic. For $i = d$, ϕ_d is trivial if $r_d = r_{d-1}$ and has depth r_d otherwise. For $i = -1$, ϕ_{-1} is trivial if $G^0 = S$ and otherwise satisfies $\phi_{-1}|_{S(F)_{0+}} = 1$.

□

The discussion of [HM08, §3.5] makes clear the direct parallel between this definition and the original notion of Howe factorization for the group GL_N, as formulated for example in [HM08, Defintion 3.33]. Our goal in this subsection is to show that Howe factorizations always exist. But we begin by reviewing the process inverse to Howe factorization alluded to above.

Lemma 3.6.3. Let $t \geq 0$ be a natural number. If $t = 0$ assume given a sequence of real numbers $s_0 \geq s_{-1} = 0$; if $t > 0$ assume given a sequence of real numbers $s_t \geq s_{t-1} > \cdots > s_0 > s_{-1} = 0$. Assume given a tower $S = H^{-1} \subsetneq H^0 \subsetneq H^1 \cdots \subsetneq H^t = G$ of tame twisted Levi subgroups. Let $\phi_i : H^i \to \mathbb{C}^\times$, $i = -1, \dots, t$ be characters. Assume that the characters ϕ_i satisfy properties 2. and 3. of Definition 3.6.2 with respect to the groups H^i and the numbers s_i. Define $\theta = \prod_{i=-1}^t \phi_i|_{S(F)}$.

Then

1. The numbers $s_{t-1} > s_{t-2} > \cdots > s_0 \geq 0$ are precisely the breaks of the filtration (3.6.1) associated to the character θ. That is, $d = t$ and $s_i = r_i$ for all $i = -1, \dots, d$.

2. For each $i = -1, \dots, d$, the subset $R_{r_{i-1}+} \subset R(S, G)$ is the root system of the group H^i. That is, $H^i = G^i$.

3. $(\phi_{-1}, \dots, \phi_d)$ is a Howe factorization of (S, θ).

□

Proof. The characters $(\phi_{-1}, \dots, \phi_t)$ satisfy part 1. of Definition 3.6.2 by definition of θ. They also satisfy parts 2. and 3. of Definition 3.6.2, but with G^i and r_i replaced by H^i and s_i. Thus, the third point of the lemma follows from the first two points. Those in turn are equivalent to the following two inclusions

$$R(S, H^{i+1}) \subset R_{s_i+}, \qquad \forall i = -1, \dots, t-1$$

and

$$R(S, H^{i+1}) \smallsetminus R(S, H^i) \subset R_{s_i+} \smallsetminus R_{s_i}, \qquad \forall i = 0, \dots, t-1,$$

where we set $s_{-1} = 0$. If $t = 0$ these inclusions are trivial, so we assume $t > 0$.

If $\alpha \in R(S, H^{i+1})$ and $j > i$ then $\phi_j \circ N_{E/F} \circ \alpha^\vee(E^\times) = 1$, because $N_{E/F} \circ \alpha^\vee$ takes image in $H^{i+1}_{\text{sc}}(F)$, while the pull-back of ϕ_j to $H^j_{\text{sc}}(F)$, and hence also to $H^{i+1}_{\text{sc}}(F)$, is trivial by assumption. Thus

$$\theta \circ N_{E/F} \circ \alpha^\vee = (\phi_{-1} \dots \phi_i) \circ N_{E/F} \circ \alpha^\vee.$$

Since $N_{E/F}(\alpha^\vee(E^\times_r)) \subset S(F)_r$ for any $r > 0$ we see $\alpha \in R_{s_i+}$, which is the first claimed inclusion. Furthermore, for $j < i$ we have $\phi_j \circ N_{E/F} \circ \alpha^\vee(E^\times_{s_i}) = 1$, because ϕ_j is trivial on $S(F)_{s_j+} \supset S(F)_{s_i}$. Thus

$$\theta(N_{E/F}(\alpha^\vee(E^\times_{s_i}))) = \phi_i(N_{E/F}(\alpha^\vee(E^\times_{s_i}))).$$

Assume now $\alpha \notin R(S, H^i)$. We will show that $\phi_i(N_{E/F}(\alpha^\vee(E_{s_i}^\times))) \neq 1$. A direct computation shows

$$\phi_i(N_{E/F}(\alpha^\vee(1+x))) = \Lambda \circ \mathrm{tr}_{E/F}(x\langle X_i^*, H_\alpha\rangle),$$

where $X_i^* \in \mathrm{Lie}^*(Z(H^i))(F)_{-s_i}$ represents ϕ_i. By assumption $\mathrm{ord}(\langle X_i^*, H_\alpha\rangle) = -s_i$, so every element of $O_E^\times$ can be written as $x\langle X_i^*, H_\alpha\rangle$ for some $x \in E_{s_i}$. The character $\Lambda \circ \mathrm{tr}_{E/F}$ is non-trivial on O_E, so the left-hand side is non-trivial for some $x \in E_{s_i}$. We conclude that $\phi_i \circ N_{E/F} \circ \alpha^\vee$, and hence also $\theta \circ N_{E/F} \circ \alpha^\vee$, is non-trivial on $E_{s_i}^\times$, as claimed. This implies $\alpha \notin R_{s_i}$. We have thus proved the second inclusion. ■

Before we discuss the existence of Howe factorizations, let us first collect some of their properties.

Fact 3.6.4. Let S_{sc}^i be the preimage of S in G_{sc}^i. Then the restrictions of ϕ_i and θ to $S_{\mathrm{sc}}^{i+1}(F)_{r_i}$ agree. □

Proof. This follows from (3.6.2), since $\phi_{i+1}, \dots, \phi_d$ restrict trivially to $G_{\mathrm{sc}}^{i+1}(F)$ and $\phi_{i-1}, \dots, \phi_{-1}$ restrict trivially to $S(F)_{r_i}$. ■

Lemma 3.6.5. Let $r = 0$ or $r = 0+$.

1. The stabilizer of $\theta|_{S(F)_r}$ in $\Omega(S, G)(F)$ lies in $\Omega(S, G^0)(F)$ and equals the stabilizer of $\phi_{-1}|_{S(F)_r}$ there.
2. The stabilizer of $\theta|_{S(F)_r}$ in $N(S, G)(F)$ lies in $N(S, G^0)(F)$ and equals the stabilizer of $\phi_{-1}|_{S(F)_r}$ there.

□

Proof. We begin with the following observation: For any $s \in S(F)$ and $w \in \Omega(S, G)(F)$ the element $wsw^{-1}s^{-1} \in S(F)$ lifts to $S_{\mathrm{sc}}(F)$. Since ϕ_d is trivial on $G_{\mathrm{sc}}(F)$ we see that $\phi_d|_{S(F)}$ is invariant under $\Omega(S, G)(F)$.

We now prove the first claim by induction on d. The case $d = 0$ follows immediately from above observation. Now assume the claim has been proved for all reductive groups and all torus-character pairs that have a Howe-factorization of length less than d. Let $w \in \Omega(S, G)(F)$ fix $\theta|_{S(F)_r}$. Applying again above observation we see that w fixes $\theta_{d-1}|_{S(F)_r}$, where $\theta_{d-1} = \theta \cdot \phi_d^{-1}$. Since $r < r_{d-1}$, w fixes $\theta_{d-1}|_{S(F)_{r_{d-1}}} = \phi_{d-1}|_{S(F)_{r_{d-1}}}$. But ϕ_{d-1} is G^d-generic, hence $w \in \Omega(S, G^{d-1})(F)$. We see that the stabilizer of $\theta|_{S(F)_r}$ in $\Omega(S, G)(F)$ belongs to $\Omega(S, G^{d-1})(F)$ and equals the stabilizer of $\theta_{d-1}|_{S(F)_r}$ there. We can now apply the induction hypothesis to the character θ_{d-1} of the maximal torus $S(F)$ in the group $G^{d-1}(F)$, with the Howe factorization $\theta_{d-1} = \phi_{-1} \dots \phi_{d-1}$. This proves the first claim.

Chasing through the following commutative diagram with exact rows

$$\begin{tikzcd}
S(F) \arrow[r] \arrow[d, equal] & N(S,G)(F) \arrow[r] & \Omega(S,G)(F) \arrow[r] & H^1(\Gamma, S) \arrow[d, equal] \\
S(F) \arrow[r] & N(S,G^0)(F) \arrow[r] \arrow[u] & \Omega(S,G^0)(F) \arrow[r] \arrow[u] & H^1(\Gamma, S)
\end{tikzcd}$$

we see that the first claim implies the second. ■

Lemma 3.6.6. If $(\phi_{-1},\dots,\phi_d)$ and $(\phi'_{-1},\dots,\phi'_d)$ are two Howe factorizations of the same pair (S,θ), then they are refactorizations of each other in the sense of [HM08, Definition 4.19] $\square$

Proof. We need to check the three properties F0,F1,F2 in [HM08, Definition 4.19], which we reviewed in §3.5. As there, we define $\chi_i : G^i(F) \to \mathbb{C}^\times$ by

$$\chi_i = \prod_{j=i}^{d} \phi_j \cdot \phi_j'^{-1},$$

for $0 \le i \le d$. For F0, the definition of Howe factorization implies that $\phi_d = 1$ if and only if $r_d = r_{d-1}$ if and only if $\phi'_d = 1$.

For F1, we observe that by (3.6.2) we have $\chi_i|_{S(F)} = \prod_{j=-1}^{i-1} \phi_j^{-1}\phi'_j|_{S(F)}$, which implies depth$(\chi_i|_{S(F)}) \le r_{i-1}$. To show that this implies depth$(\chi_i) \le r_{i-1}$ we choose a z-extension $\tilde{G} \to G$ and apply the two exact sequences of Lemma 3.5.3. The first sequence allows us to replace G by $\tilde{G}$ or, in other words, assume that G has a simply connected derived subgroup. Since ϕ_j and ϕ'_j are trivial on $G^j_{\text{sc}}(F)$ for all $j \ge i$ we see that χ_i is trivial on $G^i_{\text{sc}}(F)$ and thus descends to $D(F)$, where $D = G/G_{\text{sc}}$. The second sequence of Lemma 3.5.3 shows that it is enough to check depth$(\chi_i) \le r_{i-1}$ on $D(F)$. Finally, applying Lemma 3.1.3 to the exact sequence $1 \to S_{\text{sc}} \to S \to D \to 1$ we reduce this to depth$(\chi_i|_{S(F)}) \le r_{i-1}$.

For F2, we have again by (3.6.2) the equality $\phi'_{-1} = \phi_{-1}\chi_0$ and the statement follows from Lemma 3.4.28. ■

We will now show that Howe factorizations exist.

Proposition 3.6.7. Any pair (S,θ) consisting of a tame maximal torus $S \subset G$ and a character $\theta : S(F) \to \mathbb{C}^\times$ has a Howe factorization $(\phi_{-1},\dots,\phi_d)$. $\square$

The remainder of this section is devoted to the proof of this proposition. This proof will be constructive, i.e. we shall give an algorithm that recursively produces the Howe factorization. First, we prove some technical lemmas that will be needed for the algorithm.

Lemma 3.6.8. Let $H \subset G$ be a twisted Levi subgroup containing S and let $\phi : H(F) \to \mathbb{C}^\times$ be a character of positive depth r, trivial on $H_{\text{sc}}(F)$. Then ϕ is generic if and only if for all $\alpha \in R(S,G)\smallsetminus R(S,H)$ we have $\phi(N_{E/F}(\alpha^\vee(E_r^\times))) \ne 1$. $\square$

Proof. Assume first that the derived subgroup of G, and hence also of H, is simply connected. Put $D = H/H_{\text{sc}}$. Fix a point $x \in \mathcal{A}^{\text{red}}(S,E) \cap \mathcal{B}^{\text{red}}(H,F)$. By Kneser's theorem [BT87, §4.7] $H(F) \to D(F)$ is surjective and by Lemma 3.3.2 $H(F)_{x,s} \to D(F)_s$ remains surjective for all $s > 0$, so ϕ is inflated from a character of the torus $D(F)$ of depth r. Let $\Lambda : F \to \mathbb{C}^\times$ be a character of depth zero, i.e. trivial on F_1 but not on F_0. Recall the Moy-Prasad isomorphism

$$\text{MP}_{H,x,r} : \text{Lie}(H)(F)_{x,r}/\text{Lie}(H)(F)_{x,r+} \to H(F)_{x,r}/H(F)_{x,r+}.$$

Let $X^* \in \text{Lie}^*(D)(F)_{-r}/\text{Lie}^*(D)(F)_{-r+}$ be the element satisfying

$$\phi(\text{MP}_{D,r}(Y)) = \Lambda\langle X^*, Y\rangle, \quad \forall Y \in \text{Lie}(D)(F)_r.$$

The surjection $H \to D$ leads to an injection $\mathrm{Lie}^*(D) \to \mathrm{Lie}^*(H)$ whose image is precisely the subspace that Yu identifies with $\mathrm{Lie}^*(Z(H)^\circ)$ in [Yu01, §8]. Letting X^* stand also for its image in $\mathrm{Lie}^*(H)$ we then obtain

$$\phi(\mathrm{MP}_{H,x,r}(Y)) = \Lambda\langle X^*, Y\rangle, \quad \forall Y \in \mathrm{Lie}(H)(F)_r.$$

We can restrict this equation to $Y \in \mathrm{Lie}(S)(F)_r$. Using the fact that $\mathrm{MP}_{S,r}$ is Γ-equivariant we see that for all $Y \in \mathrm{Lie}(S)(E)_r$ we have

$$\phi(N_{E/F}(\mathrm{MP}_{S,r}(Y))) {=} \phi(\mathrm{MP}_{S,r}(\mathrm{tr}_{E/F}(Y))) {=} \Lambda\langle X^*, \mathrm{tr}_{E/F} Y\rangle {=} \Lambda \circ \mathrm{tr}_{E/F}\langle X^*, Y\rangle.$$

From the functoriality of the Moy-Prasad isomorphism in the case of the split torus $S \times E$ we have for all $y \in E_r/E_{r+}$

$$\mathrm{MP}_{S,r}(d\alpha^\vee(y)) = \alpha^\vee(y+1).$$

Combining the last two equations we obtain

$$\phi(N_{E/F}(\alpha^\vee(y+1))) = \Lambda \circ \mathrm{tr}_{E/F}(y\langle X^*, H_\alpha\rangle).$$

Therefore $\phi(N_{E/F}(\alpha^\vee(E_r^\times))) \neq 1$ is equivalent to the existence of some $y \in E_r$ s.t. $\Lambda \circ \mathrm{tr}_{E/F}(y\langle X^*, H_\alpha\rangle)$ doesn't vanish. Since the character $\Lambda \circ \mathrm{tr}_{E/F}$ is of depth zero this in turn is equivalent to $\mathrm{ord}\langle X^*, H_\alpha\rangle = -r$. This condition for all $\alpha \in R(S,G) \smallsetminus R(S,H)$ is condition GE1 of [Yu01, §8] for ϕ. By [Yu01, Lemma 8.1] condition GE1 implies GE2.

Now drop the assumption that the derived subgroup of G is simply connected. Let $1 \to K \to \tilde{G} \to G \to 1$ be a z-extension. Pull-back along the inclusion $H \to G$ gives a z-extension $1 \to K \to \tilde{H} \to H \to 1$. Let $\tilde{\phi} : \tilde{H}(F) \to \mathbb{C}^\times$ be the pull-back of ϕ. If $X^* \in \mathrm{Lie}^*(H)(F)_{-r}$ represents $\phi|_{H(F)_r}$ as above, then its image $\tilde{X}^* \in \mathrm{Lie}^*(\tilde{H})(F)_{-r}$ under the natural inclusion represents $\tilde{\phi}|_{\tilde{H}(F)_r}$. Since H_α is naturally an element of $\mathrm{Lie}(\tilde{H})$ the above argument shows $\mathrm{ord}\langle X^*, H_\alpha\rangle = \mathrm{ord}\langle \tilde{X}^*, H_\alpha\rangle = -r$. ■

Lemma 3.6.9. Let $\theta : S(F) \to \mathbb{C}^\times$ be a character of positive depth r. If for all $\alpha \in R(S,G)$ we have $\theta \circ N_{E/F} \circ \alpha^\vee|_{E_r^\times} = 1$, then there exists a character $\phi : G(F) \to \mathbb{C}^\times$ of depth r that is trivial on $G_{\mathrm{sc}}(F)$ and satisfies $\phi|_{S(F)_r} = \theta|_{S(F)_r}$. □

Proof. Let S_{der} and S_{sc} be the preimages of S in G_{der} and G_{sc}. We claim that $\theta|_{S_{\mathrm{der}}(F)_r} = 1$. Let $R = \mathrm{Res}_{E/F}(S_{\mathrm{sc}} \times E)$. We have the norm homomorphism $N_{E/F} : R \to S_{\mathrm{sc}}$. It is surjective and we call its kernel R^1. We have the exact sequence

$$1 \to R^1 \to R \to S_{\mathrm{sc}} \to 1$$

of tori defined over F and split over E. According to Lemma 3.1.3 the homomorphism $S_{\mathrm{sc}}(E)_r = R(F)_r \to S_{\mathrm{sc}}(F)_r \to S_{\mathrm{der}}(F)_r$ is surjective. The claim would thus follow from $\theta \circ N_{E/F}(S_{\mathrm{sc}}(E)_r) = 1$. However, $X_*(S_{\mathrm{sc}})$ is generated by $R(S,G)^\vee$ over $\mathbb{Z}$ so the latter follows immediately from the assumption of the lemma.

With the claim proved, we turn to the proof of the lemma. Let $D = G/G_{\mathrm{der}}$. From Lemma 3.1.3 we have the equality $D(F)_r = S(F)_r/S_{\mathrm{der}}(F)_r$ and the claim we just proved tells us that θ descends to a non-trivial character of $D(F)_r$ that is trivial on $D(F)_{r+}$. This finite order character of $D(F)_r$ can be extended by Pontryagin duality [HR79, Corollary 24.12] to a character $\phi : D(F) \to \mathbb{C}^\times$, trivial on $D(F)_{r+}$. This character pulls back to a character of $G(F)$ whose restriction to $S(F)_r$ is equal to that of θ. ■

Corollary 3.6.10. Let $\theta : S(F) \to \mathbb{C}^\times$ be a character of positive depth r. Assume that $\theta \circ N_{E/F} \circ \alpha^\vee|_{E_r^\times} = 1$ for all $\alpha \in R(S,G)$. Then there exists a character $\phi : G(F) \to \mathbb{C}^\times$ of depth r, trivial on $G_{\text{sc}}(F)$, such that $\theta' = \theta \cdot \phi^{-1}|_{S(F)}$ has depth $r' < r$. Moreover, if $r' > 0$ there exists a root $\alpha \in R(S,G)$ such that $\theta'(N_{E/F}(\alpha^\vee(E_{r'}^\times))) \neq 1$. □

Proof. We work recursively on the depth of θ. Put $\theta_0 = \theta$ and $r_0 = r$ and apply Lemma 3.6.9 and obtain $\phi_0 : G(F) \to \mathbb{C}^\times$ of depth r such that the depth r_1 of $\theta_1 := \theta_0 \cdot \phi^{-1}|_{S(F)}$ is strictly less than r_0. If $r_1 > 0$ and $\theta_1(N_{E/F}(\alpha^\vee(E_{r_1}^\times))) = 1$ for all $\alpha \in R(S,G)$ we apply the recursion step again with θ_1 and r_1. The recursion eventually stops because the set of positive numbers that are depths of characters of $S(F)$ (i.e. the breaks of the Moy-Prasad filtration of $S(F)$) has no accumulation points. Let ϕ be the product of all ϕ_i. Its depth is equal to that of ϕ_0, which is r_0. ■

Proof of Proposition 3.6.7. We first deal with the following two trivial cases: If $d = 0$ and $r_0 = r_{-1} = 0$ then the twisted Levi sequence we have is $G = G^0 \supset S$ and we set $\phi_0 = 1$ and $\phi_{-1} = \theta$ and we are done. If $d = 1$, $r_1 = r_0 > r_{-1} = 0$, and $R_{0+} = \emptyset$, then the twisted Levi sequence we have is $G = G^1 \supset G^0 = S$ and θ is a G-generic character of $S(F)$ of depth $r_1 = r_0$ according to Lemma 3.6.8, and we set $\phi_1 = 1$, $\phi_0 = \theta$, and $\phi_{-1} = 1$.

Assume now $r_d > 0$. To begin the recursion: If $r_d > r_{d-1}$ let $i = d$ and $\theta_d = \theta$, and if $r_d = r_{d-1}$ let $i = d-1$, $\phi_d = 1$, and $\theta_{d-1} = \theta_d = \theta$.

The recursion step assumes that we are in the following situation (which is true in the beginning because we have separately handled the two trivial cases above): $\theta_i : S(F) \to \mathbb{C}^\times$ is a character of depth $r_i > 0$, $R(S,G^i) \neq \emptyset$, and for any $r \geq 0$ and $\alpha \in R(S,G^i)$, we have

$$\theta_i(N_{E/F}(\alpha^\vee(E_r^\times))) = \theta(N_{E/F}(\alpha^\vee(E_r^\times))).$$

Given that, we apply Corollary 3.6.10 to G^i and θ_i and obtain a character $\phi_i : G^i(F) \to \mathbb{C}^\times$ of depth r_i and set $\theta_{i-1} = \theta_i \cdot \phi_i^{-1}|_{S(F)}$. Note that the character ϕ_i is trivial on $G^i_{\text{sc}}(F)$ and hence for all $r \geq 0$ and $\alpha \in R(S,G^i)$ we have the following strengthening of the second recursion hypothesis

$$\theta_{i-1}(N_{E/F}(\alpha^\vee(E_r^\times))) = \theta_i(N_{E/F}(\alpha^\vee(E_r^\times))) = \theta(N_{E/F}(\alpha^\vee(E_r^\times))). \quad (3.6.3)$$

We claim that $r' := \text{depth}(\theta_{i-1})$ is equal to r_{i-1} if $i > 0$, and $r' \leq r_{i-1} = r_{-1} = 0$ if $i = 0$. If we assume $r' > r_{i-1}$, then $r' > 0$ and according to Corollary 3.6.10 we would have a root $\alpha \in R(S,G^i)$ satisfying $1 \neq \theta_{i-1}(N_{E/F}(\alpha^\vee(E_{r'}^\times))) = \theta(N_{E/F}(\alpha^\vee(E_{r'}^\times)))$, contradicting the definition of G^i. Thus $r' \leq r_{i-1}$. If we in addition assume $i > 0$, then r_{i-1} is a jump of the filtration R_r and so there exists $\alpha \in R(S,G^i)$ such that $1 \neq \theta(N_{E/F}(\alpha^\vee(E_{r_{i-1}}^\times))) = \theta_{i-1}(N_{E/F}(\alpha^\vee(E_{r_{i-1}}^\times)))$, showing $r' \geq r_{i-1}$.

If $i = 0$ the recursion stops and we set $\phi_{-1} = \theta_{-1}$. If $i = 1$ but $G^0 = S$ the recursion also stops and we set $\phi_0 = \theta_0$ and $\phi_{-1} = 1$. Otherwise, we have just checked that (G^{i-1}, θ_{i-1}) meets the requirements of the recursion step, and we continue with it.

Let us now show that the characters ϕ_i obtained in this way are a Howe factorization of (S,θ). The first two parts of the definition of Howe factorization are immediate from the construction, as well as the claims about ϕ_d and ϕ_{-1}.

Let now $d > i > -1$. According to Corollary 3.6.10 we have $\operatorname{depth}(\phi_i) = \operatorname{depth}(\theta_i) = r_i$. Moreover, for $\alpha \in R(S, G^{i+1}) \smallsetminus R(S, G^i)$

$$1 \neq \theta(N_{E/F}(\alpha^\vee(E_{r_i}^\times))) = \theta_i(N_{E/F}(\alpha^\vee(E_{r_i}^\times))) = \phi_i(N_{E/F}(\alpha^\vee(E_{r_i}^\times))),$$

the first (non)equality holding by definition of $R(S, G^{i+1}) \smallsetminus R(S, G^i)$, the second by (3.6.3), and the third by the fact that $\operatorname{depth}(\theta_{i-1}) < r_i = \operatorname{depth}(\theta_i) = \operatorname{depth}(\phi_i)$. According to Lemma 3.6.8 ϕ_i is G^{i+1}-generic. ■

3.7 Regular supercuspidal representations of positive depth

In this Subsection we will introduce the notion of (extra) regular Yu-data and (extra) regular supercuspidal representations. The work of Hakim-Murnaghan reviewed in §3.5 implies that the (extra) regular supercuspidal representations are classified by the G-equivalence classes of (extra) regular Yu-data. We will show that in fact the (extra) regular supercuspidal representations are classified by much simpler data, namely $G(F)$-conjugacy classes of tame elliptic (extra) regular pairs, a concept we will also introduce.

We assume that p is not a bad prime for G.

3.7.1 Regular Yu-data

Definition 3.7.1. We shall call a reduced generic cuspidal G-datum $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \dots, \phi_d))$ *normalized* if the pull-back of ϕ_i to $G^i_{\mathrm{sc}}(F)$ is trivial, for all $0 \leq i \leq d$. □

Lemma 3.7.2. If p does not divide the order of the fundamental group of G_{der}, then every reduced generic cuspidal G-datum is G-equivalent to a normalized one. □

Proof. Let $\phi : G(F) \to \mathbb{C}^\times$ be a character of depth $r > 0$. Put $D = G/G_{\mathrm{der}}$. Let $x \in \mathcal{B}^{\mathrm{red}}(G, F)$. By Lemma 3.5.1 $\phi|_{G_{\mathrm{der}}(F)_{x,0+}}$ is trivial, while Lemma 3.3.2 implies that $G(F)_{x,0+} \to D(F)_{0+}$ is surjective. Thus ϕ induces a smooth character of $D(F)_{0+}$. It is of finite order, hence unitary, and by Pontryagin duality [HR79, Corollary 24.12] extends to a character ϕ' of $D(F)$, which we may pull back to $G(F)$. Then $\phi \cdot (\phi')^{-1}$ is a character trivial on $G(F)_{x,0+}$.

If $(\phi_0, \dots, \phi_d)$ is the sequence of characters in a reduced generic cuspidal G-datum, we can use this procedure to replace (ϕ_{d-1}, ϕ_d) by $(\phi_{d-1}\phi_d(\phi_d')^{-1}, \phi_d')$, thereby obtaining a refactorization for which ϕ_d' is trivial on $G_{\mathrm{der}}(F)$. Doing this inductively leads to the desired refactorization. ■

Definition 3.7.3. Let $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \dots, \phi_d))$ be a reduced generic cuspidal G-datum. We shall call it

1. *regular*, if π_{-1} is a regular depth-zero supercuspidal representation of $G^0(F)$ in the sense of §3.4;
2. *extra regular*, if it is normalized and π_{-1} is an extra regular depth-zero supercuspidal representation of $G^0(F)$ in the sense of §3.4.

□

The regularity of π_{-1} is a non-trivial restriction. This is seen already in the case of SL_2, where four of the supercuspidal representations of this group are not regular. Nonetheless one can say, somewhat informally, that most depth-zero supercuspidal representations are regular. Indeed, the depth-zero supercuspidal representations correspond to cuspidal representations of finite groups of Lie type by Moy-Prasad theory, which are in turn grouped into Lusztig series, indexed by semi-simple elements of the Lusztig dual group. The condition of π_{-1} being regular translates to a Zariski-open and dense condition on the semi-simple elements of the Lusztig dual group of each finite group of Lie type that is the reductive quotient of a parahoric subgroup.

According to Proposition 3.4.27, we can replace π_{-1} by a maximally unramified elliptic maximal torus S of G^0 and a depth-zero character $\phi_{-1} : S(F) \to \mathbb{C}^\times$ that is (extra) regular with respect to G^0. Recall our conventions $G^{-1} = S$ and $r_{-1} = 0$. Note that it may happen that $G^0 = G^{-1}$.

Using Lemma 3.4.28 one sees that the regularity of a G-datum is an invariant of its G-equivalence class. One also sees that the extra regularity of a normalized G-datum is an invariant of its G-equivalence class, provided we only consider normalized G-data, for then we are only allowed to replace π_{-1} by $\chi_0 \otimes \pi_{-1}$, where $\chi_0 : G^0(F) \to \mathbb{C}^\times$ is trivial on $G^0_{\mathrm{sc}}(F)$, but given $w \in \Omega(S, G^0)(F)$ and $s \in S(F)$ the element $wsw^{-1}s^{-1} \in S(F)$ lifts to $G^0_{\mathrm{sc}}(F)$. On the other hand, the π_{-1} component of a G-datum that is not normalized, but equivalent to a normalized extra regular datum, need not be an extra regular depth-zero supercuspidal representation of $G^0(F)$.

Example 3.7.4. As an example we can take $G^0(F)$ to be the norm-1 elements in a division algebra D/F of degree d and $S(F)$ to be the norm-1 elements in the unramified extension F_d of F of degree d, embedded into D. We shall show that $N(S, G^0)(F)/S(F)$ is trivial, so every character of $S(F)_0$ is regular with respect to G^0, while $\Omega(S, G^0)(F)$ is cyclic of degree d and there exist characters of $S(F)_0$ that are extra regular with respect to G^0, as well as characters that are not. Finally, we shall show that every depth-zero character of $S(F)$ extends uniquely to $G^0(F)$. As a result, in a given Yu-tower $G^0 \subset G$ an extra regular Yu-datum can be refactorized into one where $\phi_{-1} = 1$, in particular not extra regular.

We follow the exposition of [PR94, §1.4]. Let $\tilde{G}^0(F) = D^\times$ and $\tilde{S}(F) = F_d^\times$. Then $\Omega(S, G^0)(F) = \Omega(\tilde{S}, \tilde{G}^0)(F) = N(\tilde{G}^0, \tilde{S})(F)/\tilde{S}(F)$, the latter due to $\tilde{S}$ being an induced torus and thus having trivial first Galois cohomology group. Any element of $\tilde{G}^0(F)$ acts trivially on the center $F^\times$. Thus elements of $\Omega(\tilde{S}, \tilde{G}^0)(F)$ act on $\tilde{S}(F) = F_d^\times$ as Galois elements. On the other hand, it is known that the Frobenius element of F_d/F is realized by conjugation by an element $g \in \tilde{G}^0(F)$. Thus $\Omega(S, G^0)(F) = \mathrm{Gal}(F_d/F)$ is cyclic of order d.

The valuation of the reduced norm of g is an integer a coprime to d. Given an integer k, the element of $\Omega(S, G^0)(F)$ represented by g^k can be lifted to $N(S, G^0)(F)$ if and only if there exists $s \in F_d^\times$ such that $sg^k \in D^1$. The reduced norm of g^k has valuation ka, while the reduced norm of s, which coincides with the field norm for the extension F_d/F, has valuation in $d\mathbb{Z}$. So $sg^k \in D^1$ implies $ka \in d\mathbb{Z}$, hence $k \in d\mathbb{Z}$. We conclude that no non-trivial element of $\Omega(S, G^0)(F)$ lifts to $N(S, G^0)(F)$, and so $N(S, G^0)(F)/S(F)$ is trivial.

The triviality of $N(S, G^0)(F)/S(F)$ implies that all characters of $S(F)$ are regular, even the trivial character. On the other hand, extra regular is a non-trivial condition. For example, the trivial character is not extra-regular. On the other hand, there do exist extra regular characters. Indeed, $S(F)_{0:0+} = k^1_{F_d}$ is a cyclic group of order $n = \sum_{i=0}^{d-1} q^i$, so its character group is isomorphic to the group $\mu_n(\mathbb{C})$ of n-th roots of unity. The character corresponding to $\zeta \in \mu_n(\mathbb{C})$ is fixed by the k-th power of Frobenius if and only if $\zeta^{q^k-1} = 1$. There clearly exist ζ for which this equation is not true for any $k = 1, \ldots, d-1$.

Finally, let x be the unique point in the Bruhat-Tits building of G^0. We have $G^0(F) = G^0(F)_{x,0}$, since G^0 is simply connected. It is shown in the proofs of [PR94, Theorem 1.8, Proposition 1.8] that the inclusion $F_d^\times \to D^\times$ induces an isomorphism $\tilde{S}(F)_{0:0+} = O_{F_d}^\times/(1+\mathfrak{p}_{F_d}) \to O_D^\times/(1+\mathfrak{p}_D) = \tilde{G}(F)_{x,0:0+}$, and that this isomorphism restricts to an isomorphism $S(F)_{0:0+} \to G^0(F)_{x,0:0+}$. Thus every character of depth zero extends uniquely to a character of $G^0(F)$. □

3.7.2 Tame regular elliptic pairs

Definition 3.7.5. Let $S \subset G$ be a maximal torus and $\theta : S(F) \to \mathbb{C}^\times$ a character. We shall call the pair (S, θ) *tame elliptic regular* (resp. *extra regular*) if

1. S is elliptic and split over a tame extension;
2. the action of inertia on the root subsystem

$$R_{0+} = \{\alpha \in R(S, G) | \theta(N_{E/F}(\alpha^\vee(E_{0+}^\times))) = 1\}$$

   preserves a positive set of roots, where E/F is any tame Galois extension splitting S;
3. the character $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $N(S, G^0)(F)/S(F)$ (resp. $\Omega(S, G^0)(F)$), where $G^0 \subset G$ is the reductive subgroup with maximal torus S and root system R_{0+}.

□

We recall from Lemma 3.6.1 that R_{0+} is a Levi subsystem of R, and from Fact 3.4.1 that the second condition is equivalent to saying that S is a maximally unramified maximal torus of G^0. We furthermore recall from Lemma 3.6.5 that, when p does not divide the order of the fundamental group of G_{der}, one can replace G^0 by G in the third condition.

Fact 3.7.6. If (S, θ) is a regular (resp. extra regular) tame elliptic pair and $\delta_0 : S(F) \to \mathbb{C}^\times$ is a character of depth zero that is invariant under $N(S, G^0)(F)/S(F)$ (resp. under $\Omega(S, G^0)(F)$), then $(S, \theta\delta_0)$ is regular (resp. extra regular). □

Proof. This follows from the fact that neither R_{0+} nor the appropriate stabilizer of $\theta|_{S(F)_0}$ changes when we pass from θ to $\theta\delta_0$. ■

Recall that when $p \nmid N$ the supercuspidal representations of GL_N are classified by admissible characters. The notion of admissible character and the construction of a supercuspidal representation from an admissible character appears in [How77], while the exhaustion is proved in [Moy86] (under the assumption that F has characteristic zero). We will now argue that the notion of a tame elliptic (extra) regular pair is a generalization of the notion of an admissible character to an arbitrary tamely ramified reductive p-adic group. An admissible character is really a pair $(K^\times, \theta)$, where K/F is a field extension of degree N and $\theta : K^\times \to \mathbb{C}^\times$ is a character satisfying certain axioms, listed on the first page of [How77], see also [HM08, Definition 3.29]. Since the equation $S(F) = K^\times$ provides a bijection between the conjugacy classes of elliptic maximal tori S of GL_N and the isomorphism classes of field extensions K/F of degree N, admissibility can be seen as a property of a pair (S, θ), where S is an elliptic maximal torus of GL_N and $\theta : S(F) \to \mathbb{C}^\times$ is a character. Note that the splitting extension E/F of S is the normal closure of K, and in particular E/K is unramified.

Lemma 3.7.7. If $G = \mathrm{GL}_N$ and $p \nmid N$ then the notions of extra regular, regular, and admissible, pairs coincide. □

Proof. The notions of extra regular and regular coincide, because $H^1(F, S) = 0$ for every maximal torus S, hence $\Omega(S, G^0)(F) = N(S, G^0)(F)/S(F)$.

To show that a regular elliptic pair (S, θ) is admissible, let K/F be the degree-N extension such that $S(F) = K^\times$. If there exists an intermediate extension $K/L/F$ such that $\theta = \theta_L \circ N_{K/L}$ for some $\theta_L : L^\times \to \mathbb{C}^\times$, then we can consider the twisted Levi subgroup $M = \mathrm{Res}_{L/F}\mathrm{GL}_{N'}$, where $N' = [K : L]$, and realize S as a maximal torus inside of it. Then θ is the restriction to $S(F)$ of the character $\theta_L \circ \mathrm{Res}_{L/F}(\det)$ of $M(F)$. For every $\alpha \in R(S, M)$ the character $\theta \circ N_{E/F} \circ \alpha^\vee$ of $E^\times$ is trivial, in particular $R(S, M) \subset R_{0+}$. But θ is obviously invariant under $\Omega(S, M)(F)$. This contradicts Definition 3.7.5 unless $S = M$, i.e. $L = K$.

Now assume that $\theta|_{K^\times_{0+}} = \theta_L \circ N_{K/L}$ for an intermediate extension $K/L/F$ and a character $\theta_L : L^\times_{0+} \to \mathbb{C}^\times$. With M as above we see again that $R(S, M) \subset R_{0+}$. By Definition 3.7.5 the action of inertia preserves a positive subsystem of R_{0+} and hence of $R(S, M)$. This means that over F^u the torus $S = \mathrm{Res}_{K/F}\mathbb{G}_m$ becomes a minimal Levi subgroup of $M = \mathrm{Res}_{L/F}\mathrm{GL}_{N'}$, which implies that K/L is unramified.

Conversely let (S, θ) be admissible. Since $S = \mathrm{Res}_{K/F}\mathbb{G}_m$ for some field extension K/F of degree N, it is automatically a tame elliptic maximal torus. Consider the root system R_{0+}. By Lemma 3.6.1, it spans a twisted Levi subgroup $M \subset G$ whose center is contained in S and hence anisotropic mod $Z(G)$. Thus $M = \mathrm{Res}_{L/F}\mathrm{GL}_{N'}$ for some intermediate extension $K/L/F$. Let $S_{M_{\mathrm{der}}}$ be the intersection of S with the derived subgroup of M. Since the simple coroots for M form a basis for $X_*(S_{M_{\mathrm{der}}})$ the group $S_{M_{\mathrm{der}}}(E)_{0+}$ is generated by its subgroups $\alpha^\vee(E^\times_{0+})$ for $\alpha \in R_{0+}$. Applying Lemma 3.1.3 to the surjection $\mathrm{Res}_{E/F}(S_{M_{\mathrm{der}}} \times E) \to S_{M_{\mathrm{der}}}$ induced by the norm we see that the group $S_{M_{\mathrm{der}}}(F)_{0+}$ is generated by its subgroups $N_{E/F}\alpha^\vee(E^\times_{0+})$ for $\alpha \in R_{0+}$. Thus $\theta|_{S(F)_{0+}}$ factors through the exact sequence

$$1 \to S_{M_{\mathrm{der}}}(F)_{0+} \to S(F)_{0+} \to [M/M_{\mathrm{der}}](F)_{0+} \to 1$$

of Lemma 3.1.3. We have the isomorphism $\mathrm{Res}_{L/F}(\det) : M/M_{\mathrm{der}} \to \mathrm{Res}_{L/F}\mathbb{G}_m$, which restricted to $S(F)$ becomes the map $N_{K/L} : S(F) = K^\times \to L^\times$. Thus $\theta|_{S(F)_{0+}}$ factors through $N_{K/L}$ and the admissibility of θ implies that K/L is unramified. This means that $\mathrm{Res}_{K/L}\mathbb{G}_m$ splits over L^u, or equivalently the maximal torus $S = \mathrm{Res}_{K/F}\mathbb{G}_m$ of $M = \mathrm{Res}_{L/F}\mathrm{GL}_{N'}$ becomes a minimal Levi over F^u. This implies that inertia preserves a set of positive roots in R_{0+}.

Now consider the stabilizer of $\theta|_{S(F)_0}$ in $\Omega(S, G^0)(F)$. We may as well remove the restriction of scalars L/F and consider $S = \mathrm{Res}_{K/L}\mathbb{G}_m$ as a maximal torus of $G^0 = \mathrm{GL}_{N'}/L$ and the stabilizer in $\Omega(S, G^0)(L)$ of the character $\theta : S(L)_0 \to \mathbb{C}^\times$. Now $S(L) = K^\times = O_K^\times \cdot L^\times = S(L)_0 \cdot Z(G^0)(L)$. It follows that the stabilizer in $\Omega(S, G^0)(L)$ of $\theta|_{S(L)_0}$ is the same as the stabilizer of θ. Since K/L is unramified, the splitting field E of S is equal to K. The ellipticity of S then implies that the cyclic group $\mathrm{Gal}(K/L)$ acts on $X^*(S)$ via a Coxeter element of $\Omega(S, G^0)$ (since $\Omega(S, G^0)$ is the symmetric group on N' letters, this element is simply an N'-cycle), which in turn implies that $\Omega(S, G^0)(L)$ is cyclic and generated by that Coxeter element. That is, the action of $\Omega(S, G^0)(L)$ on $S(L)$ is translated via $S(L) = K^\times$ to the action of $\mathrm{Gal}(K/L)$ on $K^\times$. If some element $\sigma \in \mathrm{Gal}(K/L)$ leaves θ invariant, then θ will factor through the group of co-invariants $K_\sigma^\times$. Letting K_1 be the subfield of K fixed by σ, the vanishing of $H^{-1}(\Gamma_{K/K_1}, K^\times) = 0$ implies that the norm map $K^\times \to K_1^\times$ descends to an isomorphism $K_\sigma^\times \to K_1^\times$. Thus θ factors through this norm map and its admissibility implies $K_1 = K$, i.e. $\sigma = 1$. ■

3.7.3 Classification when p does not divide $|\pi_1(G_{\mathrm{der}})|$

In this subsubsection we assume that p is not a bad prime for G and does not divide the order of the fundamental group of the derived subgroup of G.

Proposition 3.7.8. Let $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ be a (extra) regular reduced generic cuspidal G-datum. Using Proposition 3.4.27 let $S \subset G^0$ be a maximally unramified elliptic maximal torus and $\phi_{-1} : S(F) \to \mathbb{C}^\times$ a (extra) regular depth-zero character s.t. $\pi_{-1} = \pi_{(S,\phi_{-1})}$. Let $\theta = \prod_{i=-1}^d \phi_i|_{S(F)}$. The resulting map

$$((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d)) \mapsto (S, \theta)$$

induces a bijection between the set of G-equivalence classes of (extra) regular reduced generic cuspidal G-data and the set of $G(F)$-conjugacy classes of tame elliptic (extra) regular pairs. □

Proof. We first show that (S, θ) is a (extra) regular tame elliptic pair. From Lemma 3.6.3 we know that the groups G^i from the statement of this proposition coincide with the groups G^i constructed in §3.6 in terms of (S, θ), and that furthermore $(\phi_{-1}, \ldots, \phi_d)$ is a Howe factorization of (S, θ). Since S is elliptic in G^0 and $Z(G^0)/Z(G)$ is anisotropic, S is elliptic in G. Furthermore, $S \subset G^0$ is maximally unramified. In particular, it is split over a tame extension, since G^0 is. Thus parts 1. and 2. of Definition 3.7.5 are satisfied. The third part follows from Lemma 3.6.5.

We claim that G-equivalent data lead to $G(F)$-conjugate pairs. It is enough to check this in the three cases where the two data are related by G-conjugation, elementary transformation, or refactorization. The case of G-conjugation is obvious. The case of elementary transformation means that we replace $\pi_{(S,\phi_{-1})}$ by an isomorphic representation of $G^0(F)$. By Proposition 3.4.27 such a representation is of the form $\pi_{(S',\phi'_{-1})}$, where $(S', \phi'_{-1}) = \mathrm{Ad}(g)(S, \phi_{-1})$ for some $g \in G^0(F)$. The new datum is thus given by $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{(S',\phi'_{-1})}, (\phi_0, \phi_1, \ldots, \phi_d))$ and it leads to the pair (S', θ'), where $\theta' : S'(F) \to \mathbb{C}^\times$ is given by $\prod_{i=0}^d \phi_i|_{S'(F)} \cdot \phi'_{-1}$. But $(S', \theta') = \mathrm{Ad}(g)(S, \theta)$. Finally consider a refactorization $((G^0 \subsetneq$

$G^1 \cdots \subsetneq G^d), \pi'_{(S,\phi_{-1})}, (\phi'_0, \phi'_1, \dots, \phi'_d))$. Let θ and θ' be the two corresponding characters of $S(F)$. As in §3.5 let $\chi_i : G^i(F) \to \mathbb{C}^\times$ be defined by $\chi_i(g) := \prod_{j=i}^d \phi_j(g)\phi'_j(g)^{-1}$. Then the character χ_0 of $G^0(F)$ has the property that $\theta'\theta^{-1} = \chi_0|_{S(F)}^{-1} \cdot (\phi'_{-1}\phi_{-1}^{-1})$. At the same time, property F2 of refactorization implies $\pi_{(S,\phi'_{-1})} = \pi_{(S,\phi_{-1})} \otimes \chi_0$. According to Lemma 3.4.28 we have $\pi_{(S,\phi_{-1})} \otimes \chi_0 = \pi_{(S,\phi_{-1}\cdot\chi_0|_{S(F)})}$. Proposition 3.4.27 then implies that (S, ϕ'_{-1}) is $G^0(F)$-conjugate to $(S, \phi_{-1} \cdot \chi_0|_{S(F)})$, which, combined with the regularity of ϕ'_{-1} implies $\phi'_{-1} = \phi_{-1} \cdot \chi_0|_{S(F)}$. This in turn implies $\theta'\theta^{-1} = 1$.

We thus obtain a well-defined map between G-equivalence classes of data and $G(F)$-conjugacy classes of pairs.

To show surjectivity, let (S, θ) be a (extra) regular tame elliptic pair. As described in §3.6 we obtain a sequence of tame twisted Levi subgroups $G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d = G$. By Definition 3.7.5 the torus S is a maximally unramified maximal torus of G^0. By Proposition 3.6.7 there exists a Howe-factorization $(\phi_{-1}, \dots, \phi_d)$ of (S, θ). By Lemma 3.6.5 the (extra) regularity of the pair (S, θ) implies the (extra) regularity of the depth-zero character ϕ_{-1} with respect to G^0. Proposition 3.4.27 gives an (extra) regular depth-zero supercuspidal representation $\pi_{(S,\phi_{-1})}$ of $G^0(F)$, and we obtain a reduced generic cuspidal G-datum mapping to (S, θ), that is normalized in the sense of Definition 3.7.1 and (extra) regular in the sense of Definition 3.7.3.

To show injectivity, assume $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{(S,\phi_{-1})}, (\phi_0, \phi_1, \dots, \phi_d))$ and $((G'^0 \subsetneq G'^1 \cdots \subsetneq G'^{d'}), \pi_{(S',\phi'_{-1})}, (\phi'_0, \phi'_1, \dots, \phi'_{d'}))$ lead to conjugate pairs (S, θ) and (S', θ'). The already proved part of this lemma allows us to replace both data by G-equivalent ones, so we use Lemma 3.7.2 to assume that they are both normalized. Replacing the second datum by a G-conjugate we may assume that the two pairs are actually equal. Lemma 3.6.3 shows that $d = d'$, that $G'^i = G^i$ for all $i = 0, \dots, d$, and that $(\phi_{-1}, \dots, \phi_d)$ and $(\phi'_{-1}, \dots, \phi'_d)$ are two Howe factorizations of the same pair (S, θ). Lemma 3.6.6 implies that they are refactorizations of each other. ∎

Definition 3.7.9. Under the assumption that p does not divide the order of the fundamental group of G_{der} we shall call a supercuspidal representation of $G(F)$ *(extra) regular* if it arises via Yu's construction from an (extra) regular (reduced generic cuspidal) Yu-datum. □

We will generalize this definition to the case when p is allows to divide $|\pi_1(G_{\mathrm{der}})|$ in §3.7.4.

Corollary 3.7.10. Composing the bijection of Proposition 3.7.8 with Yu's construction provides a bijection

$$(S, \theta) \mapsto \pi_{(S,\theta)}$$

between the set of $G(F)$-conjugacy classes of (extra) regular tame elliptic pairs and the set of (extra) regular supercuspidal representations. □

Proof. Definition 3.7.9 and Corollary 3.5.5 imply that the set of (extra) regular supercuspidal representations is in bijection with the set of G-equivalence classes of (extra) regular reduced generic cuspidal G-data, which in turn is in bijection with the set of $G(F)$-conjugacy classes of (extra) regular tame elliptic pairs by Proposition 3.7.8. ∎

Fact 3.7.11. The central character of $\pi_{(S,\theta)}$ is the restriction of θ to $Z(G)(F)$. $\square$

Proof. Let $(\phi_{-1},\dots,\phi_d)$ be a Howe factorization. We examine Yu's construction, following the exposition in [HM08, §3.4]. There is a sequence of compact modulo center subgroups of $G(F)$

$$G^0(F)_x = K^0 \subset K^1 \subset \cdots \subset K^d = K$$

On each K^i there is an irreducible representation κ_i. There is a natural inflation process that makes κ_i into a representation of K. The tensor product $\kappa_{-1}\otimes\cdots\otimes\kappa_d$ is irreducible and its compact induction from K to G is $\pi_{(S,\theta)}$.

Since $Z(G)(F) \subset K^0$, the inflation process doesn't disturb the central character of κ_i. For $i = 0$ the representation κ_i is $\phi_0 \otimes \mathrm{Ind}_{S(F)G^0(F)_{x,0}}^{G^0(F)_x}\kappa_{(S,\phi_{-1})}$, where $\kappa_{(S,\phi^{-1})}$ is as in Lemma 3.4.20. The central character of this representation is $\phi_0\cdot\phi_{-1}$. For the intermediate indices i the representation κ_i is described after [HM08, Remark 3.25] and its construction may or may not involve the Weil representation. In either case, its restriction to $Z(G)(F)$ is seen to act via the character ϕ_i. ∎

3.7.4 Classification when p divides $|\pi_1(G_{\mathrm{der}})|$

In the previous subsubsection we assumed that p is not a bad prime for G and does not divide the order of $\pi_1(G_{\mathrm{der}})$. In this subsection we will remove the condition that p does not divide the order of $\pi_1(G_{\mathrm{der}})$. This is only an issue for Dynkin type A_n, for which there are no bad primes while $\pi_1(G_{\mathrm{der}})$ can be any divisor of $n+1$. For all other Dynkin types a prime that divides $\pi_1(G_{\mathrm{der}})$ is automatically bad for G, and equals either 2 or 3. We keep the assumption that p is not a bad prime for G.

Fix a z-extension $1 \to K \to G_1 \to G \to 1$.

Lemma 3.7.12. Let (S,θ) be a tame elliptic (extra) regular pair for G. Let S_1 be the preimage of S in G_1 and let θ_1 be the inflation of θ to $S_1(F)$. The map $(S,\theta)\mapsto(S_1,\theta_1)$ is a depth-preserving bijection between the $G(F)$-conjugacy classes of tame elliptic (extra) regular pairs of G and those tame elliptic (extra) regular pairs of G_1 for which $\theta_1|_{K(F)} = 1$. $\square$

Proof. This follows at once from Lemma 3.5.3. ∎

Definition 3.7.13. A supercuspidal representation π of $G(F)$ will be called *(extra) regular* if its inflation π_1 to $G_1(F)$ is so. $\square$

Proposition 3.7.14. There is a bijection $(S,\theta)\mapsto\pi_{(S,\theta)}$ between the set of $G(F)$-conjugacy classes of tame elliptic (extra) regular pairs (S,θ) for $G(F)$ and the set of (extra) regular supercuspidal representations of $G(F)$. $\square$

Proof. This follows from Corollary 3.7.10, Fact 3.7.11, and Lemma 3.7.12. ∎

Lemma 3.7.15. The notion of (extra) regularity and the bijection $(S,\theta)\mapsto\pi_{(S,\theta)}$ are independent of the choice of G_1. $\square$

Proof. Choose another z-extension G_2 of G and consider the fiber product G_3 of G_1 and G_2 over G. Then G_3 is a z-extension of G, of G_1, and of G_2. If the inflation π_1 of π is (extra) regular then $\pi_1 = \pi_{(S_1,\theta_1)}$ for some (extra) regular tame elliptic pair (S_1, θ_1). Let (S, θ) be the pair of G corresponding to (S_1, θ_1). Let (S_2, θ_2) and (S_3, θ_3) be the pairs on G_2 and G_3 corresponding to (S, θ). According to Lemma 3.7.12, (S_3, θ_3) is tame elliptic (extra) regular, and hence (S_2, θ_2) is tame elliptic (extra) regular. According to Lemma 3.3.2 the pull-back to G_3 of a Howe factorization for θ_1 is a Howe factorization for θ_3. The same is true for θ_2 in place of θ_1. Thus the representation $\pi_{(S_3,\theta_3)}$ is the pull-back to $G_3(F)$ of the representation $\pi_{(S_1,\theta_1)}$ and also of the representation $\pi_{(S_2,\theta_2)}$. But then $\pi_{(S_3,\theta_3)}$ is the pull-back of π to $G_3(F)$, which then implies that $\pi_{(S_2,\theta_2)}$ is the pull-back of π to $G_2(F)$. We conclude that the pull-back of π to $G_2(F)$ is (extra) regular. ■

4 THE CHARACTER FORMULA

Let F be a non-archimedean local field and let G be a connected reductive group defined over F and splitting over a tame extension of F. Let $((G^0 \subsetneq G^1 \cdots \subsetneq G^d), \pi_{-1}, (\phi_0, \phi_1, \ldots, \phi_d))$ be a reduced generic cuspidal G-datum in the sense of [HM08, Definition 3.11]. From this datum Yu's construction produces not just a single supercuspidal representation π of depth r_d, but in fact a supercuspidal representation π_i of the group $G^i(F)$ of depth r_i, for each $0 \leq i \leq d$, and $\pi = \pi_d$.

The Harish-Chandra character Θ_π of π has been computed in the work of Adler and Spice [AS08, AS09] and later reinterpreted in the work of DeBacker and Spice [DS18]. At the moment this work has the additional technical assumption that G^{d-1}/Z_G is anisotropic, but we are hopeful that this condition will be eliminated in the future. The resulting character formula involves various roots of unity. The main purpose of this section is to provide an alternative description of these roots of unity. As we shall see, they can be interpreted in a way that ties them closely to the Langlands-Shelstad transfer factors from the theory of endoscopy [LS87]. More precisely, we shall define certain terms ϵ and Δ_{II}^{abs}, that can be seen as absolute versions of the corresponding pieces of the transfer factor, and will show that they describe the roots of unity occurring in the character formula. These terms are absolute in the sense that they depend just on the group G, a maximal torus of it, and some auxiliary data. In the presence of an endoscopic group H, the quotient of either term for G by the corresponding term for H will be equal to the analogous term occurring in the Langlands-Shelstad transfer factor.

Once the roots of unity in the character formula have been reinterpreted in this way, we will show that the resulting expression for the character of a regular supercuspidal representation evaluated at a sufficiently shallow element is precisely analogous to the character formula for discrete series representations of real reductive groups.

4.1 Hypotheses

The papers [AS08] and [AS09] impose various hypotheses on the group G under which the character formula is obtained. Besides the assumption that G^{d-1}/Z_G is compact that we already mentioned, these are Hypotheses (A)-(D) of [AS08, §2], Hypothesis 2.3 of [AS09], and the assumption [AS09, §1.1] that the residual characteristic of F is not 2. As remarked in [AS09, §1.2], Hypotheses (A) and (D) are implied by the tameness of G. Hypothesis (C) is satisfied when G has simply connected derived subgroup. According to Lemma 3.5.2 the same is also true for Hypothesis 2.3. However, the character function of a representation π of $G(F)$ can be computed by first taking a z-extension $\tilde{G}$ of G, pulling π back to a representation $\tilde{\pi}$ of $\tilde{G}(F)$, and then computing the character function of $\tilde{\pi}$. This means that Hypotheses (C) and 2.3 are in fact superfluous. Finally, Hypothesis (B) is satisfied when the residual characteristic of F is not a bad prime for G. Thus we shall make the assumption that the residual characteristic of F is not a bad prime for G whenever we apply the character formula of [AS09], in addition to our standing assumption that it is not equal to 2. On the other hand, some results of [AS08] are valid and can be used without this assumption.

4.2 Review of orbital integrals

Let $\Lambda : F \to \mathbb{C}^\times$ be a non-trivial character. Let $S \subset G$ be a maximal torus. We can view $\mathfrak{s}^* = \mathrm{Lie}^*(S)$ as a subspace of $\mathfrak{g}^* = \mathrm{Lie}^*(G)$ as explained in [Yu01, §8], namely as the trivial-weight space for the coadjoint action of S. Let $X^* \in \mathfrak{s}^*(F) \subset \mathfrak{g}^*(F)$ be an element whose stabilizer for the coadjoint action of G is S. For any function $f^* \in \mathcal{C}_c^\infty(\mathfrak{g}^*(F))$ we have the orbital integral

$$O_{X^*}(f^*) = \int_{G(F)/S(F)} f^*(\mathrm{Ad}^*(g)X^*)dg.$$

The measure used for integration is the quotient of a measure on $G(F)$ by a measure on $S(F)$, and on both groups we take the canonical measure introduced by Waldspurger in [Wal01, §I.4], as is done in [DS18, Definition 4.1.6]. For a function $f \in \mathcal{C}_c^\infty(\mathfrak{g}(F))$ we define its Fourier-transform $\widehat{f}_{\Lambda,dY} \in \mathcal{C}_c^\infty(\mathfrak{g}^*(F))$ by

$$\widehat{f}_{\Lambda,dY}(Y^*) = \int_{\mathfrak{g}(F)} f(Y)\Lambda\langle Y, Y^*\rangle dY,$$

where we have indicated as subscripts the dependence on the choices of the character Λ and the measure dY. A fundamental result of Harish-Chandra is that the distribution $f \mapsto O_{X^*}(\widehat{f}_{\Lambda,dY})$ is represented by a function, i.e. there exists a function $\widehat{\mu}_{X^*,\Lambda}$ on $\mathfrak{g}(F)$ such that

$$O_{X^*}(\widehat{f}_{\Lambda,dY}) = \int_{\mathfrak{g}(F)} \widehat{\mu}_{X^*,\Lambda}(Y)f(Y)dY$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{g}(F))$. The function $\widehat{\mu}_{X^*,\Lambda}$ does not depend on the choice of measure dY. We can renormalize it using the usual Weyl discriminants [DS18, Definition 2.2.8] and obtain

$$\widehat{\iota}_{X^*,\Lambda}(Y) = |D(X^*)|^{\frac{1}{2}}|D(Y)|^{\frac{1}{2}}\widehat{\mu}_{X^*,\Lambda}(Y).$$

The function $\widehat{\mu}_{X^*,\Lambda}$ depends on Λ via the equation

$$\widehat{\mu}_{X^*,\Lambda\cdot c} = \widehat{\mu}_{cX^*,\Lambda},$$

where $c \in F^\times$ and $[\Lambda \cdot c](x) = \Lambda(cx)$. The same is true for $\widehat{\iota}_{X^*,\Lambda}$ provided $c \in O_F^\times$.

4.3 Review of the work of Adler-Spice and DeBacker-Spice

In this subsection F is a local field of odd residual characteristic that is not a bad prime for G. Set $r = r_{d-1}$ and $\pi = \pi_d$. Let x be the unique point in the building $\mathcal{B}^{\text{red}}(G^{d-1}, F)$. The formula of Adler-Spice gives the value of the function Θ_π, at any regular semi-simple element $\gamma \in G(F)$ that has an r-approximation $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$, in terms of the value of $\Theta_{\pi_{d-1}}$, under the assumption that $G^{d-1}/Z(G)$ is anisotropic. It is more convenient to replace $\Theta_\pi(\gamma)$ by its renormalization $\Phi_\pi(\gamma) = |D_G(\gamma)|^{\frac{1}{2}} \Theta_\pi(\gamma)$. In the form presented in [DS18, Theorem 4.6.2] the formula for $\Phi_\pi(\gamma)$ is

$$\phi_d(\gamma) \sum_{\substack{g \in J^d(F) \backslash G^d(F)/G^{d-1}(F) \\ \gamma_{<r}^g \in G^{d-1}(F)}} \epsilon_{\text{sym,ram}}(\gamma_{<r}^g) \epsilon^{\text{ram}}(\gamma_{<r}^g) \tilde{e}(\gamma_{<r}^g) \cdot \Phi_{\pi_{d-1}}(\gamma_{<r}^g) \widehat{\iota}_{\mathfrak{j}^d, {}^g X_{d-1}^*}(\log(\gamma_{\geq r})) \quad (4.3.1)$$

We need to explain the notation. Fix again a non-trivial character $\Lambda : F \to \mathbb{C}^\times$, with the additional assumption that Λ is trivial on $\mathfrak{p}_F$ but non-trivial on O_F. Let $X_{d-1}^* \in \text{Lie}^*(Z(G^{d-1}))(F)_{-r}$ be a G^d-generic element (in the sense of [Yu01, §8]) that realizes (in the sense of [Yu01, §5]) the character ϕ_{d-1}. We abbreviate $\gamma_{<r}^g = g^{-1}\gamma_{<r} g$ and ${}^g X_{d-1}^* = \text{Ad}^*(g) X_{d-1}^*$. Setting $J^d = \text{Cent}(\gamma_{<r}, G^d)^\circ$, the condition $\gamma_{<r}^g$ on the summation index g implies that $\text{Ad}(g) Z(G^{d-1})$ is a subgroup of J^d and in particular ${}^g X_{d-1}^* \in \mathfrak{j}^{d,*}(F)$. Therefore, the function $\widehat{\iota}_{\mathfrak{j}^d, {}^g X_{d-1}^*}$ that represents the normalized Fourier-transform of the integral along the coadjoint orbit of ${}^g X_{d-1}^*$ in $\mathfrak{j}^{d,*}(F)$ (as recalled in the §4.2) makes sense. Moreover, since both the function itself and the element X_{d-1}^* now depend on the choice of Λ in a parallel way, the entire expression $\widehat{\iota}_{\mathfrak{j}^d, {}^g X_{d-1}^*}$ is independent of Λ. The map log is either the true logarithm function, provided it converges at $\gamma_{\geq r}$, or else the inverse of a mock-exponential map [AS09, Appendix A].

One place where the technical assumption on the compactness of $G^{d-1}(F)$ modulo $Z_G(F)$ enters is the evaluation of $\Phi_{\pi_{d-1}}(\gamma_{<r}^g)$, because the semi-simple element $\gamma_{<r}^g \in G^{d-1}(F)$ need not be regular. When $G^{d-1}(F)/Z_G(F)$ is compact, the character $\Theta_{\pi_{d-1}}$ is defined on all (semi-simple) elements of $G^{d-1}(F)$, not just the regular elements. Thus the function $\Phi_{\pi_{d-1}} = |D_{G^{d-1}}^{\text{red}}|^{\frac{1}{2}} \Theta_{\pi_{d-1}}$ is also defined on all of $G^{d-1}(F)$. When $\gamma_{<r}$ is itself regular one can hope that the above formula applies even without the compactness assumption and we shall prove in the next subsection that it does, at least in the case when $\gamma = \gamma_{<r}$ is topologically semi-simple modulo $Z(G)^\circ$.

The remaining objects in the formula: $\epsilon_{\text{sym,ram}}$, ϵ^{ram}, and $\tilde{e}$, are all complex roots of unity of order dividing 4 and will be the focus of our study. We shall now give their definition following [DS18, §4.3]. Let T be a maximal torus of G^{d-1} containing $\gamma_{<r}^g$ and such that $x \in \mathcal{A}^{\text{red}}(T, E)^\Gamma$ for some finite Galois extension E/F splitting T. We consider the following subset of the real numbers, defined for each $\alpha \in R(T, G)$ by

$$\text{ord}_x(\alpha) = \{r \in \mathbb{R} | \mathfrak{g}_\alpha(F_\alpha)_{x,r+} \neq \mathfrak{g}_\alpha(F_\alpha)_{x,r}\},$$

where we have abbreviated by $\mathfrak{g}_\alpha(F_\alpha)_{x,r}$ the intersection $\mathfrak{g}_\alpha(F_\alpha) \cap \mathfrak{g}(F_\alpha)_{x,r}$. Based on this set we define the following subsets of the root system $R(T, G)$

$$\begin{aligned} R_{\gamma^g_{<r}} &= \{\alpha \in R(T,G) \smallsetminus R(T,G^{d-1}) | \alpha(\gamma^g_{<r}) \neq 1\}, \\ R_{r/2} &= \{\alpha \in R_{\gamma^g_{<r}} | r \in 2\mathrm{ord}_x(\alpha)\}, \\ R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2} &= \{\alpha \in R_{\gamma^g_{<r}} | r - \mathrm{ord}(\alpha(\gamma^g_{<r}) - 1) \in 2\mathrm{ord}_x(\alpha)\}. \end{aligned}$$

For $\alpha \in R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2}$ symmetric and ramified we define

$$t_\alpha = \frac{1}{2} e_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha) \langle d\alpha^\vee(1), X^*_{d-1} \rangle (\alpha(\gamma^g_{<r}) - 1) \in O^\times_{F_\alpha}.$$

Here e_α is the ramification degree of F_α/F and $w_\alpha \in F^\times_\alpha$ is any element of valuation $(\mathrm{ord}(\alpha(\gamma_{<r}) - 1) - r)/2$. The existence of w_α is argued in the proof of [AS09, Proposition 5.2.13]. It also follows from Proposition 4.5.1 below. Finally, we introduce the Gauss sum

$$\mathfrak{G} = q^{-1/2} \sum_{x \in k_F} \Lambda(x^2) \in \mathbb{C}^\times.$$

With this notation at hand, we come to the definition of the three roots of unity.

$$\epsilon_{\mathrm{sym,ram}}(\gamma^g_{<r}) = \prod_{\alpha \in \Gamma \backslash (R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2})_{\mathrm{symm,ram}}} \mathrm{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})(-\mathfrak{G})^{f_\alpha} \mathrm{sgn}_{k^\times_{F_\alpha}}(t_\alpha). \tag{4.3.2}$$

The product here runs over the Γ-orbits of symmetric ramified roots belonging to $R_{(r-\mathrm{ord}(\gamma^g_{<r}))/2}$. For each such α, let $G_{\pm\alpha}$ be the subgroup of G generated by the root subgroups for the two roots α and $-\alpha$. It is a semi-simple group of rank 1 defined over $F_{\pm\alpha}$, and $\mathrm{sgn}_{F_{\pm\alpha}}$ denotes its Kottwitz sign [Kot83], which equals 1 if the $G_{\pm\alpha}$ is split and -1 if it is anisotropic. Furthermore, f_α is the degree of the field extension k_{F_α}/k_F, and $\mathrm{sgn}_{k^\times_{F_\alpha}}$ is the quadratic character of the cyclic group $k^\times_{F_\alpha}$, onto which we can project the element $t_\alpha \in O^\times_{F_\alpha}$. Both $\mathfrak{G}$ and t_α depend on the choice of Λ (the latter through X^*_{d-1}) and it is easy to check that this dependence cancels out.

$$\epsilon^{\mathrm{ram}}(\gamma^g_{<r}) = \prod_{\alpha \in \Gamma \times \{\pm 1\} \backslash (R_{r/2})^{\mathrm{sym}}} \mathrm{sgn}_{k^\times_{F_\alpha}}(\alpha(\gamma^g_{<r})) \cdot \prod_{\alpha \in \Gamma \backslash (R_{r/2})_{\mathrm{sym,unram}}} \mathrm{sgn}_{k^1_{F_\alpha}}(\alpha(\gamma^g_{<r})). \tag{4.3.3}$$

Here the superscript sym means that we are taking $\Gamma \times \{\pm 1\}$-orbits of asymmetric roots, while the subscripts sym,unram mean that we are taking Γ-orbits of roots that are symmetric and unramified. In the first product, we project $\alpha(\gamma^g_{<r}) \in O^\times_{F_\alpha}$ to $k^\times_{F_\alpha}$. In the second product, the $F_\alpha/F_{\pm\alpha}$-norm of the element $\alpha(\gamma^g_{<r}) \in O^\times_{F_\alpha}$ is trivial, because the root α is symmetric. The same is true for the projection of $\alpha(\gamma^g_{<r})$ to $k^\times_{F_\alpha}$, because the symmetric root α is unramified. The group $k^1_{F_\alpha}$ of elements of $k^\times_{F_\alpha}$ with trivial $k_{F_\alpha}/k_{F_{\pm\alpha}}$-norm is cyclic and we apply its quadratic character to the projection of $\alpha(\gamma^g_{<r})$. Finally

$$\tilde{e}(\gamma^g_{<r}) = \prod_{\alpha \in \Gamma \backslash (R_{(r-\mathrm{ord}_{\gamma^g_{<r}})/2})_{\mathrm{sym}}} (-1). \tag{4.3.4}$$

Note that while the original definition of $\tilde{e}$ does not contain the subscript sym, we may restrict the product to symmetric roots by [DS18, Remark 4.3.4].

Each of these signs implicitly depends on T, but their product is independent of T. We will soon give an alternative formula for the product $\epsilon_{\text{sym,ram}} \cdot \tilde{e}$. About ϵ^{ram} we will only need to know the following:

Fact 4.3.1. The function $\gamma \mapsto \epsilon^{\text{ram}}(\gamma)$ is an $\Omega(T, G)(F)$-invariant character of $T(F)$. □

The observation that this function is a character was already made in [DS18] and will be very useful.

4.4 Character values at shallow elements

In this subsection F is a local field of odd residual characteristic that is not a bad prime for G. In the special case $\gamma = \gamma_{<r}$ formula (4.3.1) specializes to

$$\Phi_\pi(\gamma) = \phi_d(\gamma) \sum_{\substack{g \in J^d(F) \backslash G^d(F)/G^{d-1}(F) \\ \gamma^g \in G^{d-1}(F)}} \epsilon_{\text{sym,ram}}(\gamma^g)\epsilon^{\text{ram}}(\gamma^g)\tilde{e}(\gamma^g) \cdot \Phi_{\pi_{d-1}}(\gamma^g). \tag{4.4.1}$$

This formula still requires the assumption that G^{d-1} is anisotropic modulo center. However, we expect that this condition is unnecessary. In this subsection, we will prove that this formula is valid without this condition, but under the stronger assumption on γ that it is tame elliptic and topologically semi-simple modulo $Z(G)^\circ$. That is, we are assuming that $\gamma = \gamma_0$, which is stronger than $\gamma = \gamma_{<r}$. The regularity of γ implies that J^d is a tame elliptic maximal torus. To remind ourselves of that let us write S_γ for it.

We follow the proof of [AS09, Theorem 6.4]. We have the point $x \in \mathcal{B}^{\text{red}}(G, F)$, which is called $\bar{x}$ in loc. cit. Let $\mathcal{B}^{\text{enl}}(G, F)$ denote the enlarged Bruhat-Tits building of G, i.e. $\mathcal{B}^{\text{enl}}(G, F) = \mathcal{B}^{\text{red}}(G, F) \times X_*(Z(G))^\Gamma \otimes \mathbb{R}$. Fix a preimage $\dot{x} \in \mathcal{B}^{\text{enl}}(G, F)$, which will serve as the point x in loc. cit. Recall that the representation π is compactly induced from a finite dimensional irreducible representation σ of the group $K_\sigma = G^{d-1}(F)_x \cdot G(F)_{x,0+}$. We denote by χ_σ the character of this representation, and by $\dot{\chi}_\sigma$ the extension by zero of the function χ_σ to all of $G(F)$.

We claim that the function

$$g \mapsto \dot{\chi}_\sigma({}^g\gamma)$$

on $G(F)/Z(F)$ is compactly supported. For this, note that because γ is topologically semi-simple modulo $Z(G)^\circ$ all of its root values are topologically semi-simple. It follows from [Tit79, §3.6] that the set of fixed points of γ in $\mathcal{B}^{\text{red}}(G, E)$, where E is the splitting field of S_γ, is precisely the apartment $\mathcal{A}^{\text{red}}(S_\gamma, E)$. Thus, the set of fixed points of γ in $\mathcal{B}^{\text{red}}(G, F)$ is a singleton set $\{x_\gamma\}$. The same is of course true for the element ${}^g\gamma$, which then has the unique fixed point gx_γ. Thus, unless $gx_\gamma = x$, the element ${}^g\gamma$ does not belong to $K_\sigma \subset G(F)_x$ and consequently $\dot{\chi}_\sigma({}^g\gamma)$ vanishes. This function is thus supported on a single coset of $G(F)_x/Z(F)$ in $G(F)/Z(F)$, which is compact.

According to the Harish-Chandra integral character formula, we have

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)}\phi_d(\gamma) \int_{G(F)/Z(F)} \int_K \dot{\chi}_\sigma({}^{gc}\gamma) dc dg,$$

where K is any compact open subgroup of $G(F)$ with Haar measure dc normalized so that the volume of K is equal to 1. We can take for example $K = G(F)_{x_\gamma,0}$. Since the integrand is compactly supported as a function of g, we switch the two integrals and then remove the integral over K. We arrive at

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)}\phi_d(\gamma) \sum_{g \in K_\sigma \backslash G(F)/S_\gamma(F)} \int_{K_\sigma g S_\gamma(F)/Z(F)} \dot{\chi}_\sigma({}^{kgs}\gamma) dk ds.$$

We have $\dot{\chi}_\sigma({}^{kgs}\gamma) = \dot{\chi}_\sigma({}^{g}\gamma)$ which, as we already discussed, is zero unless $gx_\gamma = x$. Recall the subset $\mathcal{B}_r(\gamma)$ of [AS08, Definition 9.5]. By [AS08, Lemma 9.6] it is equal to $\mathcal{B}^{\text{enl}}(S_\gamma, F)$, which is the preimage of x_γ in $\mathcal{B}^{\text{enl}}(G, F)$. Thus, if $gx_\gamma = x$ then $\dot{x}$ belongs to $\mathcal{B}_r({}^g\gamma)$. Because of this, the rest of the argument in the proof of [AS09, Theorem 6.4] goes through: [AS09, Corollaries 4.5,4.6] can now be used without the assumption that $G^{d-1}(F)/Z(F)$ is anisotropic, whose purpose was to guarantee, via [AS08, Lemma 9.13], that $x \in \mathcal{B}_r({}^g\gamma)$.

4.5 Computation of $\text{ord}_x(\alpha)$

In this subsection F is a local field of odd residual characteristic. The roots of unity occurring in the character formula (4.3.1) of Adler-DeBacker-Spice depend on the sets $\text{ord}_x(\alpha) \subset \mathbb{R}$ for the various roots $\alpha \in R(T, G)$. According to [DS18, Corollary 3.1.9], when α is symmetric there are only two possibilities for $\text{ord}_x(\alpha)$, namely $e_\alpha^{-1}\mathbb{Z}$ and $e_\alpha^{-1}(\mathbb{Z} + \frac{1}{2})$, where e_α is the ramification degree of the extension F_α/F. The key to our reinterpretation of these roots of unity is the exact computation of $\text{ord}_x(\alpha)$, which is as follows.

Proposition 4.5.1. Let $T \subset G$ be a maximal torus with a tamely ramified splitting field E/F and let $x \in \mathcal{B}^{\text{red}}(G, F) \cap \mathcal{A}^{\text{red}}(T, E)$. For any $\alpha \in R(T, G)_{\text{sym}}$ we have

$$\text{ord}_x(\alpha) = \begin{cases} e_\alpha^{-1}\mathbb{Z}, & \text{if } \alpha \text{ is ramified} \\ e_\alpha^{-1}\mathbb{Z}, & \text{if } \alpha \text{ is unramified and } f_{(G,T)}(\alpha) = +1 \\ e_\alpha^{-1}(\mathbb{Z} + \frac{1}{2}), & \text{if } \alpha \text{ is unramified and } f_{(G,T)}(\alpha) = -1 \end{cases}$$

where $f_{(G,T)}(\alpha)$ is the toral invariant defined in [Kal15, §4.1]. □

The proof will occupy this subsection. The crucial step is the reduction of the proof to the case of semi-simple groups of rank 1.

Let $G_{\pm\alpha}$ be the subgroup of G generated by the root subgroups for the roots α and $-\alpha$. It is a semi-simple group of absolute rank 1 and is defined over $F_{\pm\alpha}$. Its Lie-algebra is $\mathfrak{g}_{\pm\alpha} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{g}_\alpha$, where $\mathfrak{s}_\alpha$ is the 1-dimensional subspace of $\mathfrak{g}$ spanned by the coroot H_α. Let $S_\alpha \subset G_{\pm\alpha}$ be the maximal torus whose Lie-algebra is $\mathfrak{s}_\alpha$. It is a 1-dimensional anisotropic torus defined over $F_{\pm\alpha}$ and split over F_α. Let $x_{\pm\alpha} \in \mathcal{B}^{\text{red}}(G_{\pm\alpha}, F_{\pm\alpha})$ be the unique point in $\mathcal{A}^{\text{red}}(S_\alpha, F_\alpha)^{\Gamma_{\pm\alpha}}$ (we are using here again [Pra01]).

Lemma 4.5.2. The filtrations $\mathfrak{g}_\alpha(F_\alpha) \cap \mathfrak{g}(F_\alpha)_{x,r}$ and $\mathfrak{g}_\alpha(F_\alpha) \cap \mathfrak{g}_{\pm\alpha}(F_\alpha)_{x_{\pm\alpha},r}$ are equal. □

Proof. Since E/F_α is tame we have $\mathfrak{g}(F_\alpha)_{x,r} = \mathfrak{g}(E)_{x,r} \cap \mathfrak{g}(F_\alpha)$ for all $r \in \mathbb{R}$, and the same is true for $\mathfrak{g}_{\pm\alpha}$. We may thus extend scalars to E for the comparison of the filtrations. Consider the root datum $\text{RD}_G := (T, \{U_\beta\}_{\beta \in R(T,G)})$

in the sense of [BT72, §6.1.1], where we have omitted the data M_β from the notation because they are redundant in this case, see [BT84, §4.1.19(i)]. The point $x \in \mathcal{B}^{\text{red}}(G, F) \subset \mathcal{B}^{\text{red}}(G, E)$ gives a valuation ψ_x of RD_G, consisting of functions $\psi_{x,\beta} : U_\beta(E) \to \mathbb{R} \cup \{\infty\}$, one for each $\beta \in R(T, G)$, satisfying [BT72, Definition 6.2.1]. On the group $G_{\pm\alpha}$ we have the root datum $\mathrm{RD}_{G_{\pm\alpha}} := (S_\alpha, \{U_\beta\}_{\beta=\pm\alpha})$ and it is easy to see that the functions $\{\psi_{x,\alpha}, \psi_{x,-\alpha}\}$ satisfy the conditions of [BT72, Definition 6.2.1] and hence form a valuation of $\mathrm{RD}_{G_{\pm\alpha}}$, which we shall call $\psi_{x,\pm\alpha}$. We claim that this valuation corresponds to a point in $\mathcal{A}^{\text{red}}(S_\alpha, E) \subset \mathcal{B}^{\text{red}}(G_{\pm\alpha}, E)$. For this we must show that $\psi_{x,\pm\alpha}$ is equipollent to a Chevalley valuation of $\mathrm{RD}_{G_{\pm\alpha}}$ [BT84, §4.2.1]. This follows from the fact that ψ_x is equipollent to a Chevalley valuation of RD_G. Indeed, the latter statement means by definition that there exists a system of isomorphisms $(x_\beta : \mathbb{G}_a \to U_\beta)_{\beta \in R(T,G)}$ and an element $v \in X_*(T_{\text{ad}}) \otimes \mathbb{R}$ with the following properties:

1. For all $\beta \in R(T, G)$ we have $[dx_\beta(1), dx_{-\beta}(1)] = H_\beta$ in $\mathfrak{g}$;
2. For all $\beta, \gamma \in R(T, G)$ with $\beta + \gamma \in R(T, G)$ there exists $\epsilon_{\beta,\gamma} \in \{\pm 1\}$ with $[dx_\beta(1), dx_\gamma(1)] = \epsilon_{\beta,\gamma}(r_{\beta,\gamma} + 1)dx_{\beta+\gamma}(1)$, where $r_{\beta,\gamma}$ is the largest integer such that $\gamma - r\beta \in R(T, G)$;
3. For all $\beta \in R(T, G)$ and $t \in E$ we have $\psi_{x,\beta}(x_\beta(t)) = \mathrm{ord}(t) + \langle \beta, v \rangle$.

Clearly then the system of isomorphisms $(x_\beta)_{\beta=\pm\alpha}$ and the valuation $\psi_{x,\pm\alpha}$ satisfy the same properties (the second being vacuous). We only need to show that in the third property we can replace $v \in X_*(T_{\text{ad}}) \otimes \mathbb{R}$ with some $v_{\pm\alpha} \in X_*(S_\alpha) \otimes \mathbb{R}$. For this, we observe that the surjection $X^*(T_{\text{ad}}) \otimes \mathbb{R} \to X^*(S_\alpha) \otimes \mathbb{R}$ induced by the inclusion $S_\alpha \subset T_{\text{ad}}$ has a natural section, sending the image of α under this surjection back to α. This section is dual to a surjection $X_*(T_{\text{ad}}) \otimes \mathbb{R} \to X_*(S_\alpha) \otimes \mathbb{R}$ and we let $v_{\pm\alpha}$ be the image of v under this surjection. Then by definition $\langle \alpha, v \rangle = \langle \alpha, v_{\pm\alpha} \rangle$. This proves that $\psi_{x,\pm\alpha}$ is equipollent to a Chevalley valuation of $\mathrm{RD}_{G_{\pm\alpha}}$ and thus corresponds to a point in $\mathcal{A}^{\text{red}}(S_\alpha, E)$. Finally, because the point x, and hence the valuation ψ_x, are fixed by Γ, and in particular by $\Gamma_{\pm\alpha}$, the valuation $\psi_{x,\pm\alpha}$, and hence the corresponding point in $\mathcal{A}^{\text{red}}(S_\alpha, E)$, are also fixed by $\Gamma_{\pm\alpha}$. The torus S_α being anisotropic over $F_{\pm\alpha}$, the only point in $\mathcal{A}^{\text{red}}(S_\alpha, E)^{\Gamma_{\pm\alpha}}$ is $x_{\pm\alpha}$ and this implies that the point corresponding to $\psi_{x,\pm\alpha}$ is none other than $x_{\pm\alpha}$. ■

According to this lemma, we can replace G by $G_{\pm\alpha}$, T by S_α, and x by $x_{\pm\alpha}$, in the computation of $\mathrm{ord}_x(\alpha)$. At the same time, it follows directly from the definition that we can make the same replacement in the computation of $f_{(G,T)}(\alpha)$. This reduces the proof to the case when the group G is semi-simple of absolute rank 1. Such a group is a (necessarily inner) form of either SL_2 or PGL_2. Neither $\mathrm{ord}_x(\alpha)$ nor $f_{(G,T)}(\alpha)$ is affected by passing to an isogenous group, so we may assume that G is an inner form of SL_2. Then we have to contend with four cases – G is either split or not, and S is either unramified or not. Rather than going through all four cases by hand, we will use the following lemma, which reduces to the cases where $G = \mathrm{SL}_2$. We formulate it in general, as we believe this makes the proof more transparent.

Lemma 4.5.3. Let $\xi : G \to G'$ be an inner twist and $S \subset G$ a tame elliptic maximal torus. Assume that the restriction of ξ to S is defined over F, and let $S' := \xi(S)$ and $\alpha' = \xi(\alpha)$. Then Proposition 4.5.1 is true for (G, S, α) if and only if it is true for (G', S', α'). □

Proof. Let again E/F be the tame finite Galois extension splitting S. Let x and x' be the unique Γ-fixed points in the reduced apartments of S and S' over E. Then $\xi : G \to G'$ is an isomorphism defined over E that restricts to an isomorphism $S \to S'$ defined over F. We will need to control three parameters: The failure of ξ to send x to x', the failure of the isomorphism $\mathfrak{g}_\alpha \to \mathfrak{g}'_{\alpha'}$ induced by ξ to descend to F_α, and the possible inequality of $f_{(G,S)}(\alpha)$ and $f_{G',S'}(\alpha')$.

By assumption for any $\sigma \in \Gamma$ there is $t'_\sigma \in S_{\text{ad}}$ such that $\xi^{-1}\sigma(\xi) = \text{Ad}(t'_\sigma)$. Then $t'_\bullet \in Z^1(\Gamma, S_{\text{ad}})$. According to [Ste65, Theorem 1.9] the cohomology group $H^1(I, S_{\text{ad}})$ vanishes and hence there exist $t_\bullet \in Z^1(\Gamma/I, S_{\text{ad}}(F^u))$ and $t \in S_{\text{ad}}$ so that $t'_\sigma = t_\sigma \cdot t \cdot \sigma(t)^{-1}$. Replacing ξ by $\xi \circ \text{Ad}(t)$ we obtain $\xi^{-1}\sigma(\xi) = \text{Ad}(t_\sigma)$.

We have the isomorphism $\xi : \mathcal{A}^{\text{red}}(S, E) \to \mathcal{A}^{\text{red}}(S', E)$. Let $v \in X_*(S_{\text{ad}}) \otimes \mathbb{R}$ be the element satisfying $\xi(x + v) = x'$. Then the isomorphism $\mathfrak{g}_\alpha(E) \to \mathfrak{g}'_{\alpha'}(E)$ induced by ξ restricts for all $r \in \mathbb{R}$ to an isomorphism

$$\xi : \mathfrak{g}_\alpha(E)_{x+v,r} \to \mathfrak{g}'_{\alpha'}(E)_{x',r}.$$

This isomorphism is not necessarily equivariant for the action of Γ_α, but rather satisfies $\sigma(\xi(X)) = \xi(\langle\alpha, t_\sigma\rangle\sigma(X))$ for $X \in \mathfrak{g}_\alpha(E)$ and $\sigma \in \Gamma_\alpha$. Now $\sigma \mapsto \langle\alpha, t_\sigma\rangle$ is the image of t_σ under

$$Z^1(\Gamma/I, S_{\text{ad}}(F^u)) \xrightarrow{\text{Res}} Z^1(\Gamma_\alpha/I_\alpha, S_{\text{ad}}(F^u_\alpha)) \xrightarrow{\alpha} Z^1(\Gamma_\alpha/I_\alpha, F^{u,\times}_\alpha),$$

where F^u_α denotes the fixed field of I_α in F^s. Hilbert's theorem 90 implies that this cocycle takes values not just in $F^{u,\times}_\alpha$, but in $O^\times_{F^u_\alpha}$. The vanishing of $H^1(\Gamma_\alpha/I_\alpha, O^\times_{F^u_\alpha})$ implies that there exists $u \in O^\times_{F^u_\alpha}$ such that the modified isomorphism

$$u \cdot \xi : \mathfrak{g}_\alpha(E)_{x+v,r} \to \mathfrak{g}'_{\alpha'}(E)_{x',r}$$

is Γ_α-equivariant, and hence descends to an isomorphism

$$\mathfrak{g}_\alpha(F_\alpha)_{x,r-\langle\alpha,v\rangle} = \mathfrak{g}_\alpha(F_\alpha)_{x+v,r} \to \mathfrak{g}'_{\alpha'}(F_\alpha)_{x',r}.$$

This implies $\text{ord}_x(\alpha) + \langle\alpha, v\rangle = \text{ord}_{x'}(\alpha')$. In order to prove the lemma we must now compute $\langle\alpha, v\rangle$ and relate it to the invariant $f_{(G,S)}(\alpha)$.

The isomorphism $\xi : \mathcal{A}^{\text{red}}(S, E) \to \mathcal{A}^{\text{red}}(S', E)$ is not necessarily Γ-equivariant. Rather, it satisfies $\xi^{-1}\sigma(\xi) = v(t_\sigma)$, where $v(t_\sigma) \in X_*(S_{\text{ad}}) \otimes \mathbb{R}$ is characterized by $\langle\beta, v(t_\sigma)\rangle = -\text{ord}(\beta(t_\sigma))$ for all $\beta \in R(S, G)$. Applying $\sigma \in \Gamma$ to the equation $\xi(x+v) = x'$ we obtain $v - \sigma(v) = v(t_\sigma)$ and hence $-\text{ord}(\alpha(t_\sigma)) = \langle\alpha, v - \sigma(v)\rangle = \langle\alpha - \sigma^{-1}(\alpha), v\rangle$. Choosing $\sigma \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$ we then obtain $\langle\alpha, v\rangle = -\frac{1}{2}\text{ord}(\alpha(t_\sigma))$. We now use that $\sigma \mapsto \alpha(t_\sigma)$ is the image of $t_\bullet$ under

$$Z^1(\Gamma/I, S_{\text{ad}}(F^u)) \xrightarrow{\text{Res}} Z^1(\Gamma_{\pm\alpha}/I_{\pm\alpha}, S_{\text{ad}}(F^u_{\pm\alpha})) \xrightarrow{\alpha} Z^1(\Gamma_{\pm\alpha}/I_{\pm\alpha}, S_\alpha(F^u_{\pm\alpha})),$$

where S_α is the 1-dimensional anisotropic torus defined over $F_{\pm\alpha}$ and split over F_α and $F^u_{\pm\alpha}$ denotes fixed subfield in F^s of $I_{\pm\alpha}$. We have $S_\alpha(F^u_{\pm\alpha}) = S_\alpha(F^u_\alpha)^{I_{\pm\alpha}/I_\alpha}$.

Now we distinguish two cases. If α is ramified, then $I_{\pm\alpha}/I_\alpha$ is of order 2 and $S_\alpha(F^u_\alpha)^{I_{\pm\alpha}/I_\alpha}$ is the kernel of the norm $F^{u,\times}_\alpha \to F^{u,\times}_{\pm\alpha}$. It follows that $\text{ord}(\alpha(t_\sigma)) = 0$. If α is unramified, then $I_{\pm\alpha}/I_\alpha = \{1\}$ and $S_\alpha(F^u_{\pm\alpha}) = F^{u,\times}_\alpha$. The inflation map $H^1(\Gamma_{\pm\alpha}/\Gamma_\alpha, S_\alpha(F_\alpha)) \to H^1(\Gamma_{\pm\alpha}, S_\alpha(F^s))$, which is an isomorphism, factors as

$$H^1(\Gamma_{\pm\alpha}/\Gamma_\alpha, S_\alpha(F_\alpha)) \to H^1(\Gamma_{\pm\alpha}/I_{\pm\alpha}, S_\alpha(F^u_{\pm\alpha})) \to H^1(\Gamma_{\pm\alpha}, S_\alpha(F^s))$$

and both arrows are isomorphisms. The value at $\sigma \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$ of any coboundary in the middle term is of the form $x\sigma(x)$ for some $x \in F_{\pm\alpha}^{u,\times} = F_\alpha^{u,\times}$ and its valuation belongs to $2\mathrm{ord}(F_\alpha^\times)$. This implies that $\mathrm{ord}(\alpha(t_\sigma)) \in \mathrm{ord}(c_\sigma) + 2\mathrm{ord}(F_\alpha^\times)$ for any 1-cocycle $c_\bullet \in Z^1(\Gamma_{\pm\alpha}/I_{\pm\alpha}, S_\alpha(F_{\pm\alpha}^u))$ that is cohomologous to $\alpha(t_\bullet)$. But if we take $c_\bullet \in Z^1(\Gamma_{\pm\alpha}/\Gamma, S_\alpha(F_\alpha))$, then [Kal15, Prop. 4.3(1)] implies that $-\frac{1}{2}\mathrm{ord}(c_\sigma) \in \mathrm{ord}(F_\alpha^\times)$ if and only if $f_{(G,S)}(\alpha) = f_{G',S'}(\alpha')$. We conclude

$$\langle \alpha, v\rangle \in \begin{cases} \frac{1}{2}\mathrm{ord}(F_\alpha^\times) \smallsetminus \mathrm{ord}(F_\alpha^\times), & \text{if } \alpha \text{ is unramified and } f_{(G,S)}(\alpha) \neq f_{G',S'}(\alpha') \\ \mathrm{ord}(F_\alpha^\times), & \text{otherwise.} \end{cases} \qquad \blacksquare$$

This lemma reduces the proof of Proposition 4.5.1 to the case $G = \mathrm{SL}_2$ and S an anisotropic maximal torus. Moreover, we are free to change S within its stable class if we like. This case can be treated by a simple calculation as follows. We have $F_{\pm\alpha} = F$ and $F_\alpha/F_{\pm\alpha}$ is a quadratic extension that may be ramified or unramified. Let $\sigma \in \Gamma_{\pm\alpha}/\Gamma_\alpha$ denote the non-trivial element and fix an element $a \in F_\alpha^\times$ satisfying $a + \sigma(a) = 0$. Set

$$h = \begin{bmatrix} 1 & -\frac{a^{-1}}{2} \\ a & \frac{1}{2} \end{bmatrix} \in G(F_\alpha).$$

If $T \subset G$ is the split diagonal torus, then hTh^{-1} is stably conjugate to S, so we may assume that it is equal to S. It will be convenient to change coordinates by $\mathrm{Ad}(h)$ and represent S as the diagonal torus in G. This comes at the expense of replacing the usual action σ_G of σ on $G(F_\alpha)$, given by applying σ to the entries of the matrix representing a given element of $G(F_\alpha)$, by the more complicated action given by $\mathrm{Ad}(h^{-1}\sigma(h)) \rtimes \sigma_G$. A simple computation reveals

$$h^{-1}\sigma(h) = \begin{bmatrix} 0 & \frac{a^{-1}}{2} \\ -2a & 0 \end{bmatrix}.$$

According to [DS18, Corollary 3.1.8] we have $\mathrm{ord}_x(\alpha) = -\mathrm{ord}_x(-\alpha)$ and so we are free to choose either root of S as the one we study. We take the root α whose root subspace is spanned by the element

$$X_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then we see $\mathrm{Ad}(h^{-1}\sigma(h)) \rtimes \sigma_G(X_\alpha) = -4a^2 X_{-\alpha}$ and this implies $f_{(G,S)}(\alpha) = +1$. In order to understand the filtration $\mathfrak{g}_\alpha(F_\alpha)_{x,r}$ we must compute the point $x \in \mathcal{A}^{\mathrm{red}}(S, F_\alpha)^\Gamma$. Let $o \in \mathcal{A}^{\mathrm{red}}(S, F_\alpha)$ be the point given by the pinning X_α, and let $v \in X_*(S) \otimes \mathbb{R}$ be the element satisfying $o + v = x$. Applying $\mathrm{Ad}(h^{-1}\sigma(h)) \rtimes \sigma_G$ to this equation we see that $2v$ is equal to the translation on $\mathcal{A}^{\mathrm{red}}(S, F_\alpha)$ effected by the action of $\alpha^\vee(a^{-1})$, and hence

$$\langle \alpha, v\rangle = \mathrm{ord}(a).$$

It follows that $\mathfrak{g}_\alpha(F_\alpha)_{x,r} = \mathfrak{g}_\alpha(F_\alpha)_{o,r-\mathrm{ord}(a)}$. Since the filtration $\mathfrak{g}_\alpha(F_\alpha)_{o,r}$ has a break at zero and $\mathrm{ord}(a) \in \mathrm{ord}(F_\alpha^\times)$ we conclude that the filtration $\mathfrak{g}_\alpha(F_\alpha)_{x,r}$ also has a break at zero. The proof of Proposition 4.5.1 is complete.

4.6 Definition of $\Delta_{II}^{\mathrm{abs}}$

In this subsection F is a local field of odd residual characteristic. In [LS87, §3.3] Langlands and Shelstad define the term Δ_{II}, which is a component of their transfer factor. It is associated to a connected reductive group defined over a local field, an endoscopic group, a maximal torus that is common to both groups, as well as a-data and χ-data. In this subsection, we will introduce a slight variation of Δ_{II}, which we will call Δ_{II}^{abs}. It will be associated to a connected reductive group defined over a local field, a maximal torus thereof, as well as a-data and χ-data. We think of Δ_{II}^{abs} as an absolute version of Δ_{II}, in the precise sense that the original term Δ_{II} can be written as a quotient with numerator Δ_{II}^{abs} for the reductive group and denominator Δ_{II}^{abs} for its endoscopic group.

We begin by recalling the notions of a-data and χ-data from [LS87, §2]. A set of a-data consists of elements $a_\alpha \in F_\alpha^\times$, one for each $\alpha \in R(T,G)$, having the properties $a_{-\alpha} = -a_\alpha$ and $a_{\sigma(\alpha)} = \sigma(a_\alpha)$ for $\sigma \in \Gamma$. A set of χ-data consists of characters $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$, one for each $\alpha \in R(T,G)$, having the properties $\chi_{-\alpha} = \chi_\alpha^{-1}$, $\chi_{\sigma(\alpha)} = \chi_\alpha \circ \sigma^{-1}$ for each $\sigma \in \Gamma$, and $\chi_\alpha|_{F_{\pm\alpha}^\times} = \kappa_\alpha$ whenever $\alpha \in R(T,G)_{\text{sym}}$ and $\kappa_\alpha : F_\alpha^\times \to \{\pm 1\}$ is the quadratic character associated to the quadratic extension $F_\alpha/F_{\pm\alpha}$.

Sets of a-data and χ-data always exist, but there are rarely unique choices for them without further structure. It is clear from the definitions that one can choose $a_\alpha = 1$ and $\chi_\alpha = 1$ for asymmetric $\alpha \in R(T,G)$, although it is sometimes convenient not to do so. When α is symmetric and unramified, the character κ_α is unramified and there is a distinguished choice for χ_α, namely the unramified quadratic character of $F_\alpha^\times$. When α is symmetric and ramified, the character κ_α is ramified and the unramified quadratic character of $F_\alpha^\times$ is not a valid choice for χ_α. In this situation, under the assumption $p \neq 2$, there are exactly two tamely-ramified characters of $F_\alpha^\times$ that extend κ_α. Their quotient (in either order) is the unramified quadratic character of $F_\alpha^\times$, and each of the two tame choices for χ_α is characterized by the fact that its restriction to $O_{F_\alpha}^\times$ lifts the quadratic character of $k_{F_\alpha}^\times$ and its value on any uniformizer belongs to $\{i,-i\} \subset \mathbb{C}^\times$ if -1 is not a square in F_α and to $\{+1,-1\} \subset \mathbb{C}^\times$ otherwise. Regardless of the ramification of $F_\alpha/F_{\pm\alpha}$, it is often useful to allow χ_α to have arbitrary depth.

Definition 4.6.1. We will call a set of χ-data *minimally ramified*, if $\chi_\alpha = 1$ for asymmetric α, χ_α is unramified for unramified symmetric α, and χ_α is tamely ramified for ramified symmetric α. □

As just discussed, different choices of minimally ramified χ-data can differ only at ramified symmetric roots α, and only by the unramified sign character of $F_\alpha^\times$.

Definition 4.6.2. Given sets of a-data and χ-data, we define

$$\Delta_{II}^{\text{abs}}[a,\chi] : T(F) \to \mathbb{C}^\times, \qquad \gamma \mapsto \prod_{\substack{\alpha \in \Gamma\setminus R(T,G) \\ \alpha(\gamma) \neq 1}} \chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right).$$

□

We will now recall some results from [LS87] about how this term changes when the a-data or χ-data are changed. First, the a-data $(a_\alpha)_\alpha$ can only be replaced by $(b_\alpha \cdot a_\alpha)_\alpha$, where $b_\alpha \in F_\alpha^\times$ for $\alpha \in R(T,G)$ satisfies $b_{-\alpha} = b_\alpha$ and $\sigma(b_\alpha) = b_{\sigma(\alpha)}$ for all $\sigma \in \Gamma$.

Lemma 4.6.3.
$$\Delta_{II}^{\mathrm{abs}}[ba,\chi](\gamma)=\Delta_{II}^{\mathrm{abs}}[a,\chi](\gamma)\cdot\prod_{\substack{\alpha\in\Gamma\backslash R(T,G)_{\mathrm{sym}}\\ \alpha(\gamma)\neq 1}}\kappa_\alpha(b_\alpha).$$
□

Proof. Immediate. ■

Definition 4.6.4. A set of ζ-data $(\zeta_\alpha)_\alpha$ for $R(T,G)$ consists of a character $\zeta_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ for each $\alpha\in R(T,G)$ subject to the conditions $\zeta_{-\alpha}=\zeta_\alpha^{-1}$, $\zeta_{\sigma\alpha}=\zeta_\alpha\circ\sigma^{-1}$ for all $\sigma\in\Gamma$ and, in the case of symmetric α, $\zeta_\alpha|_{F_{\pm\alpha}^\times}=1$. □

It is immediate that if $(\chi_\alpha)_\alpha$ and $(\chi'_\alpha)_\alpha$ are two sets of χ-data, then $\chi'_\alpha=\chi_\alpha\cdot\zeta_\alpha$, where $(\zeta_\alpha\cdot\chi_\alpha)_\alpha$ is a (uniquely determined) set of ζ-data.

Let O be an orbit of $\Gamma\times\{\pm1\}$ in $R(T,G)$. We will define a character $\zeta_O : T(F)\to\mathbb{C}^\times$ as follows. If O consists of two distinct Γ-orbits, choose $\alpha\in O$ and let $\zeta_O=\zeta_\alpha\circ\alpha$. If O consists of a single Γ-orbit, choose $\alpha\in O$ and let ζ_O be the composition
$$T(F)\xrightarrow{\alpha}F_\alpha^1\xleftarrow{\cong}F_\alpha^\times/F_{\pm\alpha}^\times\xrightarrow{\zeta_\alpha}\mathbb{C}^\times,$$
where $F_\alpha^1=\mathrm{Ker}(N_{F_\alpha/F_{\pm\alpha}}:F_\alpha^\times\to F_{\pm\alpha}^\times)$ and the middle isomorphism sends $x\in F_\alpha^\times$ to $x/\tau(x)\in F_\alpha^1$, with $\tau\in\Gamma_{\pm\alpha}/\Gamma_\alpha$ being the non-trivial element. In both cases it is straightforward to check that ζ_O depends only on O and not on the choice of α.

Definition 4.6.5. Given ζ-data $(\zeta_\alpha)_\alpha$ let $\zeta_T : T(F)\to\mathbb{C}^\times$ be the product of the characters ζ_O, as O runs over the set of orbits in $R(T,G)$ for the action of $\Gamma\times\{\pm1\}$. □

Lemma 4.6.6.
$$\Delta_{II}^{\mathrm{abs}}[a,\zeta\cdot\chi](\gamma)=\Delta_{II}^{\mathrm{abs}}[a,\chi](\gamma)\cdot\zeta_T(\gamma).$$
□

Proof. The argument for this constitutes the proofs of [LS87, Lemma 3.3.A, Lemma 3.3.D]. ■

Lemma 4.6.7. Let $\gamma\in T(F)_{\mathrm{reg}}$ be an element having a decomposition $\gamma=\gamma_{<r}\cdot\gamma_{\geq r}$ with $\gamma_{<r},\gamma_{\geq r}\in T(F)$ satisfying $\mathrm{ord}(\alpha(\gamma_{<r})-1)<r$ and $\mathrm{ord}(\alpha(\gamma_{\geq r})-1)\geq r$ for all $\alpha\in R(T,G)$. Assume that the χ-data is tamely ramified, i.e. $\chi_\alpha|_{[F_\alpha^\times]_{0+}}=1$. Then
$$\Delta_{II}^{\mathrm{abs},G}[a,\chi](\gamma)=\Delta_{II}^{\mathrm{abs},G}[a,\chi](\gamma_{<r})\cdot\Delta_{II}^{\mathrm{abs},J}[a,\chi](\gamma_{\geq r}),$$
where $J=\mathrm{Cent}(\gamma_{<r},G)^\circ$ and the superscripts indicate the group relative to which the factor $\Delta_{II}^{\mathrm{abs}}$ is taken. □

Proof. We need to show that
$$\chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right)=\begin{cases}\chi_\alpha\left(\frac{\alpha(\gamma_{<r})-1}{a_\alpha}\right), & \text{if } \alpha(\gamma_{<r})\neq 1,\\ \chi_\alpha\left(\frac{\alpha(\gamma_{\geq r})-1}{a_\alpha}\right), & \text{if } \alpha(\gamma_{<r})=1.\end{cases}$$
The case $\alpha(\gamma_{<r})=1$ is obvious. Assume now $\alpha(\gamma_{<r})\neq1$. Write $\alpha(\gamma_{<r})=1+x$ and $\alpha(\gamma_{\geq r})=1+y$ with $\mathrm{ord}(x)<r$ and $\mathrm{ord}(y)\geq r$. Then $\alpha(\gamma)-1=(x+1)(y+1)-1=x(1+y+yx^{-1})$. Since $1+y+yx^{-1}\in[F_\alpha^\times]_{0+}$ the proof is complete. ■

We now introduce a weaker variant of the notion of a-data that can be used in conjunction with tame χ-data and is sometimes more convenient.

Definition 4.6.8. A *mod-a-data* $\{(r_\alpha, \bar{a}_\alpha)\}$ is an assignment to each $\alpha \in R(T,G)$ of a real number $r_\alpha \in \mathbb{R}$ and a non-zero element $\bar{a}_\alpha \in [F_\alpha]_{r_\alpha}/[F_\alpha]_{r_\alpha+}$ such that $r_{\sigma\alpha} = r_\alpha = r_{-\alpha}$, $\bar{a}_{\sigma\alpha} = \sigma(\bar{a}_\alpha)$, and $\bar{a}_{-\alpha} = -\bar{a}_\alpha$, for any $\sigma \in \Gamma$. $\square$

Given tame χ-data and mod-a-data, we can choose an arbitrary lift $a_\alpha \in [F_\alpha]_{r_\alpha}$ for each $\bar{a}_\alpha$ and consider the function $\Delta_{II}^{\text{abs}}[a,\chi]$ (even if the set of lifts a_α does not constitute a-data). This function is independent of the chosen lifts, because another choice would have the form $a_\alpha + a'_\alpha$, where $a'_\alpha \in [F_\alpha]_{r_\alpha+}$. Now $a_\alpha + a'_\alpha = a_\alpha b_\alpha$, where $b_\alpha = 1 + \frac{a'_\alpha}{a_\alpha}$ belongs to $[F_\alpha^\times]_{0+}$, and χ_α restricts trivially to this group. We will denote the resulting function by $\Delta_{II}^{\text{abs}}[\bar{a},\chi]$.

4.7 A formula for $\epsilon_{\text{sym,ram}} \cdot \tilde{e}$

In this subsection F is a local field of odd residual characteristic. We will use the results of Subsections 4.5 and 4.6 to give a formula for the product of the two roots of unity $\epsilon_{\text{sym,ram}}(\gamma_{<r}^g) \cdot \tilde{e}(\gamma_{<r}^g)$.

Recall that we have fixed an additive character $\Lambda : F \to \mathbb{C}^\times$ that is non-trivial on O_F and trivial on $\mathfrak{p}_F$. Recall also that the definition of the roots of unity depends on a tame maximal torus T of G^{d-1} containing $\gamma_{<r}^g$. We now choose a-data and χ-data for $R(T,G)$ as follows. If $\alpha \in R(T,G^{d-1})$ or $\alpha(\gamma_{<r}^g) = 1$, we leave the choice unspecified, as these roots will not contribute to the formula. For any other $\alpha \in R(T,G)$ we set $a_\alpha = \langle H_\alpha, X_{d-1}^*\rangle$, and we take χ_α to be the trivial character if α is asymmetric and the unramified quadratic character if α is symmetric and unramified. If α is symmetric and ramified, we choose among the two possible tamely-ramified characters by demanding

$$\chi_\alpha(2a_\alpha) = f_{(G,T)}(\alpha)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \text{tr}_{F_{\pm\alpha}/F}), \tag{4.7.1}$$

where $f_{(G,T)}(\alpha)$ is the toral invariant [Kal15, §4.1] and $\lambda_{F_\alpha/F_{\pm\alpha}}$ is Langlands' constant [Lan, Theorem 2.1], [BH05b, §1.5]. To see that this specifies a valid χ-data, note that since $a_\alpha \in F_\alpha^\times$ is an element of trace zero, we have $\text{ord}(a_\alpha) \in \text{ord}(F_\alpha^\times) \smallsetminus \text{ord}(F_{\pm\alpha}^\times)$. Thus the value of $\chi_\alpha(2a_\alpha)$ distinguishes the two possible choices of χ_α. Since the square of the right hand side of (4.7.1) equals $\kappa_\alpha(-1)$, we may indeed chose χ_α to satisfy (4.7.1). Finally, it is enough to check that $\chi_{\sigma\alpha} \circ \sigma = \chi_\alpha$ on the element $2a_\alpha$, where it is obvious.

Note that the choice of χ_α depends only on G, T, and ϕ_{d-1}, but not on Λ, because the dependence of the right side of (4.7.1) on Λ is canceled by the dependence of a_α on Λ.

The main step in our reinterpretation of the character formula is the following expression for the product $\epsilon_{\text{sym,ram}}(\gamma_{<r}^g)\tilde{e}(\gamma_{<r}^g)$.

Lemma 4.7.1. The product $\epsilon_{\text{sym,ram}}(\gamma_{<r}^g)\tilde{e}(\gamma_{<r}^g)$ is equal to

$$\prod_{\alpha \in \Gamma \backslash (R_{\gamma_{<r}^g})_{\text{sym}}} f_{(G,T)}(\alpha)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \text{tr}_{F_{\pm\alpha}/F})^{-1}\chi_\alpha\left(\frac{\alpha(\gamma_{<r}^g) - 1}{a_\alpha}\right).$$

$\square$

Proof. According to (4.3.2) and (4.3.4) we can write $\epsilon_{\text{sym,ram}}(\gamma^g_{<r})\tilde{e}(\gamma^g_{<r})$ as the product of

$$\prod_{\alpha\in\Gamma\setminus(R_{(r-\text{ord}_{\gamma^g_{<r}})/2})_{\text{symm,ram}}} \text{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})(-1)^{f_\alpha+1}\mathfrak{G}^{f_\alpha}\text{sgn}_{k^\times_{F_\alpha}}(t_\alpha)$$

with

$$\prod_{\alpha\in\Gamma\setminus(R_{(r-\text{ord}_{\gamma^g_{<r}})/2})_{\text{sym,unram}}} (-1).$$

We consider the contribution of an individual symmetric $\alpha \in R_{\gamma^g_{<r}}$. If α is unramified, then χ_α is unramified and $\text{ord}(\langle H_\alpha, X^*_{d-1}\rangle) = -r \in \text{ord}(F^\times_\alpha)$, so we have

$$\chi_\alpha\left(\frac{\alpha(\gamma^g_{<r})-1}{\langle H_\alpha, X^*_{d-1}\rangle}\right) = (-1)^{e_\alpha(\text{ord}(\alpha(\gamma^g_{<r})-1)-r)},$$

while $\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda\circ\text{tr}_{F_{\pm\alpha}/F}) = -1$ according to [BH05b, Lemma 1.5]. The total contribution of α to the right hand side of the equation of the lemma is thus

$$f_{(G,T)}(\alpha)\cdot(-1)^{e_\alpha(\text{ord}(\alpha(\gamma^g_{<r})-1)-r)+1}.$$

According to Proposition 4.5.1 this expression is equal to -1 precisely when $\alpha \in R_{(r-\text{ord}_{\gamma^g_{<r}})/2}$. The contributions of α to both sides of the equation of the lemma are thus equal.

Now let α be ramified. Since $\alpha(\gamma^g_{<r}) \in F^\times_\alpha$ is an element whose $F_\alpha/F_{\pm\alpha}$-norm is trivial, $\text{ord}(\alpha(\gamma^g_{<r})-1)$ is either zero or belongs to $\text{ord}(F^\times_\alpha)\setminus\text{ord}(F^\times_{\pm\alpha})$. At the same time, $\langle H_\alpha, X^*_{d-1}\rangle \in F^\times_\alpha$ is an element whose $F_\alpha/F_{\pm\alpha}$-trace vanishes, so $-r = \text{ord}(\langle H_\alpha, X^*_{d-1}\rangle) \in \text{ord}(F^\times_\alpha)\setminus\text{ord}(F^\times_{\pm\alpha})$. It follows from Proposition 4.5.1 that $\alpha \in R_{(r-\text{ord}_{\gamma^g_{<r}})/2}$ if and only if $\text{ord}(\alpha(\gamma^g_{<r})-1) \neq 0$. Assume first that this is the case. Then α contributes to both sides of the equation of the lemma. For the contribution of the left side, we note that by the theorem of Hasse-Davenport the term $(-1)^{f_\alpha+1}\mathfrak{G}^{f_\alpha}$ is equal to the Gauss sum

$$q^{-f_\alpha/2}\sum_{x\in k_{F_\alpha}}\Lambda(\text{tr}_{k_{F_\alpha}/k_F}(x^2)).$$

Since the character $\Lambda\circ\text{tr}_{F_{\pm\alpha}/F} : F_{\pm\alpha}\to\mathbb{C}^\times$ induces on $k_{F_\alpha} = k_{F_{\pm\alpha}}$ the character $x\mapsto\Lambda(\text{tr}_{k_{F_\alpha}/k_F}(e_{\pm\alpha}x))$, the latter Gauss sum is equal by [BH05b, Lemma 1.5] to

$$\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda\circ\text{tr}_{F_{\pm\alpha}/F})\cdot\kappa_\alpha(e_{\pm\alpha}) = \lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda\circ\text{tr}_{F_{\pm\alpha}/F})^{-1}\cdot\kappa_\alpha(-e_{\pm\alpha}).$$

Next, using that χ_α is trivial on $N_{F_\alpha/F_{\pm\alpha}}$-norms and that $\langle H_\alpha, X^*_{d-1}\rangle \in F_\alpha$ is an element whose $F_\alpha/F_{\pm\alpha}$-trace vanishes, we see

$$\text{sgn}_{k^\times_{F_\alpha}}(t_\alpha) = \chi_\alpha(t_\alpha) = \kappa_\alpha(e_{\pm\alpha})\kappa_\alpha(-1)\chi_\alpha\left(\frac{\alpha(\gamma^g_{<r})-1}{\langle H_\alpha, X^*_{d-1}\rangle}\right).$$

These computations and the fact that $f_{(G,T)}(\alpha) = \text{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})$ imply that the contributions of α to the both sides of the equation of the lemma agree.

Assume now that $\text{ord}(\alpha(\gamma^g_{<r})-1)$ is zero, so that $\alpha \notin R_{(r-\text{ord}_{\gamma^g_{<r}})/2}$ and thus α does not contribute to the left side of the equation of the lemma. To compute its contribution to the right side, we first notice that $\alpha(\gamma^g_{<r}) \in -1+\mathfrak{p}_{F_\alpha}$ and

hence $\alpha(\gamma^g_{<r}) - 1 = (-2)u$ for some $u \in 1 + \mathfrak{p}_{F_\alpha}$. Since u is in the kernel of χ_α and $\mathrm{tr}_{F_\alpha/F_{\pm\alpha}}(a_\alpha) = 0$ we see

$$\chi_\alpha\left(\frac{\alpha(\gamma^g_{<r}) - 1}{a_\alpha}\right) = \chi_\alpha\left(-2a_\alpha^{-1}\right) = \chi_\alpha(2a_\alpha),$$

and according to (4.7.1) the contribution of α to the right side of the equation of the lemma is 1. ■

Using Kottwitz's result [Kal15, §4.5] on the relationship between ϵ-factors and Weil constants, as well as the factor Δ_{II}^{abs} defined in §4.6, we can restate this lemma as follows. Let T_{G^d} denote the minimal Levi subgroup of the quasi-split inner form of G^d, and let T_{J^d} denote the minimal Levi subgroup of the quasi-split inner form of J^d.

Corollary 4.7.2. The product $\epsilon_{\text{sym,ram}}(\gamma^g_{<r})\tilde{e}(\gamma^g_{<r})$ is equal to

$$\frac{e(G^d)e(J^d)}{e(G^{d-1})e(J^{d-1})}\frac{\epsilon_L(X^*(T_{G^d})_\mathbb{C} - X^*(T_{J^d})_\mathbb{C}, \Lambda)}{\epsilon_L(X^*(T_{G^{d-1}})_\mathbb{C} - X^*(T_{J^{d-1}})_\mathbb{C}, \Lambda)}\frac{\Delta_{II}^{\text{abs},G^d}[a,\chi](\gamma^g_{<r})}{\Delta_{II}^{\text{abs},G^{d-1}}[a,\chi](\gamma^g_{<r})}$$

□

Proof. This follows immediately from [Kal15, Corollary 4.11] and the additivity of ϵ_L in degree zero. Note that there is a typo in loc.cit: $\mathrm{tr}_{F_\alpha/F_{\pm\alpha}}$ should read $\mathrm{tr}_{F_{\pm\alpha}/F}$. Note also that $J^d \cong \mathrm{Ad}(g^{-1})J^d$ and $J^{d-1} \cong \mathrm{Ad}(g^{-1})J^{d-1}$ as reductive groups over F. ■

We remark here that this expression does not depend on the choice of Λ, because both ϵ_L and the a-data $a_\alpha = \langle H_\alpha, X^*_{d-1}\rangle$ depend on Λ in a parallel way. Thus we may from now on use an arbitrary additive character Λ, i.e. remove the condition on its depth.

A slight variant of this corollary will also be useful later when we study L-packets. It involves the following modified choice of χ-data, where we use

$$\chi'_\alpha(2a_\alpha) = \lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \mathrm{tr}_{F_{\pm\alpha}/F}) \tag{4.7.2}$$

instead of (4.7.1). Then χ'_α is a valid set of χ-data for the same reasons that χ_α was. The usefulness of χ'_α comes from the fact that it depends only on the torus T and the character ϕ_{d-1}, but not on the group G in the sense that it is insensitive to replacing T by a stably conjugate torus in an inner form of G. The relationship between the two χ-data can be expressed by

$$\chi'_\alpha(x) = \chi_\alpha(x)\epsilon_\alpha(x),$$

where $\epsilon_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ is the trivial character unless α is symmetric and ramified, in which case it is given by $\epsilon_\alpha(x) = f_{(G,T)}(\alpha)^{e_\alpha \mathrm{ord}(x)}$. The collection $(\epsilon_\alpha)_\alpha$ is ζ-data for $R(T,G)$ in the sense of Definition 4.6.4.

Definition 4.7.3. Let $\epsilon_{f,\text{ram}} : T(F) \to \mathbb{C}^\times$ be the character of Definition 4.6.5 corresponding to this ζ-data. □

This character is similar, but not the same as, the one introduced in [Kal15, §4.6], the difference being that in loc. cit. ϵ_α was assigned non-trivial even when α was symmetric and unramified. However, due to [Kal15, Proposition 4.4] both definitions yield the same result in the case of epipelagic representations.

Lemma 4.7.4. For all $\gamma \in T(F)$ we have

$$\epsilon_{f,\mathrm{ram}}(\gamma) = \prod_{\substack{\alpha \in R(T,G)_{\mathrm{sym,ram}}/\Gamma \\ \alpha(\gamma) \neq 1 \\ \mathrm{ord}(\alpha(\gamma)-1)=0}} f_{(G,T)}(\alpha).$$

□

Proof. Using that ϵ_α is trivial unless α is symmetric and ramified we have

$$\epsilon_{f,\mathrm{ram}}(\gamma) = \prod_{\alpha \in R(T,G)_{\mathrm{sym,ram}}/\Gamma} \epsilon_\alpha(\delta^\alpha)^{-1},$$

where $\delta^\alpha \in F_\alpha^\times$ is any element satisfying $\delta^\alpha/\sigma(\delta^\alpha) = \alpha(\gamma)$ for the non-trivial element $\sigma \in \Gamma_{\pm\alpha}/\Gamma_\alpha$. If $\mathrm{ord}(\alpha(\gamma)-1) > 0$ we have $\mathrm{ord}(\delta^\alpha - 1) > 0$ and hence $\epsilon_\alpha(\delta^\alpha) = 1$. If $\mathrm{ord}(\alpha(\gamma)-1) = 0$ then the fact that α is symmetric and ramified implies $\alpha(\gamma) \in -1+\mathfrak{p}_{F_\alpha}$. Writing $\alpha(\gamma) = -1 \cdot u$ with $u \in 1+\mathfrak{p}_{F_\alpha}$ we may choose $\delta^\alpha = \omega \cdot v$ with $v \in 1+\mathfrak{p}_{F_\alpha}$ satisfying $v/\sigma(v) = u$ and $\omega \in F_\alpha^\times$ a uniformizer with $\sigma(\omega) = -\omega$. Then $\epsilon_\alpha(\delta^\alpha) = f_{(G,T)}(\alpha)$. ■

Fact 4.7.5. The character $\epsilon_{f,\mathrm{ram}}$ is $N(T,G)(F)$-invariant. □

Proof. This follows from the $N(T,G)(F)$-invariance of the function $f_{(G,T)}$. ■

Corollary 4.7.6. The product $\epsilon_{\mathrm{sym,ram}}(\gamma_{<r}^g)\tilde{e}(\gamma_{<r}^g)$ is equal to

$$\frac{\epsilon_{f,\mathrm{ram}}^{G^d}(\gamma_{<r}^g)e(G^d)e(J^d)}{\epsilon_{f,\mathrm{ram}}^{G^{d-1}}(\gamma_{<r}^g)e(G^{d-1})e(J^{d-1})} \frac{\epsilon_L(X^*(T_{G^d})_{\mathbb{C}} - X^*(T_{J^d})_{\mathbb{C}}, \Lambda)}{\epsilon_L(X^*(T_{G^{d-1}})_{\mathbb{C}} - X^*(T_{J^{d-1}})_{\mathbb{C}}, \Lambda)} \cdot \frac{\Delta_{II}^{\mathrm{abs},G^d}[a,\chi'](\gamma_{<r}^g)}{\Delta_{II}^{\mathrm{abs},G^{d-1}}[a,\chi'](\gamma_{<r}^g)}.$$

□

Before going further, it will be useful to express the a-data $a_\alpha = \langle H_\alpha, X_{d-1}^* \rangle$ in a way that does not reference the structure of the p-adic group G. In fact, since we are using tame χ-data, it will be enough to specify mod-a-data. For this, we consider the character

$$F_\alpha^\times \to \mathbb{C}^\times, \quad x \mapsto \phi_{d-1}(N_{F_\alpha/F}(\alpha^\vee(x))),$$

where $N_{F_\alpha/F} : T(F_\alpha) \to T(F)$ is the norm map. According to Lemma 3.6.8, restriction to $[F_\alpha^\times]_r$ provides a non-trivial character $[F_\alpha^\times]_r/[F_\alpha^\times]_{r+} \to \mathbb{C}^\times$. At the same time, we have the character

$$\Lambda \circ \mathrm{tr}_{F_\alpha/F} : F_\alpha \to \mathbb{C}^\times,$$

which factors through $[F_\alpha]_0/[F_\alpha]_{0+}$. We have the isomorphism

$$X \mapsto X+1 : [F_\alpha]_r/[F_\alpha]_{r+} \to [F_\alpha^\times]_r/[F_\alpha^\times]_{r+},$$

which is a truncated version of the exponential map. The equation

$$\phi_{d-1}(N_{F_\alpha/F}(\alpha^\vee(X+1))) = \Lambda(\mathrm{tr}_{F_\alpha/F}(\bar{a}_\alpha X)), \tag{4.7.3}$$

characterizes the image $\bar{a}_\alpha$ of $\langle H_\alpha, X_{d-1}^* \rangle$ in $[F_\alpha]_{-r}/[F_\alpha]_{-r+}$.

Fact 4.7.7. The a-data $a_\alpha = \langle H_\alpha, X_{d-1}^* \rangle$, and hence the χ-data χ', are invariant under the action of $\Omega(T, G^{d-1})(F)$. □

Proof. For the a-data this follows from the fact that X_{d-1}^* belongs to the dual Lie algebra of the center of G^{d-1}. For the χ-data this follows from its definition (4.7.2). ■

4.8 The characters of toral supercuspidal representations

In this subsection F is a local field of odd residual characteristic that is not a bad prime for G.

The supercuspidal representation π is called toral if it arises from a Yu-datum of the form $((S,G),1,(\phi,1))$, where $G^{d-1}=S$ is an elliptic maximal torus and $\phi=\phi_{d-1}$ is a generic character of $S(F)$ of positive depth. Let r be the depth of ϕ and let $X^*\in\mathrm{Lie}^*(S)(F)_{-r}$ be a generic element realizing $\phi|_{S(F)_r}$. In this special case, the Adler-DeBacker-Spice character formula (4.3.1) applies to all regular semi-simple $\gamma=\gamma_{<r}\cdot\gamma_{\geq r}\in G(F)$, because the compactness assumption is automatically satisfied. Moreover, the formula of Corollary 4.7.6 simplifies, because we have $J^{d-1}=G^{d-1}=S$. Thus we obtain

Corollary 4.8.1. The product $\epsilon_{\text{sym,ram}}(\gamma_{<r}^g)\tilde{e}(\gamma_{<r}^g)$ is equal to

$$\epsilon_{f,\text{ram}}(\gamma_{<r}^g)e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C}-X^*(T_J)_\mathbb{C},\Lambda)\cdot\Delta_{II}^{\text{abs}}[a,\chi'](\gamma_{<r}^g).$$

□

Combining this with (4.3.1), and setting $\theta=\phi_{d-1}:S(F)\to\mathbb{C}^\times$, we arrive at

Corollary 4.8.2. The value of the normalized character Φ_π at the element $\gamma=\gamma_{<r}\cdot\gamma_{\geq r}$ is given as the product

$$e(G)e(J)\epsilon_L(X^*(T_G)_\mathbb{C}-X^*(T_J)_\mathbb{C},\Lambda)$$
$$\cdot\sum_{\substack{g\in J(F)\backslash G(F)/S(F)\\ \gamma_{<r}^g\in S(F)}}\Delta_{II}^{\text{abs}}[a,\chi'](\gamma_{<r}^g)\epsilon_{f,\text{ram}}(\gamma_{<r}^g)\epsilon^{\text{ram}}(\gamma_{<r}^g)\theta(\gamma_{<r}^g)\widehat{\iota}_{\mathfrak{j},{}^gX^*}(\log(\gamma_{\geq r}))$$

□

4.9 Character values of regular supercuspidal representations at shallow elements: depth zero

In this subsection F is a local field of odd residual characteristic.

Lemma 4.9.1. Let S be a maximally unramified maximal torus of G and let T be a minimal Levi subgroup of the quasi-split inner form of G. If $\Lambda:F\to\mathbb{C}^\times$ is a character of depth zero, then

$$\epsilon(X^*(S)_\mathbb{C}-X^*(T)_\mathbb{C},\Lambda)=(-1)^{r_S-r_T},$$

where r_S and r_T are the split ranks of S and T respectively. □

Proof. By Lemma 3.2.2 the torus S transfers to the quasi-split inner form of G. The statement we are proving is invariant under replacing G by its quasi-split inner form, so we may now assume G is quasi-split. We may also assume that G is simply connected. Let $o\in\mathcal{A}^{\text{red}}(T,F)$ be the superspecial vertex associated to a Γ-invariant pinning. By Lemma 3.4.12 we may further replace S by a stable conjugate so that o is the unique point in $\mathcal{B}^{\text{red}}(G,F)\cap\mathcal{A}^{\text{red}}(S,F^u)$, and moreover so that S and T are conjugate under $G(F^u)_{o,0}$.

We will use Kottwitz's result [Kal15, Corollary 4.11] to compute the left hand side. It is the formula

$$\epsilon(X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}, \Lambda) = \prod_{\alpha \in R(S,G)_{\text{sym}}/\Gamma} f_{(G,S)}(\alpha)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \text{tr}_{F_{\pm\alpha}/F}).$$

Here $f_{(G,S)}$ is the toral invariant of $S \subset G$ [Kal15, §4], and $\lambda_{F_\alpha/F_{\pm\alpha}}$ is the Langlands constant [BH05b, §1.5]. Since S is maximally unramified, each quadratic extension $F_\alpha/F_{\pm\alpha}$ is unramified, so by [BH05b, Lemma 1.5] the corresponding Langlands constant is -1. Moreover, according to Proposition 4.5.1, $f_{(G,S)}(\alpha) = +1$ if and only if $0 \in \text{ord}_o(\alpha)$. According to [DS18, Remark 3.1.4] we may extend scalars to F^u before computing $\text{ord}_o(\alpha)$. Since S and T are conjugate under $G(F^u)_{o,0}$ we may thus replace S by T in the computation of $\text{ord}_o(\alpha)$.

We are now interested in the question of whether there is an element of $\mathfrak{g}(E)_\alpha$ whose valuation with respect to o is zero, and which is fixed by the action of $\text{Gal}(E/F^u_\alpha)$, where E/F^u is the splitting field of T. Consider the simple component of the root system $R(T,G)$ to which α belongs. Then $\text{Gal}(E/F^u_\alpha)$ is a cyclic group preserving this component and acting on it by a pinned automorphism that fixes the root α. Let's call this automorphism θ. It preserves the line $\mathfrak{g}_\alpha(E)$. The fixed pinning provides a pair of elements $\{X, -X\} \subset \mathfrak{g}_\alpha(E)$. Both of these elements have o-valuation equal to 0. Now $\theta(X) = \zeta \cdot X$, where $\zeta \in F^{u,\times}$ is a root of unity of order divisible by the order of θ. A direct examination of the simple root systems shows that $\zeta = 1$ unless α belongs to a simple component of type A_{2n} and θ is the non-trivial pinned automorphism, in which case $\zeta = -1$. In the case $\zeta = 1$ we have $X \in \mathfrak{g}_\alpha(F^u_\alpha)$ and hence $0 \in \text{ord}_x(\alpha)$. In the case $\zeta = -1$, let $\varpi \in E$ be a square root of a uniformizing element of F^u, then $\varpi X \in \mathfrak{g}_\alpha(F^u_\alpha)$ and hence $0 \notin \text{ord}_o(\alpha)$.

Returning to the original torus S, we can interpret this as follows. Let $S' \subset S$ be the maximal unramified subtorus. Let $R(S',G)$ be the corresponding relative root system. It need not be reduced. The fibers of the map $R(S,G) \to R(S',G)$ induced by the inclusion $S' \to S$ are precisely the inertial orbits in $R(S,G)$. This map then sets up a bijection between the Γ-orbits in $R(S,G)$ and the Frobenius-orbits in $R(S',G)$ and this bijection restricts to a bijection between the symmetric orbits. A root $\alpha \in R(S,G)$ restricts to a divisible root in $R(S',G)$ if and only if $0 \notin \text{ord}_o(\alpha)$. If such a root is symmetric, we have $f_{(G,S)}(\alpha)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \text{tr}_{F_{\pm\alpha}/F}) = 1$, because both factors are equal to -1. For any other symmetric root, we have $f_{(G,S)}(\alpha)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \text{tr}_{F_{\pm\alpha}/F}) = -1$, because the first factor is equal to 1 and the second is equal to -1. With this, Kottwitz's formula above becomes

$$\epsilon(X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}, \Lambda) = (-1)^{\#R(S',G)_{\text{nd,sym}}/\text{Fr}},$$

where the subscript "nd" denotes the set of non-divisible roots. On the other hand, we have

$$(-1)^{r_S - r_T} = (-1)^{\dim([X^*(S)_{\mathbb{C}}]^I)^{\text{Fr}} - \dim([X^*(T)_{\mathbb{C}}]^I)^{\text{Fr}}} = (-1)^{\dim[X^*(S)_{\mathbb{C}}]_I^{\text{Fr}} - \dim[X^*(T)_{\mathbb{C}}]_I^{\text{Fr}}}.$$

Since S is maximally unramified, the I-modules $X_*(S)_{\mathbb{C}}$ and $X_*(T)_{\mathbb{C}}$ are equal. Moreover, the action of Frobenius on $[X_*(S)_{\mathbb{C}}]_I$ is the twist of the action of Frobenius on $[X_*(T)_{\mathbb{C}}]_I$ by an unramified 1-cocycle w_σ. Letting $V = [X_*(T)_{\mathbb{C}}]_I$, ϕ be the automorphism of V by which Frobenius acts, and $w \in \Omega(T,G)$ the value of w_σ at the Frobenius element, we get

$$(-1)^{r_S - r_T} = (-1)^{\dim V^\phi - \dim V^{w\phi}}.$$

Both ϕ and w are of finite order and preserve a $\mathbb{Q}$-structure on V, so their eigenvalues are either $+1$, -1, or pairs of conjugate non-real roots of unity. From this we see

$$(-1)^{r_S - r_T} = \det(\phi|V)^{-1}\det(w\phi|V) = \det(w|V).$$

Note that V is the vector space in which the root system $R(S', G)_{\text{nd}}$ resides, and in fact is spanned by that root system, because $X^*(T)_{\mathbb{C}}$ is spanned by $R(T, G)$. Thus $\det(w|V) = (-1)^{l(w)}$. According to the argument of [Kal11, Lemma 4.0.7], $(-1)^{l(w)}$ is equal to $(-1)^N$, where N is the number of symmetric Frobenius orbits in $R(S', G)_{\text{nd}}$. ■

For a maximally unramified maximal torus $S \subset G$ all symmetric roots in $R(S, G)$ are unramified. We can thus fix unramified χ-data for $R(S, G)$ and we can fix mod-a-data consisting of units, i.e. non-zero elements of $[F_\alpha]_0/[F_\alpha]_{0+}$.

Proposition 4.9.2. Let $S \subset G$ be a maximally unramified maximal torus, $\theta : S(F) \to \mathbb{C}^\times$ a regular depth-zero character, and $\pi_{(S,\theta)}$ the corresponding regular depth-zero supercuspidal representation as in §3.4. If $\gamma \in G(F)$ is a regular topologically semi-simple element belonging to an elliptic maximally unramified maximal torus, then the character of $\pi_{(S,\theta)}$ at γ is zero, unless γ is (conjugate to) an element of $S(F)$, in which case it is given by

$$e(G)\epsilon(X^*(T)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\text{abs}}[\bar{a}, \chi](\gamma^w)\theta(\gamma^w),$$

where χ is unramified χ-data and $\bar{a}$ is any mod-a-data consisting of units. □

Proof. Recall from Lemma 3.4.20 that $\pi_{(S,\theta)} = \text{c-Ind}_{S(F)G(F)_{x,0}}^{G(F)} \kappa_{(S,\theta)}$ and that $x \in \mathcal{B}^{\text{red}}(G, F)$ is the point associated to the torus S. We will perform this induction in stages, where we let $\dot{\kappa} = \dot{\kappa}_{(S,\theta)}$ be the induction of $\kappa_{(S,\theta)}$ to $G(F)_x$. Since $S(F)G(F)_{x,0}$ is a subgroup of $G(F)_x$ of finite index, $\dot{\kappa}$ is still finite dimensional. We compute the character of $\pi_{(S,\theta)}$ in terms of that of $\dot{\kappa}$ by means of Harish-Chandra's integral character formula [DR09, §9.1] and we obtain

$$\frac{\deg(\pi; dg/dz)}{\deg(\dot{\kappa})} \int_{G(F)/Z(F)} \int_K \dot{\chi}_{\dot{\kappa}}({}^{gk}\gamma) dk dg/dz$$

where K is any compact open subgroup of $G(F)$ with Haar measure dk of normalized volume 1, Z is the center of G, and $\dot{\chi}_{\dot{\kappa}}$ is the extension by zero of the character function $\chi_{\dot{\kappa}}$ of $\dot{\kappa}$. Just as in §4.4 we can argue that the function $g \mapsto \dot{\chi}_{\dot{\kappa}}({}^{gk}\gamma)$ is compactly supported modulo center and thus remove the integral over K, which leads us to

$$\frac{\deg(\pi; dg/dz)}{\deg(\dot{\kappa})} \int_{G(F)/Z(F)} \dot{\chi}_{\dot{\kappa}}({}^{g}\gamma) dg/dz$$

The integrand is zero unless $gx_\gamma = x$, where x_γ is the unique fixed point of γ in $\mathcal{B}^{\text{red}}(G, F)$. Thus if γ is not conjugate to an element of $G(F)_x$ the character is zero. Assume now $\gamma \in G(F)_x$. Then the domain of integration reduces to $G(F)_x/Z(F)$ and since the integrand is $G(F)_x$-invariant we obtain

$$\text{vol}(G(F)_x/Z(F); dg/dz)\deg(\pi; dg/dz)\deg(\dot{\kappa})^{-1}\chi_{\dot{\kappa}}(\gamma),$$

which is equal to $\chi_{\dot{\kappa}}(\gamma)$. We compute this using the Frobenius formula and obtain

$$\sum_{[g] \in G(F)_x/[G(F)_{x,0} \cdot S(F)]} \text{tr}(\kappa_{(S,\theta)}(g^{-1}\gamma g)).$$

Note that $G(F)_{x,0} \cdot S(F)$ is a normal subgroup of $G(F)_x$, so the element $g\gamma g^{-1}$ lies in $G(F)_{x,0} \cdot S(F)$ and is regular topologically semi-simple. Corollary 3.4.26 implies that the summand $\mathrm{tr}(\kappa_{(S,\theta)}(g^{-1}\gamma g))$ is zero unless $g^{-1}\gamma g$ is $G(F)_{x,0}$-conjugate to an element of $S(F)$. If γ is not $G(F)$-conjugate to an element of $S(F)$ then all summands in the sum are zero and hence the vanishing statement of the proposition is proved.

Assume now that $\gamma \in S(F)$ and consider again the above formula. Let $[g] \in G(F)_x/[G(F)_{x,0} \cdot S(F)]$ be a coset giving a non-zero contribution to the sum. As we have just argued, this coset can be represented by $g \in G(F)_x$ such that $g^{-1}\gamma g \in S(F)$. Since γ is regular semi-simple we see $g \in N(S,G)(F)$. Thus $[g]$ lies in the subset $N(S,G)(F)/[N(S,G(F)_{x,0}) \cdot S(F)]$ of $G(F)_x/[G(F)_{x,0} \cdot S(F)]$. Lemma 3.4.10 and Corollary 3.4.26 imply that the character of $\pi_{(S,\theta)}$ at γ is given by

$$(-1)^{r_G - r_T}(-1)^{r_T - r_S} \sum_{w \in N(S,G)(F)/S(F)} \theta(\gamma^w),$$

where again T is the minimal Levi subgroup of the quasi-split inner form of G. We have $(-1)^{r_T - r_S} = \epsilon(X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}, \Lambda)$ from Lemma 4.9.1 and $(-1)^{r_G - r_T} = e(G)$ by [Kot83]. It remains to check that $\Delta_{II}^{\mathrm{abs}}[\bar{a}, \chi](\gamma^w) = 1$. The $F_\alpha/F_{\pm\alpha}$-norm of $\alpha(\gamma^w) \in F_\alpha^\times$ is equal to 1, so $\alpha(\gamma^w) \in O_{F_\alpha}^\times$. Moreover, since γ is regular and topologically semi-simple, $\alpha(\gamma^w) \notin 1 + \mathfrak{p}_{F_\alpha}$, and therefore $\alpha(\gamma^w) - 1 \in O_{F_\alpha}^\times$. Since the mod-$a$-data consists of units and the χ-data is unramified, the claim follows. ∎

Remark 4.9.3. We close this subsection with a remark about the characters of extra regular depth-zero supercuspidal representations of groups that split over F^u. These are the representations constructed in [DR09, §4.4]. DeBacker and Reeder compute in [DR09, §9,10,11,12] the characters of these representations at arbitrary regular semi-simple elements: For an element $\gamma \in G_{\mathrm{sr}}(F)_0$ with topological Jordan decomposition $\gamma = \gamma_s \cdot \gamma_u$ the character of $\pi_{(S,\theta)}$ is given by

$$(-1)^{r_G - r_J} \sum_{\substack{g \in J(F)\backslash G(F)/S(F) \\ \gamma_s^g \in S(F)}} \theta(\gamma_s^g)\widehat{\mu}_{\mathfrak{j},{}^gX}(\log(\gamma_u)),$$

where again J is the connected centralizer of γ_s in G and r_G denotes the split rank of the group G. The final paragraph of the preceding proof shows that this formula is the same as the formula of Corollary 4.8.2. We expect that the same is true for tamely ramified groups as well. □

4.10 Character values of regular supercuspidal representations at shallow elements: general depth

In this subsection F is a local field of odd residual characteristic that is not a bad prime for G.

Consider a regular supercuspidal representation $\pi_{(S,\theta)}$. Let $\tilde{G} \to G$ be a z-extension and let $\pi_{(\tilde{S},\tilde{\theta})}$ be the pull-back of $\pi_{(S,\theta)}$ to $\tilde{G}(F)$. Since the character function of $\pi_{(\tilde{S},\tilde{\theta})}$ is the pull-back to $\tilde{G}(F)$ of the character function of $\pi_{(S,\theta)}$, we may assume without loss of generality that $G = \tilde{G}$.

Let $G^0 \subsetneq \cdots \subsetneq G^d$ be the corresponding twisted Levi sequence, $(\phi_{-1}, \dots, \phi_d)$ a Howe factorization, and $(r_{-1}, r_0, \dots, r_d)$ the sequence of depths of the characters ϕ_i. Let $\gamma \in G(F)$ be regular semi-simple. If $S \neq G^0$ we will call γ

"shallow" if it is a topologically semi-simple element $\gamma = \gamma_0$. If $S = G^0$ we will call γ "shallow" if it is topologically semi-simple modulo $Z(G)^\circ$, although we believe that in this case it is enough to require $\gamma = \gamma_{<r_0}$.

We will now fix mod-a-data and χ-data for $R(S,G)$. We do this successively for $R(S,G^i) \smallsetminus R(S,G^{i-1})$ as described in §4.7, where i runs from d to 1. We also need to handle the step $i=0$ in the case where $G^0 \neq S$. Let us first focus on $i > 0$. Fix a character $\Lambda : F \to \mathbb{C}^\times$ of depth zero. For any $\alpha \in R(S,G^i) \smallsetminus R(S,G^{i-1})$ the equation

$$\phi_{i-1}(N_{F_\alpha/F}(\alpha^\vee(X+1))) = \Lambda(\mathrm{tr}_{F_\alpha/F}(\bar{a}_\alpha X)),$$

in the variable $X \in [F_\alpha]_{r_{i-1}}/[F_\alpha]_{r_{i-1}+}$ specifies $\bar{a}_\alpha \in [F_\alpha]_{-r_{i-1}}/[F_\alpha]_{-r_{i-1}+}$. In this way we obtain a set of mod-a-data for $R(S,G^i) \smallsetminus R(S,G^{i-1})$; it is the same as the one of (4.7.3). Applying Fact 3.6.4 to the Howe factorization $(\phi_{-1},\dots,\phi_d)$ of θ we see that this equation is equivalent to

$$\theta(N_{F_\alpha/F}(\alpha^\vee(X+1))) = \Lambda(\mathrm{tr}_{F_\alpha/F}(\bar{a}_\alpha X)). \tag{4.10.1}$$

We take this as the defining equation for the mod-a-data, as it clearly demonstrates that this data depends only on θ and not on the Howe factorization. From the mod-a-data we obtain χ-data χ' via (4.7.2). Now consider the case $i = 0$, which is only relevant when $G^0 \neq S$. We can still take (4.10.1) as the defining equation for mod-a-data and we obtain $\bar{a}_\alpha \in [F_\alpha]_0/[F_\alpha]_{0+}$. On the other hand, we take χ'_α to be unramified. This is possible because the action of inertia preserves a base in $R(S,G^0)$ so all symmetric roots are unramified. With the mod-a-data and χ-data fixed this way, we have the following formula.

Corollary 4.10.1. Let $\gamma \in G(F)$ be a shallow regular semi-simple element. The value of the normalized character Φ_π at γ is zero, unless γ is ($G(F)$-conjugate to) an element of $S(F)$, in which case is given by

$$e(G)\epsilon_L(X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\mathrm{abs}}[a,\chi'](\gamma^w)\epsilon_{f,\mathrm{ram}}(\gamma^w)\epsilon^{\mathrm{ram}}(\gamma^w)\theta(\gamma^w),$$

where T is a minimal Levi subgroup in the quasi-split inner form of G. □

Proof. We write (4.4.1) as

$$\phi_d(\gamma) \sum_{\substack{g \in G^d_\gamma(F)\backslash G^d(F)/G^{d-1}(F) \\ \gamma^g \in G^{d-1}(F)}} \epsilon(\pi_{d-1},\gamma^g)\Phi_{\pi_{d-1}}(\gamma^g).$$

Here we use the notation G^d_γ in place of J^d for the connected centralizer of γ in G^d, so that we can keep track of the element γ. We have combined all three roots of unity into the single term $\epsilon(\pi_{d-1},\gamma^g)$, and we have included π_{d-1} into the notation of this term. We are now going to unwind the induction inherent in this formula. To see what is going on we substitute the formula for $\Phi_{\pi_{d-1}}$ and obtain

$$\phi_d(\gamma) \sum_{\substack{g \in G^d_\gamma(F)\backslash G^d(F)/G^{d-1}(F) \\ \gamma^g \in G^{d-1}(F)}} \epsilon(\pi_{d-1},\gamma^g)\phi_{d-1}(\gamma^g) \sum_{\substack{h \in G^{d-1}_{\gamma^g}(F)\backslash G^{d-1}(F)/G^{d-2}(F) \\ \gamma^{gh} \in G^{d-2}(F)}} \epsilon(\pi_{d-2},\gamma^{gh})\Phi_{\pi_{d-2}}(\gamma^{gh}).$$

Recall [DS18, Remark 4.3.5] that the term $\epsilon(\pi_{d-1},\gamma^g)$ remains unchanged if we conjugate both π_{d-1} and γ^g by an element of $G^d(F)$. If this element happens to belong to $G^{d-1}(F)$, then π_{d-1} remains unchanged. With this we obtain

$$\sum_g \sum_h \phi_d(\gamma^{gh})\phi_{d-1}(\gamma^{gh})\epsilon(\pi_{d-1},\gamma^{gh})\epsilon(\pi_{d-2},\gamma^{gh})\Phi_{\pi_{d-2}}(\gamma^{gh}),$$

where the summation indices are as before, and after re-indexing the sum this leads to

$$\sum_{\substack{g\in G^d_\gamma(F)\backslash G^d(F)/G^{d-2}(F)\\ \gamma^g\in G^{d-2}(F)}} \phi_d(\gamma^g)\phi_{d-1}(\gamma^g)\epsilon(\pi_{d-1},\gamma^g)\epsilon(\pi_{d-2},\gamma^g)\Phi_{\pi_{d-2}}(\gamma^g).$$

We have used here the equality $G^d_{\gamma^g}\cap G^{d-1}=G^{d-1}_{\gamma^g}$, which is implied by the stronger statement $G^d_{\gamma^g}\subset G^{d-1}$ coming from the regularity of γ. We do this inductively, where at the (-1)-stage we apply Proposition 4.9.2 if $G^0\neq S$, and obtain the formula

$$\sum_{\substack{g\in G^d_\gamma(F)\backslash G^d(F)/S(F)\\ \gamma^g\in S(F)}} \prod_{i=-1}^{d}\phi_i(\gamma^g)\epsilon(\pi_i,\gamma^g).$$

In particular, we see that the result is zero unless γ is $G(F)$-conjugate to an element of $S(F)$. We can thus assume that $\gamma\in S(F)$ and then $G^d_\gamma=S$, so the summation index becomes $g\in N(S,G)(F)/S(F)$.

We now go into the roots of unity $\epsilon(\pi_i,\gamma^g)$. Their definition depends on the choice of a tame maximal torus T containing γ^g. In the current situation we have a canonical choice for T, namely $T=S$. We now apply Corollary 4.7.6 using the mod-a-data and χ-data fixed in (4.10.1) and (4.7.2). Recalling (3.6.2) that θ is the product of all ϕ_i restricted to $S(F)$ and letting ϵ^{ram} be the product of all $\epsilon^{\mathrm{ram}}(\pi_i,-)$ of (4.3.3), the proof is complete. ■

It is noted in [DS18] that the map $\gamma\mapsto\epsilon^{\mathrm{ram}}(\gamma)$ is a character of $S(F)$. If we let θ' be the character $\epsilon_{f,\mathrm{ram}}\cdot\epsilon^{\mathrm{ram}}\cdot\theta$, then the character formula takes the form

$$e(G)\epsilon_L(X^*(T)_{\mathbb{C}}-X^*(S)_{\mathbb{C}},\Lambda)\sum_{w\in N(S,G)(F)/S(F)}\Delta_{II}^{\mathrm{abs}}[a,\chi'](\gamma^w)\theta'(\gamma^w). \quad (4.10.2)$$

4.11 Comparison with the characters of real discrete series representations

In this subsection only, we let G be a connected reductive group defined over $\mathbb{R}$ and having a discrete series of representations, or equivalently having elliptic maximal tori. All elliptic maximal tori in G are conjugate under $G(\mathbb{R})$. Fix one such $S\subset G$. We also fix an element $i\in\mathbb{C}$ with $i^2=-1$.

Let $\theta: S(\mathbb{R})\to\mathbb{C}^\times$ be a character. Its differential at 1 is a homomorphism $\mathrm{Lie}(S(\mathbb{R}))\to\mathbb{C}$ of $\mathbb{R}$-vector spaces and gives rise to a $\mathbb{C}$-linear form $\mathrm{Lie}(S(\mathbb{R}))\otimes\mathbb{C}\to\mathbb{C}$. Now $\mathrm{Lie}(S(\mathbb{R}))\otimes\mathbb{C}=\mathrm{Lie}(S(\mathbb{C}))=X_*(S)\otimes\mathbb{C}$ and hence $d\theta\in X^*(S)\otimes\mathbb{C}$. Since every character of an anisotropic real torus is algebraic, we see that the image of $d\theta$ in $X^*(S_{\mathrm{sc}})\otimes\mathbb{C}$, which is the differential of the restriction of θ to $S_{\mathrm{sc}}(\mathbb{R})$, lies in the sublattice $X^*(S_{\mathrm{sc}})$ of $X^*(S_{\mathrm{sc}})\otimes\mathbb{C}$. We may thus ask whether $d\theta$ is dominant for a given choice of positive roots in $R(S,G)$.

The discrete series representations of $G(\mathbb{R})$ are parameterized by pairs (θ,ρ) (taken up to conjugation by $N(S,G)(\mathbb{R})/S(\mathbb{R})$) consisting of a character $\theta: S(\mathbb{R})\to\mathbb{C}^\times$ and a choice of positive roots ρ for $R(S,G)$ for which $d\theta$ is dominant. Given such a pair (θ,ρ) the corresponding representation is characterized by the fact that the value of its character on any regular $\gamma \in S(\mathbb{R})$ is given by

$$(-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\theta(\gamma^w)}{\prod_{\alpha>0}(1-\alpha(\gamma^w)^{-1})}, \tag{4.11.1}$$

where $q(G)$ is half of the dimension of the symmetric space of $G(\mathbb{R})$.

We now claim that this formula is the same as (4.10.2) specialized to the case $F = \mathbb{R}$. The latter formula involves a non-trivial character $\Lambda : \mathbb{R} \to \mathbb{C}^\times$, but is independent of the choice. We choose here the standard character $\Lambda(x) = \exp(2\pi i x)$. It also involves a-data, which is to be computed based on Λ and θ according to (4.10.1), namely

$$\theta(N_{\mathbb{C}/\mathbb{R}}(\alpha^\vee(\exp(z)))) = \Lambda(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(a_\alpha z))$$

for $z \in \mathbb{C}$, keeping in mind that all elements of $R(S,G)$ are symmetric, with $F_\alpha = \mathbb{C}$ and $F_{\pm\alpha} = \mathbb{R}$, because complex conjugation acts by -1 on the root system of S. Evaluating this formula we find

$$\theta(N_{\mathbb{C}/\mathbb{R}}(\alpha^\vee(\exp(z)))) = \theta(\alpha^\vee(e^{z-\bar{z}})) = e^{(z-\bar{z})\langle\alpha^\vee, d\theta\rangle},$$

while at the same time $\Lambda(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(a_\alpha z)) = e^{2\pi i(a_\alpha z + \bar{a}_\alpha \bar{z})}$. This implies

$$a_\alpha = \frac{\langle\alpha^\vee, d\theta\rangle}{2\pi i}.$$

Finally, we need to choose χ-data, and we take $\chi_\alpha(z) = \mathrm{sgn}_{\mathbb{C}}(z)$ for $z \in \mathbb{C}^\times$ and $\alpha > 0$, where $\mathrm{sgn}_{\mathbb{C}} : \mathbb{C}^\times \to \mathbb{S}^1$ denotes the argument function. We will discuss the significance of this choice at the end of this subsection.

Having made these preparations we now explicate (4.10.2) in this setting. First, we use the real case of [Kal15, Corollary 4.11], which gives us

$$e(G)\epsilon_L(X^*(T)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) = \prod_{\alpha \in R(S,G)/\Gamma} f_{(G,S)}(\alpha)\lambda_{\mathbb{C}/\mathbb{R}}(\Lambda \circ \mathrm{tr}_{\mathbb{C}/\mathbb{R}})^{-1}.$$

Now $\lambda_{\mathbb{C}/\mathbb{R}}(\Lambda \circ \mathrm{tr}_{\mathbb{C}/\mathbb{R}}) = i$ and $f_{(G,S)}(\alpha)$ equals -1 if α is compact and $+1$ if α is non-compact. It follows that

$$e(G)\epsilon_L(X^*(T)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) = (-1)^{q(G)} \prod_{\alpha \in R(S,G)/\Gamma} i.$$

On the other hand we have

$$\Delta_{II}^{\mathrm{abs}}[a,\chi](\gamma) = \prod_{\alpha<0} \mathrm{sgn}_{\mathbb{C}}\left(\frac{\alpha(\gamma)-1}{\langle\alpha^\vee, d\theta\rangle(2\pi i)^{-1}}\right)^{-1} = \prod_{\alpha<0}(-i) \cdot \prod_{\alpha<0} \mathrm{sgn}_{\mathbb{C}}\left(\frac{\alpha(\gamma)-1}{\langle\alpha^\vee, d\theta\rangle}\right)^{-1}.$$

Recall that $d\theta$ is dominant for the chosen set of positive roots, so $\langle\alpha^\vee, d\theta\rangle < 0$ whenever $\alpha < 0$, leading to

$$\mathrm{sgn}_{\mathbb{C}}\left(\frac{\alpha(\gamma)-1}{\langle\alpha^\vee, d\theta\rangle}\right)^{-1} = \mathrm{sgn}_{\mathbb{C}}(1-(-\alpha)(\gamma)^{-1})^{-1}.$$

At the same time

$$\begin{aligned}
|D(\gamma)|^{\frac{1}{2}} &= \prod_{\alpha\in R(S,G)} |\alpha(\gamma)-1|^{\frac{1}{2}} \\
&= \left(\prod_{\alpha>0} |\alpha(\gamma)-1||\alpha(\gamma)^{-1}-1|\right)^{\frac{1}{2}} \\
&= \left(\prod_{\alpha>0} |\alpha(\gamma)^{\frac{1}{2}}-\alpha(\gamma)^{-\frac{1}{2}}||\alpha(\gamma)^{-\frac{1}{2}}-\alpha(\gamma)^{\frac{1}{2}}|\right)^{\frac{1}{2}} \\
&= \prod_{\alpha>0} |\alpha(\gamma)^{\frac{1}{2}}-\alpha(\gamma)^{-\frac{1}{2}}| \\
&= \prod_{\alpha>0} |1-\alpha(\gamma)^{-1}|
\end{aligned}$$

where the last equality follows form the fact that $\alpha(\gamma)$ is of absolute value 1. Combining these calculations we see that

$$e(G)\epsilon_L(X^*(T_0)_{\mathbb{C}}-X^*(S)_{\mathbb{C}},\Lambda)|D_G(\gamma)|^{-\frac{1}{2}}\Delta_{II}^{\text{abs}}[a,\chi'](\gamma)$$

is equal to

$$(-1)^{q(G)}\prod_{\alpha>0}(1-\alpha(\gamma)^{-1})^{-1}$$

and we conclude that the formula (4.10.2), interpreted for the ground field $\mathbb{R}$, evaluates to

$$(-1)^{q(G)}\sum_{w\in N(S,G)(\mathbb{R})/S(\mathbb{R})}\frac{\theta'(\gamma^w)}{\prod\limits_{\alpha>0}(1-\alpha(\gamma^w)^{-1})},$$

which is indeed the character formula for a discrete series representation (4.11.1).

Finally, we make a comment on the choice of χ-data used here. The choice we have just used is well-known by the name of "based" χ-data from the work of Shelstad [She08a, §9], and is intimately connected with the local Langlands correspondence. In the p-adic case, our choice was made so as to encode the roots of unity that occur in the character formula of Adler-DeBacker-Spice. If we speculatively view $\mathbb{C}/\mathbb{R}$ as an analog of a ramified quadratic extension of non-archimedean local fields and apply formula (4.7.2), we would get the inverse of the based χ-data used here. So it appears that the real and p-adic case are very closely related.

The particular choice of χ-data is however of minor importance. Indeed, making a different choice of χ-data has the effect of multiplying the term Δ_{II}^{abs} by a character of $S(F)$. This character can then be absorbed into θ, and so can be the characters $\epsilon_{f,\text{ram}}$ and ϵ^{ram}. In fact, the particular representation of $G(F)$ that the character θ of $S(F)$ leads to depends on the details of the construction that is used, and different constructions could lead to slightly different representations. What is important for us here is that the formula for the character of regular supercuspidal representations at shallow elements has the same structure, including the roots of unity that cannot be absorbed into a character of $S(F)$, as the character formula for real discrete series. This fact will be our guide to the construction and study of L-packets in what follows.

Let G be a connected reductive group defined and quasi-split over F and split over a tame extension of F. Let $\widehat{G}$ be a Langlands dual group for G and ${}^LG = \widehat{G} \rtimes W_F$ the Weil-form of the corresponding L-group.

In this section we are going to construct those L-packets of all inner forms of G that consist entirely of regular supercuspidal representations and assign to each such L-packet a Langlands parameter. The construction will allow an explicit passage from parameters to representations and conversely. Each of the L-packets will contain extra regular supercuspidal representations, which is the reason for their name.

We will eventually assume that the residual characteristic p of F is not a bad prime for G and does not divide $|\pi_0(Z(G))|$. Note that the bad primes for G and $\widehat{G}$ are the same, and that $\pi_0(Z(G))$ has the same order as the fundamental group of the derived subgroup of $\widehat{G}$.

5.1 Admissible embeddings

We recall here some basic facts about the relationship between G and $\widehat{G}$. For this, F can be any field, but $\widehat{G}$ is taken over $\mathbb{C}$.

Let S be a torus defined over F of dimension equal to the rank of G, and let J be a Γ-stable $G(\bar{F})$-conjugacy class of embeddings $j : S \to G$ defined over $\bar{F}$. From J we obtain a Γ-stable $\widehat{G}$-conjugacy class $\widehat{J}$ of embeddings $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ as follows. Fix Γ-invariant pinnings $(T, B, \{X_\alpha\})$ of G and $(\widehat{T}, \widehat{B}, \{Y_{\widehat{\alpha}}\})$ of $\widehat{G}$. Any $j \in J$ embeds S as a maximal torus of G, so we may choose $j \in J$ such that $j(S) = T$ and define $\widehat{\jmath}$ to be the inverse of the isomorphism $\widehat{T} \to \widehat{S}$ of complex tori induced by j. Then the $\widehat{G}$-conjugacy class $\widehat{J}$ of $\widehat{\jmath}$ is Γ-stable. Indeed, $w : \sigma \mapsto j \circ \sigma(j^{-1})$ is an element of $Z^1(\Gamma, \Omega(T, G))$, which under the isomorphism $\Omega(T, G) \cong \Omega(\widehat{T}, \widehat{G})$ corresponds to an element $\widehat{w} \in Z^1(\Gamma, \Omega(\widehat{T}, \widehat{G}))$, and we have $\widehat{\jmath} \circ \sigma(\widehat{\jmath}^{-1}) = \widehat{w}_\sigma$. The choice of $j \in J$ can only be altered to $v \circ j$ for some $v \in \Omega(T, G)$, but then $\widehat{\jmath}$ becomes $\widehat{v} \circ \widehat{\jmath}$ and leads to the same $\widehat{J}$. The choices of pinning also have no influence, because any two F-pinnings of G are conjugate by $G_{\mathrm{ad}}(F)$ and any two Γ-stable pinnings of $\widehat{G}$ are conjugate by $\widehat{G}^\Gamma$ [Kot84, Corollary 1.7].

The same procedure can be performed in the opposite direction and produces J from $\widehat{J}$. Since G is quasi-split, there exist Γ-fixed elements $j \in J$ by [Kot82, Corollary 2.2], which applies also in positive characteristic due to [BS68, §8.6].

From J we obtain the following structure on S.

- An embedding $Z(G) \to S$ over F, by choosing $j : S \to T$ and restricting j^{-1} to $Z(G)$;
- A Γ-invariant subset $R(S, G) \subset X^*(S)$, by choosing $j : S \to T$ and pulling back $R(T, G)$ along j;
- A Γ-invariant subgroup $\Omega(S, G) \subset \mathrm{Aut}_{\mathrm{alg.grp}}(S)$, by choosing $j : S \to T$ and pulling back $\Omega(T, G)$.

Again it is clear that this structure depends only on J and not on the choices of pinning of G or $j \in J$. Moreover, if $j \in J$ is Γ-fixed, then it provides Γ-equivariant isomorphisms $R(S,G) \to R(jS,G)$ and $\Omega(S,G) \to \Omega(jS,G)$.

We follow standard terminology and call the embeddings belonging to J *admissible*. More generally, if (G',ξ) is an inner twist of G, we will call an embedding $j : S \to G'$ *admissible* if $\xi^{-1} \circ j \in J$. The notion of admissible does depend on the datum of J. Outside of this subsection, we will not use the symbol J for a conjugacy class of embeddings, but will rather keep it reserved for connected centralizers of semi-simple elements of G. The notion of admissible will be taken with respect to a distinguished conjugacy class of embeddings that will be clear from the context. An example of such a context is given by an endoscopic datum for G. This datum identifies a maximal torus in the dual group of the endoscopic group with a maximal torus of $\widehat{G}$, so we can speak of admissibility of embeddings of a maximal torus of the endoscopic group into G or inner forms of G.

If $j : S \to G'$ is an admissible embedding and S' is its image, we shall call $j : S \to S'$ an admissible isomorphism. Two elements $\gamma \in S$ and $\gamma' \in S'$ are called *related* if there exists an admissible isomorphism $j : S \to S'$ with $j(\gamma) = \gamma'$. There exists exactly one such isomorphism, and we will write $f_{\gamma,\gamma'}$ for it. If γ and γ' are Γ-fixed, then so is $f_{\gamma,\gamma'}$ due to its uniqueness.

5.2 Construction of L-packets

We now introduce the Langlands parameters that correspond to regular supercuspidal L-packets. We will give two definitions – the first one (Definition 5.2.1) is easier to state and describes most of the parameters we need. It also generalizes many of the parameters that have previously been studied. The second definition (Definition 5.2.3) is slightly more general and turns out to be the one that we need.

From now on we assume that the residual characteristic of F is not 2 and is not a bad prime for G. We also assume that the characteristic of F is zero. While this latter assumption is not needed for any of the arguments here, it is assumed in [Kal16], which we will use. We are convinced that the constructions and arguments of [Kal16] are also valid in positive characteristic, so the adventurous reader is encouraged to think about the positive characteristic case as well. Alternatively, if one replaces $H^1(u \to W, -)$ by $B(-)_{\text{bas}}$ of [Kot], this assumption can be dropped, at the expense of possibly not reaching all inner forms.

Definition 5.2.1. A *strongly regular supercuspidal parameter* is a discrete Langlands parameter $\varphi : W_F \to {}^LG$ such that $\varphi(P_F)$ is contained in a torus of $\widehat{G}$ and $\mathrm{Cent}(\varphi(I_F),\widehat{G})$ is abelian. □

Special cases of such parameters are those discussed in [DR09, §4.1], [Roe11], [Ree08, §6.3], and [Kal15, §5.1]. In fact, these examples are special cases of a class of parameters which one might call toral, that is much smaller than the class of strongly regular parameters. The case of positive depth toral parameters is treated in more detail in §6, because they are much easier to deal with and because the current state of the Adler-DeBacker-Spice character formula allows us to obtain additional results for them.

Before coming to the second definition, we collect some basic facts.

Lemma 5.2.2. Let $\varphi : W_F \to {}^L G$ be a Langlands parameter.

1. If $\varphi(P_F)$ is contained in a torus of $\widehat{G}$, then $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$ is a Levi subgroup of $\widehat{G}$. If p does not divide $|\pi_0(Z(G))|$, then $\mathrm{Cent}(\varphi(P_F), \widehat{G})$ is connected.

2. If $\varphi(P_F)$ is contained in a torus of $\widehat{G}$ and $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ is a torus, then $\widehat{T} = \mathrm{Cent}(\widehat{C}, \widehat{M})$ is a maximal torus of $\widehat{G}$ normalized by $\varphi(W_F)$ and contained in a Borel subgroup of $\widehat{M}$ normalized by $\varphi(I_F)$. Furthermore, $\widehat{T}$ is normalized by $\mathrm{Cent}(\varphi(I_F), \widehat{G})$.

□

Proof. By continuity, $\varphi(P_F)$ is a finite p-subgroup of $\widehat{G}$, let $x_1, \dots, x_n$ be its elements. We work by induction on n. By [AS08, Proposition A.7] $\mathrm{Cent}(x_1, \widehat{G})^\circ$ is a Levi subgroup of $\widehat{G}$. Any torus of $\widehat{G}$ containing $x_1, \dots, x_n$ is contained in $\mathrm{Cent}(x_1, \widehat{G})^\circ$. If p does not divide $|\pi_0(Z(G))|$, then it does not divide the order of the fundamental group of $\widehat{G}_{\mathrm{der}}$ and $\mathrm{Cent}(x_1, \widehat{G})$ is connected by [SS70, Corollary 4.6]. Being a Levi subgroup of $\widehat{G}$, the fundamental group of its derived subgroup is a subgroup of the fundamental group of $\widehat{G}_{\mathrm{der}}$. In either case, replace $\widehat{G}$ by $\mathrm{Cent}(x_1, \widehat{G})^\circ$ and proceed with x_2. This proves the first point.

For the second point, we have $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ = \mathrm{Cent}(\varphi(I_F), \widehat{M})^\circ$. The action of I_F on $\widehat{M}$ by $\mathrm{Ad}(\varphi(-))$ restricts trivially to P_F. Since I_F/P_F is pro-cyclic, the centralizer of $\varphi(I_F)$ in $\widehat{M}$ is the fixed-point set of a single automorphism θ of $\widehat{M}$, namely $\mathrm{Ad}(\varphi(x))$, where $x \in I_F$ projects onto a topological generator of I_F/P_F. The automorphism θ is semi-simple (in fact of finite order) and by [Ste68, Theorem 7.5] it preserves a Borel pair of $\widehat{M}$. Let $\widehat{T}$ be the maximal torus in that Borel pair. From [KS99, Theorem 1.1.A] we know that $[\widehat{T} \cap \widehat{C}]^\circ$ is a maximal torus of $\widehat{C}$ and hence must equal $\widehat{C}$, and moreover $\widehat{T} = \mathrm{Cent}(\widehat{C}, \widehat{M})$. Since $\widehat{M}$ is a Levi subgroup of $\widehat{G}$, $\widehat{T}$ is also a maximal torus of $\widehat{G}$. Finally, since both $\widehat{M}$ and $\widehat{C}$ are normalized by $\varphi(W_F)$ as well as by $\mathrm{Cent}(\varphi(I_F), \widehat{G})$, so is $\widehat{T}$. ■

Definition 5.2.3. A *regular supercuspidal parameter* is a discrete Langlands parameter $\varphi : W_F \to {}^L G$ satisfying the following:

1. $\varphi(P_F)$ is contained in a torus of $\widehat{G}$; set $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$.

2. $C := \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ is a torus; let $\widehat{S}$ be the Γ-module with underlying abelian group $\widehat{T} := \mathrm{Cent}(C, \widehat{M})$ and Γ-action given by $\mathrm{Ad}(\varphi(-))$.

3. If $n \in N(\widehat{T}, \widehat{M})$ projects onto a non-trivial element of $\Omega(\widehat{S}, \widehat{M})^\Gamma$, then $n \notin \mathrm{Cent}(\varphi(I_F), \widehat{G})$.

□

We note that if φ is a strongly regular supercuspidal parameter then

$$N(\widehat{T},\widehat{M})\cap\mathrm{Cent}(\varphi(I_F),\widehat{G})\subset\mathrm{Cent}(\varphi(I_F),\widehat{M})\subset\mathrm{Cent}(C,\widehat{M})=\widehat{T},$$

where the second containment follows from the fact that $\mathrm{Cent}(\varphi(I_F),\widehat{M})$ is abelian and contains C, thus φ is regular. Conversely, almost all regular supercuspidal parameters are strongly regular.

Our task in this subsection is to construct for each regular supercuspidal parameter and each inner form of G the corresponding L-packet. The first step is to identify the set of equivalence classes of such parameters with a set of equivalence classes of another kind of data. We find it most convenient to organize this new data into a category, which we call the category of regular supercuspidal L-packet data. The objects in the category will be tuples $(S,\widehat{\jmath},\chi,\theta)$, where

1. S is a torus of dimension equal to the absolute rank of G, defined over F and split over a tame extension of F;
2. $\widehat{\jmath}:\widehat{S}\to\widehat{G}$ is an embedding of complex reductive groups whose $\widehat{G}$-conjugacy class is Γ-stable;
3. χ is minimally ramified χ-data for $R(S,G)$ in the sense of Definition 4.6.1;
4. $\theta:S(F)\to\mathbb{C}^\times$ is a character.

We require of this data that the χ-data be $\Omega(S,G^0)(F)$-invariant, $S/Z(G)$ be anisotropic, and (S,θ) be a tame extra regular elliptic pair in the sense of Definition 3.7.5. Here we are using the structure on S that is given to us by $\widehat{\jmath}$ as described in §5.1, and moreover $\Omega(S,G^0)$ is the subgroup of $\Omega(S,G)$ generated by the reflection along the sub-root system $R_{0+}\subset R(S,G)$ as in Definition 3.7.5.

A morphism $(S,\widehat{\jmath},\chi,\theta)\to(S',\widehat{\jmath}',\chi',\theta')$ is a triple (ι,g,ζ), where $\iota:S\to S'$ is an isomorphism of F-tori, $g\in\widehat{G}$, and $\zeta=(\zeta_{\alpha'})_{\alpha'}$ is a collection of characters $\zeta_{\alpha'}:F_{\alpha'}^\times\to\mathbb{C}^\times$, one for each $\alpha'\in R(S',G)$, satisfying the conditions listed after Lemma 4.6.3. We require that $\widehat{\jmath}\circ\widehat{\iota}=\mathrm{Ad}(g)\circ\widehat{\jmath}'$, that $\chi_{\alpha'\circ\iota}=\chi'_{\alpha'}\cdot\zeta_{\alpha'}$, and that $\zeta_{S'}^{-1}\cdot\theta'\circ\iota=\theta$, where $\zeta_{S'}$ is the character of $S'(F)$ corresponding to ζ as in Definition 4.6.5. Composition of morphisms is defined in the obvious way. Note that every morphism is an isomorphism and that ζ is determined by χ and χ'.

While not needed for the purposes of this paper, the following result might be worth recording.

Lemma 5.2.4. The map $s\mapsto(1,\widehat{\jmath}(s),1)$ is an isomorphism from $\widehat{S}$ to the group of automorphisms of $(S,\widehat{\jmath},\chi,\theta)$. □

Proof. It is enough to show that if (ι,g,ζ) is an automorphism of $(S,\widehat{\jmath},\chi,\theta)$, then $\iota=\mathrm{id}$ and $\zeta_\alpha=1$. From $\mathrm{Ad}(g)\circ\widehat{\jmath}=\widehat{\jmath}\circ\widehat{\iota}$ we see that $g\in N(\widehat{\jmath}(\widehat{S}),\widehat{G})$ and this implies that ι is given by an element $w\in\Omega(S,G)(F)$. Now $\zeta_\alpha=\chi_{w\alpha}\cdot\chi_\alpha^{-1}$. Since χ is minimal the character ζ_S restricts trivially to $S(F)_{0+}$, so the equation $\zeta_S\cdot\theta\circ w=\theta$ implies that w fixes $\theta|_{S(F)_{0+}}$. Lemma 3.6.5 then implies that $w\in\Omega(S,G^0)(F)$. The $\Omega(S,G^0)(F)$-invariance of $(\chi_\alpha)_\alpha$ now implies that $\zeta_\alpha=1$. But this in turn leads to the equation $\theta\circ w=\theta$, and the extra regularity of θ now implies $w=1$, i.e. $\iota=\mathrm{id}$. ■

Proposition 5.2.5. There is a natural 1-1 correspondence between the $\widehat{G}$-conjugacy classes of regular supercuspidal parameters and the isomorphism classes of regular supercuspidal L-packet data. □

The proof of this Proposition will use the following supplementary results.

Lemma 5.2.6. Let $M \subset G$ be a tame twisted Levi. Let $\widehat{M} \to \widehat{G}$ be the natural inclusion, well-defined up to $\widehat{G}$-conjugacy. There exists an extension of $\widehat{M} \to \widehat{G}$ to a tame L-embedding ${}^LM \to {}^LG$. □

Proof. Fix a Γ-invariant pinning $(\widehat{T}, \widehat{B}, \{X_{\widehat{\alpha}}\})$ of $\widehat{G}$. The unique standard Levi subgroup of $\widehat{G}$ dual to the Levi subgroup $M \times \bar{F}$ of $G \times \bar{F}$ is a dual group of M, so we can take it as $\widehat{M}$. The natural inclusion $\widehat{M} \to \widehat{G}$ lies in the canonical $\widehat{G}$-conjugacy class.

We have an action of Γ on $\widehat{G}$ coming from the fact that $\widehat{G}$ is the dual group of G. We also have an action of Γ on $\widehat{M}$, preserving the pinning of $\widehat{M}$ induced by the fixed pinning of $\widehat{G}$, coming from viewing $\widehat{M}$ as the dual group of M. The inclusion $\widehat{M} \to \widehat{G}$ need not be equivariant for these actions, and in fact the Γ-action on $\widehat{G}$ need not even preserve $\widehat{M}$. The restriction to $\widehat{T}$ of the Γ-action on $\widehat{M}$ differs from the restriction to $\widehat{T}$ of the Γ-action on $\widehat{G}$ by an element $w_M \in Z^1(\Gamma_{K/F}, \Omega(\widehat{T}, \widehat{G}))$, where K/F is a finite Galois extension, tame because the actions of Γ_F on $\widehat{G}$ and $\widehat{M}$ are tame. We will find a homomorphism $\xi : W_F/P_F \to N(\widehat{T}, \widehat{G}) \rtimes W_F/P_F$ such that each $\xi(w)$ preserves the pinning of $\widehat{M}$ and acts on $\widehat{T}$ via $w_M(\sigma_w) \rtimes \sigma_w$, where $\sigma_w \in \Gamma$ is the image of w. This ξ will then give us the L-embedding

$$\widehat{M} \rtimes W_F \to \widehat{G} \rtimes W_F, \qquad m \rtimes w \mapsto m\xi(w).$$

For this, let $n_M(\sigma) \in N(\widehat{T}, \widehat{G})$ be the Tits lift [Spr81, 11.2.9] of $w_M(\sigma)$ relative to the fixed pinning of $\widehat{G}$. For $\widehat{\alpha} \in \Delta_M^\vee \subset \Delta^\vee$ we have $\mathrm{Ad}(n_M(\sigma))\sigma(X_{\widehat{\alpha}}) = X_{\widehat{\alpha}}$ by [Spr81, 11.2.11]. The map $\sigma \mapsto n_M(\sigma) \rtimes \sigma \in N(\widehat{T}, \widehat{G}) \rtimes \Gamma$ is not necessarily a homomorphism. We have by [LS87, Lemma 2.1.A] that

$$[n_M(\sigma) \rtimes \sigma] \cdot [n_M(\tau) \rtimes \tau] = t(\sigma, \tau) \cdot n_M(\sigma\tau) \rtimes \sigma\tau,$$

where $t(\sigma, \tau) = \alpha_{\sigma,\tau}(-1)$ and $\alpha_{\sigma,\tau} \in X_*(\widehat{T})$ is the sum of all members of the set

$$\{\beta \in R(\widehat{T}, \widehat{G})^\vee | \beta > 0, [w_M(\sigma)\sigma]^{-1}\beta < 0, [w_M(\sigma)\sigma w_M(\tau)\tau]^{-1}\beta > 0\}. \tag{5.2.1}$$

We claim that $t(\sigma, \tau) \in Z(\widehat{M})^\circ$. Since $X_*(Z(\widehat{M})^\circ)$ is the annihilator in $X_*(\widehat{T})$ of the root lattice $Q(\widehat{M}) \subset X^*(\widehat{T})$, it will be enough to show that $\alpha_{\sigma,\tau}$ annihilates $Q(\widehat{M})$. This is equivalent to showing that $\alpha_{\sigma,\tau}$ is fixed by $\Omega(\widehat{T}, \widehat{M})$, because for any $\widehat{\beta} \in R(\widehat{T}, \widehat{M})$ we have

$$\langle \alpha_{\sigma,\tau}, \beta^\vee \rangle = 0 \Leftrightarrow \langle \alpha_{\sigma,\tau}, \widehat{\beta} \rangle = -\langle \alpha_{\sigma,\tau}, \widehat{\beta} \rangle = \langle s_{\widehat{\beta}} \alpha_{\sigma,\tau}, \widehat{\beta} \rangle.$$

Now observe that any member β of (5.2.1) must be outside of $R(\widehat{T}, \widehat{M})^\vee$, because otherwise $[w_M(\sigma)\sigma]^{-1}$ would not make it negative. The action of $\Omega(\widehat{T}, \widehat{M})$ on $R(\widehat{T}, \widehat{G})$ preserves the set of positive roots in $R(\widehat{T}, \widehat{G}) \smallsetminus R(\widehat{T}, \widehat{M})$. It follows that if $u \in \Omega(\widehat{T}, \widehat{M})$, then $u\beta > 0$ and for the same reason $[w_M(\sigma)\sigma]^{-1}u\beta =$

$v[w_M(\sigma)\sigma]^{-1}\beta < 0$, with $v = [w_M(\sigma)\sigma]^{-1}u[w_M(\sigma)\sigma] \in \Omega(\widehat{T}, \widehat{M})$. This shows that the set (5.2.1) is $\Omega(\widehat{T}, \widehat{M})$-invariant, hence its sum $\alpha_{\sigma,\tau}$ is $\Omega(\widehat{T}, \widehat{M})$-fixed.

We have thus proved $t(\sigma, \tau) \in Z(\widehat{M})^\circ$, i.e. $t \in Z^2(\Gamma_{K/F}, Z(\widehat{M})^\circ)$. But then [Lan79, Lemma 4] implies that there is $r \in C^1(W_{K/F}, Z(\widehat{M})^\circ)$ whose differential is the inflation of t. This means that

$$\xi : w \mapsto r(w)n_M(\sigma_w) \rtimes \sigma_w$$

is a homomorphism $W_{K/F} \to N(\widehat{T}, \widehat{G}) \rtimes W_F$. Since $r(w) \in Z(\widehat{M})^\circ$, it acts trivially on the root spaces of $\widehat{M}$ and thus $\xi(w)$ preserves the pinning of $\widehat{M}$.

We claim that after inflating ξ to W_F, its restriction to P_F is trivial. Since K/F is tame, the image of P_F in $W_{K/F}$ is equal to $K_{0+}^\times$, so we must check that $r(w) = 1$ when $w \in K_{0+}^\times$. This can be extracted from the proof of [Lan79, Lemma 4]. It proceeds by embedding $Z(\widehat{M})^\circ$ into an exact sequence

$$1 \to Z(\widehat{M})^\circ \to \widehat{S}_1 \to \widehat{S}_2 \to 1,$$

where S_1 and S_2 are tori defined over F and split over K and S_1 is induced. Then $r(w)$ is expressed as $d^{-1}(w)c(\sigma_w)a^{-1}\sigma_w(a)$, where $c \in C^1(\Gamma_{K/F}, \widehat{S}_1)$ is chosen so that its co-boundary is t (it exists because S_1 is induced and furthermore $H^2(\Gamma, \mathbb{C}^\times) = H^3(\Gamma, \mathbb{Z}) = 0$, the latter because Γ has strict cohomological dimension 2) and $d \in Z^1(W_{K/F}, \widehat{S}_1)$ and $a \in \widehat{S}_1$ are chosen so that the equation $d(w) = c(\sigma_w)a^{-1}\sigma_w(a)$ holds in $\widehat{S}_2$ (they exist because $H^1(W_{K/F}, \widehat{S}_1) \to H^1(W_{K/F}, \widehat{S}_2)$ is surjective, which follows from the injectivity of $S_2(F) \to S_1(F)$ and the Langlands correspondence for tori).

For $w \in K_{0+}^\times$ we have $\sigma_w = 1 \in \Gamma_{K/F}$ and therefore both $c(\sigma_w)$ and $a^{-1}\sigma_w(a)$ are trivial. To show that d is trivial on $K_{0+}^\times$, we use Lemma 3.1.3 which implies the injectivity of $S_2(F)/S_2(F)_{0+} \to S_1(F)/S_1(F)_{0+}$ and hence by [Yu09, Theorem 7.10] the surjectivity of $H^1(W_{K/F}/K_{0+}^\times, \widehat{S}_1) \to H^1(W_{K/F}/K_{0+}^\times, \widehat{S}_2)$. ■

Let $\varphi : W_F \to \widehat{G} \rtimes W_F$ be a Langlands parameter such that $\varphi(P_F)$ is contained in a torus and $\mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ is a torus. Let $\widehat{T} \subset \widehat{M} \subset \widehat{G}$ be the maximal torus and Levi subgroup from Lemma 5.2.2, normalized by $\mathrm{Ad}(\varphi(-))$, and let $\widehat{S}$ be the Γ-module with underlying abelian group $\widehat{T}$ and Γ-action given by $\mathrm{Ad}(\varphi(-))$. By construction $\varphi(P_F) \subset Z(\widehat{M}) \subset \widehat{T}$, so the Γ-module $\widehat{S}$ is tame. Since $\varphi(I_F)$ preserves a Borel subgroup of $\widehat{M}$ containing $\widehat{T}$, the action of I_F on $R(\widehat{S}, \widehat{M})$ preserves a positive chamber. We let $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ be the tautological embedding of the abelian group $\widehat{T}$ underlying $\widehat{S}$ into $\widehat{G}$. Let S be the algebraic torus over F dual to $\widehat{S}$. Write $R(S, M) \subset R(S, G) \subset X_*(\widehat{S}) = X^*(S)$ for the dual root systems to $R(\widehat{S}, \widehat{M})$ and $R(\widehat{S}, \widehat{G})$.

Lemma 5.2.7. There exists χ-data for $R(S, G)$ that is minimal and $\Omega(S, M)^\Gamma$-invariant. □

Proof. We mimic some of the arguments of the Howe factorization algorithm of §3.6 and begin by considering the filtration

$$R_r = \{\alpha \in R(S, G) | \widehat{\alpha}(\varphi(I^r)) = 1\}.$$

This is well-defined because $\varphi(P_F) \subset Z(\widehat{M}) \subset \widehat{T}$. Let $r_{d-1} > \cdots > r_0 > 0$ be the jumps of this filtration. Set in addition $r_d = \text{depth}(\varphi)$ and $r_{-1} = 0$. Thus $R_{0+} = R(S, M)$ and $R_{r_{d-1}+} = R(S, G)$. Fix an additive character $\Lambda : F \to \mathbb{C}^\times$. For notational convenience, we assume Λ is of depth zero, i.e. trivial on F_{0+} but not on F_0. Given $\alpha \in R(S, G) \smallsetminus R(S, M)$, let r_α be the unique r_i, $d > i \geq 0$, such that $\alpha \in R_{r_\alpha+} \smallsetminus R_{r_\alpha}$.

We define a character $\zeta_\alpha : [F_\alpha^\times]_{r_\alpha}/[F_\alpha^\times]_{r_\alpha+} \to \mathbb{C}^\times$ as follows. The composition $\widehat{\alpha} \circ \varphi$ gives a homomorphism $I^{r_\alpha} \to \mathbb{C}^\times$ that is trivial on $I^{r_\alpha+}$. We claim that this homomorphism can be extended to W_{F_α}. Indeed, if we fix arbitrary tame χ-data for $R(S, G)$ we obtain a tame L-embedding ${}^L j : \widehat{S} \rtimes W_F \to \widehat{G} \rtimes W_F$ containing the image of φ and hence a factorization $\varphi = {}^L j \circ \varphi_S$ with $\varphi_S : W_F \to \widehat{S} \rtimes W_F$ having the property $\varphi_S|_{P_F} = \varphi|_{P_F}$, and then $\widehat{\alpha} \circ \varphi_S : W_{F_\alpha} \to \widehat{S} \rtimes W_{F_\alpha} \to \widehat{S}_{W_{F_\alpha}} \to \mathbb{C}^\times$ is an extension of $\widehat{\alpha} \circ \varphi|_{P_F}$ to W_{F_α}. Thus $\widehat{\alpha} \circ \varphi$ corresponds to a character $\zeta_\alpha : [F_\alpha^\times]_{r_\alpha}/[F_\alpha^\times]_{r_\alpha+} \to \mathbb{C}^\times$, which of course does not depend on the extension of $\widehat{\alpha} \circ \varphi$ to W_{F_α}. The equation

$$\zeta_\alpha(X + 1) = \Lambda(\text{tr}_{F_\alpha/F}(\bar{a}_\alpha X)), \qquad X \in [F_\alpha]_{r_\alpha},$$

specifies an element $\bar{a}_\alpha \in [F_\alpha]_{-r_\alpha}/[F_\alpha]_{-r_\alpha+}$. We claim that $\{(-r_\alpha, \bar{a}_\alpha)\}$ is a set of $\Omega(S, M)^\Gamma$-invariant mod-a-data for $R(S, G) \smallsetminus R(S, M)$. The Γ-invariance of the filtration R_r and $R_{0+} = R(S, M)$ implies the equations $r_{\tau\alpha} = r_{-\alpha} = r_{w\alpha} = r_\alpha$ for $\tau \in \Gamma$ and $w \in \Omega(S, M)^\Gamma$. We further need to show $\tau(\bar{a}_\alpha) = \bar{a}_{\tau\alpha}$, $-\bar{a}_\alpha = \bar{a}_{-\alpha}$, and $\bar{a}_{w\alpha} = \bar{a}_\alpha$, for $\tau \in \Gamma$ and $w \in \Omega(S, M)^\Gamma$. This in turn translates to $\zeta_{\tau\alpha} = \zeta_\alpha \circ \tau^{-1}$, $\zeta_{-\alpha} = \zeta_\alpha^{-1}$ and $\zeta_{w\alpha} = \zeta_\alpha$. Note that $\tau : F_\alpha \to F_{\tau\alpha}$ is an isomorphism of F-algebras and $F_{w\alpha} = F_\alpha$, so these formulas make sense. The claimed properties of the characters ζ_α are seen as follows: Going from α to $-\alpha$ is trivial, going from α to $w\alpha$ comes from the fact that w can be represented by conjugation by an element of $\widehat{M}$, which centralizes $\varphi(P_F)$, and going from α to $\tau\alpha$ comes from

$$\tau(\varphi(w)) = \tau(\varphi_S(w)) = \varphi_S(\tau)\varphi_S(w)\varphi_S(\tau)^{-1} = \varphi_S(\tau w \tau^{-1}) = \varphi(\tau w \tau^{-1}),$$

for $w \in P_F$. From the $\Omega(S, M)^\Gamma$-invariant mod-a-data we obtain $\Omega(S, M)^\Gamma$-invariant tame χ-data for $R(S, G) \smallsetminus R(S, M)$ by (4.7.2). We augment this with unramified χ-data for $R(S, M)$, which suffices since $R(S, M)$ has no ramified symmetric roots, and which is automatically $\Omega(S, M)^\Gamma$-invariant. ■

Proof of Proposition 5.2.5. Given a datum $(S, \widehat{\jmath}, \chi, \theta)$ we use the χ-data to extend $\widehat{\jmath}$ to an L-embedding ${}^L j : {}^L S \to {}^L G$ as explained in [LS87, §2.6] and let $\varphi_S : W_F \to {}^L S$ be the parameter for θ. Define $\varphi = {}^L j \circ \varphi_S$.

Let us first check that the $\widehat{G}$-conjugacy class of φ depends only on the isomorphism class of the datum $(S, \widehat{\jmath}, \chi, \theta)$. Keeping this datum fixed, the parameter φ_S is determined by θ up to $\widehat{S}$-conjugacy and the embedding ${}^L j$ is determined by χ and $\widehat{\jmath}$ also up to $\widehat{S}$-conjugacy, so the $\widehat{G}$-conjugacy class of φ does not depend on these choices. Now we vary the datum $(S, \widehat{\jmath}, \chi, \theta)$ within its isomorphism class. It is enough to check the three basic cases of an isomorphism: $(\iota, 1, 1)$, $(1, g, 1)$, and $(1, 1, \zeta)$. In the first case we have ${}^L j \circ {}^L \iota = {}^L j'$ and ${}^L \iota \circ \varphi_{S'} = \varphi_S$, hence ${}^L j \circ \varphi_S = {}^L j' \circ \varphi_{S'}$. In the second case we have $S = S'$ and ${}^L j = \text{Ad}(g) \circ {}^L j'$. In the third case we have ${}^L j' = {}^L j \cdot c^{-1}$, where c is the 1-cocycle of [LS87, Corollary 2.5.B], as well as $\theta' = \theta \cdot \zeta_S$. The proof of [LS87, Lemma 3.5.A] shows that c is the Langlands parameter for the character ζ_S, so that $\varphi_{S'} = \varphi_S \cdot c$.

We now check that φ satisfies the conditions of Definition 5.2.3. Since S is tamely ramified and the χ-data is at most tamely ramified, we have $\varphi|_{P_F} = \widehat{\jmath} \circ \varphi_S|_{P_F}$, so $\varphi(P_F)$ is contained in the image of $\widehat{\jmath}$, which is a maximal torus of $\widehat{G}$. Before we can discuss $\mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ we need some preparation. Fix a Γ-invariant pinning $(\widehat{T}, \widehat{B}, \{X_{\widehat{\alpha}}\}_{\widehat{\alpha}\in\Delta^\vee})$ of $\widehat{G}$ and replace $(S, \widehat{\jmath}, \chi, \theta)$ by an isomorphic datum so that $\widehat{\jmath}(\widehat{S}) = \widehat{T}$.

Let $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$. According to Lemma 5.2.2 this is a Levi subgroup of $\widehat{G}$. It is normalized by the action of $\varphi(W_F)$ and the resulting homomorphism $W_F \to \mathrm{Aut}(\widehat{M}) \to \mathrm{Out}(\widehat{M})$ extends to Γ_F, because its target is finite. We have arranged that $\widehat{T} \subset \widehat{M}$, so the fixed pinning of $\widehat{G}$ induces the pinning $(\widehat{T}, \widehat{B} \cap M, \{X_{\widehat{\alpha}}\}_{\widehat{\alpha}\in\Delta_M^\vee})$ of $\widehat{M}$. This pinning gives a splitting $\mathrm{Out}(\widehat{M}) \to \mathrm{Aut}(\widehat{M})$ of the natural projection, so we obtain an action $\Gamma_F \to \mathrm{Aut}(\widehat{M})$ preserving the pinning. Note that the original action of Γ_F on $\widehat{G}$ need not preserve $\widehat{M}$, so there is no potential for confusion whenever we speak about "the" Γ_F-action on $\widehat{M}$. The group $\widehat{M}$ endowed with this Γ_F-action is the dual group of a quasi-split F-group M. We now claim that under the identification $R(\widehat{S}, \widehat{G}) = R^\vee(S, G)$ the root system

$$R(\widehat{S}, \widehat{M}) = \{\widehat{\alpha} \in R(\widehat{S}, \widehat{G}) | \widehat{\alpha}(\varphi(P_F)) = 1\}$$

becomes identified with the coroot system of the subsystem R_{0+} of Definition 3.7.5. For any $\widehat{\alpha} \in R(\widehat{S}, \widehat{G})$ let $\alpha^\vee \in R^\vee(S, G)$ be the corresponding cocharacter. Letting E/F be the tame Galois extension splitting S, the parameter of the character $\theta \circ N_{E/F} \circ \alpha^\vee$ is equal to the restriction to W_E of $\widehat{\alpha} \circ \varphi_S$. Since $P_F = P_E \subset W_E$, we see using [Yu09, Theorem 7.10] that $R(\widehat{S}, \widehat{M})$ is the subset of $R^\vee(S, G)$ consisting of those $\alpha^\vee$ for which $\theta \circ N_{E/F} \circ \alpha^\vee$ restricts trivially to $E^\times_{0+}$ and the claim is proved. This claim implies, in particular, that the twisted Levi subgroup of G containing S and having R_{0+} as root system is an inner form of the quasi-split group M, and thus $\widehat{M}$ is a dual group of that twisted Levi subgroup.

Consider the embedding $\widehat{\jmath} : \widehat{S} \to \widehat{M}$. It is trivially W_F-equivariant if we endow both sides with the action of W_F given by $\mathrm{Ad}(\varphi(-))$. It is no longer W_F-equivariant if we endow $\widehat{M}$ with the action of W_F via which $\widehat{M}$ becomes the dual group of M, but this latter action differs from the previous action only by inner automorphisms of $\widehat{M}$, so the $\widehat{M}$-conjugacy class of $\widehat{\jmath} : \widehat{S} \to \widehat{M}$ is still Γ-stable. As discussed in §5.1 this gives us the notion of admissible embeddings $S \to M$. Tracking through the definitions we see that the subset $R(S, M)$ of $X^*(S)$ arising from this notion is a subset of $R(S, G)$ and the identification $R(\widehat{S}, \widehat{G}) = R^\vee(S, G)$ identifies $R(\widehat{S}, \widehat{M})$ with $R^\vee(S, M)$.

By assumption on θ the action of I_F on $R(\widehat{S}, \widehat{M})$ leaves a basis invariant. This implies that all symmetric roots in $R(\widehat{S}, \widehat{M})$ are unramified and hence there is a canonical (up to $\widehat{S}$-conjugation) extension of the embedding $\widehat{\jmath} : \widehat{S} \to \widehat{M}$ to an L-embedding ${}^L j_{S,M} : {}^L S \to {}^L M$, namely the one given by the construction [LS87, §2.6] for unramified χ-data.

Composing the unramified L-embedding ${}^L j_{S,M} : {}^L S \to {}^L M$ with a tame L-embedding ${}^L j_{M,G} : {}^L M \to {}^L G$ supplied by Lemma 5.2.6 we obtain an L-embedding ${}^L j_1 : {}^L S \to {}^L G$, which extends $\widehat{\jmath} : \widehat{S} \to \widehat{G}$. Then ${}^L j_1 = {}^L j \cdot b$, for some $b \in Z^1(W_F, \widehat{S})$. Let $\theta_b : S(F) \to \mathbb{C}^\times$ be the character corresponding to b. By [Yu09, Theorem 7.10] and the fact that ${}^Lj_1$ and ${}^Lj$ agree on P_F we see that θ_b is trivial on $S(F)_{0+}$. We claim that θ_b is $\Omega(S,M)^\Gamma$-invariant. If $w \in \Omega(S,M)^\Gamma$ then [LS87, (2.6.2)] and the fact that ${}^Lj$ is produced from $\Omega(S,M)^\Gamma$-invariant χ-data imply the existence of $n \in N(\widehat{T},\widehat{M})$ representing w such that ${}^Lj \circ w = \mathrm{Ad}(n) \circ {}^Lj$. At the same time ${}^Lj_1 \circ w = \mathrm{Ad}(n') \circ {}^Lj_1$ – we have ${}^Lj_{S,M} \circ w = \mathrm{Ad}(n') \circ {}^Lj_{S,M}$, for some possibly different $n' \in N(\widehat{T},\widehat{M})$ lifting w, for the same reason as for ${}^Lj$, namely the w-invariance of the unramified χ-data, and we have ${}^Lj_{M,G} \circ \mathrm{Ad}(n') = \mathrm{Ad}(n') \circ {}^Lj_{M,G}$ tautologically, because ${}^Lj_{M,G}$ is a group homomorphism extending the identity $\widehat{M} \to \widehat{G}$. We conclude that b and $w \circ b$ are cohomologous, hence $\theta_b = \theta_b \circ w$.

We now have $\varphi = {}^Lj \circ \varphi_S = {}^Lj_1 \circ b \cdot \varphi_S$ and can proceed with the computation of $\mathrm{Cent}(\varphi(I_F),\widehat{G})^\circ$. Clearly $\mathrm{Cent}(\varphi(I_F),\widehat{G})^\circ = \mathrm{Cent}(\varphi(I_F),\widehat{M})^\circ = \mathrm{Cent}({}^Lj_1 \circ b \cdot \varphi_S(I_F),\widehat{M})^\circ = \mathrm{Cent}({}^Lj_{S,M} \circ b \cdot \varphi_S(I_F),\widehat{M})^\circ$. Since ${}^Lj_{S,M} \circ b \cdot \varphi_S(P_F) \subset Z(\widehat{M})$ the action of ${}^Lj_{S,M} \circ b \cdot \varphi_S(I_F)$ on $\widehat{M}$ factors through the pro-cyclic quotient I_F/P_F. Letting $x \in I_F$ be a pre-image of a generator of this quotient, we are considering the automorphism ${}^Lj_{S,M} \circ b \cdot \varphi_S(x)$ of $\widehat{M}$. It is semi-simple (in fact of finite order) and by [Ste68, Theorem 7.5] it preserves a Borel pair of $\widehat{M}$. Conjugating within $\widehat{M}$ we may assume that it preserves the Borel pair belonging to the fixed pinning of $\widehat{M}$. But then it must be given by an element $t \rtimes x \in \widehat{T} \rtimes I_F$. The connected centralizer in $\widehat{M}$ of this automorphism is a reductive subgroup $\widehat{M}^{t \rtimes x,\circ} \subset \widehat{M}$ with maximal torus $\widehat{T}^{x,\circ}$. We recall the description of its root system from [KS99, §1.3]. One divides the roots $R(\widehat{T},\widehat{M})$ in three types, depending on their image in the relative (and possibly non-reduced) root system $R(\widehat{T}^{x,\circ},\widehat{M})$. One says that $\widehat{\alpha} \in R(\widehat{T},\widehat{M})$ is of type R1/R2/R3, if its image $\widehat{\alpha}_{\text{res}} \in R(\widehat{T}^{x,\circ},\widehat{M})$ is a relative root that: is neither divisible nor multipliable/is multipliable/is divisible. For any $\widehat{\alpha} \in R(\widehat{T},\widehat{M})$ we denote by $N\widehat{\alpha}$ the sum of all elements of the orbit of $\widehat{\alpha}$ under the automorphism x. Then $\widehat{\alpha}_{\text{res}}$ is a root of $\widehat{M}^{t \rtimes x,\circ}$ if and only if it is either of type R1 or R2 and $N\widehat{\alpha}(t) = 1$ or if it is of type R3 and $N\widehat{\alpha}(t) = -1$.

Our goal is to show that neither of these cases occurs. For any $\widehat{\alpha} \in R(\widehat{T},\widehat{M})$ the homomorphism $N\widehat{\alpha} : \widehat{T} \to \mathbb{C}^\times$ is I-invariant and descends to a homomorphism $\widehat{T}_I \to \mathbb{C}^\times$. Note here that we are using the Γ-action on $\widehat{T}$ inherited from $\widehat{M}$ and not from $\widehat{G}$. In particular, ${}^Lj_{S,M}$ restricts to an isomorphism $\widehat{S} \rtimes I \to \widehat{T} \rtimes I$. Consider the composition

$$I_F \xrightarrow{b\varphi_S} \widehat{S} \rtimes I \xrightarrow{{}^Lj_{S,M}} \widehat{T} \rtimes I \to \widehat{T}_I \xrightarrow{N\widehat{\alpha}} \mathbb{C}^\times. \tag{5.2.2}$$

The restriction of this homomorphism to P_F is trivial and the image of $x \in I_F$ under this homomorphism is equal to the value $N\widehat{\alpha}(t)$. We want to show that this image is not equal to 1 when $\widehat{\alpha}$ is of type R1 or R2 and is not equal to -1 when $\widehat{\alpha}$ is of type R3.

First let us assume that $\widehat{\alpha}$ is of type R1. Then $N\widehat{\alpha}(t) \neq 1$ is equivalent to the homomorphism (5.2.2) being non-trivial. We shall interpret that homomorphism in terms of the character $\theta_b \cdot \theta$. Since M is quasi-split there exists by Lemma 3.2.2 an admissible embedding $S \to M$ defined over F. By assumption on θ the image is a maximally unramified maximal torus and using Lemma 3.4.12 we may choose the admissible embedding so that the associated point $o \in \mathcal{B}^{\text{red}}(M,F)$ is the superspecial point associated to a Chevalley valuation. Let F'/F be an unramified extension over which S becomes a minimal Levi subgroup of M.

The point o is still special over F' and the root system of M_o° is the subsystem of non-divisible roots in $R(A_S, M)$. The bijection $R(S,M) \leftrightarrow R(\widehat{S},\widehat{M})$ sending $\widehat{\alpha}$ to $\alpha = \widehat{\alpha}^\vee$ restricts to a bijection $R(A_S,M) \leftrightarrow R(\widehat{S}^{x,\circ},\widehat{M})$ that preserves types. Thus $\widehat{\alpha}$ corresponds to a root $\alpha \in R(S,M)$ whose restriction to A_S is neither divisible nor multipliable. This root is then also an element of $R(\mathsf{S}',\mathsf{M}_o^\circ)$ and the corresponding coroot is $N\alpha^\vee = N\widehat{\alpha}$. The L-embedding ${}^Lj_{S,M}$ restricts to an isomorphism $\widehat{S} \rtimes W_{F'} \to \widehat{T} \rtimes W_{F'}$ and

$$W_{F'} \xrightarrow{b\varphi_S} \widehat{S} \rtimes W_{F'} \xrightarrow{{}^Lj_{S,M}} \widehat{T} \rtimes W_{F'} \to \widehat{T}_{W_F'} \xrightarrow{N\widehat{\alpha}} \mathbb{C}^\times$$

is the parameter of the character $[\theta_b\cdot\theta]\circ N_{F'/F}\circ[N\alpha^\vee]$. The character $[\theta_b\cdot\theta]_{S(F)_0}$ has trivial stabilizer in $\Omega(S,M)^\Gamma$ and thus reduces to a character of $\mathsf{S}'(k_F)$ in regular position. According to Lemma 3.4.14 the composition $[\theta_b\cdot\theta]\circ N_{F'/F}\circ [N\alpha^\vee]$ is a non-trivial character of $k_{F'}^\times$, or, seen as a character of $[F']^\times$, has non-trivial restriction to $O_{F'}^\times$. This in turn is equivalent to the claim that its parameter restricts non-trivially to $I_{F'} = I_F$. But that restriction is exactly (5.2.2).

We now turn to the cases where $\widehat{\alpha}$ is of type R2 or R3. These cases are linked together – if $\widehat{\alpha}$ is a root of type R2 and l is the smallest positive number such that $x^l\widehat{\alpha} = \widehat{\alpha}$, then l is even and $\widehat{\beta} = \widehat{\alpha} + x^{l/2}\widehat{\alpha}$ is a root of type R3. Conversely, every root of type R3 occurs in this way. In this situation, we have $N\widehat{\beta} = N\widehat{\alpha}$. The cases of $\widehat{\alpha}$ and $\widehat{\beta}$ will be handled simultaneously if we can show $(2N\widehat{\alpha})(t) \neq 1$. But the bijection $R(A_S,M) \leftrightarrow R(\widehat{S}^{x,0},\widehat{M})$ sends $\widehat{\alpha}$ to a non-divisible relative root α which then occurs in $R(\mathsf{S}',\mathsf{M}_x^\circ)$. Its coroot is $2N\alpha^\vee = 2N\widehat{\alpha}$. The same argument now shows that the homomorphism (5.2.2), where we replace $N\widehat{\alpha}$ by $2N\widehat{\alpha}$, is non-trivial.

We have thus shown that $C := \mathrm{Cent}(\varphi(I_F),\widehat{M})^\circ = \widehat{M}^{t\rtimes x,\circ}$ is a reductive group with maximal torus $\widehat{T}^{x,\circ}$ and an empty root system, so it equals $T^{x,\circ}$ and is thus contained in $\widehat{T}$.

It now remains to check the third property in Definition 5.2.3. Let $n' \in N(\widehat{T},\widehat{M})$ project to $w \in \Omega(S,M)^\Gamma$ and centralize $\varphi(I_F)$. Write $n' = s^{-1}n$, where $s \in \widehat{S}$ and $n \in N(\widehat{T},\widehat{M})$ satisfies $\mathrm{Ad}(n)\circ {}^Lj = {}^Lj \circ w$. The latter equality together with $\mathrm{Ad}(n)\circ\varphi|_{I_F} = \mathrm{Ad}(s)\circ\varphi|_{I_F}$ implies $w\circ\varphi_S|_{I_F} = \mathrm{Ad}(s)\circ\varphi_S|_{I_F}$. By Lemma 3.1.8 this means that w stabilizes $\theta|_{S(F)_0}$. The regularity of θ implies $w=1$.

We now give the inverse construction. Let $\varphi : W_F \to \widehat{G} \rtimes W_F$ be a regular supercuspidal parameter. Let $\widehat{T} \subset \widehat{M} \subset \widehat{G}$ be the maximal torus and Levi subgroup from Lemma 5.2.2, normalized by $\mathrm{Ad}(\varphi(-))$, and let $\widehat{S}$ be the Γ-module with underlying abelian group $\widehat{T}$ and Γ-action given by $\mathrm{Ad}(\varphi(-))$. By construction $\varphi(P_F) \subset Z(\widehat{M}) \subset \widehat{T}$, so the Γ-module $\widehat{S}$ is tame. Since $\varphi(I_F)$ preserves a Borel subgroup of $\widehat{M}$ containing $\widehat{T}$, the action of I_F on $R(\widehat{S},\widehat{M})$ preserves a positive chamber. We let $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ be the tautological embedding of the abelian group $\widehat{T}$ underlying $\widehat{S}$ into $\widehat{G}$.

Lemma 5.2.7 allows us to fix $\Omega(S,M)^\Gamma$-invariant minimal χ-data for $R(S,G)$. From it we obtain a $\widehat{G}$-conjugacy class of L-embeddings $\widehat{S}\rtimes W_F \to \widehat{G}\rtimes W_F$. Fix ${}^Lj_\chi$ within this conjugacy class by demanding ${}^Lj_\chi|_{\widehat{S}} = \widehat{\jmath}$. The image of ${}^Lj_\chi$ contains the image of φ and we obtain the factorization

$$\varphi = {}^Lj_\chi \circ \varphi_{S,\chi},$$

for some Langlands parameter $\varphi_{S,\chi} : W_F \to {}^L S$. Let $\theta = \theta_\chi : S(F) \to \mathbb{C}^\times$ be the corresponding character. Since any L-embedding that is $\widehat{G}$-conjugate to ${}^L j_\chi$ and also restricts to $\widehat{\jmath}$ must be conjugate to ${}^L j_\chi$ by an element of $\widehat{T}$, the $\widehat{S}$-conjugacy class of $\varphi_{S,\chi}$, and hence the character θ_χ, are independent of the choice of ${}^L j_\chi$. They depend only on the choice of χ.

Lemma 5.2.8. The stabilizer of $\theta|_{S(F)_0}$ in $\Omega(S,M)^\Gamma$ is trivial. □

Proof. This is equivalent to $[\theta \circ w/\theta]|_{S(F)_0} \neq 1$ for all $w \in \Omega(S,M)^\Gamma$. By Lemma 3.1.8 this is equivalent to $w \circ \varphi_S \neq \varphi_S$ in $H^1(I_F, \widehat{S})$. Explicitly we need to show that for any $w \in \Omega(S,M)^\Gamma$ there does not exist $s_w \in \widehat{S}$ with $w \circ \varphi_S|_{I_F} = \mathrm{Ad}(s_w)\varphi_S|_{I_F}$. Assume this fails for some w and let s_w be the corresponding element. We have the equality

$$w \circ \varphi_S|_{I_F} = \mathrm{Ad}(s_w) \circ \varphi_S|_{I_F}.$$

Composing both sides of the displayed equation with ${}^L j_\chi$ we obtain

$${}^L j_\chi \circ w \circ \varphi_S|_{I_F} = \mathrm{Ad}(s_w)\varphi|_{I_F}.$$

We may now use [LS87, (2.6.2)] together with the w-invariance of χ-data to get ${}^L j \circ w = \mathrm{Ad}(n) \circ {}^L j$, where $n \in N(\widehat{T}, \widehat{M})$ is a suitable lift of w. This leads to

$$\mathrm{Ad}(n)\varphi|_{I_F} = \mathrm{Ad}(s_w)\varphi|_{I_F},$$

i.e. $s_w^{-1} n \in \mathrm{Cent}(\varphi(I_F), \widehat{M})$ contradicting part 3 of Definition 5.2.3. ■

We now form $(S, \widehat{\jmath}, \chi, \theta_\chi)$ and claim that its isomorphism class depends only on the $\widehat{G}$-class of φ. Keeping φ fixed, recall from §4.6 that the χ-data can only be changed to $\zeta \cdot \chi$ and then according to [LS87, (2.6.3)] we have ${}^L j_{\zeta\chi} = {}^L j_\chi \cdot c$, where c is the element of $Z^1(W_F, \widehat{S})$ defined in [LS87, Corollary 2.5.B]. Thus $\varphi_{S,\zeta\chi} = \varphi_S \cdot c^{-1}$. As we already remarked, c is the Langlands parameter for the character ζ_S. Thus we obtain the isomorphic object $(S, \widehat{\jmath}, \zeta \cdot \chi, \theta_\chi \cdot \zeta_S^{-1})$.

If we replace φ by $\varphi' = \mathrm{Ad}(g) \circ \varphi$ for some $g \in \widehat{G}$ then $\widehat{T}' = \mathrm{Ad}(g)\widehat{T}$, where $\widehat{T}'$ is the maximal torus analogous to $\widehat{T}$ but corresponding to φ', because $\widehat{T}$ can be recovered from φ as $\mathrm{Cent}(C, \widehat{M})$, with $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$ and $C = \mathrm{Cent}(\varphi(I_F), \widehat{M})^\circ$. If we let $\widehat{S}'$ be the Γ-module with abelian group $\widehat{T}'$ and Γ-action given by $\mathrm{Ad}(\varphi'(-))$, then we see that $\mathrm{Ad}(g) : \widehat{S} \to \widehat{S}'$ is an isomorphism of Γ-modules. It gives rise to an isomorphism $\iota : S' \to S$ of algebraic tori. Choose minimal $\Omega(S, \widehat{M})^\Gamma$-invariant χ-data on $R(S,G)$ and transport it via ι to χ-data χ' on $R(S',G)$, which is minimal and $\Omega(S', \widehat{M})^\Gamma$-invariant. Use it to obtain a character $\theta_{\chi'} : S'(F) \to \mathbb{C}^\times$. One then checks immediately that the isomorphism ι identifies the characters θ_χ and $\theta_{\chi'}$. The proof of Proposition 5.2.5 is now complete. ■

Let now $(S, \widehat{\jmath}, \chi, \theta)$ be a regular supercuspidal L-packet datum. We will define a function $\Theta : S(F)_{\mathrm{reg}} \to \mathbb{C}$ as follows. Choose a non-trivial character $\Lambda : F \to \mathbb{C}^\times$. For any $\alpha \in R(S,G)$ we have the character $\Lambda \circ \mathrm{tr}_{F_\alpha/F} : F_\alpha \to \mathbb{C}^\times$; let $r_{\Lambda,\alpha}$ be its depth. On the other hand, we have the character $\theta \circ N_{F_\alpha/F} \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times$; let $r_{\theta,\alpha}$ be its depth. By restriction we obtain a character

$$[F_\alpha]_{r_{\theta,\alpha}}/[F_\alpha]_{r_{\theta,\alpha}+} \to \mathbb{C}^\times, X \mapsto \theta \circ N_{F_\alpha/F} \circ \alpha^\vee(X+1).$$

Let $\bar{a}_\alpha \in [F_\alpha]_{(r_{\Lambda,\alpha}-r_{\theta,\alpha})}/[F_\alpha]_{(r_{\Lambda,\alpha}-r_{\theta,\alpha})+}$ be the unique element satisfying

$$\theta \circ N_{F_\alpha/F} \circ \alpha^\vee(X+1) = \Lambda \circ \mathrm{tr}_{F_\alpha/F}(\bar{a}_\alpha X).$$

It is immediate to check that $\{(r_{\Lambda,\alpha} - r_{\theta,\alpha}, \bar{a}_\alpha)\}$ is a set of mod-a-data. Define

$$\Theta(\gamma) := \epsilon_L(X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda)\Delta_{II}^{\mathrm{abs}}[\bar{a}, \chi](\gamma)\theta(\gamma). \tag{5.2.3}$$

Lemma 5.2.9. The function Θ depends only on the datum $(S, \widehat{\jmath}, \chi, \theta)$. Any isomorphism $(S, \widehat{\jmath}, \chi, \theta) \to (S', \widehat{\jmath}', \chi', \theta')$ carries Θ over to the corresponding function Θ' on $S'(F)_{\mathrm{reg}}$. □

Proof. The character Λ can be replaced by $\Lambda \cdot c$ for some $c \in F$, where we recall that $[\Lambda \cdot c](x) = \Lambda(cx)$. Then $\bar{a}_\alpha$ is replaced by $c^{-1}\bar{a}_\alpha$. Invoking [Kal15, Corollary 4.11], using that $\lambda_{F_\alpha/F_{\pm\alpha}}([\Lambda \cdot c] \circ \mathrm{tr}_{F_{\pm\alpha/F}}) = \kappa_\alpha(c)\lambda_{F_\alpha/F_{\pm\alpha}}(\Lambda \circ \mathrm{tr}_{F_{\pm\alpha}/F})$, and appealing to Lemma 4.6.3, we see that (5.2.3) is unchanged.

We now discuss the isomorphisms $(S, \widehat{\jmath}, \chi, \theta) \to (S', \widehat{\jmath}', \chi', \theta')$. Again we can treat the three basic isomorphism types $(\iota, 1, 1)$, $(1, g, 1)$ and $(1, 1, \zeta)$ separately. For the first two the statement is trivial. For the third, we have by definition $\chi = \chi' \cdot \zeta$ and $\theta = \theta' \cdot \zeta_S^{-1}$ and the statement follows from Lemma 4.6.6. ■

It is clear from Proposition 5.2.5 that the isomorphism classes of regular supercuspidal L-packet data will correspond to L-packets. We now introduce another category, which we call the category of regular supercuspidal data, whose isomorphism classes of object will correspond to the individual supercuspidal representations that are to be organized into L-packets. The objects in this category are tuples $(S, \widehat{\jmath}, \chi, \theta, (G_1, \xi, z), j)$, where $(S, \widehat{\jmath}, \chi, \theta)$ is a regular supercuspidal L-packet datum, (G', ξ, z) is a rigid inner twist of G in the sense of [Kal16, §5.1], and $j : S \to G'$ is an admissible embedding defined over F. A morphism $(S_1, \widehat{\jmath}_1, \chi_1, \theta_1, (G_1', \xi_1, z_1), j_1) \to (S_2, \widehat{\jmath}_2, \chi_2, \theta_2, (G_2', \xi_2, z_2), j_2)$ in this category is given by (ι, g, ζ, f), where (ι, g, ζ) is an isomorphism of the underlying regular supercuspidal L-packet data, $f : (G_1', \xi_1, z_1) \to (G_2', \xi_2, z_2)$ is an isomorphism of rigid inner twists, and $j_2 \circ \iota = f \circ j_1$. There is an obvious forgetful functor from the category of regular supercuspidal data to the category of regular supercuspidal L-packet data. If we fix a regular supercuspidal L-packet datum $(S, \widehat{\jmath}, \chi, \theta)$, the set of isomorphism classes of regular supercuspidal data mapping to it is a torsor under $H^1(u \to W, Z(G) \to S)$. This torsor is given by the relation

$$x \cdot (G_1', \xi_1, z_1, j_1) = (G_2', \xi_2, z_2, j_2) \Leftrightarrow x = \mathrm{inv}(j_1, j_2),$$

see [Kal16, §5.1] for this statement and the notation involved.

We will now attach to each regular supercuspidal datum $(S, \widehat{\jmath}, \chi, \theta, (G', \xi, z), j)$ a regular supercuspidal representation of $G'(F)$. For this we take our lead from the construction of L-packets of real discrete series representations [Lan89]. Ideally we would like to take "the" regular supercuspidal representation of $G'(F)$ whose Harish-Chandra character, evaluated at shallow regular elements of $S(F)$, is given by the formula

$$e(G')|D_{G'}(\gamma')|^{-\frac{1}{2}} \sum_{w \in \Omega(jS(F), G'(F))} \Theta(j^{-1}(\gamma'^w)), \tag{5.2.4}$$

where Θ is the function (5.2.3). We don't know yet quite enough about the Harish-Chandra character of regular supercuspidal representations to know whether this would specify a unique representation. However, we can achieve the same result by the following construction, which, while less elegant, has the virtue of describing explicitly the inducing datum of the representation.

From θ we construct mod-a-data by (4.10.1) and then χ-data for $R(S,G)$ by (4.7.2). The mod-a-data depends on the choice of an additive character $\Lambda : F \to \mathbb{C}^\times$, but the resulting χ-data does not. Replace $(S,\hat{\jmath},\chi,\theta,(G',\xi,z),j)$ by an isomorphic object in which the χ-data is the one just constructed. Note that if we constructed mod-a-data and χ-data with respect to the new θ, we'd obtain an equivalent result, because the difference between the new and old θ is tamely ramified, see the next paragraph. Consider the maximal torus $jS \subset G'$ and the character on it given by $j\theta' := \theta \circ j^{-1} \cdot \epsilon_{f,\text{ram}} \cdot \epsilon^{\text{ram}}$. Here $\epsilon_{f,\text{ram}}$ is the character of $jS(F)$ defined in Definition 4.7.3, and ϵ^{ram} is given by (4.3.3). The character $j\theta'$ is regular according to Facts 4.3.1 and 4.7.5, but may fail to be extra regular due to the occurrence of $\epsilon_{f,\text{ram}}$. The representation of $G'(F)$ corresponding to the regular supercuspidal datum $(S,\hat{\jmath},\chi,\theta,(G',\xi,z),j)$ is then $\pi_{(jS,j\theta')}$. It is regular, but may fail to be extra regular. However, it will be extra regular at least when the point of $\mathcal{B}^{\text{red}}([G']^0,F)$ associated to jS is superspecial, by Lemma 3.4.10.

We claim that the character of this representation, evaluated at shallow regular elements of $S(F)$, is given by (5.2.4). This follows at once from Corollary 4.10.1 once the following remark has been made: The mod-a-data and χ-data occurring in that formula are computed from the character $j\theta'$, while the mod-a-data and χ-data we used here were computed from θ. For all $\alpha \in R(S,G) \smallsetminus R(S,G^0)$, the mod-$a$-data depends only on $j\theta'|_{jS(F)_{0+}}$ respectively $\theta|_{S(F)_{0+}}$. Since $\epsilon_{f,\text{ram}} \cdot \epsilon^{\text{ram}}$ has depth zero, the two restrictions are identified by j and the corresponding mod-a-data are equal. The same is then true for the χ-data, which are computed in terms of the mod-a-data via (4.7.2). For $\alpha \in R(S,G^0)$, the mod-a-data might be different, but they are units in both cases, and since we are taking unramified χ-data in both cases, this difference is irrelevant.

From now on we assume that p does not divide $|\pi_0(Z(G))|$.

We define the compound L-packet Π_φ to be the following set of equivalence classes of representations of rigid inner twists. Fix a regular supercuspidal L-packet datum $(S,\hat{\jmath},\chi,\theta)$ corresponding to φ. For each regular supercuspidal datum $(S,\hat{\jmath},\chi,\theta,(G',\xi,z),j)$ let π_j be the representation of $G'(F)$ just constructed. Then

$$\Pi_\varphi = \{(G',\xi,z,\pi_j)\}, \tag{5.2.5}$$

where $(S,\hat{\jmath},\chi,\theta,(G',\xi,z),j)$ runs over all regular supercuspidal data mapping to the regular supercuspidal L-packet datum $(S,\hat{\jmath},\chi,\theta)$. By Lemma 3.4.12 there exists at least one regular supercuspidal datum for which the point associated to jS in $\mathcal{B}^{\text{red}}([G']^0,F)$ is superspecial, which shows that Π_φ contains extra regular supercuspidal representations.

5.3 Parameterization of L-packets

As in the previous subsection, we are assuming that the residual characteristic of F is odd, is not a bad prime for G, and does not divide $|\pi_0(Z(G))|$. We also keep the assumption that the characteristic of F is zero due to our usage of [Kal16].

Let $\varphi : W_F \to {}^LG$ be a regular supercuspidal parameter and let $(S, \widehat{\jmath}, \chi, \theta)$ be a corresponding regular supercuspidal L-packet datum. Let $S_\varphi = \mathrm{Cent}(\varphi, \widehat{G})$. We apologize for the double usage of the letter S, which seems unavoidable given the standard notation.

Lemma 5.3.1. The embedding $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ induces an isomorphism $\widehat{S}^\Gamma \to S_\varphi$. For any finite subgroup $Z \subset Z(G)$ defined over F this isomorphism extends to an isomorphism $[\widehat{\bar{S}}]^+ \to S_\varphi^+$. □

Proof. Let $s \in S_\varphi$. Then $s \in \mathrm{Cent}(\varphi(P_F), \widehat{G}) = \widehat{M}$, and furthermore, $s \in \mathrm{Cent}(\varphi(I_F), \widehat{G})$. Thus s normalizes $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ and then also $\widehat{T} = \mathrm{Cent}(\widehat{C}, \widehat{M})$. The projection of $s \in N(\widehat{T}, \widehat{M})$ to $\Omega(\widehat{S}, \widehat{M})$ is Γ-fixed, so by Definition 5.2.3 it must be trivial, i.e. $s \in \widehat{T}$.

We have thus shown $S_\varphi \subset \widehat{T}$. Since $\widehat{\jmath}$ maps $\widehat{S}$ isomorphically to $\widehat{T}$ and we have the equation ${}^Lj_\chi \circ \varphi_{S,\chi} = \varphi$, we conclude that $\widehat{\jmath}$ maps $\widehat{S}^\Gamma$ isomorphically to S_φ, as claimed.

Via the canonical embedding $Z(G) \to S$ we can view Z as a subgroup of S and form $\bar{S} = S/Z$. The isomorphism $\widehat{\jmath} : \widehat{S} \to \widehat{T}$ extends uniquely to an isomorphism $\widehat{\bar{S}} \to \widehat{\bar{T}}$. But $[\widehat{\bar{S}}]^+$ is the preimage of $\widehat{S}^\Gamma$ in $\widehat{\bar{S}}$, while S_φ^+ is the preimage of S_φ in $\widehat{\bar{G}}$, thus by what was shown above also in $\widehat{\bar{T}}$. It follows that the isomorphism $\widehat{\bar{S}} \to \widehat{\bar{T}}$ identifies $[\widehat{\bar{S}}]^+$ with S_φ^+. ■

Consider the composition $H^1(u \to W, Z \to S) \to \pi_0([\widehat{\bar{S}}]^+)^D \to \pi_0(S_\varphi^+)^D$, where the first arrow is the isomorphism of [Kal18b, Proposition 5.3], and the second arrow is the isomorphism obtained from the above Lemma. The constituents of Π_φ are in canonical bijection with the set of isomorphism classes of regular supercuspidal data that map to the isomorphism class of $(S, \widehat{\jmath}, \chi, \theta)$ under the forgetful functor. We have already argued that this set is a torsor under $H^1(u \to W, Z \to S)$. In this way, we obtain a canonical simply transitive action of $\pi_0(S_\varphi^+)^D$ on Π_φ.

In order to obtain a bijection $\Pi_\varphi \to \pi_0(S_\varphi^+)^D$ from this simply transitive action, we need to fix a base point. Fix a Whittaker datum $\mathfrak{w}$ for G. According to the strong tempered L-packet conjecture there should exist a unique constituent of Π_φ that is $\mathfrak{w}$-generic. At the moment we can prove this conjecture only in the case of toral representations, see Lemma 6.2.2. The same argument should go through without much modification once the character formula for regular supercuspidal representations, which is currently being developed in the work of Spice and others, is known. Granted this result, let $j_\mathfrak{w}$ be the admissible embedding $S \to G$ so that $\pi_{j_\mathfrak{w}}$ is the generic constituent. Then we obtain the perfect pairing

$$\langle -, - \rangle_\mathfrak{w} : \Pi_\varphi \times \pi_0(S_\varphi^+) \to \mathbb{C}$$

by

$$\langle (G', \xi, z, \pi_j), s \rangle_\mathfrak{w} = \langle \mathrm{inv}(j_\mathfrak{w}, j), s \rangle, \tag{5.3.1}$$

where on the right the pairing comes from the isomorphism $H^1(u \to W, Z \to S) \to \pi_0(S_\varphi^+)^D$. Then the map $s \mapsto \langle (G', \xi, z, \pi_j), s \rangle_\mathfrak{w}$ is a character of $\pi_0(S_\varphi^+)$, while the map $(G', \xi, z, \pi_j) \mapsto \langle (G', \xi, z, \pi_j), - \rangle_\mathfrak{w}$ is a bijection identifying Π_φ with $\pi_0(S_\varphi^+)^D$.

If $\mathfrak{w}'$ is another Whittaker datum, then we have

$$\langle (G', \xi, z, \pi_j), s\rangle_{\mathfrak{w}'} = \langle (G', \xi, z, \pi_j), s\rangle_{\mathfrak{w}} \cdot \langle \mathrm{inv}(j_{\mathfrak{w}'}, j_{\mathfrak{w}}), s\rangle. \tag{5.3.2}$$

5.4 Comparison with the case of real groups

Continuing the theme of §4.11 we will now show that the construction of the regular supercuspidal part of the local Langlands correspondence given in §5.2 and §5.3 is a direct generalization of Langlands' construction [Lan83] of real discrete series L-packets and Shelstad's [She82], [She10], [She08b] parameterization of these.

In this subsection only, let G be a connected reductive group defined and quasi-split over $\mathbb{R}$ and let $\varphi : W_\mathbb{R} \to {}^LG$ be a discrete Langlands parameter. We briefly recall the construction of the correspondence, following the exposition in [Kal16, §5.6]. One chooses a Borel pair $(\widehat{T}, \widehat{B})$ in $\widehat{G}$ and modifies φ in its conjugacy class so that $\varphi(\mathbb{C}^\times) \subset \widehat{T}$. Write $\varphi(z) = z^\mu \bar{z}^\nu$, with $\mu, \nu \in X_*(\widehat{T})_\mathbb{C}$, $\mu - \nu \in X_*(\widehat{T})$. One shows that the image of μ in $X_*(\widehat{T}_{\mathrm{ad}})_\mathbb{C}$ is integral, i.e. belongs to $X_*(\widehat{T}_{\mathrm{ad}})$, and moreover regular [Lan83, Proof of Lemma 3.3]. One then modifies φ again within its conjugacy class so that this image is $\widehat{B}$-dominant. The parameter φ is now pinned down within its $\widehat{G}$-conjugacy class up to conjugation by $\widehat{T}$. The action of $W_\mathbb{R}$ on $\widehat{T}$ via $\mathrm{Ad}(\varphi(w))$ factors through $\Gamma_\mathbb{R}$ and gives a twist $\widehat{S}$ of the Γ-structure on $\widehat{T}$. The real torus S dual to $\widehat{S}$ comes equipped with a stable class of embeddings $S \to G'$ (note that the images of any two such embeddings are conjugate under $G(\mathbb{R})$, but the embeddings themselves need not be) into any inner form G' of G (this follows from [Kot82, Corollary 2.2] in the case of the quasi-split group G and from [She79, Lemma 2.8] for its inner forms G'). By construction there is a distinguished Weyl-chamber in $X^*(S)$. Using based χ-data [She08a, §9] for $R(S, G)$ with respect to that chamber, we obtain an L-embedding ${}^Lj : {}^LS \to {}^LG$ whose image contains the image of φ. We write $\varphi = {}^Lj \circ \varphi_S$, for $\varphi_S : W_\mathbb{R} \to {}^LS$. The local Langlands correspondence for S produces from φ_S a character $\theta : S(\mathbb{R}) \to \mathbb{C}^\times$. For any embedding $j : S \to G$ we let π_j be the unique discrete series representation of $G(\mathbb{R})$ whose character evaluates at a strongly regular element $\gamma \in jS(\mathbb{R})$ to the function (4.11.1), where we are to replace S and θ in this formula with jS and $\theta \circ j^{-1}$. The L-packet on any inner form G' of G is defined to be the set $\{\pi_j\}$ where j runs over the rational classes of embeddings $j : S \to G'$ in the given stable class.

Fixing a Whittaker datum $\mathfrak{w}$, there is a unique embedding $j_\mathfrak{w} : S \to G$ such that the corresponding representation $\pi_{j_\mathfrak{w}}$ is $\mathfrak{w}$-generic [Kos78], [Vog78]. For a canonical internal parameterization of the L-packets we use rigid inner twists. Fix a finite subgroup $Z \subset G$, a rigid inner twist (G', ξ, z) realized by Z, and an admissible rational embedding $j : S \to G'$. Given $s \in S_\varphi^+ = [\widehat{\bar{S}}]^+$, we define

$$\langle (G', \xi, z, \pi_j, s\rangle_\mathfrak{w} = \langle \mathrm{inv}(j_\mathfrak{w}, j), s\rangle,$$

where the pairing on the right is the one from [Kal16, Corollary 5.4].

This exposition makes the direct analogy with the constructions of §5.2 and §5.3 almost obvious. In fact, the exposition here is already slightly different from the one presented in [Kal16, §5.6] in that it uses L-embeddings and factorization of parameters, where in [Kal16, §5.6] we kept more closely to the original construction in [Lan83]. That the two presentations are equivalent is explained in [She10, §7b]. With §4.11 in mind, the only point where the construction of regular supercuspidal L-packets may seem to differ from that of real discrete series L-packets is that in the real case one chooses a specific parameter within its $\widehat{G}$-conjugacy class based on a pinning of $\widehat{G}$ and the notion of dominance. This choice is also used to construct the L-embedding ${}^Lj$ using based χ-data with respect to the same Weyl chamber that gives the notion of dominance. But if we use the argument of §4.11 to rewrite the real discrete series character formula (4.11.1) as (4.10.2), then Lemma 5.2.9 tells us that we can use arbitrary χ-data, at which point the $\widehat{B}$-dominance of μ becomes irrelevant and we recognize the construction in the p-adic case as a direct generalization of the construction in the real case.

6 TORAL L-PACKETS

In this section we will consider the special case of those regular supercuspidal L-packets whose constituents are toral supercuspidal representations. These are the representations arising from Yu-data of the form $(S \subset G, 1, (\phi_0, 1))$, where $\phi_0 : S(F) \to \mathbb{C}^\times$ is a G-generic character of positive depth. These representations were constructed by Adler in the paper [Adl98], which, as far as we know, was the first construction of supercuspidal representations for general reductive p-adic groups, and whose approach formed the basis of Yu's more general construction.

The class of toral supercuspidal representations is general enough to include the epipelagic representations [RY14] (such representations always have depth $1/m$ for some natural number m) when p does not divide $2m$ [Kal15], and the representations considered by Reeder [Ree08]. It is at the same time special enough so that the construction of L-packets simplifies considerably. The biggest advantage of this class of representations is that, from the current standpoint, they are the only ones of the regular supercuspidal representations for which the full character formula is known for all members of the L-packet. This will allow us to sharpen and extend our results – we will prove the existence and uniqueness of a generic constituent in each toral L-packet, as well as the stability and endoscopic transfer of these packets.

6.1 Construction and exhaustion

In this subsection we assume that the residual characteristic of F is odd, not a bad prime for G, and not a divisor of $|\pi_0(Z(G))|$. We further assume that the characteristic of F is zero due to our use of [Kal16], but as we already mentioned this assumption is likely unnecessary and can be avoided if one uses [Kot] instead of [Kal16], at the expense of not treating all reductive groups.

Definition 6.1.1. A *toral supercuspidal parameter of generic depth $r > 0$* is a discrete Langlands parameter $\varphi : W_F \to {}^LG$ satisfying the following conditions.

1. $\mathrm{Cent}(\varphi(I^r), \widehat{G})$ is a maximal torus and contains $\varphi(P_F)$;
2. $\varphi(I^{r+})$ is trivial.

□

Since $\text{Cent}(\varphi(I), \widehat{G}) \subset \text{Cent}(\varphi(I^r), \widehat{G})$, the toral supercuspidal parameters are a special case of the strongly regular supercuspidal parameters of Definition 5.2.1 and hence their L-packets have already been constructed in §5.2. However, since the construction in this special case is considerably simpler, we shall examine it in detail, with the hope that it will be more useful to the readers who are only interested in this special case, and will also serve as an introduction to the more general construction.

The first step is to give the corresponding subcategory of the category of regular supercuspidal L-packet data. We will call it the category of toral L-packet data of generic depth r. A regular supercuspidal L-packet datum $(S, \widehat{\jmath}, \chi, \theta)$ will belong to this subcategory precisely when θ is a G-generic character of depth r.

Proposition 6.1.2. The construction of Proposition 5.2.5 restricts to a bijection between the $\widehat{G}$-conjugacy classes of toral supercuspidal parameters of generic depth r and the isomorphism classes of toral L-packet data of generic depth r. □

Proof. We fix a Γ-stable pinning $(\widehat{T}, \widehat{B}, \{X_{\widehat{\alpha}}\})$ of $\widehat{G}$. Let $\varphi : W_F \to {}^LG$ be a toral supercuspidal Langlands parameter of generic depth r. We conjugate φ so that $\text{Cent}(\varphi(I^r), \widehat{G}) = \widehat{T}$. The composition

$$W_F \xrightarrow{\varphi} N(\widehat{T}, \widehat{G}) \rtimes W_F \to \Omega(\widehat{T}, \widehat{G}) \rtimes W_F \to \text{Aut}_{\text{alg}}(\widehat{T})$$

factors through a finite quotient of W_F and endows $\widehat{T}$ with a new Γ-module structure, which we will record by using the name $\widehat{S}$. The assumption that $\varphi(P_F) \subset \widehat{T}$ ensures that P_F acts trivially on $\widehat{S}$. The Γ-module $\widehat{S}$ is the complex dual torus to a torus S defined over F. Let $\widehat{\jmath} : \widehat{S} \to \widehat{G}$ be the embedding coming from the equality $\widehat{S} = \widehat{T}$ of complex tori. The $\widehat{G}$-conjugacy class of the embedding $\widehat{\jmath}$ is Γ-stable and we obtain a Γ-stable G-conjugacy class of embeddings $S \to G$ as in §5.1, which we will call admissible. Choose minimal tame χ-data for $R(S, G)$. Via the construction of [LS87, §2.6] it gives a $\widehat{G}$-conjugacy class of L-embeddings ${}^LS \to {}^LG$ whose elements extend the $\widehat{G}$-conjugates of $\widehat{\jmath}$. We choose one particular L-embedding ${}^Lj_\chi$ within this conjugacy class whose restriction to $\widehat{S}$ is equal to $\widehat{\jmath}$. By definition, the projections of ${}^Lj_\chi(1 \rtimes w)$ and $\varphi(w)$ in $\Omega(\widehat{T}, \widehat{G}) \rtimes W_F$ are equal for any $w \in W_F$. This implies that the image of φ is contained in the image of ${}^Lj_\chi$, which in turn leads to a factorization

$$\varphi = {}^Lj_\chi \circ \varphi_{S,\chi},$$

for some Langlands parameter $\varphi_{S,\chi} : W_F \to {}^LS$. Let $\theta_\chi : S(F) \to \mathbb{C}^\times$ be the corresponding character. Since any L-embedding that is $\widehat{G}$-conjugate to ${}^Lj_\chi$ and also restricts to $\widehat{\jmath}$ must be conjugate to ${}^Lj_\chi$ by an element of $\widehat{T}$, the $\widehat{S}$-conjugacy class of $\varphi_{S,\chi}$, and hence the character θ_χ, are independent of the choice of ${}^Lj_\chi$. They depend only on the choice of χ.

We claim that θ_χ is generic of depth r. Let E/F be the splitting field of S. By [Kal15, Lemma 3.2] we need to check that for each root $\alpha \in R(S, G)$ the character $E_r^\times / E_{r+}^\times \to \mathbb{C}^\times$ given by $\theta_\chi \circ N_{E/F} \circ \alpha^\vee$ is non-trivial and that the stabilizer of $\theta_\chi \circ N_{E/F}|_{S(E)_r}$ in $\Omega(S, G)$ is trivial. For the first point, the parameter of $\theta_\chi \circ N_{E/F} \circ \alpha^\vee$ is the homomorphism $\widehat{\alpha} \circ \varphi_S|_{W_E}$. By [Yu09, Theorem 7.10] the character restricts non-trivially to $E_r^\times$ if and only if its parameter restricts non-trivially to $I_E^r = I_F^r$. But the restriction of $\widehat{\alpha} \circ \varphi_S$ to I_F^r is equal to the restriction of $\widehat{\alpha} \circ \varphi$, by the tameness of χ-data, and the latter is non-trivial, due to $\mathrm{Cent}(\varphi(I^r), \widehat{G}) = \widehat{T}$. The second point follows from the same reasoning – the stabilizer in $\Omega(S,G)$ of $\theta_\chi \circ N_{E/F}|_{S(E)_r}$ is equal to the stabilizer of $\varphi|_{I^r}$, which is trivial by assumption.

The object $(S, \widehat{\jmath}, \chi, \theta_\chi)$ we thus obtain belongs to the category of toral L-packet data of generic depth r. The proof that its isomorphism class depends only on the $\widehat{G}$-conjugacy class of φ is exactly the same as in Proposition 5.2.5.

We now give the converse construction. Given a toral L-packet datum $(S, \widehat{\jmath}, \chi, \theta)$ of generic depth r we use the χ-data to extend $\widehat{\jmath}$ to an L-embedding ${}^Lj : {}^LS \to {}^LG$ and let $\varphi_S : W_F \to {}^LS$ be the parameter for θ. Define $\varphi = {}^Lj \circ \varphi_S$. We claim that φ satisfies the conditions of Definition 6.1.1. Since P_F acts trivially on $\widehat{S}$, we can regard $\varphi_S|_{P_F}$ as a homomorphism $P_F \to \widehat{S}$. We use again [Kal15, Lemma 3.2] and see that the genericity of θ implies that the restriction $\varphi_S|_{I^{r+}}$ is trivial; the centralizer of $\varphi_S|_{I^r}$ in $\Omega(\widehat{S}, \widehat{G})$ is trivial; and for each $\widehat{\alpha} \in R(\widehat{S}, \widehat{G})$ the composition $\widehat{\alpha} \circ \varphi_S|_{I^r}$ is non-trivial. The tameness of the χ-data and of G implies that we can replace φ_S with φ in these statements, from which we obtain that the homomorphism $\widehat{\jmath} \circ \varphi|_{P_F}$ is trivial on I^{r+} and $\mathrm{Cent}(\widehat{\jmath} \circ \varphi(I^r), \widehat{G}) = \widehat{\jmath}(\widehat{S})$.

Thus φ is a toral supercuspidal parameter of generic depth r. The proof that its $\widehat{G}$-conjugacy class depends only on the isomorphism class of $(S, \widehat{\jmath}, \chi, \theta)$ is again the same as for Proposition 5.2.5. ■

We define the category of toral supercuspidal data of generic depth r as a subcategory of the category of regular supercuspidal data in the same way: A regular supercuspidal datum $(S, \widehat{\jmath}, \chi, \theta, (G_1, \xi, z), j)$ belongs to the subcategory precisely when θ is G-generic of depth r. For any such datum the representation associated to it in §5.2 is a toral representation of depth r. Indeed, it is by construction the regular supercuspidal representation $\pi_{(jS, j\theta')}$ of $G'(F)$, where we recall that $j\theta' := \theta \circ j^{-1} \cdot \epsilon_{f,\mathrm{ram}} \cdot \epsilon^{\mathrm{ram}}$. Since both $\epsilon_{f,\mathrm{ram}}$ and ϵ^{ram} are of depth zero, $j\theta'$ is still G-generic of depth r. In the Howe factorization algorithm of §3.6 this is the second "trivial" case, i.e. in the notation of that subsection we have $d = 1, r_1 = r_0 > r_{-1} = 0, S = G^0 \subsetneq G^1 = G$, so we obtain the twisted Levi sequence $S \subset G$ and the Howe factorization $(1, \theta, 1)$. The resulting Yu-datum then reduces to an Adler datum.

We conclude that the compound L-packet Π_φ consists of toral supercuspidal representations of depth r and moreover every such representation is contained in one of these L-packets.

The internal parameterization of Π_φ is as described in §5.3, but with the added precision that we are now in the position to prove the existence and uniqueness of a generic constituent. This will be done in the next subsection.

6.2 Characters and genericity

We keep the assumptions on F from the previous subsection: the residual characteristic of F is odd, not a bad prime for G, and not a divisor of $|\pi_0(Z(G))|$, and furthermore the characteristic of F is zero. From now on this latter assumption becomes more significant, as it is made in the various references we cite (although we do expect that most of the results we cite remain valid in sufficiently large positive characteristic).

In this and the following subsections we will use the character formula for toral supercuspidal representations of §4.8. For this, fix a character $\Lambda : F \to \mathbb{C}^\times$ of depth zero. We will use the following short-hand notation: $\epsilon(T_G - T_J) = \epsilon_L(X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}, \Lambda)$, $\gamma^j = j^{-1}(\gamma)$, and ${}^jX^* = dj(X^*)$.

Lemma 6.2.1. Let $(S, \widehat{\jmath}, \chi, \theta, (G', \xi, z), j)$ be a toral supercuspidal datum of generic depth r and let π be the corresponding representation of $G'(F)$. The character of π at a regular semi-simple element $\gamma' = \gamma'_{<r} \cdot \gamma'_{\geq r} \in G'(F)$ is given by

$$\frac{e(G')}{e(J')} \frac{\epsilon_L(T_G - T_J)}{|D_{G'}(\gamma')|^{\frac{1}{2}}} \sum_{\substack{g \in J'(F) \backslash G'(F)/jS(F) \\ \gamma'^{g}_{<r} \in jS(F)}} \Delta_{II}^{\text{abs}}[a, \chi'](\gamma'^{gj}_{<r}) \theta(\gamma'^{gj}_{<r}) \widehat{\iota}_{\mathfrak{j}', {}^{jg}X^*}(\log(\gamma'_{\geq r})),$$

where $J' = \text{Cent}(\gamma'_{<r}, G')^\circ$, and T_G and T_J are the minimal Levi subgroups in the quasi-split inner forms of G' and J'. □

Proof. This follows directly from Corollary 4.8.2. ■

Let $(T, B, \{X_\alpha\})$ be an F-pinning of G. Together with the character Λ, it determines a Whittaker datum $\mathfrak{w}$ for G.

Lemma 6.2.2. Let $(S, \widehat{\jmath}, \chi, \theta)$ be a toral L-packet datum of generic depth r. There exists a unique (up to $G(F)$-conjugacy) admissible rational embedding $j_\mathfrak{w} : S \to G$ such that the representation corresponding to $(S, \widehat{\jmath}, \chi, \theta, (G, \text{id}, 1), j_\mathfrak{w})$ is $\mathfrak{w}$-generic. Moreover, the splitting invariant [LS87, §2.3] for the torus $j_\mathfrak{w} S \subset G$ relative to $(T, B, \{X_\alpha\})$ and the mod-a-data constructed in (4.7.3) is trivial. □

Proof. The statement about genericity is a result of DeBacker and Reeder, [DR10, Proposition 4.10]. In that reference the statement is formulated only for the case that S is unramified, but the same argument goes through in general. We limit ourselves to a sketch:

Let $j : S \to G$ be a rational admissible embedding, let π_j be the representation of $G(F)$ corresponding to the supercuspidal toral datum $(S, \widehat{\jmath}, \chi, \theta, (G, \text{id}, 1), j)$, and let Θ_j be its character. According to the Harish-Chandra local character expansion, for strongly regular semi-simple elements $\gamma \in G(F)$ that are sufficiently close to the identity we have

$$\Theta_j(\gamma) = \sum_{\mathcal{O}} c(\mathcal{O}) \widehat{\mu}_\mathcal{O}(\log(\gamma)),$$

where the sum runs over the set of nilpotent orbits of the adjoint action of $G(F)$ on $\mathfrak{g}(F)$, $c(\mathcal{O})$ are complex constants, and $\widehat{\mu}_\mathcal{O}$ are the Fourier transforms of the invariant integrals along these orbits.

Fix a $G(F)$-invariant non-degenerate symmetric bilinear form β on $\mathfrak{g}(F)$. Define an element $f_\mathfrak{w} \in \mathfrak{u}^-(F)$, where U^- is the unipotent radical of the Borel subgroup of G that is T-opposite to B, and $\mathfrak{u}^-$ is its Lie-algebra, by $f_\mathfrak{w} = \sum_\alpha \beta(X_\alpha, X_{-\alpha})^{-1} \cdot X_{-\alpha}$, where the sum runs over the B-simple roots of T. This element has the property that the character of $\mathfrak{u}(F)$ given by $X \mapsto \Lambda(\beta(f_\mathfrak{w}, X))$, when composed with $\exp$, is equal to the generic character of $U(F)$ determined by the splitting $(T, B, \{X_\alpha\})$ and the character Λ. The main result of [MW87] then states that the representation π_j is $\mathfrak{w}$-generic if and only if the constant $c(\mathrm{Ad}(G(F))f_{\mathfrak{w}})$ is non-zero.

According to Lemma 6.2.1, if $g = g_{\geq r} \in G(F)$ is a strongly regular semi-simple element, then

$$\Theta_j(g) = |D_G(g)|^{-\frac{1}{2}} \widehat{\iota}_{\mathfrak{g},{}^jX^*}(\log(g)) = |D_G({}^jX^*)|^{\frac{1}{2}} \widehat{\mu}_{\mathfrak{g},{}^jX^*}(\log(g)).$$

Equating the last two displayed formulas and using a result of Shelstad [She89], reinterpreted as [DR10, Proposition 4.2], we see that $c(\mathrm{Ad}(G(F))f_{\mathfrak{w}})$ is non-zero precisely when the $G(F)$-orbit of jX^* meets the Kostant section $f_{\mathfrak{w}} + \mathrm{Cent}(e_{\mathfrak{w}}, \mathfrak{g})$, where $e_{\mathfrak{w}} = \sum_\alpha \beta(X_\alpha, X_{-\alpha})X_\alpha$, and where we are interpreting $jX^* \in \mathfrak{g}^*(F)$ as an element of $\mathfrak{g}(F)$ via β. From this, the uniqueness of $j_{\mathfrak{w}}$ follows.

We turn to the triviality of the splitting invariant. Let $X'_\alpha = \beta(X_\alpha, X_{-\alpha})X_\alpha$. The main result of [Kot99] asserts that the splitting invariant of $j_{\mathfrak{w}}S$ vanishes, if it is computed with respect to the pinning $(T, B, \{X'_\alpha\})$ and the a-data $a_\gamma = d\gamma(j_{\mathfrak{w}}X^*)$, for $\gamma \in R(S, G)$. Now $d\gamma(j_{\mathfrak{w}}X^*) = \beta(H_\gamma, j_{\mathfrak{w}}X^*) \cdot \beta(X_\gamma, X_{-\gamma})^{-1}$. The function $\alpha \mapsto \beta(X_\alpha, X_{-\alpha})$ extends to a $\Omega(T, G) \rtimes \Gamma$-equivariant function and then [Kal13, Lemma 5.1] implies that the splitting invariant of $j_{\mathfrak{w}}S$ vanishes, if it is computed with respect to the splitting $(T, B, \{X_\alpha\})$ and the a-data $\beta(H_\gamma, j_{\mathfrak{w}}X^*)$. But the a-data $\beta(H_\gamma, j_{\mathfrak{w}}X^*) = \langle H_\gamma, X^*\rangle$ projects to the mod-a-data of (4.7.3). ■

6.3 Stability and transfer

In this subsection we assume that F has characteristic zero and sufficiently large residual characteristic, so that the logarithm map is defined on $G(F)_{0+}$. It is shown in [DR09, App. B] that this is true provided $p \geq (2 + e)n$, where e is the ramification degree of $F/\mathbb{Q}_p$ and n is the dimension of a faithful rational representation of G.

We continue with a toral supercuspidal parameter φ of generic depth r with associated L-packet Π_φ. For any rigid inner twist (G', ξ, x) and any $s \in S_\varphi^+$ define the function

$$\Theta^s_{\varphi,\mathfrak{w},x} = e(G') \sum_{(G',\xi,x,\pi)\in\Pi_\varphi} \langle (G', \xi, x, \pi), s\rangle_{\mathfrak{w}} \cdot \Theta_\pi$$

on $G'(F)$. According to (5.3.2), when $s = 1$ this function does not depend on the choice of $\mathfrak{w}$ and we can denote it by $S\Theta_{\varphi,x} = \Theta^1_{\varphi,\mathfrak{w},x}$.

Lemma 6.3.1. The value of $\Theta^s_{\varphi,\mathfrak{w},x}$ at a regular semi-simple element $\gamma' = \gamma'_{<r} \cdot \gamma'_{\geq r} \in G'(F)$ is given by

$$e(J') \frac{\epsilon(T_G - T_J)}{|D_{G'}(\gamma')|} \sum_j \Delta^{\mathrm{abs}}_{II}[a, \chi'](\gamma'^j_{<r}) \theta(\gamma'^j_{<r}) \sum_k \langle \mathrm{inv}(j_{\mathfrak{w}}, k), s\rangle \widehat{\iota}_{\mathfrak{j}',{}^kX^*}(\log(\gamma'_{\geq r})),$$

where J', T_G, and T_J are as in Lemma 6.2.1, j runs over the set of J'-stable classes of embeddings $S \to J'$, whose composition with $J' \subset G'$ is admissible, k runs over the set of $J'(F)$-rational classes inside the stable class j, and $j_{\mathfrak{w}} : S \to G$ is the admissible embedding given by Lemma 6.2.2. □

Proof. Let $(S, \hat{\jmath}, \chi, \theta)$ be a toral L-packet datum of generic depth r in the isomorphism class associated to φ by Proposition 6.1.2. According to Lemma 6.2.1, for any admissible embedding $j : S \to G'$ the value at γ' of the character Θ_j of the corresponding representation is given by

$$\frac{e(G')}{e(J')}\frac{\epsilon(T_G - T_J)}{|D_{G'}(\gamma')|}\sum_k \Delta_{II}^{\text{abs}}[a,\chi'](\gamma'^k_{<r})\theta(\gamma'^k_{<r})\hat{\iota}_{\mathrm{j}',{}^kX^*}(\log(\gamma_{\geq r})),$$

where k runs over the set of $J'(F)$-conjugacy classes of embeddings $S \to J'$ that are $G'(F)$-conjugate to j. We have $\Theta^s_{\varphi,\mathfrak{w},x} = \sum_j \langle \mathrm{inv}(j_\mathfrak{w}, j), s\rangle\Theta_j$ according to (5.3.1), where the sum runs over the $G'(F)$-conjugacy classes of admissible embeddings $j : S \to G'$ defined over F. Putting both sums together and re-indexing we see that $\Theta^s_{\varphi,\mathfrak{w},x}(\gamma')$ is equal to

$$e(J')\frac{\epsilon(T_G - T_J)}{|D_{G'}(\gamma')|}\sum_j\sum_k \langle \mathrm{inv}(j_\mathfrak{w}, k), s\rangle \Delta_{II}^{\text{abs}}[a,\chi'](\gamma'^k_{<r})\theta(\gamma'^k_{<r})\hat{\iota}_{\mathrm{j}',{}^kX^*}(\log(\gamma'_{\geq r})),$$

where now j runs over the set of J-stable conjugacy classes of G'-admissible embeddings $S \to J'$ defined over F and k runs over the set of $J'(F)$-conjugacy classes of embeddings $S \to J'$ in the J'-stable class of j. Since $\gamma'_{<r}$ is central in J', this expression is equal to the one in the statement of the lemma. ∎

Before we begin the study of stability and endoscopic transfer, we make the following convention. Let T be a maximal torus of G and $\gamma \in T(F)$ a strongly regular semi-simple element with a normal r-approximation $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$. If T' is a maximal torus in some inner form of G or in an endoscopic group of G and $f : T \to T'$ is an admissible isomorphism, then $f(\gamma) = f(\gamma_{<r}) \cdot f(\gamma_{\geq r})$ is a normal r-approximation. This is proved in [DS18, Lemma 5.2] for the case of stable conjugacy, but the argument works without change for the case of transfer to an endoscopic group. This fixes the approximations of all stable conjugates and transfers of γ. It is well-defined, because the only admissible automorphism of T carrying γ to itself is the identity.

Theorem 6.3.2. The function $S\Theta_{\varphi,*}$ is stable across inner forms. That is, for any two rigid inner twist (G'_1, ξ_1, x_1) and (G'_2, ξ_2, x_2) and stably conjugate strongly regular semi-simple elements $\gamma'_1 \in G'_1(F)$ and $\gamma'_2 \in G'_2(F)$ we have

$$S\Theta_{\varphi,x_1}(\gamma'_1) = S\Theta_{\varphi,x_2}(\gamma'_2).$$

□

Proof. It is enough to consider the case where one of the two rigid inner twists is trivial. Thus let (G', ξ, x) be a rigid inner twist of G, $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ a strongly regular semi-simple element of $G(F)$ and $\gamma' = \gamma'_{<r} \cdot \gamma'_{\geq r} \in G'(F)$ stably conjugate to γ. Let $J = \mathrm{Cent}(\gamma_{<r}, G)^\circ$ and $J' = \mathrm{Cent}(\gamma'_{<r}, G')^\circ$. The admissible isomorphism $f_{\gamma,\gamma'}$ (recall notation from §5.1) provides an inner twist $J \to J'$ which carries $\gamma_{>r}$ to $\gamma'_{>r}$. Moreover, for every J-stable class of G-admissible rational embeddings $j : S \to J$, $j' = f_{\gamma,\gamma'} \circ j$ is a J'-stable class of G'-admissible rational embeddings $S \to J'$, and $j \leftrightarrow j'$ is a 1-1 correspondence, under which we have $j' \circ j^{-1}(\gamma_{<r}) = \gamma'_{<r}$. For each pair $j \leftrightarrow j'$ of corresponding stable classes of embeddings, the result of Waldspurger [Wal06, Theoreme 1.5] and Kottwitz's computation of ϵ-factors [Kal15, Theorem 4.10] imply

$$e(J)\sum_k \hat{\iota}_{\mathrm{j},{}^kX^*}(\log(\gamma_{\geq r})) = e(J')\sum_{k'} \hat{\iota}_{\mathrm{j}',{}^{k'}X^*}(\log(\gamma'_{\geq r})).$$ ∎

We will now prove the endoscopic character identities for toral L-packets. Let $\mathfrak{e} = (H, s, {}^L\eta)$ be an extended endoscopic triple for G. This means that $(H, s, \widehat{\eta})$ is an endoscopic triple and ${}^L\eta : {}^LH \to {}^LG$ is an L-embedding extending $\widehat{\eta} : \widehat{H} \to \widehat{G}$. While an extension of $\widehat{\eta}$ to ${}^L\eta$ need not always exist, the argument for the slightly more general case where ${}^L\eta$ does not exist is the same, but the notation is more cumbersome, so we leave it to the reader and refer to [Kal16, (5.11)] for a formulation of these identities in this general case. We may further assume that H splits over a tame extension and the 1-cocycle $W_F \to \widehat{G}$, given by restricting ${}^L\eta$ to W_F and projecting to $\widehat{G}$, is tame. Without this assumption, our problem would be vacuous, as a regular supercuspidal parameter, in particular a toral parameter, would not factor through ${}^L\eta$.

For any rigid inner twist (G', ξ, x) we have the normalized transfer factor $\Delta = \Delta_{\mathfrak{w},x}$ defined in [Kal18b, (5.10)]. This factor was decorated with a prime symbol in loc. cit., because it is a normalization of the factor Δ' of [KS, §5.1], which itself is slightly different from the factor Δ of [LS87]. Nonetheless, to aid readability, we will drop the prime decoration here. As in [Kal15, §6.4], we denote by $\mathring{\Delta}$ the transfer factor Δ with its part Δ_{IV} removed.

Lemma 6.3.3. Let $\gamma^H \in H(F)$ and $\gamma' \in G'(F)$ be strongly regular semi-simple elements. For any sufficiently large natural number k we have

$$\mathring{\Delta}(\gamma^H_{<r} \cdot [\gamma^H_{\geq r}]^{p^{2k}}, \gamma'_{<r} \cdot [\gamma'_{\geq r}]^{p^{2k}}) = \mathring{\Delta}(\gamma^H, \gamma').$$

□

Proof. Since we have arranged that an admissible isomorphism carrying γ^H to γ' carries $\gamma^H_{<r}$ to $\gamma'_{<r}$ and $\gamma^H_{\geq r}$ to $\gamma'_{\geq r}$, the notion of relatedness (see §5.1) is unchanged.

We must compare the terms Δ_I, Δ_{II}, Δ_{III_1} and Δ_{III_2} of both sides. For each root α of $T' = \mathrm{Cent}(\gamma', G')$ we have $\mathrm{ord}(\alpha(\gamma'_{<r}) - 1) < r$ (or $\alpha(\gamma'_{<r}) = 1$) and $\mathrm{ord}(\alpha(\gamma'_{\geq r}) - 1) \geq r$. It follows that $\gamma'_{<r} \cdot [\gamma'_{\geq r}]^{p^{2k}}$ is still a regular element of T'. Thus Δ_I and Δ_{III_1} don't change. To treat the other two, we choose tamely ramified χ-data. Then Δ_{III_2} is a tamely ramified character of $T'(F)$ and thus any power of $\gamma'_{\geq r}$ belongs to its kernel. For Δ_{II}, we apply Lemma 4.6.7 and see that the contributions of those roots α with $\alpha(\gamma'_{<r}) \neq 1$ to both sides are the same. If α is a root with $\alpha(\gamma'_{<r}) = 1$, let $y = \alpha(\gamma'_{\geq r}) \in [F_\alpha^\times]_r$. Then the contribution of α to the left-hand side is $\chi_\alpha(a_\alpha^{-1}(y^{p^{2k}} - 1))$. According to [Hal93, Lemma 3.1] and the tameness of χ_α, this is equal to $\chi_\alpha(a_\alpha^{-1}p^{2k}(y-1))$, which is equal to the product of the contribution of α to the right-hand side with $\kappa_\alpha(p)^{2k} = 1$. ■

Theorem 6.3.4. Let $\gamma' \in G'(F)$ be a strongly regular semi-simple element with a normal r-approximation $\gamma = \gamma'_{<r} \cdot \gamma'_{\geq r}$. Assume that $\varphi = {}^L\eta \circ \varphi^H$ for $\varphi^H : W_F \to {}^LH$. Then

$$\Theta^s_{\varphi,\mathfrak{w},x}(\gamma') = \sum_{\gamma^H \in H(F)/\mathrm{st}} \mathring{\Delta}_{\mathfrak{w},x}(\gamma^H, \gamma') \frac{D^H(\gamma^H)}{D^{G'}(\gamma')} S\Theta_{\varphi^H,1}(\gamma^H).$$

□

Proof. Let $T' = \mathrm{Cent}(\gamma', G')$. We follow the beginning of the proof of [Kal15, Theorem 6.6]. In doing so, we will make active use of the descent lemmas established in [Kal15, §6.3]. Rather than recalling their fairly technical statements, we refer the reader to the cited exposition, which is self-contained.

Let Y be a set of representatives for the stable classes of preimages in $H(F)$ of $\gamma_{<r}$ chosen so that the connected centralizer $H_y = \mathrm{Cent}(y,H)^\circ$ is quasi-split for each $y \in Y$. According to [Kal15, Lemma 6.4] we can write the right hand side of the equality in the statement as

$$\sum_{y \in Y} |\pi_0(H^y)(F)|^{-1} \sum_{z \in H_y(F)_1/\mathrm{st}} \mathring{\Delta}_{\mathfrak{w},x}(yz,\gamma') \frac{D^H(yz)}{D^{G'}(\gamma')} S\Theta_{\varphi^H,1}(yz),$$

where H^y denotes the (possibly disconnected) centralizer of y in H, and $H_y(F)_1$ is the subset of $H_y(F)$ consisting of those elements z for which yz is strongly regular semi-simple and has normal r-approximation with head y and tail z. Applying Lemma 6.3.3, we can rewrite this as

$$\sum_{y \in Y} |\pi_0(H^y)(F)|^{-1} \sum_{z \in H_y(F)_1/\mathrm{st}} \mathring{\Delta}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}) \frac{D^H(yz)}{D^{G'}(\gamma')} S\Theta_{\varphi^H,1}(yz). \quad (6.3.1)$$

As before let $J' = \mathrm{Cent}(\gamma'_{<r},G')^\circ$. Recall the set $\Xi(H_y,J')$ from [Kal15, §6.3]. It encodes the different inequivalent ways in which H_y can be realized as an endoscopic group of J' via descent. There exists a unique $\xi \in \Xi(H_y,J')$ for which the element $yz^{p^{2k}} \in H_y(F)$ is related to the element $\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}$ (for every value of k). We apologize here for the double use of ξ, but the inner twist $\xi : G \to G'$ will not be used in this proof. Taking k large enough, we can apply the Langlands-Shelstad descend theorem [LS90, Theorem 1.6] and conclude that there is a unique normalization $\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}$ of the transfer factor for the group J' and its endoscopic group H_y, realized by descent according to ξ, with the property

$$\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}) = \mathring{\Delta}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}).$$

For any other ξ we take $\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}$ to be an arbitrary normalization of the transfer factor for J' and H_y and have

$$\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}) = 0.$$

This discussion allows us to rewrite (6.3.1) as

$$\sum_{y \in Y} |\pi_0(H^y)(F)|^{-1} \sum_{\xi} \sum_{z \in H_y(F)_1/\mathrm{st}} \mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}) \frac{D^H(yz)}{D^{G'}(\gamma')} S\Theta_{\varphi^H,1}(yz), \quad (6.3.2)$$

where ξ runs over the set $\Xi(H_y,J')$. The sum over z can be extended to $H_y(F)_{\mathrm{sr}}$, since for elements outside of $H_y(F)_1$ the transfer factor will be zero. Furthermore, since y is central in H_y and $\gamma'_{<r}$ is central in J', we may apply [LS90, Lemma 3.5.A] and obtain

$$\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(yz^{p^{2k}},\gamma'_{<r}(\gamma'_{\geq r})^{p^{2k}}) = \lambda_{J',\xi}(\gamma'_{<r}) \mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(z^{p^{2k}},(\gamma'_{\geq r})^{p^{2k}}),$$

where $\lambda_{J',\xi}$ is the character of $Z(J')(F)$ denoted by λ_G in [LS90]. Increasing k if necessary we have

$$\mathring{\Delta}^{\mathrm{desc},\xi}_{\mathfrak{w},x}(z^{p^{2k}},(\gamma'_{\geq r})^{p^{2k}}) = \mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(\log(z^{p^{2k}}),\log((\gamma'_{\geq r})^{p^{2k}})),$$

where on the right we have the transfer factor for the Lie-algebra of J' that is compatibly normalized with the one on the left. Since the Lie-algebra transfer factor is invariant under multiplication by $F^{\times,2}$, we can remove the p^{2k}-power.

Plugging this into (6.3.2), replacing $S\Theta_{\varphi^H,1}$ with the formula from Lemma 6.3.1, and rearranging terms, we arrive at

$$\frac{\lambda_{J',\xi}(\gamma'_{<r})}{D^{G'}(\gamma')}\sum_{y\in Y}|\pi_0(H^y)(F)|^{-1}\sum_{\xi}\sum_{j_H}\Delta_{II}^{\mathrm{abs},H}[a^H,\chi^H](y^{j_H})\theta^H(y^{j_H}) \tag{6.3.3}$$
$$\epsilon(T_H-T_{H_y})\sum_{Z\in\mathfrak{h}_y(F)_{\mathrm{rs}}/\mathrm{st}}\mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(Z,\log((\gamma'_{\geq r})))\sum_{k_H}\widehat{\iota}_{\mathfrak{h}_y,{}^{k_H}X^*}(Z).$$

Here j_H and k_H run as in Lemma 6.3.1 but with target H instead of G', and we have fixed a toral L-packet datum $(S^H,\hat{\jmath}^H,\chi^H,\theta^H)$ for φ^H.

Fix a triple (y,ξ,j_H) contributing to the upper line of (6.3.3). Via [Kal15, Lemma 6.5] this triple corresponds to a J'-stable class of rational G'-admissible embeddings $j:S\to J'$.

Fix a $J'(F)$-invariant non-degenerate symmetric bilinear form β on the Lie-algebra $\mathfrak{j}'(F)$ and use it to identify this Lie-algebra with its dual. The results of Waldspurger [Wal97], [Wal06], and Ngô [Ngô10], imply that then

$$\sum_{Z\in\mathfrak{h}_y(F)_{\mathrm{rs}}/\mathrm{st}}\mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(Z,\log((\gamma'_{\geq r})))\sum_{k_H}\widehat{\iota}_{\mathfrak{h}_y,{}^{k_H}X^*}(Z)$$

is equal to

$$\gamma_\Lambda(\mathfrak{j}',\beta)\gamma_\Lambda(\mathfrak{h}_y,\beta)^{-1}\sum_k\mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(j_HX^*,kX^*)\widehat{\iota}_{\mathfrak{j}',{}^kX^*}(\log(\gamma'_{\geq r})),$$

where now k runs over the set of $J'(F)$-conjugacy classes in the J'-stable class of j. According to [Kal15, Theorem 4.10, Lemma 4.8] we have

$$\gamma_\Lambda(\mathfrak{j}',\beta)\gamma_\Lambda(\mathfrak{h}_y,\beta)^{-1}=e(J')\epsilon(T_{H_y}-T_J)\prod_{\alpha\in R(jS,J'-H_y)_{\mathrm{sym}}/\Gamma}\kappa_\alpha(\beta_\alpha),$$

where α runs over the Γ-orbits of symmetric roots of jS in J' that are outside of H_y. Appealing to the correspondence $(y,\xi,j_H)\leftrightarrow j$ of [Kal15, Lemma 6.5] we can rewrite (6.3.3) as

$$\frac{\lambda_{J',\xi}(\gamma'_{<r})}{D^{G'}(\gamma')}\sum_j\Delta_{II}^{\mathrm{abs},H}[a^H,\chi^H](y^{j_H})\theta^H(y^{j_H})e(J')\epsilon(T_H-T_J) \tag{6.3.4}$$
$$\prod_{\alpha\in R(jS,J'-H_y)_{\mathrm{sym}}/\Gamma}\kappa_\alpha(\beta_\alpha)\sum_k\mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(j_HX^*,kX^*)\widehat{\iota}_{\mathfrak{j}',{}^kX^*}(\log(\gamma'_{\geq r})).$$

Selecting a small $z\in F^\times$ we undo the descent of the transfer factor by

$$\begin{aligned}&\lambda_{J',\xi}(\gamma'_{<r})\mathring{\Delta}^{\mathfrak{j}'}_{\mathfrak{w},x}(j_HX^*,kX^*)\\ =\;&\mathring{\Delta}_{\mathfrak{w},x}(y\exp(z^2j_HX^*),\gamma'_{<r}\exp(z^2kX^*))\\ =\;&\mathring{\Delta}_{\mathfrak{w},1}(y\exp(z^2j_HX^*),\gamma_{<r}\exp(z^2j_{\mathfrak{w}}X^*))\langle\mathrm{inv}(j_{\mathfrak{w}},k),s\rangle,\end{aligned}$$

where $\gamma_{<r}=j_{\mathfrak{w}}k^{-1}(\gamma'_{<r})$. Then (6.3.4) becomes

$$\frac{e(J')}{D^{G'}(\gamma')}\sum_j\Delta_{II}^{\mathrm{abs},H}[a^H,\chi^H](y^{j_H})\theta^H(y^{j_H})\epsilon(T_H-T_J)\prod_{\alpha\in R(jS,J'-H_y)_{\mathrm{sym}}/\Gamma}\kappa_\alpha(\beta_\alpha) \tag{6.3.5}$$
$$\mathring{\Delta}_{\mathfrak{w},1}(y\exp(z^2j_HX^*),\gamma_{<r}\exp(z^2j_{\mathfrak{w}}X^*))\sum_k\langle\mathrm{inv}(j_{\mathfrak{w}},k),s\rangle\widehat{\iota}_{\mathfrak{j}',{}^kX^*}(\log(\gamma'_{\geq r})).$$

Comparing this with Lemma 6.3.1 we see that the theorem will be proved once we prove that $\mathring{\Delta}_{\mathfrak{w},1}(y\exp(z^2 j_H X^*), \gamma_{<r}\exp(z^2 j_{\mathfrak{w}} X^*))$ is equal to

$$\epsilon(T_G - T_H)\frac{\Delta_{II}^{\text{abs}}(\gamma_{<r}'^{j})\theta(\gamma_{<r}'^{j})}{\Delta_{II}^{\text{abs},H}(y^{j_H})\theta^H(y^{j_H})}\prod_{\alpha\in R(jS,J'-H_y)_{\text{sym}}/\Gamma}\kappa_\alpha(\beta_\alpha),$$

where $(S,\widehat{\jmath},\chi,\theta)$ is a toral L-packet datum corresponding to φ.

For this we examine the structure of $\mathring{\Delta}_{\mathfrak{w},1}$. Its first argument belongs to the maximal torus $j_H S^H \subset H$ and its second argument belongs to the maximal torus $j_{\mathfrak{w}} S \subset G$. Modifying $(S^H, \widehat{\jmath}^H, \chi^H, \theta^H)$ within its isomorphism class, we may assume that the isomorphism $\widehat{\jmath}^{-1}\circ\widehat{\eta}\circ\widehat{\jmath}^H : \widehat{S}^H \to \widehat{S}$ is Γ-equivariant. Using the dual of this isomorphism we identify S^H and S and also obtain an admissible isomorphism $j_H S \to j_{\mathfrak{w}} S$ that we use in the discussion of the transfer factor. We select as χ-data for $j_{\mathfrak{w}} S$ the transport via $j_{\mathfrak{w}}$ of the χ-data from the toral L-packet datum $(S,\widehat{\jmath},\chi,\theta)$, and as a-data we select the one used in the character formula of Lemma 6.2.1, namely the one from (4.7.3). The admissible isomorphism $j_{\mathfrak{w}}\circ j_H^{-1}$ transports this to a-data and χ-data for S^H. By modifying the toral L-packet datum $(S^H,\widehat{\jmath}^H,\chi^H,\theta^H)$ within its isomorphism class we may assume that the resulting χ-data is the transport via j_H of the χ-data χ^H.

Recall that $\mathring{\Delta} = \epsilon(T_G - T_H)\Delta_I\Delta_{II}\Delta_{III_2}$, the term Δ_{III_1} being trivial by our choice of admissible isomorphism. According to Lemma 6.2.2 we have $\Delta_I = 1$. By definition, Δ_{III_2} is the value at $\gamma_{<r}$ of the character of $j_{\mathfrak{w}} S$ given by $\theta\circ j_{\mathfrak{w}}^{-1}/\theta^H\circ j_{\mathfrak{w}}^{-1}$. Taking z small enough and using $j^{-1}(\gamma'_{<r}) = j_{\mathfrak{w}}^{-1}(\gamma_{<r}) = j_H^{-1}(y)$ we get $\Delta_{III_2} = \theta(j^{-1}(\gamma_{<r}))/\theta^H(j_H^{-1}(y))$.

To handle the term Δ_{II} we apply Lemma 4.6.7, which reduces the proof to the claim that for small z we have

$$\prod_{\alpha\in R(jS,J'-H_y)_{\text{sym}}/\Gamma}\kappa_\alpha(\beta_\alpha) = \frac{\Delta_{II}^{\text{abs},J}(\exp(z^2 j_{\mathfrak{w}} X^*))}{\Delta_{II}^{\text{abs},H_y}(z^2 j_H X^*)}.$$

Indeed, we have

$$\lim_{z\to 0}\frac{\alpha(\exp(z^2 j_{\mathfrak{w}} X^*)) - 1}{z^2} = d\alpha(X^*) = \langle H_\alpha, X^*\rangle\beta_\alpha^{-1}$$

and recalling that $a_\alpha = \langle H_\alpha, X^*\rangle$ we see

$$\chi_\alpha\left(\frac{\alpha(\exp(z^2 j_{\mathfrak{w}} X^*)) - 1}{a_\alpha}\right) = \kappa_\alpha(\beta_\alpha). \qquad \blacksquare$$

References

[ABD+64] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud, and J.-P. Serre, *Schémas en groupes.*, Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, Institut des Hautes Études Scientifiques, Paris, 1963/1964.

[Adl98] Jeffrey D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math. **185** (1998), no. 1, 1–32. MR 1653184 (2000f:22019)

[Adr10] Moshe Adrian, *A new construction of the tame local Langlands correspondence for GL(n,F), n a prime*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–University of Maryland, College Park. MR 2941466

[Adr13] ______, *A new realization of the Langlands correspondence for* $\mathrm{PGL}(2, F)$, J. Number Theory **133** (2013), no. 2, 446–474. MR 2994367

[AS08] Jeffrey D. Adler and Loren Spice, *Good product expansions for tame elements of p-adic groups*, Int. Math. Res. Pap. IMRP (2008), no. 1, Art. ID rpn 003, 95. MR 2431235 (2009m:22020)

[AS09] ______, *Supercuspidal characters of reductive p-adic groups*, Amer. J. Math. **131** (2009), no. 4, 1137–1210. MR 2543925 (2011a:22018)

[BH05a] Colin J. Bushnell and Guy Henniart, *The essentially tame local Langlands correspondence. I*, J. Amer. Math. Soc. **18** (2005), no. 3, 685–710. MR 2138141 (2006a:22014)

[BH05b] ______, *The essentially tame local Langlands correspondence. II. Totally ramified representations*, Compos. Math. **141** (2005), no. 4, 979–1011. MR 2148193 (2006c:22017)

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 (91i:14034)

[Bor98] Mikhail Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50. MR 1401491

[BS68] A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups. II*, Tôhoku Math. J. (2) **20** (1968), 443–497. MR 0244259

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251. MR 0327923 (48 #6265)

[BT84] ______, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316 (86c:20042)

[BT87] ______, *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 671–698. MR 927605 (89b:20099)

[Car93] Roger W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993, Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication. MR 1266626 (94k:20020)

[DeB06] Stephen DeBacker, *Parameterizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory*, Michigan Math. J. **54** (2006), no. 1, 157–178. MR 2214792 (2007d:22012)

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161. MR 0393266 (52 #14076)

[DR09] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901. MR 2480618 (2010d:22023)

[DR10] ———, *On some generic very cuspidal representations*, Compos. Math. **146** (2010), no. 4, 1029–1055. MR 2660683 (2011j:20114)

[DS18] Stephen DeBacker and Loren Spice, *Stability of character sums for positive-depth, supercuspidal representations*, J. Reine Angew. Math. **742** (2018), 47–78. MR 3849622

[Gil04] P. Gille, *Type des tores maximaux des groupes semi-simples*, J. Ramanujan Math. Soc. **19** (2004), no. 3, 213–230. MR 2139505

[Hak17] Jeffrey Hakim, *Constructing tame supercuspidal representations*, arXiv:1701.07533, 2017.

[Hal93] Thomas C. Hales, *A simple definition of transfer factors for unramified groups*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 109–134. MR 1216184 (94e:22020)

[He08] Xuhua He, *On the affineness of Deligne-Lusztig varieties*, J. Algebra **320** (2008), no. 3, 1207–1219. MR 2427638 (2009c:20085)

[HM08] Jeffrey Hakim and Fiona Murnaghan, *Distinguished tame supercuspidal representations*, Int. Math. Res. Pap. IMRP (2008), no. 2, Art. ID rpn005, 166. MR 2431732 (2010a:22022)

[HN10] Lars Halvard Halle and Johannes Nicaise, *The Néron component series of an abelian variety*, Math. Ann. **348** (2010), no. 3, 749–778. MR 2677903

[How77] Roger E. Howe, *Tamely ramified supercuspidal representations of* Gl_n, Pacific J. Math. **73** (1977), no. 2, 437–460. MR 0492087 (58 #11241)

[HR79] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin-New York, 1979, Structure of topological groups, integration theory, group representations. MR 551496

[Kal11] Tasho Kaletha, *Endoscopic character identities for depth-zero supercuspidal L-packets*, Duke Math. J. **158** (2011), no. 2, 161–224. MR 2805068 (2012f:22031)

[Kal13] ———, *Genericity and contragredience in the local Langlands correspondence*, Algebra Number Theory **7** (2013), no. 10, 2447–2474. MR 3194648

[Kal14] ———, *Supercuspidal L-packets via isocrystals*, Amer. J. Math. **136** (2014), no. 1, 203–239. MR 3163358

[Kal15] ———, *Epipelagic L-packets and rectifying characters*, Invent. Math. **202** (2015), no. 1, 1–89. MR 3402796

[Kal16] ———, *Rigid inner forms of real and p-adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632. MR 3548533

[Kal18a] ______, *Global rigid inner forms and multiplicities of discrete automorphic representations*, Invent. Math. **213** (2018), no. 1, 271–369. MR 3815567

[Kal18b] ______, *Rigid inner forms vs isocrystals*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 1, 61–101. MR 3743236

[Kim07] Ju-Lee Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc. **20** (2007), no. 2, 273–320 (electronic). MR 2276772 (2008c:22014)

[Kos78] Bertram Kostant, *On Whittaker vectors and representation theory*, Invent. Math. **48** (1978), no. 2, 101–184. MR 507800 (80b:22020)

[Kot] Robert E. Kottwitz, *$B(G)$ for all local and global fields*, arXiv:1401.5728.

[Kot82] ______, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785–806. MR 683003 (84k:20020)

[Kot83] ______, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297. MR 697075 (84i:22012)

[Kot84] ______, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650. MR 757954 (85m:11080)

[Kot86] ______, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR 858284 (88d:22027)

[Kot97] ______, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. MR 1485921 (99e:20061)

[Kot99] ______, *Transfer factors for Lie algebras*, Represent. Theory **3** (1999), 127–138 (electronic). MR 1703328 (2000g:22028)

[KS] Robert E. Kottwitz and Diana Shelstad, *On splitting invariants and sign conventions in endoscopic transfer*, arXiv:1201.5658.

[KS99] ______, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190. MR 1687096 (2000k:22024)

[Kut77] P. C. Kutzko, *Mackey's theorem for nonunitary representations*, Proc. Amer. Math. Soc. **64** (1977), no. 1, 173–175. MR 0442145 (56 #533)

[Lan] Robert P. Langlands, *On the functional equation of the Artin L-functions*, `http://publications.ias.edu/rpl/paper/61`.

[Lan79] R. P. Langlands, *Stable conjugacy: definitions and lemmas*, Canad. J. Math. **31** (1979), no. 4, 700–725. MR 540901 (82j:10054)

[Lan83] ______, *Les débuts d'une formule des traces stable*, Publications Mathématiques de l'Université Paris VII [Mathematical Publications of the University of Paris VII], vol. 13, Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983. MR 697567 (85d:11058)

[Lan89] ______, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., vol. 31, Amer. Math. Soc., Providence, RI, 1989, pp. 101–170. MR 1011897 (91e:22017)

[Lan13] Robert P. Langlands, *Singularités et transfert*, Ann. Math. Qué. **37** (2013), no. 2, 173–253. MR 3117742

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271. MR 909227 (89c:11172)

[LS90] R. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 485–563. MR 1106907 (92i:22016)

[Mor89] Lawrence Morris, *P-cuspidal representations of level one*, Proc. London Math. Soc. (3) **58** (1989), no. 3, 550–558. MR 988102 (90c:22056)

[Moy86] Allen Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), no. 4, 863–930. MR 853218 (88b:11081)

[MP94] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408. MR 1253198 (95f:22023)

[MP96] ______, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR 1371680 (97c:22021)

[Mur11] Fiona Murnaghan, *Parametrization of tame supercuspidal representations*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 439–469. MR 2767524 (2012e:22024)

[MW87] C. Mœglin and J.-L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques*, Math. Z. **196** (1987), no. 3, 427–452. MR 913667 (89f:22024)

[Ngô10] Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169. MR 2653248 (2011h:22011)

[NX91] Enric Nart and Xavier Xarles, *Additive reduction of algebraic tori*, Arch. Math. (Basel) **57** (1991), no. 5, 460–466. MR 1129520 (92m:14056)

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263 (95b:11039)

[PR08] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198, With an appendix by T. Haines and Rapoport. MR 2435422 (2009g:22039)

[Pra01] Gopal Prasad, *Galois-fixed points in the Bruhat-Tits building of a reductive group*, Bull. Soc. Math. France **129** (2001), no. 2, 169–174. MR 1871292 (2002j:20088)

[Rag04] M. S. Raghunathan, *Tori in quasi-split-groups*, J. Ramanujan Math. Soc. **19** (2004), no. 4, 281–287. MR 2125504

[Rap05] Michael Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque (2005), no. 298, 271–318, Automorphic forms. I. MR 2141705 (2006c:11071)

[Ree08] Mark Reeder, *Supercuspidal L-packets of positive depth and twisted Coxeter elements*, J. Reine Angew. Math. **620** (2008), 1–33. MR 2427973 (2009e:22019)

[Rie70] Carl Riehm, *The norm* 1 *group of a* $\mathfrak{p}$*-adic division algebra*, Amer. J. Math. **92** (1970), 499–523. MR 0262250

[Roe11] David Lawrence Roe, *The Local Langlands Correspondence for Tamely Ramified Groups*, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–Harvard University. MR 2898606

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR 1411570

[RY14] Mark Reeder and Jiu-Kang Yu, *Epipelagic representations and invariant theory*, J. Amer. Math. Soc. **27** (2014), no. 2, 437–477. MR 3164986

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)

[Sha90] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330. MR 1070599 (91m:11095)

[She79] D. Shelstad, *Characters and inner forms of a quasi-split group over* $\mathbf{R}$, Compositio Math. **39** (1979), no. 1, 11–45. MR 539000 (80m:22023)

[She82] ———, *L-indistinguishability for real groups*, Math. Ann. **259** (1982), no. 3, 385–430. MR 661206 (84c:22017)

[She89] ———, *A formula for regular unipotent germs*, Astérisque (1989), no. 171-172, 275–277, Orbites unipotentes et représentations, II. MR 1021506 (91b:22012)

[She08a] ———, *Tempered endoscopy for real groups. I. Geometric transfer with canonical factors*, Representation theory of real reductive Lie groups, Contemp. Math., vol. 472, Amer. Math. Soc., Providence, RI, 2008, pp. 215–246. MR 2454336 (2011d:22013)

[She08b] ———, *Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure*, Represent. Theory **12** (2008), 369–402. MR 2448289 (2010c:22016)

[She10] ———, *Tempered endoscopy for real groups. II. Spectral transfer factors*, Automorphic forms and the Langlands program, Adv. Lect. Math. (ALM), vol. 9, Int. Press, Somerville, MA, 2010, pp. 236–276. MR 2581952

[Spi08] Loren Spice, *Topological Jordan decompositions*, J. Algebra **319** (2008), no. 8, 3141–3163. MR 2408311

[Spr81] T. A. Springer, *Linear algebraic groups*, Progress in Mathematics, vol. 9, Birkhäuser, Boston, Mass., 1981. MR 632835 (84i:20002)

[SS70] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266. MR 0268192 (42 #3091)

[Ste65] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80. MR 0180554 (31 #4788)

[Ste68] ______, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR 0230728 (37 #6288)

[Tit78] Jacques Tits, *Groupes de Whitehead de groupes algébriques simples sur un corps (d'après V. P. Platonov et al.)*, Séminaire Bourbaki, 29e année (1976/77), Lecture Notes in Math., vol. 677, Springer, Berlin, 1978, pp. Exp. No. 505, pp. 218–236. MR 521771

[Tit79] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR 546588 (80h:20064)

[Vog78] David A. Vogan, Jr., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), no. 1, 75–98. MR 0506503 (58 #22205)

[Wal97] J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), no. 2, 153–236. MR 1440722 (98h:22023)

[Wal01] Jean-Loup Waldspurger, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*, Astérisque (2001), no. 269, vi+449. MR 1817880 (2002h:22014)

[Wal06] J.-L. Waldspurger, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525. MR 2241929 (2007h:22007)

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622 (electronic). MR 1824988 (2002f:22033)

[Yu09] ______, *On the local Langlands correspondence for tori*, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 177–183. MR 2508725 (2009m:11201)

[Yu15] ______, *Smooth models associated to concave functions in Bruhat-Tits theory*, Autour des schémas en groupes. Vol. III, Panor. Synthèses, vol. 47, Soc. Math. France, Paris, 2015, pp. 227–258. MR 3525846