Corrections and additions in the book "Bruhat-Tits theory: a new approach"

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Contents

Introduction

- At the end of the second full paragraph on p. xix, replace Φ_x by Ψ_x.

- At the beginning of p. xxv, in item (3) replace "anisotropic tori" with "anisotropic maximal tori".

Chapter 1

Section 1.1.

- In the statement of Lemma 1.1.13 and Remark 1.1.14 replace "metric space" with "geodesic space".

- In Definition 1.1.12 (1), "A subset" should be replaced by "A nonempty subset".

- In the second paragraph of the proof of Theorem 1.1.15, "Let ϵ be a positive" should be replaced by "Let $\epsilon < r$ be a positive". In the first sentence of the third paragraph of the proof of this theorem, replace the word "positive" with "nonnegative".

- In the proof of Proposition 1.1.20 (1), replace "$d(y_1, y_2)^2 < 16r\epsilon$" with "$d(y_1, y_2)^2 \leqslant 2\epsilon(2r + \epsilon)$", and delete "hence $d(x, m) < r - \epsilon$,".

- The proof of assertion (2) of Proposition 1.1.20 is incomplete. Add the following at the end of this proof.

  Therefore, there exists a sequence $\{\epsilon_n\}$ of nonnegative real numbers that conerges to 0, such that $d(x, \pi(x_n)) = (1 + \epsilon_n)d(x, \pi(x))$. Now let m_n be the mid-point of the geodesic $[\pi(x_n), \pi(x)]$. Then

  $$2d(x, m_n)^2 + (1/2)d(\pi(x_n), \pi(x))^2 \leqslant d(x, \pi(x))^2((1 + \epsilon_n)^2 + 1),$$

  and as $d(x, m_n) \geqslant d(x, \pi(x))$ from the definition of $\pi(x)$, we conclude that $d(\pi(x_n), \pi(x))^2 \leqslant 2\epsilon_n(2+\epsilon_n)d(x, \pi(x))^2$. This implies that $d(\pi(x_n), \pi(x)) \to 0$.

Section 1.2.

- In the third paragraph of §1.2 (which is just a sentence), replace "a vector space" by "a finite dimensional vector space".

- In the last line of Definition 1.2.3, replace $f = \nabla F$ by $W = \nabla B$.

- In the last line of Definition 1.2.4, append "and write $f = \nabla F$".

- In paragraph after Definition 1.2.7, replace "k is infinite" by "k has characteristic zero".

- In the line above Definition 1.2.8, replace "$a \in A$" by "$x \in A$".

- Proposition 1.2.10, all affine spaces and vector spaces are assumed finite-dimensional. The part of the proof that begins with "When $\dim(W) < \infty$ this is enough" can be removed.

Section 1.3.

- At the end of the proof of Proposition 1.3.12, replace "means $\dot{\psi}(\dot{\eta}^\vee) = \dot{\psi}(\dot{\eta}^\vee)$ which is" with "implies $\dot{\psi}(\dot{\eta}^\vee)$ is".

- Lemma 1.3.14(3), in statement, replace "irreducible" by "absolutely irreducible"; in the proof, replace "Chapter 6" by "Chapter VI" in the reference to Bourbaki.

- On p. 19, in line 4 from the bottom, replace "Figure. 1.3.1." with "Figures 1.3.1 and 1.3.2.".

- Proposition 1.3.22(3) proof: The chamber $\mathcal{C}'$ should be chosen so that H_ψ is a wall of it and $\psi(\mathcal{C}') > 0$.

- Proposition 1.3.26: We give the details of the proof of (3): First note that we can rewrite (1) as the equivalence of $\ell(ws) > \ell(w)$ and $w\alpha \in \Psi(\mathcal{C})^+$ using the fact that ℓ is invariant under inversion.

  Define $\beta_i := s_q \cdots s_{i+1}(\alpha_i)$. Since $s_1 \cdots s_q$ is a reduced expression, so is the expression $s_q \cdots s_i$, and applying (1) with $w = s_q \ldots s_{i+1}$ and $\alpha = \alpha_i$ we see that β_i is positive. Since $w\beta_i = s_1 \cdots s_{i-1} s_i(\alpha_i) = -s_1 \cdots s_{i-1}(\alpha_i)$, and $s_1 \cdots s_{i-1}(\alpha_i)$ is positive again by (1), we conclude that $w\beta_i$ is negative.

  We have thus shown that the set $\{\beta_1, \ldots, \beta_q\}$ consists of positive non-divisible roots that are mapped by w to negative roots, i.e. it is a subset of $\Psi(\mathcal{C})^{\mathrm{nd},+} \cap w^{-1}\Psi(\mathcal{C})^{\mathrm{nd},-}$. According to (2) the latter set has cardinality q, so it remains to show that the roots $\beta_1, \ldots, \beta_q$ are pairwise distinct.

  To that end, assume $\beta_i = \beta_j$ for some $i < j$. Multiplying both sides by $s_j \cdots s_q$ we arrive at $s_{j-1} \cdots s_{i+1}\alpha_i = -\alpha_j$, where the product $s_{j-1} \cdots s_{i+1}$ is understood to be empty if $j-1 < i+1$. But $-\alpha_j$ is negative by assumption, while $s_{j-1} \cdots s_{i+1}\alpha_i$ is positive by (1), contradiction.

- In the last sentence of 1.3.29, delete "affine and extended".

- Remark 1.3.32: It should in addition be assumed that the scalar product $\langle -, - \rangle$ is chosen so that the short roots of $\Phi^\vee$ have length 1, and hence ℓ is the square of the length of the long roots of $\Phi^\vee$.

- Proposition 1.3.35:
  - A paragraph explaining the notions of a "parabolic", "symmetric", and "closed" subset of roots was supposed to appear before the statement of this proposition, but due to a copy-editing error now appears before the statement of Lemma 1.3.77.

  - We record a basic fact about parabolic subsets of a finite root system. Let V be a finite-dimensional **R**-vector space and let $\Phi \subset V^*$ be a finite root system. Let $P \subset \Phi$ be a parabolic subset. Set $M = P \cap -P$ and $U = P \setminus M$, so that $P = M \cup U$. According to [Bou02, Chapter VI, §1, no. 7, Proposition 20] there exists a basis B of Φ and a subset $S \subset B$ such that $P = \Phi^+ \cup \Phi_S^-$, where Φ^+ is the subset of Φ consisting of non-negative linear combinations of elements of B, and Φ_S^- is the subset of Φ consisting of non-positive linear combinations of elements of S. According to [Bou02, Chapter VI, §1, no. 7, Corollary 6], M is a subrootsystem of Φ and S is a basis for M; we can also denote M by Φ_S, and have the decomposition $\Phi_S = \Phi_S^+ \cup \Phi_S^-$, with $\Phi_S^+ = -\Phi_S^-$ the subset of Φ consisting of non-negative linear combinations of elements of S. Moreover $U = \Phi^+ \smallsetminus \Phi_S^+$ is the set of non-negative linear combinations of elements of B to which at least one element of $B \smallsetminus S$ contributes non-trivially. Note however that while B and S are non-unique choices, M and U are uniquely determined by P.

– We further recall here from [Bou02, Chapter V, §1, no. 2] the notion of a "facet" with respect to any locally finite set of hyperplanes $\mathcal{H}$ of a real affine space E: it is an equivalence class of points of E for the equivalence relation with respect to which two points of E are equivalent if for every hyperplane $H \in \mathcal{H}$ these points either both lie in H or both lie on the same side of H. Then E is the disjoint union of facets, and for any $x \in E$ the unique facet that contains x, respectively the closure of that facet, is given by

$$\mathcal{F} = \bigcap_{x \in H \in \mathcal{H}} H \cap \bigcap_{x \notin H \in \mathcal{H}} D_H(x), \qquad \overline{\mathcal{F}} = \bigcap_{x \in H \in \mathcal{H}} H \cap \bigcap_{x \notin H \in \mathcal{H}} \overline{D_H(x)},$$

where $D_H(x)$ denotes the unique connected component of $E \smallsetminus H$ that contains x, and $\overline{D_H(x)}$ is the closure of $D_H(x)$, also equal to $D_H(x) \cup H$. One can conversely obtain $\mathcal{F}$ from $\overline{\mathcal{F}}$ as the topological interior of $\overline{\mathcal{F}}$ relative to the topology of the affine span of $\overline{\mathcal{F}}$ in E.
The facets in A that we have been discussing so far are those for the set $\mathcal{H}$ of all affine root hyperplanes.

– The statement of part (8) of this proposition should be replaced with the following:
(8) Let $\mathcal{H}$ denote the set of all affine root hyperplanes in A, and let $\mathcal{H}_0$ denote the subset of those affine root hyperplanes that contain $\mathcal{F}$.
  (i) Every $\mathcal{H}$-facet whose closure contains $\mathcal{F}$ is contained in a unique $\mathcal{H}_0$-facet, and every $\mathcal{H}_0$-facet contains a unique $\mathcal{H}$-facet whose closure contains $\mathcal{F}$. This sets up order-preserving mutually-inverse bijections between these two sets of facets.

  (ii) Taking the derivative produces an order-preserving bijection from the set of $\mathcal{H}_0$-facets to the set of facets for the hyperplane arrangement of the finite root system $\Psi_{\mathcal{F}}$; the inverse of this bijection is also order-preserving.

(iii) If W is a finite-dimensional $\mathbf{R}$-vector space and $\Phi \subset W^*$ is a finite root system, the maps

$$\overline{\mathcal{F}} \mapsto \{a \in \Phi \mid a(\overline{\mathcal{F}}) \geqslant 0\} \qquad \text{and} \qquad P \mapsto \bigcap_{a \in P} W^{a \geqslant 0}$$

are mutually-inverse order-reversing bijections between the set of closures of facets for the hyperplane arrangement of Φ and the set of parabolic subsets of Φ. Writing $P = M \cup U$ as above, the facet $\mathcal{F}$ whose closure is $\overline{\mathcal{F}}$ is given by

$$\bigcap_{a \in M} H_a \cap \bigcap_{a \in U} W^{a>0}.$$

(iv) Combining the bijections of (i)-(iii) one obtains mutually-inverse order-reversing bijections between the set of $\mathcal{H}$-facets whose closure contains $\mathcal{F}$ and the set of parabolic subsets of $\Psi_{\mathcal{F}}$, one of which is given by

$$\mathcal{F}' \mapsto \Psi_{\mathcal{F}}(\mathcal{F}')^+ := \{\psi \in \Psi \mid \psi(\mathcal{F}) = 0, \psi(\mathcal{F}') \geqslant 0\}.$$

– The proof of part (8) should be replaced by the following.

(i) Let $\mathcal{F}'$ be an $\mathcal{H}$-facet containing $\mathcal{F}$ in its closure. As above we have

$$\mathcal{F}' = \bigcap_{x \in H \in \mathcal{H}} H \cap \bigcap_{x \notin H \in \mathcal{H}} D_H(x)$$

for any $x \in \mathcal{F}$. Then

$$\mathcal{F}'_0 = \bigcap_{x \in H \in \mathcal{H}_0} H \cap \bigcap_{x \notin H \in \mathcal{H}_0} D_H(x)$$

is an $\mathcal{H}_0$-facet, and contains $\mathcal{F}'$, hence is also independent of the choice of $x \in \mathcal{F}'$, and is therefore the unique $\mathcal{H}_0$-facet containing $\mathcal{F}'$. Conversely, if $\mathcal{F}'_0$ is an $\mathcal{H}_0$-facet, the second formula above holds for any $x \in \mathcal{F}'_0$, and then the first formula defines an $\mathcal{H}$-facet $\mathcal{F}'$. In general $\mathcal{F}'$ will depend on x and will not contain $\mathcal{F}$ in its closure. But when x is close enough to $\mathcal{F}$ (i.e. in a certain open neighborhood of $\mathcal{F}$) then the closure of $\mathcal{F}'$ will contain $\mathcal{F}$. Moreover, any $x \in \mathcal{F}'_0$ that is close enough to $\mathcal{F}$ will lie in the same $\mathcal{F}'$, so this $\mathcal{F}'$ depends only on $\mathcal{F}'_0$. It is clear from the two formulas above that these assignments are mutually inverse bijections. These bijections respect the closure operations, hence also the orders.

(ii) Let $A'_{\mathcal{F}} \subset A^*$ be the subspace spanned by $\Psi_{\mathcal{F}}$ and let $V^*_{\mathcal{F}} \subset V^*$ be its isomorphic image under the derivative map. We identify

$\Psi_{\mathcal{F}}$ with its image in $V_{\mathcal{F}}^*$, where it is a finite root system. Let $V_{\mathcal{F}}$ be the quotient of V that is dual to $V_{\mathcal{F}}^*$. The hyperplane arrangement of $\Psi_{\mathcal{F}}$ lies in $V_{\mathcal{F}}$: for each $\alpha \in \Psi_{\mathcal{F}}$ one has the hyperplane $\{x \in V_{\mathcal{F}} \,|\, \dot{\alpha}(x) = 0\}$. For each $\alpha \in \Psi_{\mathcal{F}}$ the affine hyperplane H_α has as its derivative the hyperplane $H_{\dot{\alpha}} \subset V^*$. This hyperplane contains the kernel of $V \to V_{\mathcal{F}}$, by virtue of being the vanishing hyperplane of an element of $V_{\mathcal{F}}^*$. Thus, the operation of taking derivative provides a bijection from $\mathcal{H}_0$ to the set of hyperplanes in $V_{\mathcal{F}}$ for the finite root system $\Psi_{\mathcal{F}}$ (note that two hyperplanes with the same derivative are parallel, and if they both contain $\mathcal{F}$ then they must be equal). The claim about facets follows from this.

(iii) For any $w \in W$ the set $\{a \in \Phi \,|\, a(w) \geqslant 0\}$ is evidently parabolic. This set depends only on the facet containing a. Conversely, given a parabolic set P we write $P = M \cup U$ as discussed above, and choose a basis B of Φ and a subset $S \subset \Phi$ such that $M = \Phi_S$ and $U = \Phi^+ \smallsetminus \Phi_S^+$. Write $T = B - S$. Now

$$\bigcap_{a\in P} W^{a\geqslant 0} = \bigcap_{a\in M} W^{a\geqslant 0} \cap \bigcap_{a\in U} W^{a\geqslant 0} = \bigcap_{a\in S} H_a \cap \bigcap_{a\in T} W^{a\geqslant 0}$$

and this is the closure of

$$\bigcap_{a\in S} H_a \cap \bigcap_{a\in T} W^{a>0} = \bigcap_{a\in M} H_a \cap \bigcap_{a\in U} W^{a>0}.$$

Since the elements of $B = S \cup T$ are linearly independent, this intersection is non-empty, and equals the facet determined by any point in that intersection. We have thus shown that both maps are well-defined. It is immediate that they reverse order (given by inclusion on both sides).

To show that they are mutually inverse, consider first a facet $\mathcal{F}$ and let P be the corresponding parabolic subset. One the one hand, if we take $w \in \mathcal{F}$ and write $\mathcal{H}$ for the set of root hyperplanes, we have

$$\mathcal{F} = \bigcap_{w\in H\in\mathcal{H}} H \cap \bigcap_{w\notin H\in\mathcal{H}} D_H(w) = \bigcap_{\substack{a\in\Phi\\ a(w)=0}} H_a \cap \bigcap_{\substack{a\in\Phi\\ a(w)>0}} W^{a>0}.$$

On the other hand, as noted above, we have the decomposition $P = M \cup U$. Since $P = \{a \in \Phi \,|\, a(w) \geqslant 0\}$ we see $M = P \cap -P = \{a \in \Phi \,|\, a(w) = 0\}$ and $U = P \smallsetminus M = \{a \in \Phi \,|\, a(w) > 0\}$. Then

$$\bigcap_{a\in P} W^{a\geqslant 0} = \bigcap_{a\in M} H_a \cap \bigcap_{a\in U} W^{a\geqslant 0},$$

which we recognize as the closure of $\mathcal{F}$.
Conversely, start with a parabolic subset P and write it as $P = M \cup U$. The associated facet is then $\mathcal{F} = \bigcap_{a\in M} H_a \cap \bigcap_{a\in U} W^{a>0}$. Then any root in M is zero on $\mathcal{F}$, while any root in U is strictly positive on $\mathcal{F}$, so the set $\{a \in \Phi \,|\, a(\mathcal{F}) \geqslant 0\} = \{a \in \Phi \,|\, a(\overline{\mathcal{F}}) \geqslant 0\}$ equals precisely $M \cup U = P$.

- Proposition 1.3.47:
  - In the statement and proof of (1), replace " common denominator" with " common factor". Also, replace the fourth line of the proof with " collection of non-negative integers $n_1, \ldots, n_\ell$ ".
  - Add the following statement (3′): Assume $\Psi = \Psi_\Phi$ arises from Construction 1.3.27 with Φ reduced, $\Delta_0 = \{a_1, \ldots, a_r\}$ is a basis of Φ, and $\Delta = \{1 - a_0, a_1, \ldots, a_r\}$ is the corresponding basis of Ψ as in Remark 1.3.48 below. Then a vertex $x \in \mathcal{C}$ is special (equivalently, by Proposition 1.3.43(4), extra special), if and only if $\psi \in \Delta$ corresponding to x (cf. Proposition 1.3.22(6)) satisfies $n_\psi = 1$. The roots ψ corresponding to such special x are $1 - a_0$ and those a_i $(1 \leq i \leq r)$ whose multiplicity in a_0 is 1.
  - Add the following proof (3′): This follows from [Bou02, Chapter VI, §2, no. 2, Propositions 3, 5].
  - Add the following statement (3″): The three affine root systems Ψ, $\Psi^\vee$, and Ψ^{nd}, have $\mathcal{C}$ as chamber, and a point $x \in \overline{\mathcal{C}}$ is special (resp. extra special) with respect to one of them if and only if it is so with respect to all of them.
  - Add the following proof (3″): The three affine root systems Ψ, $\Psi^\vee$, and Ψ^{nd}, share the same set of hyperplanes. This implies that $\mathcal{C}$ is a chamber for both Ψ and $\Psi^\vee$, and that a vertex of $\mathcal{C}$ is special for one of them if and only if it is special for all of them. If $\{\psi_1, \ldots, \psi_\ell\} \subset \Psi_x$ is such that $\dot\psi_1, \ldots, \dot\psi_\ell$ is a basis for $\Phi = \nabla\Psi$, then $\{\psi_1^\vee, \ldots, \psi_\ell^\vee\} \subset \Psi_x^\vee$ and the identity $(\nabla\psi)^\vee = \nabla(\psi^\vee)$ shows that $\dot\psi_1^\vee, \ldots, \dot\psi_\ell^\vee$ is a basis for $\Phi^\vee = \nabla(\Psi^\vee)$. Thus being extra special for Ψ is equivalent to being extra special for $\Psi^\vee$. Moreover $\dot\psi_1, \ldots, \dot\psi_\ell$ necessarily lie in Φ^{nd}, which implies that $\psi_1, \ldots, \psi_\ell$ lie in Ψ^{nd}, and thus being extra special for Ψ implies being extra special for Ψ^{nd}. The converse implication is trivial.
- Lemma 1.3.62: The statement of this lemma should read

$$\overline{\Psi^\vee} = \overline{\Psi}.$$

In other words, the claim is not that the finite root system $\overline{\Psi}$ is self-dual, but rather that applying the construction $\Psi \mapsto \overline{\Psi}$ to both Ψ and $\Psi^\vee$ yields the same result.

- Remark 1.3.68: Append to the last sentence: ", and a bond with label ∞ with a quadrupple edge, possibly with orientation. Note that the Coxeter graph of the Coxeter system (W, S) coincides with the graph underlying the affine Dynkin diagram of Ψ. This follows from [Bou02, Chapter V, §3, no. 4, Proposition 3]."

- Theorem 1.3.69: Replace the proof with the following.

  We first exhibit explicitly the four families of non-reduced affine root systems. In all cases, we take $A = V = \mathbf{R}^n$, so that $A^* = \mathbf{R}^n \oplus \mathbf{R}$, where the summand $\mathbf{R}^n$ of A^* is identified with the linear dual space to $\mathbf{R}^n$ via the standard scalar product, and we write $\{e_1, \dots, e_n\}$ for the standard basis of $V = \mathbf{R}^n$, and $\{\varepsilon_1, \dots, \varepsilon_n\}$ for its dual basis, which is again the standard basis of $V^* = \mathbf{R}^n$.

$$\begin{aligned}
(B_n, B_n^\vee): \quad \Psi &= \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}. \\
(C_n^\vee, C_n): \quad \Psi &= \{\pm\varepsilon_i + \tfrac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}. \\
(C_n^\vee, BC_n): \quad \Psi &= \{\pm\varepsilon_i + \tfrac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\} \\
&\simeq \{\pm\varepsilon_i + \tfrac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + (2\mathbf{Z} + 1)\}. \\
(BC_n, C_n): \quad \Psi &= \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}.
\end{aligned}$$

  Here i and j run over $1, \dots, n$ and $i < j$. To switch between the two presentations of $(C_n^\vee, BC_n)$ one applies the translation $\frac{1}{2}(e_1 + \cdots + e_n) \in V$ to A.

  Using Proposition 1.3.12 it is easy to check that these are affine root systems. First, note that it is enough to handle the first three cases, the fourth being the dual of the third. Assumptions (1) and (3) of that proposition are then clear, and assumption (2) follows from the fact that the hyperplanes (and hence also the reflections) for Ψ are the same as for Ψ^{nd}, whose type is the first member of the type pair, and then one inspects that $\Psi \smallsetminus \Psi^{\mathrm{nd}}$ is preserved by these reflections.

  It now remains to show that, up to isomorphism, any irreducible non-reduced affine root system Ψ is isomorphic to a member of one of the above four families. For this, consider $\Phi = \nabla\Psi$, and irreducible non-reduced finite root system, thus of type BC_n for some n, so Φ^{nd} is of type B_n and Φ^{nm} is of type C_n. We have $\Phi^{\mathrm{nd}} \subset \nabla(\Psi^{\mathrm{nd}}) \subset \Phi$ and $\Phi^{\mathrm{nm}} \subset \nabla(\Psi^{\mathrm{nm}}) \subset \Phi$. This limits the possible types of Ψ^{nd} to B_n, $C_n^\vee$, and BC_n, and the possible types of Ψ^{nm} to C_n, $B_n^\vee$, and BC_n.

Write further $\Psi^{\mathrm{nd,nm}} = \Psi^{\mathrm{nd}} \cap \Psi^{\mathrm{nm}}$ for the subset of affine roots that are neither divisible nor multipliable, $\Psi^{\mathrm{m}} = \Psi^{\mathrm{nd}} \smallsetminus \Psi^{\mathrm{nd,nm}}$ for the multipliable affine roots, and $\Psi^{\mathrm{d}} = \Psi^{\mathrm{nm}} \smallsetminus \Psi^{\mathrm{nd,nm}}$ for the divisible affine roots. Then $\Psi = \Psi^{\mathrm{m}} \cup \Psi^{\mathrm{nd,nm}} \cup \Psi^{\mathrm{d}}$ is a disjoint union decomposition, and $\Psi^{\mathrm{d}} = 2\Psi^{\mathrm{m}}$. It follows that Ψ is determined by Ψ^{nd} and the decomposition $\Psi^{\mathrm{nd}} = \Psi^{\mathrm{m}} \cup \Psi^{\mathrm{nd,nm}}$.

Choose an isomorphism of affine spaces $A \to \mathbf{R}^n$ which transports Ψ^{nd} to the standard presentation of the root system B_n, $C_n^\vee$, or BC_n, respectively. It remains to isolate the subset $\Psi^{\mathrm{m}} \subset \Psi^{\mathrm{nd}}$. Consider the analogous decomposition $\Phi = \Phi^{\mathrm{m}} \cup \Phi^{\mathrm{nd,nm}} \cup \Phi^{\mathrm{d}}$. The derivative restricts to a map $\Psi^{\mathrm{m}} \to \Phi^{\mathrm{m}}$ that is $W(\Psi)$-equivariant, where $W(\Psi)$ acts on the target by its quotient $W(\Phi)$. The set Φ^{m} is the subset of short roots in Φ^{nd} and hence is a single $W(\Phi)$-orbit (there is a unique short node in the finite Dynkin diagram of type B_n), and we conclude that $\Psi^{\mathrm{m}} \to \Phi^{\mathrm{m}}$ is surjective.

Consider first the case that Ψ^{nd} is of type B_n or BC_n. Then the preimage of Φ^{m} in Ψ^{nd} is $\{\pm\varepsilon_i + \mathbf{Z}\}$, which is again the set of short roots in Ψ^{nd} and hence a single $W(\Psi)$-orbit (there is a unique short node in the affine Dynkin diagram of type B_n), and we conclude that it equals Ψ^{m}. This shows that the isomorphism $A \to \mathbf{R}^n$ carries Ψ to the explicit root system of type $(B_n, B_n^\vee)$ or (BC_n, C_n) above.

Next consider the case that Ψ^{nd} is of type $C_n^\vee$. Then the preimage of Φ^{m} in Ψ^{nd}, which now is $\{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}\}$, consists of two $W(\Psi)$-orbits (corresponding to the two short nodes in the affine Dynkin diagram of type $C_n^\vee$), namely $\{\pm\varepsilon_i + \mathbf{Z}\}$ and $\{\pm\varepsilon_i + (\frac{1}{2}\mathbf{Z} \smallsetminus \mathbf{Z})\}$. If we take the first orbit as Ψ^{m}, we obtain the system of type $(C_n^\vee, BC_n)$ in its first presentation above, while if we take the second orbit as Ψ^{m}, we obtain the system of type $(C_n^\vee, BC_n)$ in its second presentation above. If we take the union of both orbits as Ψ^{m}, we obtain the system of type $(C_n^\vee, C_n)$.

- Table 1.3.3: While the diagrams are correct as given, it would be more suggestive, in type $(C_n^\vee, BC_n)$, to place the circle around the right-most node, rather than the left-most. In this way the diagram would be consistent with the picture of the extended Dynkin diagram of type C_n in [Bou02, Chapter VI, Plate III], as well as with the explicit description of the affine root systems given further down in this erratum/addendum, in which ε_n is the multipliable root, and is listed last (rather than first).

- Remark 1.3.70: Add the following sentence: "For checking the extra special nodes, Propositions 1.3.22(6) and 1.3.47(3), as well as Table 1.3.5, can be helpful."

- Lemma 1.3.75: Replace statement and proof with the following:

  Lemma 1.3.75 *Let Ψ be an irreducible affine root system.*
  (1) $W(\Psi)^{\mathrm{ext}} = W(\Psi^\vee)^{\mathrm{ext}} = W(\Psi^{\mathrm{nd}})^{\mathrm{ext}} \cap W(\Psi^{\mathrm{nm}})^{\mathrm{ext}}$.
  (2) If Φ is reduced, and $\Psi = \Psi_\Phi$ arises from Construction 1.3.27, then the translation subgroup of $W(\Psi)^{\mathrm{ext}}$ equals the coweight lattice in V.
  (3) Assume that Ψ is reduced and let $\mathcal{C}$ a chamber. The stabilizier Ξ of $\mathcal{C}$ in $W(\Psi)^{\mathrm{ext}}$ acts simply transitively on the set of extra special vertices of $\mathcal{C}$.

  Proof: (1) An element of $W(\Psi)^{\mathrm{ext}}$ is an affine automorphism of A and hence induces a linear automorphism of A^*. Its derivative lies in $W(\Phi)$, so preserves the scalar product used to construct $\Psi^\vee$ out of Ψ. Therefore $W(\Psi)^{\mathrm{ext}}$ preserves $\Psi^\vee$, hence $W(\Psi)^{\mathrm{ext}} \subset W(\Psi^\vee)^{\mathrm{ext}}$. The reverse inclusion is obtained by interchanging Ψ and $\Psi^\vee$. The identity $W(\Psi)^{\mathrm{ext}} = W(\Psi^{\mathrm{nd}})^{\mathrm{ext}} \cap W(\Psi^{\mathrm{nm}})^{\mathrm{ext}}$ follows by observing that $\Psi = \Psi^{\mathrm{nd}} \cup \Psi^{\mathrm{nm}}$ and $W(\nabla\Psi) = W(\nabla\Psi^{\mathrm{nd}}) = W(\nabla\Psi^{\mathrm{nm}})$, because these three root systems are equal when Ψ is reduced, and otherwise $\nabla\Psi = BC_n$ and the other two systems are among B_n, C_n, BC_n, all of which have the same Weyl group.

  (2) The translation t_v by $v \in V$ belongs to $W(\Psi)^{\mathrm{ext}}$ if and only if $t_v(\Psi) = \Psi$. Since $\Psi = \{a + \mathbf{Z} \,|\, a \in \Phi\}$, that condition is equivalent to $a(v) \in \mathbf{Z}$ for all $a \in \Phi$, i.e. to v lying in the coweight lattice for Φ.

  (3) According to Proposition 1.3.47(3") (in this erratum) the set of extra special vertices for Ψ is the same as that for $\Psi^\vee$, which in combination with (1) reduces the proof to the case $\Psi = \Psi_\Phi$ for a finite root system Φ. When Φ is reduced, the claim follows from [Bou02, Chapter VI, §2, no. 3, Remark 1], which builds on the preceding Proposition and Corollary. For the convenience of the reader we match our notation with that of loc. cit.: our $W(\Psi)$, $W(\Psi)^{\mathrm{ext}}$, and Ξ are denoted by W_a, W'_a, and Γ_C in loc. cit. Moreover, $0 \in \overline{C}$ in loc. cit. is an extra special vertex by construction. It is shown in [Bou02, Chapter VI, §2, no. 2, Proposition 3] that the special (automatically extra special) points in $\overline{C}$ are $\overline{C} \cap P(\Phi^\vee)$.

  When Φ is not reduced, then Ψ is of type BC_n and has a unique extra special vertex, while $W(\Psi)^{\mathrm{ext}} = W(\Psi)$, so the stabilizer of $\mathcal{C}$ is trivial. □

  We remark that the identity $W(\Psi^{\mathrm{nd}}) = W(\Psi^{\mathrm{nm}})$ holds for every irreducible affine root system Ψ except the those of type $(C_n^\vee, BC_n)$ and (BC_n, C_n).

- Remark 1.3.76: The reader may find it useful to look at the material in [Bou02, Chapter VI, §2, no. 3], which identifies Ξ with the group $P(\Phi^\vee)/Q(\Phi^\vee)$ when $\Psi = \Psi_\Phi$ for a reduced irreducible finite root system

Φ. That group and its action on the affine Dynkin diagram can be read off from part (XII) of the Plates in [Bou02, Chapter VI].

- Replace "$W(\Psi)^{\mathrm{aff}}$" by "$W(\Psi)$" everywhere.

- Add the following at the end of §1.3.

  In the rest of this section, we will list all irreducible affine root systems in standard coordinates and give their Weyl groups and some other information. We will have $A = V$ and $A^* = V^* \oplus \mathbf{R}$, and elements of A^* will be written as $\eta + r$ with $\eta \in V^*$ and $r \in \mathbf{R}$. We will identify the dual of $\mathbf{R}^n$ with $\mathbf{R}^n$ via the standard scalar product. We write $e_1, \ldots, e_n$ for the standard basis of $\mathbf{R}^n$ considered as V, and $\varepsilon_1, \ldots, \varepsilon_n$ for the standard basis of $\mathbf{R}^n$ considered as V^*. We will write $\mathbf{R}^n_0$ for the hyperplane in $\mathbf{R}^n$ of vectors whose coordinates sum to zero, $\mathbf{Z}^n_0 = \mathbf{R}^n_0 \cap \mathbf{Z}^n$, and $\mathbf{Z}^n_e$ for the subset of vectors whose coodinates have even sum. In all classical types except A_n, the index i runs over $1, \ldots, n$, and the pair i, j runs over $1 \leqslant i < j \leqslant n$. Thus, for example, $\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$ is a short-hand notation for $\{\pm\varepsilon_i \,|\, 1 \leqslant i \leqslant n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \leqslant i < j \leqslant n\}$.

  The Weyl group $W(\Psi)$ will be presented in the notation $T \rtimes W(\Phi)$, where T is the translation subgroup and $W(\Phi)$ is identified with the subgroup of $W(\Psi)$ that fixes the point $0 \in V = A$, which will be extra special. The computation of $W(\Psi)$ will use Proposition 1.3.61, which is applicable for all reduced types except BC_n, in which case a small additional (ad-hoc) argument is needed. In the same vein, we will present the extended affine Weyl group $W(\Psi)^{\mathrm{ext}}$ as $T^{\mathrm{ext}} \rtimes W(\Phi)$. For the computation of $W(\Psi)^{\mathrm{ext}}$ we will use Lemma 1.3.75 (in the version of this erratum). We will write Δ for a convenient choice of a set of simple affine roots, $\Delta_{\mathrm{es}} \subset \Delta_{\mathrm{s}}$ for the sets of extra special, and special but not extra special, elements in Δ, and Δ_{m} for the subset of multipliable elements in Δ (empty unless Ψ is non-reduced).

  - Type A_n, $n \geqslant 1$: $V = \mathbf{R}^{n+1}_0$, $\Psi = \{\pm(\varepsilon_i - \varepsilon_j) + \mathbf{Z}, 1 \leqslant i < j \leqslant n+1\}$, $\Phi = \{\pm(\varepsilon_i - \varepsilon_j), 1 \leqslant i < j \leqslant n+1\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = \mathbf{Z}^{n+1}_0 \rtimes S_{n+1}$, $W(\Psi)^{\mathrm{ext}} = \langle (-1, (n+1)^{-1}, \ldots, (n+1)^{-1}), \mathbf{Z}^{n+1}_0 \rangle \rtimes S_{n+1}$, $\Delta = \{1 + \varepsilon_{n+1} - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_n - \varepsilon_{n+1}\}$, $\Delta_{\mathrm{es}} = \Delta_{\mathrm{s}} = \Delta$.

  - Type B_n, $n \geqslant 3$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = \mathbf{Z}^n_e \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\mathrm{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$, $\Delta_{\mathrm{es}} = \Delta_{\mathrm{s}} = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2\}$.

- Type $B_n^\vee$, $n \geqslant 3$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $\overline{\Psi} = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$, $W(\Psi) = \mathbf{Z}_e^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\text{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$, $\Delta_{\text{es}} = \Delta_{\text{s}} = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2\}$.

- Type C_n, $n \geqslant 2$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\text{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\text{ext}} = (\frac{1}{2}(1, \ldots, 1) + \mathbf{Z}^n) \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1 - 2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$, $\Delta_{\text{es}} = \Delta_{\text{s}} = \{1 - 2\varepsilon_1, 2\varepsilon_n\}$.

- Type $C_n^\vee$, $n \geqslant 2$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$, $\overline{\Psi} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $W(\Psi) = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\text{ext}} = (\frac{1}{2}(1, \ldots, 1) + \mathbf{Z}^n) \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{\frac{1}{2} - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$, $\Delta_{\text{es}} = \Delta_{\text{s}} = \{\frac{1}{2} - \varepsilon_1, \varepsilon_n\}$.

- Type BC_n, $n \geqslant 1$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + (2\mathbf{Z} + 1)\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = W(\Psi)^{\text{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1 - 2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$, $\Delta_{\text{es}} = \{1 - 2\varepsilon_1\}$, $\Delta_{\text{s}} = \{1 - 2\varepsilon_1, \varepsilon_n\}$.

  One can also present BC_n as $\{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z} \smallsetminus \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$. Then Φ, $\overline{\Psi}$, $W(\Psi)$, and $W(\Psi)^{\text{ext}}$ remain unchanged, and the set of simple roots becomes $\Delta = \{-2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n + \frac{1}{2}\}$, where now $-2\varepsilon_1$ is extra special and $\varepsilon_n + \frac{1}{2}$ is special but not extra special.

  These two presentations are related to each other either by taking the dual root system with respect to an inner product on V^* for which the roots $\pm\varepsilon_i$ have length 1, or by applying the translation $\frac{1}{2}(e_1 + \cdots + e_n) \in V$ to A.

- Type D_n, $n \geqslant 4$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = \mathbf{Z}_e^n \rtimes ((\mathbf{Z}/2\mathbf{Z})_0^n \rtimes S_n)$, $W(\Psi)^{\text{ext}} = (\frac{1}{2}(1, \ldots, 1) + \mathbf{Z}^n) \rtimes ((\mathbf{Z}/2\mathbf{Z})_0^n \rtimes S_n)$, $\Delta = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$, $\Delta_{\text{es}} = \Delta_{\text{s}} = \{1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$.

- Types E_6, E_7, E_8, F_4, G_2: For the definition of V^* and $\Phi \subset V^*$ we refer to [Bou02, Chapter VI, Plates V–IX], where our V^* is denoted by V. Then $\Psi \subset A^* = V^* \oplus \mathbf{R}$ is given by $\Psi = \{a + \mathbf{Z} \,|\, a \in \Phi\}$, $\overline{\Psi} = \Phi$, $W(\Psi) = Q^\vee \rtimes W(\Phi)$, $W(\Psi)^{\text{ext}} = P^\vee \rtimes W(\Phi)$ where $Q^\vee \subset P^\vee \subset V$ denote the coroot and the coweight lattices of Φ. A set of simple roots is given by $\Delta = \{1 - \alpha_0, \alpha_1, \ldots, \alpha_r\}$, where $\{\alpha_1, \ldots, \alpha_r\}$ is a set of simple roots for Φ (the standard choices are listed in [Bou02, Chapter VI, Plates V–IX]), and α_0 is the highest root (denoted by $\tilde{\alpha}$ in loc. cit.) Using the enumeration in loc. cit., the extra special members are $1-\alpha_0, \alpha_1, \alpha_6$ for E_6, $1-\alpha_0, \alpha_7$ for E_7, $1-\alpha_0$ for E_8 and F_4 and G_2.

– Type $F_4^\vee$: $V = \mathbf{R}^4$, $\Psi = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}, \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 + 2\mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i, \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4\}$, $\overline{\Psi} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i, \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$, $W(\Psi) = W(\Psi)^{\mathrm{ext}} = \mathbf{Z}_e^4 \rtimes W(\Phi)$, $\Delta = \{1-\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_4, 2\varepsilon_4, \varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4\}$, $\Delta_{\mathrm{es}} = \Delta_{\mathrm{s}} = \{1-\varepsilon_1-\varepsilon_2\}$.

– Type $G_2^\vee$: $V = \mathbf{R}^3/\{\mathbf{R}\cdot(1,1,1)\}$, $V^* = \{(\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3 \mid \xi_1+\xi_2+\xi_3 = 0\}$, $\Psi = \{\pm 3(\varepsilon_1-\varepsilon_2)+3\mathbf{Z}, \pm 3(\varepsilon_1-\varepsilon_3)+3\mathbf{Z}, \pm 3(\varepsilon_2-\varepsilon_3)+3\mathbf{Z}, \pm(2\varepsilon_1-\varepsilon_2-\varepsilon_3)+\mathbf{Z}, \pm(2\varepsilon_2-\varepsilon_1-\varepsilon_3)+\mathbf{Z}, \pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)+\mathbf{Z}\}$, $\Phi = \{\pm 3(\varepsilon_1-\varepsilon_2), \pm 3(\varepsilon_1-\varepsilon_3), \pm 3(\varepsilon_2-\varepsilon_3), \pm(2\varepsilon_1-\varepsilon_2-\varepsilon_3), \pm(2\varepsilon_2-\varepsilon_1-\varepsilon_3), \pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)\}$, $\overline{\Psi} = \{\pm(\varepsilon_1-\varepsilon_2), \pm(\varepsilon_1-\varepsilon_3), \pm(\varepsilon_2-\varepsilon_3), \pm(2\varepsilon_1-\varepsilon_2-\varepsilon_3), \pm(2\varepsilon_2-\varepsilon_1-\varepsilon_3), \pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)\}$, $W(\Psi) = W(\Psi)^{\mathrm{ext}} = T \rtimes W(\Phi)$, where $T \subset V$ is the lattice generated by $\{\frac{1}{3}(2\varepsilon_1-\varepsilon_2-\varepsilon_3), \frac{1}{3}(2\varepsilon_2-\varepsilon_1-\varepsilon_3), \frac{1}{3}(2\varepsilon_3-\varepsilon_1-\varepsilon_2)\}$, $\Delta = \{1+\varepsilon_1+\varepsilon_2-2\varepsilon_3, 3(\varepsilon_1-\varepsilon_2), -2\varepsilon_1+\varepsilon_2+\varepsilon_3\}$, $\Delta_{\mathrm{es}} = \Delta_{\mathrm{s}} = 1+\varepsilon_1+\varepsilon_2-2\varepsilon_3\}$.

– Type (BC_n, C_n), $n \geqslant 1$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}$, $\Psi^{\mathrm{nd}} = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + (2\mathbf{Z}+1)\}$, $\Psi^{\mathrm{nm}} = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $W(\Psi) = W(\Psi)^{\mathrm{ext}=}\mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1-2\varepsilon_1, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \ldots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$, $\Delta_{\mathrm{m}} = \{\varepsilon_n\}$.

– Type $(C_n^\vee, BC_n)$, $n \geqslant 1$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$, $\Psi^{\mathrm{nd}} = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Psi^{\mathrm{nm}} = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $W(\Psi) = W(\Psi)^{\mathrm{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{\frac{1}{2}-\varepsilon_1, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \ldots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$, $\Delta_{\mathrm{m}} = \{\varepsilon_n\}$.

– Type $(B_n, B_n^\vee)$, $n \geqslant 2$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$, $\Psi^{\mathrm{nd}} = \{\pm\varepsilon_i + \mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Psi^{\mathrm{nm}} = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + 2\mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $W(\Psi) = \mathbf{Z}_e^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\mathrm{ext}} = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{1-\varepsilon_1-\varepsilon_2, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \ldots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$, $\Delta_{\mathrm{m}} = \{\varepsilon_n\}$.

– Type $(C_n^\vee, C_n)$, $n \geqslant 1$: $V = \mathbf{R}^n$, $\Psi = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}$, $\Psi^{\mathrm{nd}} = \{\pm\varepsilon_i + \frac{1}{2}\mathbf{Z}, \pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}\}$, $\Psi^{\mathrm{nm}} = \{\pm\varepsilon_i \pm \varepsilon_j + \mathbf{Z}, \pm 2\varepsilon_i + \mathbf{Z}\}$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}$, $W(\Psi) = \mathbf{Z}^n \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $W(\Psi)^{\mathrm{ext}} = (\frac{1}{2}(1,\ldots,1)+\mathbf{Z}^n) \rtimes ((\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n)$, $\Delta = \{\frac{1}{2}-\varepsilon_1, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \ldots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$, $\Delta_{\mathrm{m}} = \{\frac{1}{2}-\varepsilon_1, \varepsilon_n\}$.

Section 1.4. Add the following at the end of §1.4.

In the rest of this section, we will review the notion of Generalized Tits System introduced and studied by N. Iwahori.

Definition 1.4.15 A *generaized Tits system* is a quadruple (G, B, N, S) consisting of a group G, subgroups B and N of G, and a subset $S \subset N/(B \cap N)$, subject to the conditions

(1) The set $B \cup N$ generates G and $T := B \cap N$ is a normal subgroup of N.
(2) Every element of the set S is of order 2 and these elements generate a normal subgroup W' of the quotient $W := N/T$.
(3) Given $s \in S$ and $w \in W$ one has $sBw \subset BwB \cup BswB$
(4) Given $s \in S$ one has $sBs \neq B$.
(5) The stabilizer Ω of B in W is a complement to W'.
(6) Given $1 \neq \rho \in \Omega$ one has $B\rho \neq B$.

Lemma 1.4.16 *Let (G, B, N, S) be a generalized Tits system. Let W' and Ω be as in Definition 1.4.15. Let $G' = BW'B$ and let N' and Γ respectively be the inverse images of W' and Ω in N. Then*

(1) (G', B, N', S) is a Tits system with Weyl group W'.
(2) The group G' is normal in G and $G = G' \cdot \Gamma$.
(3) The conjugation action of G preserves the G'-conjugacy class of the pair (B, N').

Proof. (1) The conditions defining generalized Tits system immediately imply that (G', B, N', S) is a Tits system.

(2) Condition (5) in the above definition implies that Γ normalizes B and $\Gamma \cdot N' = N$. As $B \cup N'$ generates G', we see that Γ normalizes G' so $\Gamma \cdot G' = G' \cdot \Gamma$ is a subgroup, and since this subgroup contains $B \cup N$, it equals G. This in particular implies that G' is a normal subgroup of G.

(3) The third assertion follows from the fact that $G = G' \cdot \Gamma$ and Γ fixes the pair (B, N'). $\square$

Remark 1.4.17 Using the above Lemma and the well-known fact (see Remark 1.4.2) that the set S is uniquely determined as the set of those $w \in W'$ for which $B \cup BwB$ is a subgroup of G' (equivalently of G), we may also refer to the pair (B, N) as a generalized Tits system (or a generalized BN-pair). Since B is stable under the action of Ω, this description of S implies that it is stable under the conjugation action of Ω on W'.

Lemma 1.4.18 *Let (G', B, N', S) be a Tits system and assume that G' is a normal subgroup of a group G. Assume that the conjugation action of G*

preserves the G'-conjugacy class of the pair (B, N'). Let $\widehat{B}$ and $\widehat{N'}$ respectively be the normalizers of B and N' in G and let $\Gamma = \widehat{B} \cap \widehat{N'}$. Let $N = \Gamma \cdot N'$. Then (G, B, N, S) is a generalized Tits system.

Proof. Since Γ normalizes both B and N', it normalizes G' and also $T' := B \cap N'$. We will now show that $G = G' \cdot \Gamma$. To see this, let $g \in G$, then there is a $g' \in G'$ such that $g \circ (B, N') = g' \circ (B, N')$. Then $(g')^{-1} g$ normalizes both B and N' and hence it is contained in Γ. This implies that $g \in G' \cdot \Gamma$, so $G = G' \cdot \Gamma = \Gamma \cdot G'$. Hence, G is generated by $B \cup N$.

As $\Gamma \subset \widehat{B}$, $\widehat{B} = \Gamma \cdot (\widehat{B} \cap G')$. Since $\widehat{B} \cap G'$ is the normalizer of B in G', it equals B, so $\widehat{B} = \Gamma \cdot B$ and $\Gamma \cap G' = \Gamma \cap B \subset B$. Since $T' = B \cap N' \subset \widehat{B} \cap \widehat{N'} = \Gamma$, $T' \subset \Gamma \cap N' \subset B \cap N' = T'$. So $\Gamma \cap N' = B \cap N' = T'$.

Let $T := B \cap N = B \cap (\Gamma \cdot N')$. We wish now to show that $T = \Gamma \cap B$. For this purpose, let $\gamma n' \in B$, with $\gamma \in \Gamma$ and $n' \in N'$. Since both $\gamma n' (\in B)$ and γ normalize B, we conclude that n' normalizes B, hence it lies in $B \cap N' = T' \subset \Gamma \cap B$. So γ lies in $\Gamma \cap B$ which implies that $T = \Gamma \cap B$.

Now we want to show that $T (= \Gamma \cap B)$ is a normal subgroup of $N (= \Gamma \cdot N')$. As Γ normalizes $\Gamma \cap B (= \Gamma \cap G')$, it would suffice to show that for $\gamma \in \Gamma \cap B$ and $n \in N'$, $n \gamma n^{-1} \in T$. Since γ normalizes N', we see that the commutator $n \gamma n^{-1} \gamma^{-1} \in N'$, say $n \gamma n^{-1} \gamma^{-1} = n' \in N'$. Then $\gamma n^{-1} \gamma^{-1} = n^{-1} n'$. Now using the Bruhat decomposition for the pair (G', B), we see (since $\gamma \in B$) that $n' \in T'$. Hence, $n \gamma n^{-1} \in T' \gamma = \gamma T' \subset T$. This proves that T is a normal subgroup of N and we have verified condition (1) of the definition of the generalized Tits system for (G, B, N, S).

We have shown above that $G = \Gamma \cdot G'$, $\widehat{B} = \Gamma \cdot B$ and $T = \Gamma \cap B = \Gamma \cap G'$. Hence, $\Omega := \Gamma / T$, G/G' and $\widehat{B}/B$ are canonically isomorphic to each other.

As Γ contains T' and $T = \Gamma \cap B$, we find

$$N' \cap T = N' \cap (\Gamma \cap B) = \Gamma \cap B \cap N' = T'.$$

Now the inclusion of N' (resp. Γ) in N permits identification of $W' = N'/T'$ (resp. $\Omega = \Gamma/T$) as a subgroup of $W := N/T$. Moreover, since N' is a normal subgroup of N, W' is a normal subgroup of W, and as $\Gamma \cap N' \subset T' (\subset T)$, we see that $\Omega \cap W'$ is trivial, so W is the semidirect product $W = \Omega \ltimes W' = W' \rtimes \Omega$. Note that Γ normalizes B and $T \subset B$. Now condition (3) of the definition of the generalized Tits system holds for (G, B, N, S) is seen using the corresponding property of Tits system (G', B, N', S). The fact that $\Gamma \cap B = T$ verifies condition (6).

We have verified above all the conditions of the definition of generalized Tits system for (G, B, N, S). □

Lemma 1.4.19 *Let (G, B, N, S) be a generalized Tits system. Let N' and Γ be as in Lemma 1.4.16.*

(1) *G is equal to the disjoint union $\bigcup_{w\in W} B\,w\,B$.*
(2) *$N_G(B) = \bigcup_{\rho\in\Omega} B\,\rho\,B = B\,\Omega\,B = B\cdot\Omega = \Omega\cdot B$.*
(3) *$N_G(B)/B = \Omega$.*
(4) *The inclusion $\Gamma \to G$ induces an isomorphism $\Omega \to G/G'$.*

Proof. Since $W = W' \rtimes \Omega$ and Ω stabilizes B, assertion (1) follows at once from the Bruhat decomposition for the Tits system (G', B, N', S).

Assertion (2) follows from the facts that Ω stabilizes B and B is its own normalizer in G'.

Assertion (3) is clear from (2) and the fact that $\Gamma \cap B = T$.

From the facts that $G = \Gamma \cdot G'$ (Lemma 1.4.16(2)) and $\Gamma \cap G' = T'$ assertion (4) follows. □

Lemma 1.4.20 *Let (G, B, N, S) be a generalized Tits system.*

(1) *For any subgroup P_1 of G containing B there exists a unique subgroup $\Omega_1 \subset \Omega$ and a unique subset $S_1 \subset S$ such that $P_1 = BW_1'\Omega_1 B$, where $W_1' \subset W'$ is the subgroup generated by S_1. The subset S_1 of S is normalized by Ω_1.*
(2) *Given a subset $S_1 \subset S$ and a subgroup $\Omega_1 \subset \Omega$ that normalizes S_1, let $W_1' \subset W'$ be the subgroup generated by S_1 and let $W_1 = W_1' \rtimes \Omega_1$. Then $P_1 = BW_1B$ is a subgroup of G containing B.*
(3) *The previous two points give mutually inverse Ω-equivariant and inclusion-preserving bijections between the set of subgroups $P_1 \subset G$ that contain B and the set of pairs (S_1, Ω_1) consisting of a subset $S_1 \subset S$ and a subgroup $\Omega_1 \subset \Omega$ that normalizes S_1. Here Ω acts on the set of subgroups P_1 via its identification with $N_G(B)/B$.*
(4) *Let $P_1 \subset G$ be a subgroup containing B. Let $P_2 = N_G(P_1)$ and $P_1' = P_1 \cap G'$. Let (S_1, Ω_1), (S_2, Ω_2), and (S_1', Ω_1') be the pairs corresponding to P_1, P_2, and P_1', by the above bijection. Then $S_2 = S_1 = S_1'$, $\Omega_2 = N_\Omega(S_1, \Omega_1)$, and $\Omega_1' = \varnothing$.*
(5) *Let $P_1, P_2 \subset G$ be subgroups containing B and let (S_1, Ω_1) and (S_2, Ω_2) the associated pairs. The following are equivalent*
   (a) *P_1 and P_2 are G-conjugate.*
   (b) *P_1 and P_2 are $N_G(B)$-conjugate.*
   (c) *(S_1, Ω_1) and (S_2, Ω_2) are Ω-conjugate.*

Proof. We will use the Tits system (G', B, N', S) of Lemma 1.4.16.

(1) Let $P_1' = P_1 \cap G'$. Then $B \subset P_1'$ and according to the basic properties of a Tits system (Proposition 1.4.5) there exists a unique subset $S_1 \subset S$ such that $P_1' = BW_1'B$, where W_1' is the subgroup of W' generated by S_1. The normality of G' in G implies the normality of P_1' in P_1. Let $\Omega_1 = P_1/P_1' \subset G/G' = \Omega$, where we have used Lemma 1.4.19(4).

Let $\Delta_1 = N_{P_1}(B)$. Since P_1' is normal in P_1 and S_1 is uniquely determined by P_1', the action of Δ_1 on S stabilizes S_1.

The inclusion $\Delta_1 \subset P_1$ composed with the surjection $P_1 \to \Omega_1$ induces a map $\Delta_1 \to \Omega_1$. We claim this map is surjective. Indeed, given $p \in P_1$ mapping to an element of Ω_1 Lemma 1.4.16(3) implies the existence of $g \in G'$ such that $gpB(gp)^{-1} = B$. Thus $g^{-1}Bg = pBp^{-1}$ and both B and $g^{-1}Bg$ are contained in P_1'. Proposition 1.4.5(4) implies $g \in P_1'$, hence $gp \in \Delta_1$ and its image in Ω_1 is the same as that of p, proving the claim that $\Delta_1 \to \Omega_1$ is surjective. This claim implies that the action of Ω_1 on S stabilizes S_1. It also implies that $P_1 = P_1' \cdot \Delta_1 = BW_1'B\Delta_1 = BW_1'\Delta_1 B = BW_1'\Delta_1 B = BW_1'\Omega_1 B$, where the last identity follows from $\Delta_1 \cap P_1' = N_{P_1'}(B) = B$, i.e. $\Delta_1/B = \Omega_1$.

We have thus proved the existence of S_1 and Ω_1 with the desired properties. If S_2 and Ω_2 are such that $BW_1'\Omega_1 B = BW_2'\Omega_2 B$, then projecting modulo G' shows $\Omega_1 = \Omega_2$, and intersecting with G' shows $S_1 = S_2$.

(2) We know that $P_1' = BW_1'B$ is a subgroup of G' containing B. Letting $\Delta_1 \subset N_G(B)$ be the preimage of Ω_1, we know that Δ_1 stabilizes S_1, hence normalizes W_1', so $BW_1'B\Delta_1$ is also a group (a subgroup of G), and equals $BW_1'\Delta_1 B = BW_1 B$.

(3) From (2) we obtain the map $(S_1, \Omega_1) \mapsto BW_1B$, and (1) shows that this map is bijective. The Ω-equivariance is immediate from the construction.

(4) Since $N_{G'}(P_1') = P_1'$ (Proposition 1.4.5(3)) we see that $P_1 \cap G' = P_2 \cap G' = P_1'$. The construction in (1) shows $S_1 = S_2 = S_1'$. This construction also shows $\Omega_1' = \varnothing$. The Ω-equivariance of the bijection in (3) shows that $\Omega_2 \subset N_\Omega(S_1, \Omega_1)$. On the other hand, setting $\Omega_3 = N_\Omega(S_1, \Omega_1)$ and applying this bijection to (S_2, Ω_3) we obtain a subgroup $P_3 \subset G$ that contains P_2 and normalizes P_1'. But then $P_3 = P_2$, hence $\Omega_3 = \Omega_2$.

(5) The equivalence of (b) and (c) follows from the Ω-equivariance of the bijection of (3). That (b) implies (a) is tautological. Assuming (a) let $g \in G$ be such that $gP_1g^{-1} = P_2$. Then P_2 contains B and gBg^{-1}. But $gBg^{-1} \subset G'$, so $P_2' = P_2 \cap G'$ contains B and gBg^{-1}. By Proposition 1.4.5(4) there exists $p \in P_2'$ such that $B = pgB(pg)^{-1}$. Now $pgP_1(pg)^{-1} = P_2$ and $pg \in N_G(B)$, hence (b). $\square$

Even though the group W is not a Coxeter group, some of the properties of Coxeter groups can be extended to it. For example, the length function $\ell : W' \to \mathbf{N}$ extends to $W = W' \rtimes \Omega$ by defining $\ell(w'\rho) = \ell(w')$ with $w' \in W'$ and $\rho \in \Omega$. A reduced expression of $w \in W$ is an expression of the form $s_1 \dots s_n \rho$ with $s_i \in S$, $\rho \in \Omega$, and $\ell(w) = n$. Note that ρ is uniquely determined since W is semi-direct product of W' and Ω. As Ω normat of W', one can also write a reduced expression for $w \in W$ as $\rho s_1 \dots s_n$, where the $s_i \in S$ ocurring in this expression are the ρ^{-1}-conjugates of those occurring in the previous expression.

The Bruhat order also extends from W' to W, by

$$w_1'\rho_1 \leqslant w_2'\rho_2 \Leftrightarrow w_1' \leqslant w_2', \ \ \rho_1 = \rho_2.$$

Lemma 1.4.21 *Let $s \in S$ and $w \in W$. Then*

$$BsBwB = BswB \Leftrightarrow \ell(w) < \ell(sw) \Leftrightarrow w < sw,$$

$$BsBwB = BsB \cup BswB \Leftrightarrow \ell(w) > \ell(sw) \Leftrightarrow w > sw.$$

Proof. Write $w = w'\rho$ with $w' \in W'$ and $\rho \in \Omega$. Using the fact that ρ normalizes B the assertions follow from the properties of the Tits system (G', B, N', S). □

Corollary 1.4.22 *Let $w_1, w_2 \in W$. If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ then*

$$Bw_1Bw_2B = Bw_1w_2B.$$

Proof. Write $w_1 = \rho s_1 \ldots s_n$ as a reduced expression and use the previous lemma. □

Corollary 1.4.23 *Let $w_1, w_2 \in W$. Then*

$$Bw_1Bw_2B \subset \bigcup_{z \leqslant w_1} Bzw_2B, \qquad Bw_1Bw_2B \subset \bigcup_{z \leqslant w_2} Bw_1zB.$$

Proof. The second inclusion follows from the first by inversion (and replacing w_1, w_2 by w_2^{-1}, w_1^{-1}). The first inclusion follows by writing $w_1 = \rho s_1 \ldots s_n$ as a reduced expression and using the previous Lemma. □

Corollary 1.4.24 *Let $P_1, P_2 \subset G$ be subgroups containing B. Using Lemma 1.4.20(3) we associate pairs (S_1, Ω_1) and (S_2, Ω_2), and let $W_1, W_2 \subset W$ be the subgroups as in Lemma 1.4.20(2).*

(1) For $w \in W$ we have $P_1wP_2 = BW_1wW_2B$.
(2) The natural inclusion $N \to G$ induces a bijection $W_1\backslash W/W_2 \to P_1\backslash G/P_2$.

Proof. (1) The inclusion $BW_1wW_2B \subset P_1wP_2$ is obvious. For the converse inclusion, we use $P_i = BW_iB$, which reduces the proof to showing $Bw_1BwBw_2B \subset BW_1wW_2B$ for any $w_1 \in W_1$ and $w_2 \in W_2$. The latter follows from Corollary 1.4.23.

(2) Using $P_i = BW_iB$ we see that the natural inclusion $N \to G$ induces a map $W_1\backslash W/W_2 \to P_1\backslash G/P_2$. Using $G = BWB$ we see that this map is surjective. Injectivity is equivalent to the equivalence of $P_1aP_2 = P_1bP_2$ and $a \in W_1bW_2$ for $a, b \in W$. This latter equivalence follows from (1). □

Section 1.5.

- Just before the statement of Proposition 1.5.6, the appeal to Proposition 1.5.13 should be replaced by Proposition 1.5.6(5), with $\mathcal{A} = \mathcal{A}_1$, $g\mathcal{A} = \mathcal{A}_2$, $\mathcal{F} = \mathcal{F}_1$, and $n\mathcal{F}' = \mathcal{F}_2$.

- At the end of the statement of Proposition 1.5.6, add the following as a new item.

  (8) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two apartments that share a common chamber, then there exists a $g \in G$ such that $g\mathcal{A}_1 = \mathcal{A}_2$ and g fixes every facet contained in $\mathcal{A}_1 \cap \mathcal{A}_2$.*

- Just before the end of the proof of Proposition 1.5.6, add the following as two new paragraphs.

  (8) Since according to (7), G acts transitively on the set of pairs consisting of an apartment and a chamber contained in it, we may assume, after replacing $\mathcal{A}_1$ and $\mathcal{A}_2$ by their conjugates under an element of G, that $\mathcal{A}_1 = \mathcal{A}$ and the standard chamber $\mathcal{C}$ lies in $\mathcal{A}_2$. Now suppose for $i = 1, 2$, $g_i \in G$ is such that it fixes $\mathcal{C}$ and carries $\mathcal{A}$ to $\mathcal{A}_2$. Then $g_1^{-1}g_2$ carries $\mathcal{A}$ to itself and fixes $\mathcal{C}$. Hence, $g_1^{-1}g_2 \in B \cap N$. Now we will show that every element of $B \cap N$ fixes every facet lying in $\mathcal{A}$, so its action on $\mathcal{A}$ is trivial. To see this, we note that $B \cap N$ is a normal subgroup of N and every facet of $\mathcal{A}$ is of the form $n\mathcal{F}'$ for $n \in N$ and $\mathcal{F}'$ a (not necessarily proper) facet of $\mathcal{C}$. Now
  $$B \cap H \subset B = P_{\mathcal{C}} \subset P_{\mathcal{F}'},$$
  and hence $B \cap H \subset nP_{\mathcal{F}'}n^{-1} = P_{n\mathcal{F}'}$. This shows that $B \cap H$ fixes $n\mathcal{F}'$.

  We conclude from the above consideration that the action of g_1 and g_2 on $\mathcal{A}$ coincide. Now as we observed, every facet of $\mathcal{A}$, and so also that of $\mathcal{A} \cap \mathcal{A}_2$, is of the form $n\mathcal{F}'$, with $n \in N$, and $\mathcal{F}'$ is a (not necessarily proper) face of $\mathcal{C}$. Using (5) with $\mathcal{F} = \mathcal{C}$, we conclude that any $g \in G$ that carries $\mathcal{A}$ to $\mathcal{A}_2$ and fixes $\mathcal{C}$, fixes every facet lying in $\mathcal{A} \cap \mathcal{A}_2$.

- At the end of item (2) of Definition 1.5.17 add the following sentence. "Note that the notion of admissibilty of a facet, or a parabolic subgroup, is defined with respect to a given Tits system."

- In the proof of Proposition 1.5.18, replace $P_{S_i^c}$ with $G_{S_i^c}$.

- At the end of Definition 1.5.20 add the following two sentences. " The notion of restricted building makes sense only when we are given a Tits system (G, B, N, S). Then the restricted building of Proposition 1.5.18 is called *the restricted building of the Tits system* (G, B, N, S)."

- Replace the word "simplicial" with "polysimplicial" in the first sentence of Proposition 1.5.28 and in the paragraph appearing just after the statement of this proposition.

- At the end of the statement of Proposition 1.5.28, add the following as a new paragraph.

  Let $S \subset B/B\cap N$ be the set of reflections as in Remark 1.4.2 (S is the set of reflections in the codimension-1 faces of $\mathcal{C}$). If S is not irreducible, then $S = S_1 \cup \cdots \cup S_n$, is a disjoint union of mutually commuting irreducible subsets. Then the restricted building $\mathcal{B}'$ determined by (G, B, N, S) is the set of admissible (with respect to (G, B, N, S)) facets of $\mathcal{B}$.

- Just before the end of the proof of Proposition 1.5.28, add the following two new paragraphs.

  We will now prove the last assertion. Note that by the definition of Tits systems, B is a minimal parabolic subgroup of G. We assume in the rest of this paragraph that S is irreducible. Then every proper parabolic subgroup is admissible and set of all proper parabolic subgroups of G containing B is parametrized by the set of proper subsets of S, with B corresponding to the empty subset. The chamber $\mathcal{C}$ is a simplex. Its faces are also in bijective correspondence with proper subsets of S and the stabilizer of any such face contains B and it is the proper parabolic subgroup corresponding to the subset of S that corresponds to the face. Thus we see that proper parabolic subgroups containing B are in natural bijective correspondence with the faces of $\mathcal{C}$, and the parabolic subgroup corresponding to a face of $\mathcal{C}$ is the stabilizer of that face. Now note that any facet of $\mathcal{B}$ is conjugate (under G) to a unique face of $\mathcal{C}$ and any proper parabolic subgroup of G is conjugate (under G) to a unique proper parabolic subgroup containing B. Thus $\mathcal{B}$ is the (restricted) building of the Tits system (G, B, N, S), and it is moreover a simplicial complex.

  Now we assume that S is not irreducible. Then as in the proof of Proposition 1.5.18, $\mathcal{B}' \to \prod_i \mathcal{B}_i$ is an isomorphism, where $\mathcal{B}'$ is the restricted building of the Tits system (G, B, N, S) and $\mathcal{B}_i$ is the building of the Tits system $(G, G_{S_i^c}, N, S_i)$. It is easily seen that $\mathcal{B}$ contains the product $\prod_i \mathcal{B}_i$ (see Proposition 1.4.10) as the set of admissible facets (with respect to the Tits system (G, B, N, S)) of $\mathcal{B}$.

Section 1.6. Replace the text of that section with the following.

This section does not belong to the discussion of affine root systems and abstract buildings. It discusses filtrations indexed by the real numbers, a notion that is instrumental in Bruhat–Tits theory.

Definition 1.6.1 Let X be a pointed set.

(1) A *descending* $\mathbf{R}$-*filtration* on X assigns to each $r \in \mathbf{R}$ a subset $X_r \subset X$ containing the distinguished point, such that $X_s \subset X_r$ whenever $s > r$.
(2) The filtration is called *separated* if $\bigcap_{r\in\mathbf{R}} X_r = \{*\}$, where $* \in X$ is the distinguished point.

Remark 1.6.2 This definition has some obvious variations. We can work with ordinary sets instead of pointed sets, in which case separatedness should mean $\bigcap_{r\in\mathbf{R}} X_r = \varnothing$. Or we could work with groups (or vector spaces, algebras, etc), rather than general pointed sets, in which case each X_r should be required to be a subgroup (or subspace, subalgebra, etc). We could also consider filtrations indexed by $\mathbf{R}_{\geqslant 0}$ instead of $\mathbf{R}$. We will usually say just "filtration" in place of "$\mathbf{R}$-filtration".

It is oftentimes useful to consider the subset $X_{r+} = \bigcup_{s>r} X_s$, which is contained in X_r and could be a proper subset. We can think of $r+$ as a number that is infinitesimally larger than r; it is larger than r, but smaller than any real number that is larger than r. This leads us to introduce the set $\tilde{\mathbf{R}} = (\mathbf{R} \times \{0, 1\}) \cup \{\infty\}$. We will write r in place of $(r, 0)$ and $r+$ in place of $(r, 1)$, and we think of $r+$ as a number infinitesimally larger than r.

The set $\tilde{\mathbf{R}}$ is made into a totally ordered commutative monoid that contains $\mathbf{R}$ as an ordered submonoid via the following rules.

(1) $r + (s+) = (r+) + s = (r+) + (s+) = (r + s)+$.
(2) $r + \infty = (r+) + \infty = \infty$.
(3) $\infty > r+ > r$ for any $r \in \mathbf{R}$.
(4) $r+ > s+$ if $r > s$.

We define an operation $\tilde{r} \mapsto \tilde{r}+$ on $\tilde{\mathbf{R}}$ by setting $(r+)+ = r+$ and $\infty+ = \infty$.

Another way to think about the monoid $\tilde{\mathbf{R}}$ is as the monoid of intervals of $\mathbf{R}$ of the form $[r, \infty)$ or (r, ∞) for $-\infty < r \leqslant \infty$. Then $r \in \mathbf{R}$ corresponds to $[r, \infty)$, $r+$ corresponds to (r, ∞), and ∞ corresponds to the empty interval. Addition corresponds to pointwise addition of intervals. The operation $\tilde{r} \to \tilde{r}+$ corresponds to taking the interior. The order is the opposite of the inclusion order.

A descending filtration X_r of a set X indexed by $\mathbf{R}$ extends, as was just discussed, to a descending filtration indexed by $\tilde{\mathbf{R}}$, with X_{r+} defined as above.

Definition 1.6.3 Let X be a pointed set equipped with a descending separated filtration.

(1) A real number $r \in \mathbf{R}$ is called a *jump* of the filtration if $X_{r+} \neq X_r$.
(2) The filtration is called *continuous*, if $X_{r+} = X_r$ for all $r \in \mathbf{R}$.

(3) The filtration is called *totally discontinuous*, if for every $* \neq x \in X$ there exists $r \in \mathbf{R}$ such that $x \in X_r \setminus X_{r+}$, i.e. if $X \setminus \{*\} = \bigcup_{r \in \mathbf{R}} (X_r \setminus X_{r+})$.

If we introduce the convention $X_{\infty+} = \varnothing$ then we have $x \in X_\infty \setminus X_{\infty+}$ and $X = \bigcup_{r \in \mathbf{R} \cup \{\infty\}} (X_r \setminus X_{r+})$.

Example 1.6.4 Consider the pointed set $X = \mathbf{R} \cup \{\infty\}$ with distinguished point ∞. The filtration $X_r = (r, \infty]$ is continuous. The filtration $X_r = [r, \infty]$ is totally discontinuous and has $X_{r+} = (r, \infty]$.

Given a descending separated filtration on the pointed set X we can define the function

$$\varphi : X \to \mathbf{R} \cup \{\infty\}, \qquad \varphi(x) = \sup\{r \mid x \in X_r\}.$$

It has the property that $\varphi^{-1}(\infty) = \{*\}$.

Conversely, given any function $\varphi : X \to \mathbf{R} \cup \{\infty\}$ with $\varphi^{-1}(\infty) = \{*\}$ we can define the descending separated filtration

$$X_r = \{x \in X \mid \varphi(x) \geqslant r\}.$$

Fact 1.6.5 Let A be the set of all descending separated filtrations on X and let B be the set of all functions $X \setminus \{*\} \to \mathbf{R}$, and let $\alpha : A \to B$ and $\beta : B \to A$ be the maps defined above.

(1) $\alpha \circ \beta = \mathrm{id}_B$. Thus α is surjective and β is injective.
(2) The image of β is the subset of totally discontinuous filtrations.
(3) Given a totally discontinuous filtration, the set of jumps of that filtration is equal to $\varphi(X \setminus \{*\})$, and for $r \in \mathbf{R}$ we have $X_{r+} = \{x \in X \mid \varphi(x) > r\}$.

In this book, we will be interested only in totally discontinuous filtrations, and in fact only in such whose set of jumps is discrete. We refer to Definition 6.1.2 and Section 7.2 for the key instances of such filtrations in Bruhat–Tits theory.

CHAPTER 2

Section 2.2.

- In the first paragraph of 2.2, replace the last sentence with " But we assume that k is Henselian." In the italicized paragraph at the end of p. 71, delete the first sentence.

Section 2.5.

- In the fifth line of 2.5(b) and in the fourth line of 2.6(d), ω_T and ω_G should be defined by $t \mapsto (\chi \mapsto \omega(\chi(t)))$ and $g \mapsto (\chi \mapsto \omega(\chi(g)))$ respectively.

- In Remark 2.5.11, replace "§7" by "§6".

- Remark 2.5.15 should be replaced by the following.

  The inclusion $T(k)^0 \subset T(k)^1$ may be strict. For example, let T be the kernel of the norm map $\mathrm{R}_{\ell/k}\mathbf{G}_m \to \mathbf{G}_m$ for some separable quadratic extension ℓ/k. Then $T(k)^1 = T(k) = \ell^1$ is the subgroup of $\ell^\times$ consisting of elements whose norm in k equals 1. The map $\ell^\times \to \ell^1$ sending $z \in \ell^\times$ to $z/\bar{z}$ is surjective according to Hilbert's theorem 90, where $\bar{z}$ denotes the Galois conjugate of z, thus giving the isomorphism $\ell^\times/k^\times \to \ell^1 = T(k)$. In fact, the map $\ell^\times \to \ell^1$ is the norm map $T(\ell) = \mathrm{R}_{\ell/k}(T_\ell)(k) \to T(k)$. Since $T(\ell)_0 = \mathfrak{o}_\ell^\times$, we see that $T(k)^0 = \{z/\bar{z} \,|\, z \in \mathfrak{o}_\ell^\times\}$. We conclude that we have the identifications

  $$T(k)^0 = \mathfrak{o}_\ell^\times/\mathfrak{o}_k^\times \subset \ell^\times/k^\times = T(k)^1 = T(k).$$

  When ℓ/k is unramified the inclusion $\mathfrak{o}_\ell^\times/\mathfrak{o}_k^\times \subset \ell^\times/k^\times$ is an equality, but when ℓ/k is ramified this inclusion is of index 2 and coincides with the kernel of the surjection $\ell^\times/k^\times \to \mathbf{Z}/2\mathbf{Z}$ given by the normalized valuation of ℓ.

  When ℓ/k is tamely ramified we can also identify $T(k)^0 = (1+\mathfrak{p}_\ell)\cap\ell^1 \subset T(k) = \ell^1$. Indeed, we can choose a uniformizer $\varpi \in \ell^\times$ such that $\overline{\varpi} = -\varpi$ and use it to write $\ell^\times = \mathfrak{f}^\times \times (1+\mathfrak{p}_\ell) \times \varpi^{\mathbf{Z}}$, where the residue field $\mathfrak{f}$ is embedded into ℓ via Teichmüller representatives. The map $z/\bar{z}$ is trivial on $\mathfrak{f}^\times$, and we conclude that it sends $\mathfrak{o}_\ell^\times$ into $(1+\mathfrak{p}_\ell)\cap\ell^1$. To see that the latter is exhausted by the image of $\mathfrak{o}_\ell$, we note that if we compose $z/\bar{z} : \ell^\times \to \ell^1 \subset \mathfrak{o}_\ell^\times$ with the projection $\mathfrak{o}_\ell^\times \to \mathfrak{f}^\times$, then elements of even valuation map to 1, while elements of odd valuation map to -1. Therefore the preimage of $(1+\mathfrak{p}_\ell)\cap\ell^1$ consists of the elements of even valuation, but any such element is the product of an element of $\mathfrak{o}_\ell^\times$ and a power of $\varpi^2 \in k$.

Section 2.6.

- Add the following two sentences just before Definition 2.6.20.

  In the rest of this section, k is a field endowed with a valuation. We assume that k is Henselian and its residue field is perfect.

- In the first sentence of Definition 2.6.23, replace "reductive group" by "reductive k-group".

Section 2.11.

- In §2.11, replace " $\mathscr{P}_{\mathscr{G}}(\lambda)$, $\mathscr{U}_{\mathscr{G}}(\lambda)$ *and* $\mathscr{Z}_{\mathscr{G}}(\lambda)$" with "$P_{\mathscr{G}}(\lambda)$. $U_{\mathscr{G}}(\lambda)$ *and* $Z_{\mathscr{G}}(\lambda)$".

- In assertions (3) and (6) of Proposition 2.11.1, replace " $\mathscr{P}_{\mathscr{G}}$, $\mathscr{U}_{\mathscr{G}}$ *and* $\mathscr{Z}_{\mathscr{G}}$" with "$P_{\mathscr{G}}(\lambda)$, $U_{\mathscr{G}}(\lambda)$ *and* $Z_{\mathscr{G}}(\lambda)$".

- In assertions (4) and (5) of Proposition 2.11.1, replace "$\mathscr{P}_{\mathscr{H}}(\lambda)$" with " $P_{\mathscr{H}}(\lambda)$" and "$\mathscr{U}_{\mathscr{H}}$ and $\mathscr{Z}_{\mathscr{H}}$" , as well as, "$\mathscr{U}_{\mathscr{H}}(\lambda)$ and $\mathscr{Z}_{\mathscr{H}}(\lambda)''$ by "$U_{\mathscr{H}}(\lambda)$ and $Z_{\mathscr{H}}(\lambda)$" respectively.

Chapter 3

- On p.142, A few lines before Notation 3.1.1, there is the following incomplete sentence “There is a unique maximal bounded subgroup in $T(k)$, called.” It shold be replaced with “There is a unique maximal bounded subgroup in $T(k)$, it is denoted $T(k)_0$.”

- Remark 3.2.3 is incorrect as stated, and is now superseded by the new version of Section 1.6 (in this erratum).

Section 4.1.

- In AS 3 on the top of p. 160, add $\cup\{1\}$ in the definitions of U_ψ and $U_{\psi+}$.

- In the last line of Remark 4.1.7, replace "$\psi_a^u = -\psi_a^{u'} = -\psi_a^{u''}$" with "$\psi_a^u = -\psi_{-a}^{u'} = -\psi_{-a}^{u''}$".

- In Axiom 4.1.8(2), it should be emphasized that Ω is assumed non-empty and bounded.

- In the first sentence of Axiom 4.1.9, immediately after $\mathcal{B}(G)$, add "($\mathcal{B}(G)$ as in Axiom 4.1.1)", in the last sentence of this axiom, replace "is precisely the" by "is a".

- Replace the third sentence of the proof of Fact 4.1.25 with the following:

  The Bruhat decomposition (cf. Proposition 1.4.5) for this Tits system implies that $\mathcal{G} = \mathcal{I} \cdot \mathcal{N} \cdot \mathcal{I}$. Now let $G(k)_\mathcal{C}$ be the stabilizer of $\mathcal{C}$ in $G(k)$. Then $\mathcal{I} = G(k)_\mathcal{C} \cap \mathcal{G}$, hence $\mathcal{I}$ is a normal subgroup of $G(k)_\mathcal{C}$. As both $N(k)$ and $\mathcal{N}$ act transitively on the set of chambers contained in $\mathcal{A}$, we see that $N(k) \subset G(k)_\mathcal{C} \cdot \mathcal{N}$. Hence, $N(k) = N(k)_\mathcal{C} \cdot \mathcal{N}$, where $N(k)_\mathcal{C} = G(k)_\mathcal{C} \cap N(k)$. Now as $G(k) = Z(k) \cdot \mathcal{G}$ (Fact 2.6.22), and $Z(k) \subset N(k)$, we see that

  $$G(k) = N(k)_\mathcal{C} \cdot \mathcal{I} \cdot \mathcal{N} \cdot \mathcal{I} = \mathcal{I} \cdot N(k)_\mathcal{C} \cdot \mathcal{N} \cdot \mathcal{I} = \mathcal{I} \cdot N(k) \cdot \mathcal{I}.$$

- Please replace the second line of Definition 4.1.26 by "Axioms 4.1.1, 4.1.2, 4.1.4, 4.1.6, 4.1.8, 4.1.9, 4.1.16, 4.1.17, 4.1.20, and 4.1.22 hold."

- Add the following remark at the end of §4.1.

  Remark 4.1.30 Let $S \subset G$ be a maximal split torus, $N = N_G(S)$, $\mathcal{C}$ a chamber in the apartment corresponding to S, and $\mathcal{I} = G(k)^0_\mathcal{C}$ the associated Iwahori subgroup (Definition 4.1.3). According to Axiom 4.1.1 the group $G(k)$ acts on the building $\mathcal{B}(G)$, which in turn is the restricted building of the Iwahori–Tits system, i.e. the Tits system of Axiom 4.1.9. Since $G(k)^0$ acts transitively on the set of pairs consisting of an apartment and a chamber of it (Proposition 1.5.6), the $G(k)$-action preserves the $G(k)^0$-conjugacy class of this Tits system, and Lemma 1.4.18 implies that $(\mathcal{I}, N(k))$ is a generalized Tits system (Definition 1.4.15, this erratum) in $G(k)$. Indeed, if $(\mathcal{I}, \mathcal{N})$ is the Iwahori–Tits system, then the normalizer of $\mathcal{N}$ in $G(k)$ equals $N(k)$, while the normalizer of $\mathcal{I}$ in $G(k)$ equals the stabilizer $G(k)_\mathcal{C}$ of the chamber $\mathcal{C}$ corresponding to $\mathcal{I}$. The intersection

$N(k) \cap G(k)_{\mathcal{C}}$ equals $N(k)_{\mathcal{C}}$. Now $N(k) = \mathcal{N} \cdot N(k)_{\mathcal{C}}$, since $N(k)$ preserves $\mathcal{A}$ and $\mathcal{N}$ acts transitively on the set of chambers in $\mathcal{A}$.

The group $\Omega = G(k)_{\mathcal{C}}/\mathcal{I} = G(k)/G(k)^0$ of this generalized Tits system (cf. Lemma 1.4.19(3)) is identified by the Kottwitz homomorphism with the group $(\pi_1(G)_I)^\Gamma$ when the residue field has dimension $\leqslant 1$, in particular, when it is algebraically closed, in which case $\Omega = \pi_1(G)_I$, see Corollary 11.7.5.

Lemma 1.4.20 now gives a complete description of all (possibly disconnected) "parahoric" subgroups, i.e. all subgroups of $G(k)$ that contain an Iwahori subgroup. Among those containing the fixed Iwahori subgroup $\mathcal{I}$, those that are "connected" (in the sense that their integral models have connected special fiber, or equivalently, are contained in $G(k)^0$) correspond to subsets of the set Ψ^0 of simple affine roots corresponding to the chamber $\mathcal{C}$. Allowing disconnectedness, these subgroups are parameterized by pairs (S_1, Ω_1) consisting of a subset $S_1 \subset \Psi^0$ and a subgroup $\Omega_1 \subset \Omega$ such that the action of Ω_1 on Ψ^0 preserves S_1. If $P \subset G(k)$ corresponds to (S_1, Ω_1) then the subgroup $P^0 \subset P$ that corresponds to $(S_1, \{1\})$ is the "identity component of P", in the sense that the special fiber of the integral model of P^0 is the identity component of the special fiber of the integral model of P, and Ω_1 is the "finite étale component group of P", in the sense that when the residue field of k has dimension $\leqslant 1$ then Ω_1 is isomorphic to the group of rational points of the component group of the special fiber of the integral model for P.

Section 4.2.

- Add 4.1.6 after 4.1.4 in the first sentence of 4.2.

- At the beginning of the second line of the statement of Proposition 4.2.1, add "which is G-invariant and".

- In the proof of Proposition 4.2.1, the appeal to Proposition 1.5.13 should be replaced by an appeal to Proposition 1.5.6(5). Indeed, 1.5.13 requires that x, y belong to the closure of the same chamber, which cannot be assumed in the given situation. On the other hand, 1.5.6(5) can be used by taking $\mathcal{A} = \mathcal{A}_1$, $g\mathcal{A} = \mathcal{A}_2$ (using that all apartments are conjugate under $G(k)^0$), $\mathcal{F}$ the facet containing x, $\mathcal{C}$ a chamber whose closure contains $\mathcal{F}$, and n in the normalizer of the maximal k-split torus corresponding to $\mathcal{A}_1$ that transports $\mathcal{C}$ to a chamber whose closure contains y.

- The third sentence of the second paragraph of the proof of Proposition 4.2.1 should be replaced with the following.

If $\mathcal{A}'$ is another apartment containing x and y, then there exists a $g \in G(k)^0$ that fixes both x and y and $g\mathcal{A} = \mathcal{A}'$ (see Proposition 4.2.24 for a more general result). To see this, let $\mathcal{C}$ (resp. $\mathcal{C}'$) be a chamber in $\mathcal{A}$ (resp. $\mathcal{A}'$) such that x (resp. y) lies in the closure of $\mathcal{C}$ (resp. $\mathcal{C}'$). Let $\mathcal{A}''$ be an apartment that contains the chambers $\mathcal{C}$ and $\mathcal{C}'$. Using Proposition 1.5.6(8), we see that there exist g' and g'' in $G(k)^0$ such that $g'\mathcal{A}'' = \mathcal{A}'$, $g''\mathcal{A} = \mathcal{A}''$, and g' (resp. g'') fixes every element of $\mathcal{A}' \cap \mathcal{A}''$ (resp. $\mathcal{A} \cap \mathcal{A}''$). Now let $g = g'g''$. Then g fixes every element of $\mathcal{A} \cap \mathcal{A}' \cap \mathcal{A}''$, and so it fixes both x and y, and it carries $\mathcal{A}$ to $\mathcal{A}'$.

- In the first sentence of the proof of Proposition 4.2.18, replace "since $G(k)$ is second countable and $\mathcal{B}$ is a metric space" by "since both $G(k)$ and $\mathcal{B}$ are metric spaces".

- Replace the proof of Corollary 4.2.21 with the following:

  Let $f : \mathcal{B} \to \mathcal{B}$ be a $G(k)$ equivariant isometry and let x be a vertex of $\mathcal{B}$. Then according to 4.1.12 (2), x is the unique point of $\mathcal{B}$ fixed under $G(k)^0_{\{x\}}$. This implies that $f(x) = x$. Now Lemma 4.2.20 implies that f is the identity.

- Replace the last three lines of the proof of Lemma 4.2.22 by the following.

  this facet $\mathcal{F}_n'$ is then the convex hull of $\mathcal{F}_n$ and $\mathcal{F}_q$. Moreover, the facet $\mathcal{F}$ is the convex hull of the union $\bigcup_{n\leqslant q-1} \mathcal{F}_n$. So Ω is the convex hull of $\bigcup_{n\leqslant q-1} \mathcal{F}_n'$. Hence, Ω is contained in the same side of each root hyperplane, so it is contained in the closure of a facet in $\mathcal{A}$ and the corresponding parahoric subgroup is contained in $\bigcap_{n\leqslant q} \mathcal{P}_n$. For otherwise, for some $n \leqslant q-1$, $\mathcal{F}_n'$ is intersected by a root hyperplane, but this is impossible since $\mathcal{F}_n'$ is a facet of $\mathcal{B}$.

Section 4.4.

- In the first line of the statement of Corollary 4.4.5, replace "If there exists a" with "If for every", and in the second line delete "such that".

- In the second paragraph of the proof of Proposition 4.4.6, replace $T(k)^0$ by $Z(k)^0$; in the second line of p. 185, replace $N(k)_x$ by $N(k)$.

Chapter 5

- In §5.2, reverse the inequality in (5.2.2).

- On the top of p. 189, add the following after the first sentence.

  The following alternative descriptions of M_a, for non-divisible a, are useful. For a non-divisible $a \in \Phi$, let S_a be the torus contained in $\operatorname{Ker} a\,(\subset S)$. Then M_a is the centralizer of S_a in G_a. If λ is a 1-parameter k-subgroup of S_a such that $\langle \lambda, b\rangle \neq 0$ for all $b \in \Phi$ which restricts non-trivially to S_a. Then M_a is the centralizer of λ in G.

- On the top of p. 190, replace "Also, we know from Axiom 4.1.16 and the compatibility of the open cell with closed subgroups (cf. Proposition 2.11.4(4))" with the following

  "Let λ be a 1-parameter k-subgroup of S such that M_a is the centralizer of λ in G. Then as the factors of the decomposition of the open cells of G with respect to (S, Φ) and $(S, -\Phi)$ are stable under conjugation by λ, we see using the decomposition of $\mathcal{P}_{\mathcal{C}}$ given by Axiom 4.1.16"

- Replace Lemma 5.3.5 and its proof with the following:

  Lemma 5.3.5 *We use the notation as in Theorem 5.3.3. For n, $n' \in N(k)$, $n \in U(k)\,n'\,\mathcal{P}_{\mathcal{C}}$ if and only if $n'\,n^{-1} \in Z(k)^0$.*

  Proof. Since $Z(k)^0$ is contained in $\mathcal{P}_{\mathcal{C}}$, it is obvious that if $n'\,n^{-1} \in Z(k)^0$, then $n \in U(k)\,n'\,\mathcal{P}_{\mathcal{C}}$. To prove the converse, let $\mathcal{C}' = n \cdot \mathcal{C}$. Then $n \in U(k)\,n'\,\mathcal{P}_{\mathcal{C}}$ if and only if $un'\,n^{-1} \in \mathcal{P}_{\mathcal{C}'}$ for some $u \in U(k)$. We will use the Iwahori decomposition (Axiom 4.1.16) of $\mathcal{I} := \mathcal{P}_{\mathcal{C}'}$ to show that if $un'\,n^{-1} \in \mathcal{I}$, then $n'\,n^{-1} \in Z(k)^0$. According to this decomposition,

  $$\mathcal{I} = (U(k) \cap \mathcal{I}) \times Z(k)^0 \times (U^-(k) \cap \mathcal{I});$$

  where U^- is the unipotent k-subgroup opposite to U determined by the set $-\Phi^+$ of roots. Using this, we see that the assertion $un'\,n^{-1} \in \mathcal{I}$ implies that $n'\,n^{-1} \in U(k)\,Z(k)\,U^-(k)$. Now the usual Bruhat decomposition of $G(k)$ in double cosets of $\{U(k)\,Z(k),\ U^-(k)\,Z(k)\}$ implies that $z := n'\,n^{-1} \in Z(k)$. Now again using the Iwahori decomposition of $\mathcal{I}$, we conclude that $u \in U(k) \cap \mathcal{I}$ and $z \in Z(k)^0$. Thus we have proved the lemma. □

- Add the following at the end of §5.4

  The following argument appears in the proof of Lemma 1.6.1 of the paper by Haines-Kottwitz-Amritanshu Prasad on Iwahori-Hecke algebras.

Lemma 5.4.7 Let S be a maximal k-split torus of G and I the Iwahori subgroup corresponding to a chamber in the apartment of S. Let $\widetilde{W}^0$ be the Iwahori–Weyl group (defined in 6.6.1). Let $w_1, w_2 \in \widetilde{W}^0$. Let U be the unipotent radical of any minimal parabolic subgroup with Levi factor $Z = Z_G(S)$. If $U(k)w_1I$ intersects Iw_2I, then $w_1 \leqslant w_2$ in the Bruhat order of $\widetilde{W}^0$.

Proof. Let $u \in U(k)$ be such that $uw_1 \in Iw_2I$. Choose a uniformizer $\pi \in k$, and $\lambda \in X_*(S)$ such that $\lambda(\pi)u\lambda(\pi)^{-1} \in I$. Then $(\lambda(\pi)u\lambda(\pi)^{-1})\,\lambda(\pi)w_1 \in \lambda(\pi)Iw_2I$, and therefore,

$$I\lambda(\pi)w_1I \subset I\lambda(\pi)Iw_2I \subset \bigcup_{\substack{z\in\widetilde{W}^0 \\ z\leqslant w_2}} I\lambda(\pi)zI.$$

Hence, for some $z \leqslant w_2$, we have $I\lambda(\pi)w_1I = I\lambda(\pi)zI$, thus $w_1 = z$. $\square$

Chapter 6

- In the proof of Lemma 6.1.12 the first string of equations should read

$$\begin{aligned}
[n(\varphi + v)]_a(u) &= (\varphi + v)_{w^{-1}a}(n^{-1}un) \\
&= \varphi_{w^{-1}a}(n^{-1}un) + w^{-1}a(v) \\
&= [n\varphi]_a(u) + a(wv) \\
&= [n\varphi + wv]_a(u).
\end{aligned}$$

- In the third line of §6.6, replace the sentence "Let N and Z be as above." with "Let N be the normalizer and Z be the centralizer of S in G."

- In the line just before the proof of Proposition 6.6.2, replace "is part of Theorem 7.5.1" with "this is Proposition 9.4.35".

- At the end of the second paragraph of (the first page of) Chapter 7, add the following sentences "As discussed in §6.3, this endows the affine space $\mathcal{A}(S)$ with two (closely related) affine root systems $\Psi \subset \Psi' \subset \mathcal{A}(S)^*$ and the action of $N(k)$ on $\mathcal{A}(S)$ provides a map from $N(k)$ to the extended affine Weyl group $W(\Psi)^{\text{ext}}$ whose image contains the affine Weyl group $W(\Psi)$ (cf. §1.3 for definitions). It is proved in §6.3 that assertions AS 1 – AS 4 of Axiom 4.1.6 hold. *We will assume in addition that assertion AS 5 holds*. Given Propositions 6.3.13 and 6.6.2, this is known when G is quasi-split, and for a general G it amounts to the assumption that, when G is semi-simple and simply connected, the image of the map $N(k) \to W(\Psi)^{\text{ext}}$ is precisely $W(\Psi)$. This is proved in this generality in Proposition 9.4.35, assuming that the residue field of k is perfect.

- Proof of Lemma 7.1.4:
  (1) In the second paragraph replace $-s$ with s.
  (2) In the fourth paragraph of the proof replace s by $-s$ and $\psi(\varphi)$ by $-\psi(\varphi)$.
  (3) In the proof of part (4), the definition of L_2 involves an element m whose definition is given two lines below.
  (4) In the proof of part (4), the penultimate line claims that U_{ψ_2} commutes with $U_{-\psi_3}$. This is not literally correct. The correct statement is $U_{-\psi_3} \cdot U_{\psi_2} \subset U_{\psi_1} \cdot U_{\psi_2} \cdot Z(k)^0 \cdot U_{-\psi_3}$. This statement is sufficient to imply the last line of the proof, because it shows that $U_{\psi_2} \cdot U_{\psi_1} \cdot Z(k)^0 \cdot U_{-\psi_3} \cdot U_{\psi_2} \cdot m \cdot Z(k)^0 \cdot U_{\psi_2}$ is contained in $U_{\psi_2} \cdot U_{\psi_1} \cdot Z^0 \cdot U_{-\psi_3} \cdot m \cdot U_{\psi_2}$, and using that $m\psi_3 = -\psi_1$ we see that this becomes L_2.
  To prove the claim $U_{-\psi_3} \cdot U_{\psi_2} \subset U_{\psi_1} \cdot U_{\psi_2} \cdot Z^0 \cdot U_{-\psi_3}$, we begin with $U_{-\psi_3} = U_{-a,x,j_1-j_2}$ and $U_{\psi_2} = U_{2a,x,j_2}$. Taking $u \in U_{-\psi_3}$ and $u' \in U_{\psi_2}$ and applying Lemma 7.1.1(1) we see that $uu' = x'zx$, with $z \in Z(k)^0$, $x' \in U_{a,x,j_2/2}$, and $x \in U_{-a,x,j_1-j_2}$. Lemma 7.1.1(3) produces the more precise information $x' \in U_{a,x,j_1} \cdot u'$. Thus $uu' \in U_{a,x,j_1} \cdot U_{2a,x,j_2} \cdot Z(k)^0 \cdot U_{-a,x,j_1-j_2} = U_{\psi_1} \cdot U_{\psi_2} \cdot Z(k)^0 \cdot U_{-\psi_3}$.

- The following part (6) and its proof should be added to proposition 7.3.12
  (6) *Assume $f(0) > 0$. If $\{a_1, \ldots, a_{2n}\}$ is an arbitrary ordering of Φ, then the product map*

$$\prod_{i=1}^{n} U_{a_i,x,f} \times Z^{\sharp}_{x,f} \to G(k)^{\sharp}_{x,f}$$

  is an isomorphism.

Proof. Part (5) gives the desired result in the special case when $\{a_1, \dots, a_n\}$ is a system of positive roots. To obtain the general case one uses that

$$U_{a,x,f} \cdot U_{-a,x,f} \subset U_{-a,x,f} \cdot Z^{\sharp}_{x,f} \cdot U_{a,x,f}$$

by Lemma 7.1.1(1). □

- The following lemma should be added at the end of §7.3.

Lemma 7.3.24 *Let $n \in N(k)$ and let $P = ZU$ be* any *minimal prabolic subgroup with Levi factor $Z = Z_G(S)$ and let $\mathcal{I}$ be the Iwahori subgroup associated to a chamber contained in the apartment of S. Then*

$$U(k)\, n \cap \mathcal{I}n\mathcal{I} \subset \mathcal{I}^+ n \mathcal{I}^+.$$

Proof. Let Φ^+ denote the subset of roots determined by U and let $\Phi^- = -\Phi^+$. Let U^- be the unipotent subgroup generated by the root groups U_a for $a \in \Phi^-$. Let $U_1 = n^{-1} U^- n$ and $U_2 = n^{-1} U\, n$. In the remainder of the proof we will write $\mathcal{U} = U(k)$ (and analogously for the other unipotent groups) and $\mathcal{Z} = Z(k)^0$ to lighten notation.

By Proposition 7.3.12 (5),

$$\mathcal{I} = (\mathcal{U} \cap \mathcal{I}) \cdot (\mathcal{U}^- \cap \mathcal{I}) \cdot \mathcal{Z} = (\mathcal{U}_1 \cap \mathcal{I}) \cdot \mathcal{Z} \cdot (\mathcal{U}_2 \cap \mathcal{I}).$$

Using these decompositions of $\mathcal{I}$ and the fact that n normalizes $\mathcal{Z}$, we rewrite $\mathcal{U}\, n \cap \mathcal{I}\, n\, \mathcal{I}$ as

$$\mathcal{U}\, n \cap (\mathcal{U} \cap \mathcal{I}) \cdot (\mathcal{U}^- \cap \mathcal{I}) \cdot n \cdot (\mathcal{U}_1 \cap \mathcal{I})\, \mathcal{Z} \cdot (\mathcal{U}_2 \cap \mathcal{I}).$$

Using the fact that $\mathcal{U}\, n$ is stable under multiplication by $\mathcal{U} \cap \mathcal{I}$ on the left and multiplication by $\mathcal{U}_2 \cap \mathcal{I}$ on the right, we conclude that $\mathcal{U}\, n \cap \mathcal{I}\, n\, \mathcal{I}$ is contained in

$$(\mathcal{U} \cap \mathcal{I}) \cdot [\mathcal{U}\, n \cap \{(\mathcal{U}^- \cap \mathcal{I})\, n\, (\mathcal{U}_1 \cap \mathcal{I})\, \mathcal{Z}\}] \cdot (\mathcal{U}_2 \cap \mathcal{I}).$$

But

$$\begin{aligned} (\mathcal{U}^- \cap \mathcal{I})\, n\, (\mathcal{U}_1 \cap \mathcal{I})\, \mathcal{Z} &= (\mathcal{U}^- \cap \mathcal{I})\, n\, (\mathcal{U}_1 \cap \mathcal{I})\, n^{-1} \cdot n\, \mathcal{Z} \\ &\subset \mathcal{U}^- \cdot n\, \mathcal{Z} = \mathcal{U}^-\, \mathcal{Z}\, n. \end{aligned}$$

As $U\, n \cap (U^-\, Z\, n) = \{n\}$, we see that

$$\mathcal{U}\, n \cap \mathcal{I}\, n\, \mathcal{I} \subset (\mathcal{U} \cap \mathcal{I})\, n\, (\mathcal{U}_2 \cap \mathcal{I}).$$

Since both $\mathcal{U} \cap \mathcal{I}$ and $\mathcal{U}_2 \cap \mathcal{I}$ are contained in $\mathcal{I}^+$, it follows that

$$\mathcal{U}\, n \cap \mathcal{I}\, n\, \mathcal{I} \subset \mathcal{I}^+ n\, \mathcal{I}^+.$$

□

- The sentence appearing just before Remark 7.4.2 should be replaced with "If $\Omega = \{x\}$, then instead of the subscript $\{x\}$ we will use x for simplicity."

- In Remark 7.4.2 replace "a chamber" by "the standard chamber (that is, the chamber fixed by the subgroup $G(k)_x^\#$ described in the remark)."

- In assertion (2) of Lemma 7.4.4, replace $f_{x,\Omega}$ with $f_{\Omega,x}$.

- Just after the first sentence of §7.5 add "Then $G(k)^0 = G(k)$." and replace $G(k)^0_{\mathcal{C}}$ with $G(k)_{\mathcal{C}}$ in the next sentence.

- Replace the second sentence of the proof of Theorem 7.5.1 with "According to Axiom 4.1.6 AS 5, the quotient $\mathcal{N}/\mathcal{Z}$ is isomorphic to W^{aff}"

- In the fourth paragraph of the proof of Theorem 7.5.1, replace ψ by $\alpha \in \Psi'$ and replace k at four places by r in the same paragraph. In the third paragraph from the bottom of the proof of this theorem, replace $\cap_{n\in\mathcal{N}} \mathcal{I}$ by $\cap_{n\in\mathcal{N}} n\mathcal{I}n^{-1}$.

- Replace the sentence appearing just before the last paragraph of the proof of Theorem 7.5.1, by "That the Weyl group of the Tits system is isomorphic to W^{aff} is simply our assumption that Axiom 4.1.6 AS 5 holds".

- In the proof of Lemma 7.5.2, replace Z_{sc} by Z', and replace "its preimage in $Z'(k)$" by "the preimage of $\mathcal{I}^\# \cap Z(k)$ in $Z'(k)$".

- Proof of Theorem 7.8.1(2): The proof of injectivity is insufficient and should be replaced with the following: Consider two elements of $\tilde{W}^0$ mapping to the same element of $\mathcal{I}\backslash G(k)/\mathcal{I}$. As remarked in 7.8.2, we can use Lemma 1.3.17 to write these elements uniquely as $w_1 \cdot \rho_1$ and $w_2 \cdot \rho_2$ with $w_i \in W^{\mathrm{aff}}$ and $\rho_i \in \tilde{W}^0_{\mathcal{C}}$, where $\mathcal{C}$ is the chamber corresponding to the Iwahori subgroup $\mathcal{I}$. Using that $\mathcal{I} \subset G(k)^0$ we see from 7.8.1.(1) (as remarked in 7.8.2) that $\rho_1 = \rho_2$. Since these elements normalize $\mathcal{I}$, we conclude that w_1 and w_2 have the same image in $\mathcal{I}\backslash G(k)^0/\mathcal{I}$. The Bruhat decomposition (Proposition 1.4.5) for the Iwahori-Tits system (Theorem 7.5.3) implies $w_1 = w_2$.

- Add the following assertion and its proof to Theorem 7.8.1
  (4)The inclusion $N \to G$ induces a bijection $\widetilde{W}^0 \to U(k)\backslash G(k)/I = U(k)Z(k)^0\backslash G(k)/I$, where U is the unipotent radical of a minimal parabolic subgroup containing S, and $Z = Z_G(S)$ is the minimal Levi subgroup determined by S.

Proof. The Iwasawa decomposition (Theorem 5.3.3) shows that the map $N(k) \to U(k)\backslash G(k)/I$ is surjective. Since $Z(k)^0$ lies in I this map descends to a surjection $\widetilde{W}^0 \to U(k)\backslash G(k)/I$, whose injectivity follows from Lemma 5.3.5 (in this erratum). $\square$

- In case (1) of Remark 7.11.9, replace "ℓ'/ℓ is ramified and there" by "There".

Chapter 8

- In the second paragraph of Chapter 8, replace "the first three sections" with "the first two sections".

- In the statement and proof of Lemma 8.1.4, replace $\bar{\mathfrak{f}}$ with $\mathfrak{f}_s$ everywhere.

- Replace the first sentence of §8.3 with the following:

  We will assume in this and the subsequent paragraphs that the residue field $\mathfrak{f}$ of k is perfect. Then the residue field of K is an algebraic closure of $\mathfrak{f}$ and by Corollary 2.3.8, G_K is quasi-split.

- Replace the second paragraph of 8.3.6 with the following:

  Let T be the centralizer of S in G. As G is quasi-split, T is a maximal K-torus of G. Let $\mathscr{T}^0$ be the connected N'éron model of T. The inclusion $S \to T$ extends to a closed immersion $\mathscr{S} \to \mathscr{T}^0$ due to Lemma B.7.11.

- Delete the last two sentences of the proof of Theorem 8.3.13 and in their place insert the following:

  As the group schemes $\mathscr{U}_{a,\Omega,0}$, $a \in \Phi$, and $\mathscr{T}^0$ have connected fibers, the homomorphisms of these group schemes into $\mathscr{G}^1_\Omega$ factor through $\mathscr{G}^0_\Omega$. Therefore, the subgroup $G(K)^0_\Omega$ generated by $U_{a,\Omega,0}\,(= \mathscr{U}_{a,\Omega,0}(\mathcal{O}))$ for $a \in \Phi$, and $T(K)^0\,(= \mathscr{T}^0(\mathcal{O}))$ is contained in $\mathscr{G}^0_\Omega(\mathcal{O})$. Thus

  $$G(K)^0_\Omega \subset \mathscr{G}^0_\Omega(\mathcal{O}) \subset \mathscr{G}^1_\Omega(\mathcal{O}) = G(K)^1_\Omega.$$

  As $G(K)^0_\Omega$ is of finite index in $G(K)^1_\Omega$, we see that $G(K)^0_\Omega$ is an open subgroup of finite index in $\mathscr{G}^0_\Omega(\mathcal{O})$. Now Lemma A.4.26 shows that $G(K)^0_\Omega = \mathscr{G}^0_\Omega(\mathcal{O})$.

- The first sentence of §8.4 should be replaced with:

  As in the preceding section, we assume here that the residue field $\mathfrak{f}$ of k is perfect.

- In the second paragraph of §8.4, replace $\mathfrak{o}$ with $\mathcal{O}$ at two places.

- At the beginning of the third paragraph of 8.4.2, insert the following sentence:

  For $a \in \Phi$, let $\mathscr{U}_{a,\Omega,0}$ be the a-root group of $\mathscr{G}^0_\Omega$.

- Delete the first sentence of the proof of Lemma 8.4.4 and replace k with K at two places in the proof.

- Just before the statement of Proposition 8.4.15, insert the following:

  Let $\Omega \prec \Omega'$ be two nonempty bounded subsets of $\mathcal{A}$. Then the inclusion $G(K)^0_{\Omega'} \subset G(K)^0_{\Omega}$ gives rise to a $\mathcal{O}$-group scheme homomorphism $\rho_{\Omega,\Omega'} : \mathscr{G}^0_{\Omega'} \to \mathscr{G}^0_{\Omega}$. We denote by $\overline{\rho}_{\Omega,\Omega'}$ the induced map $\overline{\mathscr{G}}^0_{\Omega'} \to \overline{\mathscr{G}}^0_{\Omega}$ between the special fibers.

- Delete the first sentence of Proposition 8.4.15.

Chapter 9

- Replace the second full paragraph on p. 326 with the following:

  We claim that the orbit of each point $x \in \mathcal{B}(G_K)$ under Γ is finite. To see this, let A be an apartment containing x and T be the corresponding maximal K-split torus of G_K. Let $k' \subset K$ be a finite Galois extension of k over which T is defined and split. Then the apartment A is fixed pointwise under the action of $\mathrm{Gal}(K/k')$. So the orbit map of Γ through x factors through the finite group $\mathrm{Gal}(k'/k)$ and hence this orbit is finite.

- Add the following paragraph to the end of 9.2.1 (on p. 326):

  We claim that a bounded and Γ-stable subgroup $\mathcal{G} \subset G(K)$ has a Γ-stable fixed point in $\mathcal{B}(G_K)$. To see this, let $\Omega \subset \mathcal{B}(G_K)$ be the set of all points fixed by $\mathcal{G}$. It is non-empty by Corollary 4.2.14, closed by the continuity of the action of $G(K)$ on $\mathcal{B}(G_K)$ (Proposition 4.2.18), convex by the uniqueness of geodesics (Proposition 4.2.7 and Lemma 1.1.11), and Γ-stable because $\mathcal{G}$ is so. Since the action of Γ has finite orbits, the closure Ω' of the convex hull of any Γ-orbit in Ω is non-empty closed, convex, and bounded. The Fixed Point Theorem (Theorem 1.1.15) implies that Γ has a fixed point in $\Omega' \subset \Omega$.

- In the second paragraph of the proof of Proposition 9.3.11, replace "Proposition 8.2.1(1)" with "Proposition 8.2.1(1), (5)" and "Proposition 9.3.5(2)" with "Proposition 9.3.5(1)".

- In the second sentence of the statement of Proposition 9.3.12 add "contained in an apartment of $\mathcal{B}(G)$" towards the end of that sentence. Also, delete Γ appearing twice as the superscript towards the end of the statement of this proposition as well as towards the end of its proof.

- Just before the statement of Proposition 9.3.13 add the following new paragraphs:

  There is a natural action of $V(S')$ on $\mathcal{B}(G_K)_S^\Gamma$. To show this, we recall that every element of $\mathcal{B}(G_K)_S^\Gamma$ lies in a special apartment corresponding to a special torus that contains S (Proposition 9.3.11). So there is a natural action of $V(S')\,(= V(S'_K))$ on this apartment. Now we need to show that given a point $x \in \mathcal{B}(G_K)_S^\Gamma$, $v \in V(S')$, and special apartments A_i, $i = 1, 2$, in $\mathcal{B}(G_K)_S$ containing x, the point $v +_{A_1} x$ of A_1 equals the point $v +_{A_2} x$ of A_2. To prove this, for $i = 1, 2$, let S_i be the special k-torus corresponding to the special apartment A_i. These tori contain S. Now Proposition 9.3.12 (over K in place of k, and $\Omega = \{x\}$) implies that there is a $g \in G(k)^0_x$ which commutes with S and multiplication by g gives an affine isomorphism $f : A_1 \to A_2$. Let $g_* : V(S'_1) \to V(S'_2)$ be the map induced by the conjugation action of g. Then as g commutes with S, $g_*|_{V(S')}$ is the identity map. Hence $f(v +_{A_1} x) = v +_{A_2} x$ for all $v \in V(S')$. On the other hand, the set of points of A_1 fixed under f is a closed convex set, which clearly contains the points $s \cdot x$, for all $s \in S'(k)$ since $f(s \cdot x) = g \cdot (s \cdot x) = s \cdot x$ as g commutes with S and it fixes x. Now since the closed convex hull of $\{s \cdot x \,|\, s \in S'(k)\}$ in A_1 is $V(S') +_{A_1} x$, we conclude that $v +_{A_2} x = f(v +_{A_1} x) = v +_{A_1} x$.

The following more direct argument was suggested by Kazuki Tokimoto.

Let $\Lambda\, (\subset V(S'))$ be the image of $S'(k)$ under the valuation homomorphism (cf. (2.6.4) for S' in place of G). Then Λ is a lattice in $V(S')$ and the natural action of $S'(k)$ on $\mathcal{B}(G_K)_S$ factors through Λ, and gives a well-defined action of Λ on $\mathcal{B}(G_K)_S$. Thus if $v \in \Lambda$ is the image of $s \in S'(k)$, and $x \in \mathcal{B}(G_K)_S$, then $v + x = s \cdot x$. Now we wish to show that this action of Λ extends to give an action of $V(S')$ on $\mathcal{B}(G_K)_S$.

We will first show that for any positive integer n, there is a natural extension to $\frac{1}{n}\Lambda$ of the action of Λ on $\mathcal{B}(G_K)_S$. For $v \in \Lambda$, and $x \in \mathcal{B}(G_K)_S$, let $x_{v,n} \in \mathcal{B}(G_K)_S$ be the point on the geodesic $[v + x, x]$ such that $d(x, x_{v,n}) = \frac{1}{n} d(x, x + v)$. We define $\frac{1}{n} v + x = x_{v,n}$. This way we have extended the action of Λ to $\frac{1}{n}\Lambda$. For positive integers m, n, the extended actions of $\frac{1}{m}\Lambda$ and $\frac{1}{n}\Lambda$ are compatible is seen by considering the extended action of $\frac{1}{mn}\Lambda$. Thus we have a natural action of $\mathbf{Q} \cdot \Lambda = \bigcup_n \frac{1}{n}\Lambda$ on $\mathcal{B}(G_K)_S$. As $\mathbf{Q} \cdot \Lambda$ is dense in $V(S')$, we get a natural action of $V(S')$ on $\mathcal{B}(G_K)_S$ by continuity.

- Delete the last paragraph of the proof of Proposition 9.3.13.
- Replace the first five lines in the proof of Proposition 9.3.24 with the following: “Let $S \subset G$ be a maximal k-split torus and let $Z = Z_G(S)$ be it centralizer. The”
- From the last sentence of the fifth paragraph of the proof of Proposition 9.4.8, delete the word ”non-divisible”.
- In the second paragraph of the proof of Lemma 9.8.1 replace “$\mathcal{B}(G) \to \mathcal{B}(Z)$” by “$\mathcal{B}(G)_M \to \mathcal{B}(Z)$”.

- Immediately after the proof of Corollary 9.9.4 add the following example that shows that a non-quasi-split (even an anisotropic) group may admit a reductive model.

  Example Let $k = f((t))$, where f is a field. Let G be a connected semi-simple f-group and T be a maximal f-torus of G containing a maximal f-split torus of G. Let F be the splitting field of T. Then F/f is a finite Galois extension. Let Γ be the Galois group of F/f. Let $\ell = F((t))$. Then ℓ/k is an unramified Galois extension of k with Galois group Γ. Moreover, as T splits over F so does G. Therefore, G_ℓ is a ℓ-split semi-simple group and hence Bruhat–Tits theory is available for this group. As ℓ/k is an unramified extension, by unramified descent (which holds also when the residue field f is imperfect, see [BT84a] or [Pra20b]) Bruhat–Tits theory is also available for G_k.

  We will denote $f[[t]]$ and $F[[t]]$ by $\mathfrak{o}$ and $\mathfrak{o}_\ell$ respectively. As G_F is a Chevalley group, $\mathscr{G}_{\mathfrak{o}_\ell} := G_F \times_F F[[t]] = G \times_f \mathfrak{o}_\ell$ is a Chevalley $\mathfrak{o}_\ell$-group scheme. Hence, $G(F[[t]])$ is a hyperspecial parahoric subgroup of $G(\ell)$ and the corresponding point x in the Bruhat–Tits building $\mathcal{B}(G_\ell)$ of $G(\ell)$ is a hyperspecial vertex. The $\mathfrak{o}$-group scheme $\mathscr{G} := G \times_f f[[t]]$ is clearly the descent of the Chevalley $\mathfrak{o}_\ell$-group scheme $\mathscr{G}_{\mathfrak{o}_\ell}$. Hence, $\mathscr{G}$ is a reductive $\mathfrak{o}$-group scheme, and $\mathscr{G}(\mathfrak{o}) = G(f[[t]])$ is a hyperspecial parahoric subgroup of $G(k)$. Moreover, the point x is a hyperspecial vertex in the building $\mathcal{B}(G_k) = \mathcal{B}(G_\ell)^\Gamma$.

  Since f is the residue field of k, we see that the special fiber of the hyperspecial parahoric group scheme $\mathscr{G} = G \times_f f[[t]]$ is G, and its generic fiber is G_k. As $f \hookrightarrow k = f((t))$, f-rank $G \leqslant k$-rank G_k. We assert that f-rank $G = k$-rank G_k. To see this, we recall from 8.3.6 and 9.2.5 that every parahoric $\mathfrak{o}$-group scheme associated to G_k contains an $\mathfrak{o}$-split torus whose generic fiber is a maximal k-split torus of G_k. Now let $\mathscr{S}$ be a $\mathfrak{o}$-split torus of $\mathscr{G}$ such that the generic fiber $\mathscr{S} \times_{\mathfrak{o}} k$ is a maximal k-split torus of G_k. The special fiber S of $\mathscr{S}$ is a f-split torus of G. This implies that f- rank $G \geqslant k$-rank G_k and our assertion is proved.

  Therefore, if G is anisotropic over f, then G_k is anisotropic over k. In this case, the building $\mathcal{B}(G_k) = \{x\}$, and according to Theorem 2.2.9, $G(f((t)))$ is bounded. Since the hyperspecial subgroup $G(f[[t]])$ is a maximal bounded subgroup of $G(k) = G(f((t)))$, we conclude that if G is f-anisotropic, then $G(f((t))) = G(f[[t]])$, and replacing t with $1/t$, we see that $G(f((1/t))) = G(f[[1/t]])$. Now, in case G is f-anisotropic, we conclude the following:

  $$G(f[t]) = G(f[t]) \cap G(f((1/t))) = G(f[t]) \cap G(f[[1/t]]) = G(f).$$

CHAPTER 10

Section 10.4.

- In the third line of the proof of Proposition 10.4.2, replace “but it” by “but it is fixed by $\mathcal{S}$ and it”.

Section 10.8: Pro-unipotent radicals and maximal pro-p subgroups.

This is a new section. It contains a result that appears as Proposition 13.5.2. Becuse of the usefulness of this result we state it here to increase its visibility. We also provide a slightly modified proof, following a suggestion by Brian Conrad, which avoids the use of Moy–Prasad filtrations.

In this section we assume that k is complete and has finite residue field of characteristic p and size q.

Proposition 10.8.1 *Let $\mathscr{G}$ be a smooth affine $\mathfrak{o}$-group scheme with connected fibers. The kernel of the reduction map $\mathscr{G}(\mathfrak{o}) \to \mathscr{G}(\mathfrak{f})$ is an open and pro-p (in particular, compact) subgroup of $\mathscr{G}(k)$.*

Proof Let G be the generic fiber of $\mathscr{G}$. Then $\mathscr{G}(\mathfrak{o}) \subset \mathscr{G}(k) = G(k)$ is open, closed, and bounded by Remark 2.10.3. Since k is compact, $G(k)$ is locally compact, and therefore the closed subgroup $\mathscr{G}(\mathfrak{o})$ is also compact. The remainder of the proof will show that this group is, more precisely, a pro-p group.

Choose a faithful (finite-dimensional) algebraic representation $\rho : G \to \mathrm{GL}(V)$. The subgroup $\mathscr{G}(\mathcal{O}) \subset G(K)$ is bounded (Remark 2.10.3). Its image $\mathcal{G} = \rho(\mathscr{G}(\mathcal{O}))$ is therefore bounded (Fact 2.2.4) and also Γ-stable. By the final paragraph of 9.2.1 (in this erratum), $\mathcal{G}$ fixes a Γ-stable point x of the building of $\mathrm{GL}(V)$. Let $\mathscr{GL}(V)$ be associated parahoric $\mathfrak{o}$-integral model (9.2.5) of $\mathrm{GL}(V)$. By Corollary 2.10.10 ρ extends to a morphism of $\mathfrak{o}$-group schemes $\mathscr{G} \to \mathscr{GL}(V)$. Hence, ρ maps the kernel of the reduction map $\mathscr{G}(\mathfrak{o}) \to \mathscr{G}(\mathfrak{f})$ to the kernel of the reduction map $\mathscr{GL}(V)(\mathfrak{o}) \to \mathscr{GL}(V)(\mathfrak{f})$. Since a closed subgroup of a pro-p group is pro-p, this reduces the claim to the case of parahoric integral models of $\mathrm{GL}(V)$.

We may thus assume that $\mathscr{G}$ is the parahoric integral models of $\mathrm{GL}(V)$ associated to a facet in the building. Let x be a vertex lying in the closure of that facet and let $\mathscr{G}'$ be the parahoric integral model associated to x. The identity on $\mathrm{GL}(V)$ extends, again by Corollary 2.10.10, to a morphism of integral models $\mathscr{G} \to \mathscr{G}'$, and therefore the kernel of the reduction map for $\mathscr{G}$ lies in the kernel of the reduction map for $\mathscr{G}'$.

We may thus assume that $\mathscr{G} = \mathscr{G}'$. By Theorem 4.2.15(1) and the fact that $G(k)^1 = G(k)^0$ for $G = \mathrm{GL}(V)$, $\mathscr{G}(\mathfrak{o})$ is a maximal bounded subgroup of $G(k)$ and $\mathscr{G}(\mathcal{O})$ is a maximal bounded subgroup of $G(K)$. By Proposition 2.2.13(3) there exists a lattice $\Lambda \subset V$ such that $\mathscr{G}(\mathfrak{o}) = \mathrm{GL}(\Lambda)$ and $\mathscr{G}(\mathcal{O}) = \mathrm{GL}(\Lambda \otimes_{\mathfrak{o}} \mathcal{O})$. Choosing an $\mathfrak{o}$-basis of Λ and using it to identify V with k^n we obtain an identification $\mathscr{G} = \mathrm{GL}_n/\mathfrak{o}$. This identifies the kernel of the reduction map with the group of matrices in $\mathrm{GL}_n(\mathfrak{o})$ that are congruent to the identity matrix modulo q. It is well-known that this is a pro-p group. □

Theorem 10.8.2 Let G be a connected reductive k-group. Recall the assumption that k is complete and has finite residue field of characteristic p.

(1) For any $x \in \mathcal{B}(G)$ the group $G(k)_x^{0+}$ is an open pro-p subgroup of $G(k)$. It is maximal if and only if x lies in a chamber, that is, $G(k)_x^0$ is an Iwahori subgroup.
(2) Every maximal open pro-p subgroup of $G(k)^0$ equals $G(k)_x^{0+}$ for an Iwahori subgroup $G(k)_x^0$, and $G(k)_x^0$ is the normalizer of $G(k)_x^{0+}$ in $G(k)^0$.

Proof (1) Let $\mathscr{G}$ be the parahoric integral model associated to x, G its special fiber, and R the unipotent radical of G. By definition $G(k)_x^{0+}$ is the preimage of $\mathsf{R}(\mathfrak{f})$ under the reduction map $\mathscr{G}(\mathfrak{o}) \to \mathscr{G}(\mathfrak{f})$. Let $\mathcal{G}$ be the kernel of that reduction map. We thus have the exact sequence

$$1 \to \mathcal{G} \to G(k)_x^{0+} \to \mathsf{R}(\mathfrak{f}) \to 1.$$

Now R is a unipotent $\mathfrak{f}$-group, hence a successive extension of additive groups, so $\mathsf{R}(\mathfrak{f})$ is a finite p-group. Since an extension of pro-p groups is pro-p, Proposition 10.8.1 shows that $G(k)_x^{0+}$ is indeed a pro-p group.

If $G(k)_x^0$ is *not* an Iwahori subgroup, the reductive quotient of G will contain a proper parabolic subgroup. The preimage $\mathcal{H} \subset G(k)_x^0$ of the group of $\mathfrak{f}$-points of its unipotent radical is an extension of the pro-p group $G(k)_x^{0+}$ by the group of $\mathfrak{f}$-points of that unipotent radical. The latter is a finite p-group, so $\mathcal{H}$ is an open pro-p subgroup of $G(k)^0$ that contains $G(k)_x^{0+}$ properly. To complete the proof of (1) it remains to show that $G(k)_x^{0+}$ is a maximal pro-p open subgroup if x lies in a chamber. This will be done in the course of proving (2).

(2) Consider a maximal pro-p subgroup $\mathcal{H}$ of $G(k)^0$. It is compact, hence bounded, so fixes a point in $\mathcal{B}(G)$ due to Corollary 4.2.14. Any facet of $\mathcal{B}(G)$ invariant under $\mathcal{H}$ is pointwise fixed by $\mathcal{H}$ by Proposition 1.5.13(3). Let $\mathcal{F}$ be a facet that is maximal among the facets fixed by $\mathcal{H}$. Thus $\mathcal{H} \subset G(k)_{\mathcal{F}}^0$. Since $G(k)_{\mathcal{F}}^{0+}$ is an open pro-p subgroup by (1), normalized by $\mathcal{H}$, the product $G(k)_{\mathcal{F}}^{0+} \cdot \mathcal{H}$ is an open pro-p subgroup of $G(k)^0$, thus equal to $\mathcal{H}$ by maximality of $\mathcal{H}$. Thus $G(k)_{\mathcal{F}}^{0+} \subset \mathcal{H}$. The image of $\mathcal{H}$ in

$G(k)^0_{\mathcal{F}}/G(k)^{0+}_{\mathcal{F}}$ is a maximal p-subgroup of the group of $\mathfrak{f}$-points of the quasi-split connected reductive $\mathfrak{f}$-group which is the maximal reductive quotient of the parahoric integral model associated G associated to $\mathcal{F}$. Therefore, this image is the group of $\mathfrak{f}$-points of the unipotent radical of a Borel $\mathfrak{f}$-subgroup of G. According to Theorem 8.4.19(4) there exists a chamber $\mathcal{C}$ with $\mathcal{F} \prec \mathcal{C}$ such that this Borel subgroup is the image of $\overline{\rho}_{\mathcal{F},\mathcal{C}}$. We see that $\mathcal{H} \subset G(k)^0_{\mathcal{C}}$.

The maximality of $\mathcal{F}$ implies $\mathcal{F} = \mathcal{C}$. The root system of the reductive quotient is then empty according to Theorem 8.4.10, so this reductive quotient is a torus and has no non-trivial unipotent subgroups. We conclude that $\mathcal{H} = G(k)^{0+}_{\mathcal{C}}$. This proves that every maximal pro-p subgroup of $G(k)^0$ is the pro-unipotent radical of an Iwahori subgroup. Since all Iwahori subgroups are conjugate under $G(k)^0$, we see that the pro-unipotent radical of every Iwahori subgroup is a maximal pro-p subgroup of $G(k)^0$. This completes the proof of (1).

To complete the proof of (2) it remains to show that $\mathcal{I} = G(k)^0_{\mathcal{C}}$ is the normalizer of $G(k)^{0+}_{\mathcal{C}}$ in $G(k)^0$. Let $\mathcal{N}$ be that normalizer. According to Proposition 2.2.16, $\mathcal{N}$ is a compact open subgroup of $G(k)^0$. As it contains $\mathcal{I}$, it is a parahoric subgroup, since the parahoric subgroups of $G(k)$ are precisely the parabolic subgroups of the Iwahori–Tits system of $G(k)^0$, that is, those subgroups of $G(k)^0$ that contain an Iwahori subgroup (Proposition 7.6.3). Let $\mathcal{F}$ be the corresponding facet of $\mathcal{B}(G)$. It is contained in the closure of $\mathcal{C}$. Then $G(k)^{0+}_{\mathcal{F}} \subset G(k)^{0+}_{\mathcal{C}}$ by 6.3.21. The image of $G(k)^{0+}_{\mathcal{C}}$ in $G(k)^0_{\mathcal{F}}/G(k)^{0+}_{\mathcal{F}}$ is the group of $\mathfrak{f}$-points of the unipotent radical of a Borel subgroup of the maximal reductive quotient of G. But it is also normal in $G(k)^0_{\mathcal{F}}/G(k)^{0+}_{\mathcal{F}}$. This is impossible unless $G(k)^0_{\mathcal{F}}/G(k)^{0+}_{\mathcal{F}}$ is (the group of $\mathfrak{f}$-points of a) torus. Thus $\mathcal{F} = \mathcal{C}$. □

Chapter 11

Section 11.7.

- Replace Theorem 11.7.7 and its proof by the following (which does not assume that k is a local field except where explicitly stated).

Theorem 11.7.7 *The following statements hold.*

(1) $H^1(\Gamma, G(K)^0) = \{0\}$.

(2) The Kottwitz homomorphism induces a bijection $H^1(\Gamma, G(K)) \to H^1(\Gamma, \pi_1(G)_I)$.

(3) If $\Gamma = \widehat{\mathbf{Z}}$ *(this is the case if, for example, the residue field of k is finite), the homomorphism* $H^1(\Gamma, \pi_1(G)_I) \to \pi_1(G)_{\Theta, tor}$*, that evaluates cocycles at* $Fr \in \Gamma$*, is an isomorphism.*

(4) The inflation map $H^1(\Gamma, G(K)) \to H^1(k, G)$ *is bijective.*

Thus one obtains functorial bijections

$$H^1(k,G) \to H^1(\Gamma, G(K)) \to H^1(\Gamma, \pi_1(G)_I).$$

These endow $H^1(k,G)$ and $H^1(\Gamma,G(K))$ with functorial structures of an abelian group, compatible with the inflation map. When k is a local field, one obtains further the identification of this abelian group with $\pi_1(G)_{\Theta,tor}$.

Proof. (4) follows from the inflation-restriction sequence and Steinberg's theorem in the form of Proposition 2.3.5.

(3) follows from the elementary observation that for any smooth abelian $\widehat{\mathbf{Z}}$-module M the map that evaluates cocycles at $1 \in \widehat{\mathbf{Z}}$ induces an isomorphism $H^1(\widehat{\mathbf{Z}}, M) \to M_{\widehat{\mathbf{Z}},\mathrm{tor}}$. This observation is applied to $M = \pi_1(G)_I$ and $\Gamma = \widehat{\mathbf{Z}}$ (with the Frobenius element identified with 1).

(2) let $\widetilde{\mathcal{G}} = G(K)$ and let $\mathcal{N} \subset \widetilde{\mathcal{G}}$ be the stabilizer of a Γ-stable chamber $\mathcal{C}$ in $\mathcal{B}(G_K)$. Let $\mathcal{G} = G(K)^0$ and $\mathcal{I} = \mathcal{G} \cap \mathcal{N}$. Proposition 7.6.4(1) implies that $\mathcal{I}$ is the Iwahori subgroup associated to $\mathcal{C}$, i.e. the group. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \widetilde{\mathcal{G}} \arrow[d] \\ \mathcal{N}/\mathcal{I} \arrow[r] & \widetilde{\mathcal{G}}/\mathcal{G}. \end{tikzcd}$$

The bottom map is injective by construction and surjective because $\mathcal{G}$ acts trasitively on the set of chambers in $\mathcal{B}(G_K)$ (Theorem 7.5.3(1) and Proposition 1.5.6(7)). Thus, it is bijective, and also induces a bijection after application of $H^1(\Gamma,-)$. According to Theorem 8.3.13, $\mathcal{I} = \mathscr{G}^0_{\mathcal{C}}(\mathcal{O})$ for a smooth $\mathcal{O}$-model $\mathscr{G}^0_{\mathcal{C}}$ of G with connected fibers. Theorem 10.6.1 implies that the left and top maps induce bijections after application of $H^1(\Gamma,-)$. We conclude that the right map also induces a bijection after application of $H^1(\Gamma,-)$. Finally, the Kottwitz homomorphism induces an isomorphism $\widetilde{\mathcal{G}}/\mathcal{G} \to \pi_1(G)_I$ by Corollary 11.6.2, Γ-equivariant by construction, hence an isomorphism after application of $H^1(\Gamma,-)$.

(1) We apply $H^i(\Gamma,-)$ to the exact sequence $1 \to \mathcal{G} \to \widetilde{\mathcal{G}} \to \pi_1(G)_I \to 1$. By (2) the map $H^1(\Gamma,\widetilde{\mathcal{G}}) \to H^1(\Gamma,\pi_1(G)_I)$ is injective. By Corollary 11.7.6 the map $H^0(\Gamma,\mathcal{G}) \to H^0(\Gamma,\pi_1(G)_I)$ is surjective. Thus $H^1(\Gamma,\mathcal{G}) = \{0\}$. □

- Replace Remark 11.7.8 by the following.

Remark 11.7.8 Theorem 11.7.7 gave identifications of the cohomology set $H^1(k,G) = H^1(\Gamma,G(K))$ with the abelian group $H^1(\Gamma,\pi_1(G)_I)$. We will now provide further ways to express all these equal abelian groups.

Let $\mathcal{A} \subset \mathcal{B}(G_K)$ be a special k-apartment, let $S \subset G$ be the corresponding special k-torus, and let $T \subset G$ be its centralizer, a maximal k-torus. Let $\mathcal{C} \subset \mathcal{A}$ be a Γ-stable chamber. We use notation similar to the proof of (2) above: $\widetilde{\mathcal{G}} = G(K)$, $\mathcal{N} \subset \widetilde{\mathcal{G}}$ is the stabilizer of $\mathcal{C}$, $\mathcal{G} = G(K)^0$ and $\mathcal{I} = \mathcal{G} \cap \mathcal{N}$ the Iwahori subgroup of $G(K)$ associated to $\mathcal{C}$. In addition, we consider the K-Iwahori-Weyl group $\widetilde{W}^0 = N_G(T)(K)/T(K)^0$. Then we have the commutative diagram

$$\begin{array}{ccccc} N_G(T)(K)_{\mathcal{C}} & \longrightarrow & \mathcal{N} & \longrightarrow & \widetilde{\mathcal{G}} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{W}^0_{\mathcal{C}} & \longrightarrow & \mathcal{N}/\mathcal{I} & \longrightarrow & \widetilde{\mathcal{G}}/\mathcal{G} \end{array}$$

that extends the diagram considered in the above proof, and where the subscript $\mathcal{C}$ indicates that we are taking the stabilizer of $\mathcal{C}$ in the corresponding group. We concluded in the proof of Theorem 11.7.7(2) above that the right square of this diagram induces bijections after application of $H^1(\Gamma, -)$, and moreover all four of its corners are in bijection with $H^1(k, G)$ as well as with $H^1(\Gamma, \pi_1(G)_I)$. We claim that the same is true for the left square of this enlarged diagram and its two new corners.

We repeat a few arguments from the proof of Theorem 10.6.1: the bottom left map is bijective (since $\mathcal{I}$ acts transitively on the set of apartments containing $\mathcal{C}$ by Proposition 7.7.5(4) and as $N_G(T)(K) \cap \mathcal{I} = T(K)^0$ by Lemma 7.5.2(1)), hence remains bijective after applying $H^1(\Gamma, -)$. The left vertical map becomes bijective after application of $H^1(\Gamma, -)$, because both $H^1(\Gamma, {}_cT(K)^0)$ and $H^2(\Gamma, {}_cT(K)^0)$ vanish due to Corollary B.10.14, where c is any element of $Z^1(\Gamma, \widetilde{W}^0_{\mathcal{C}})$ and ${}_cT(K)^0$ denotes the abelian group $T(K)^0$ with Γ-action twisted by c. To see this, note that ${}_cT(K)^0$ is the K-Iwahori subgroup of the k-torus ${}_cT$ obtained from T by twisting the Γ-action by c.

We have so far established the bijectivty, after applying $H^1(\Gamma, -)$, of all maps in the above diagram except possibly the top left map. Its bijectivity now follows formally.

- Add the following at the end of section 11.7.

Proposition 11.7.9 *Let $T \subset G$ be the centralizer of a special k-torus of G, let $\mathcal{A}(T_K) \subset \mathcal{B}(G_K)$ be the apartment associated to the maximal split torus in T_K, let $\mathcal{C} \subset \mathcal{A}(T_K)$ be a Γ-stable chamber, and let $z \in Z^1(\Gamma, N_G(T)(K)_{\mathcal{C}})$. Let T_z and G_z be the twists of T and G by z, so that $T_z \subset G_z$ is a maximal k-torus.*

(1) The maximal unramified subtorus in T_z is a special k-torus in G_z.

(2) *Letting Γ act on $\mathcal{A}(T_K) \subset \mathcal{B}(G_K)$ via the action twisted by z, the set of fixed points in $\mathcal{A}(T_K)$ equals the apartment $\mathcal{A}(T_z) \subset \mathcal{B}(G_z)$ associated to the maximal split torus in T_z, and the set of fixed points in $\mathcal{C}$ is a chamber in $\mathcal{A}(T_z)$.*

Proof. The statement does not change if we replace G by its quotient modulo the maximal split central torus. This allows us to assume that the center of G is anisotropic. Let us write Γ_z for the group Γ whose action on $\mathcal{B}(G_K)$ has been twisted by z.

By unramified descent we have $\mathcal{B}(G_z) = \mathcal{B}(G_K)^{\Gamma_z}$ and $\mathcal{C}_z := \mathcal{C}^{\Gamma_z}$ is a chamber in $\mathcal{B}(G_z)$. Therefore the dimension of $\mathcal{C}_z$ equals the dimension of any apartment in $\mathcal{B}(G_z)$, and also the dimension of any maximal k-split torus in G_z. The set $\mathcal{A}_z := \mathcal{A}(T_K)^{\Gamma_z}$ is an affine subspace of $\mathcal{A}(T_K)$ whose space of translations is the subspace $V_z = V(T_K)^{\Gamma_z}$. We recall that $V(T_K)$ is the real vector space spanned by the cocharacter module of the maximal K-split subtorus of T_K. Therefore, V_z is the subspace spanned by the cocharacter module of the maximal k-split subtorus of T_z. Since $\mathcal{C}_z$ is contained in the set $\mathcal{A}_z$ we conclude that the dimension of $\mathcal{A}_z$, hence also of the maximal k-split subtorus of T_z, is at least that of $\mathcal{C}_z$, hence that of any apartment of $\mathcal{B}(G_z)$. This shows that the maximal unramified subtorus of T_z is a special k-torus in G_z. By unramified descent we have $\mathcal{B}(G_z) = \mathcal{B}(G_K)^{\Gamma_z}$ and $\mathcal{C}_z := \mathcal{C}^{\Gamma_z}$ is a chamber in $\mathcal{B}(G_z)$. Therefore the dimension of $\mathcal{C}_z$ equals the dimension of any apartment in $\mathcal{B}(G_z)$, and also the dimension of any maximal k-split torus in G_z. The set $\mathcal{A}_z := \mathcal{A}(T_K)^{\Gamma_z}$ is an affine subspace of $\mathcal{A}(T_K)$ whose space of translations is the subspace $V_z = V(T_K)^{\Gamma_z}$. We recall that $V(T_K)$ is the real vector space spanned by the cocharacter module of the maximal K-split subtorus of T_K. Therefore, V_z is the subspace spanned by the cocharacter module of the maximal k-split subtorus of T_z. Since $\mathcal{C}_z$ is contained in the set $\mathcal{A}_z$ we conclude that the dimension of $\mathcal{A}_z$, hence also of the maximal k-split subtorus of T_z, is at least that of $\mathcal{C}_z$, hence that of any apartment of $\mathcal{B}(G_z)$. This shows that the maximal unramified subtorus of T_z is a special k-torus in G_z. $\square$

Chapter 12

- In many places in this chapter, the typesetter has used Θ where the lower case θ should have been used. This error is easy to spot.
- In item (3) on p. 441, replace §11.5 by Proposition 12.6.1.
- In the third full paragraph on p. 445, in the second line replace $\mathcal{B}(H)^{\Theta}$ with $\mathcal{B}(H)$.
- Delete "and Corollary 4.4.5" from the fifth line of the first paragraph of 12.4.
- Remove the last sentence of the second paragraph of §12.4.
- Proposition 12.4.1 and its proof should be replaced with the following:

  Proposition 12.4.1. *Let S and S' be as above. Let x be a Θ-fixed point of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ and A be any apartment of this building containing x. Then*

  $$A^{\Theta} = V(S') + x = \widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta}.$$

  Proof. Let $Z\,(\supset S')$ and $Z_{H'}(S)'$ be the maximal K-split central torus and the derived subgroup of $Z_{H'}(S)$ respectively. Then Z and $Z_{H'}(S)'$ are stable under Θ; moreover, $(Z^{\Theta})^0 = S'$.

  There is an action of $V(Z)$ on each apartment of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ by translations since every maximal K-split torus of $Z_{H'}(S)$ contains Z. Moreover, the parahoric subgroup associated to x acts transitively on the set of apartments of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ containing x, we see that the action of $V(Z)$ on the apartments of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ combine to give a natural action of $V(Z)$ on the entire $\widetilde{\mathcal{B}}(Z_{H'}(S))$. Hence, there is a natural action of $V(S') = V(Z)^{\Theta}$ on $\widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta}$.

  Let T be the maximal K-split torus of H' containing S corresponding to the apartment A. Then $A = V(T)+x$, hence $A^{\Theta} = V(T)^{\Theta}+x = V(S')+x$. As the building $\widetilde{\mathcal{B}}(Z_{H'}(S))$ is the union of apartments containing x, we conclude that $\widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta} = V(S') + x$. □

- Replace the first sentence of the statement of Proposition 12.4.2 with the following:

  "*Let S_1 and S_2 be maximal K-split tori of G and Ω be a non-empty bounded subset of $\widetilde{\mathcal{B}}(Z_{H'}(S_1))^{\Theta} \cap \widetilde{\mathcal{B}}(Z_{H'}(S_2))^{\Theta}$.*"

- Replace the first paragraph on p. 459 with the following:

We have thus shown that the bijection $\mathcal{A}(S) \to \widetilde{\mathcal{B}}(Z_{H'}(S))^\Theta$ maps vertices to vertices. Since every facet $\mathcal{F}$ of $\mathcal{A}(S)$ is the interior of the convex hull of set $\mathcal{A}(S) \cap V$ of its vertices, where V is the set of vertices of $\mathcal{B}(G)$ that correspond to the maximal parahoric subgroups of $G(k)$ containing the parahoric subgroup corresponding to $\mathcal{F}$, it now follows that the bijection $\mathcal{A}(S) \to \widetilde{\mathcal{B}}(Z_{H'}(S))^\Theta$ carries facets to facets.

- The last sentence of the first paragraph of the statement of Proposition 12.8.5 should be placed immediately after the first sentence of the second paragraph.

Chapter 13

- At the end of of 13.6.2, replace 1.3.35 with 8.4.8(3).
- In the proof of Theorem 13.8.5, line -5, replace $H^1(\mathrm{Gal}(\ell/k), G(\ell)_{x,r})$ by $H^n(\mathrm{Gal}(\ell/k), G(\ell)_{x,r})$.

Chapter 18

- In the fourth line in the second paragraph of 18.4.2, replace "K_v will denote its maximal unramified extension" with "K_v will denote the maximal unramified extension of k_v".
- In the second line of the first full paragraph on p. 575, replace "$\mathscr{P}_v \to \overline{\mathscr{B}}_v(\mathfrak{f}_v)$ and $P_v \to \overline{B}_v(\mathfrak{f}_v)$" by "$\mathscr{P}_v \to \overline{\mathscr{M}}_v(\mathfrak{f}_v)$ and $P_v \to \overline{M}_v(\mathfrak{f}_v)$".

Appendix A

- In line 10 (from the top) of p. 604, insert the word "perfect" before the word "residue".

Section B.10.

- In Definition B.10.8(2), it must be clarified that R is an induced K-torus, i.e. a finite product of tori of the form $\mathrm{R}_{E/K}\mathbf{G}_{m,K}$ for finite separable extensions E/K.

- In Definition B.10.8, add the following point (4): For any r define $T(k)_r^{\mathrm{mc}} = T(K)_r^{\mathrm{mc}} \cap T(k)$.

- In the proof of Corollary B.10.14, the incomplete sentence "Set $T(k)_r^{\mathrm{mc}} \cap T(k')$" on line 3 of the proof should be replaced by "Recall that $T(k')_r^{\mathrm{mc}} = T(K)_r^{\mathrm{mc}} \cap T(k')$."

- In the proof of Corollary B.10.14, the justification that $T(k')_r^{\mathrm{mc}}/T(k')_{r+1}^{\mathrm{mc}}$ is the group of $\mathfrak{f}'$-points of a connected commutative unipotent $\mathfrak{f}$-group should be augmented by appealing to Remark B.10.15, which is added in this erratum.

- After the proof of Corollary B.10.14, add the following.
  Remark B.10.15 The filtration subgroup $T(K)_r^{\mathrm{mc}}$ is Γ-stable, and hence the smooth $\mathcal{O}$-model $\mathscr{T}_r^{\mathrm{mc}}$ descends to $\mathfrak{o}$ by Fact 2.10.16. To see that, it is enough to show that if a point $p : \mathrm{Spec}(K) \to T_K$ factors through the r-th standard filtration of an induced K-torus, then so does $\sigma(p)$ for any $\sigma \in \Gamma = \mathrm{Gal}(K/k)$; here $\sigma(p) = \sigma_{T_K} \circ p \circ \sigma_{\mathbf{G}_{m,K}}^{-1}$. We may assume without loss of generality that the induced torus is of the form $\mathrm{R}_{E/K}\mathbf{G}_{m,K}$ for a finite separable extension E/K. Then $\sigma(p)$ factors through $\mathrm{R}_{\sigma(E)/K}\mathbf{G}_{m,K}$. This uses the map $\sigma : \mathrm{R}_{E/K}\mathbf{G}_{m,K} \to \mathrm{R}_{\sigma(E)/K}\mathbf{G}_{m,K}$ which covers the map $\sigma : \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ and is defined as follows. We have $X_*(\mathrm{R}_{E/K}\mathbf{G}_{m,K}) = \{f : \mathrm{Gal}(k_s/E)\backslash\mathrm{Gal}(k_s/K) \to \mathbf{Z}\}$ and the map σ induces on the level of X_* the map sending f to the function $\sigma f : \tau \mapsto f(\sigma^{-1}\tau\sigma)$, and we are using an arbitrary lift of σ to an element of $\mathrm{Gal}(k_s/k)$. One checks at once that for any $\nu \in \Gamma$ one has $\sigma\nu f = (\sigma\nu\sigma^{-1})\sigma f$, and hence we obtain the desired map $\sigma : \mathrm{R}_{E/K}\mathbf{G}_{m,K} \to \mathrm{R}_{\sigma(E)/K}\mathbf{G}_{m,K}$. On the level of K-points this map induces the map $E^\times \to \sigma(E^\times)$ induced by acting on $E^\times$ by $\sigma \in \mathrm{Gal}(k_s/k)$. It follows that this map respects the standard filtrations on both sides.