**Proceedings of TMCE 2006**, April 18–22, 2006, Ljubljana, Slovenia, edited by I. Horváth and J. Duhovnik
© Organizing Committee of TMCE 2006, ISBN 961-6536-04-4

# RAPID CONCEPT GENERATION AND EVALUATION BASED ON VAGUE DISCRETE INTERVAL MODEL AND VARIATIONAL ANALYSIS

**Kazuhiro Saitou**
Dept. of Mechanical Engineering
University of Michigan
USA
kazu@umich.edu

**Zoltán Rusák**
**Imre Horváth**
Faculty of Industrial Design Engineering
Delft University of Technology
the Netherlands
{z.rusak, i.horvath}@io.tudelft.nl

## ABSTRACT

*This paper presents a computational method for rapidly generating alternative product shapes and evaluating their structural responses during conceptual design. It integrates the vague discrete interval models (VDIM) as the representation of a set of alternative product shapes, and the variational analysis (VA) technique as the method for quickly approximating structural responses of alternative product shapes generated from the VDIM, based on the finite element analysis results of a nominal product shape. A simple case study on the two-dimensional cross section of a chair demonstrates that the method is capable of evaluating the structural responses of shape alternatives with a reasonable accuracy and negligible computational overhead.*

## KEYWORDS

Conceptual design, concept evaluation, shape instantiation, structural analysis, approximation

## 1. INTRODUCTION

During the conceptual design phase, a designer explores alternative design concepts and select one or several concept(s) to be examined further during the detailed design phase (Ullman, 2003). This two-phase exploration process is widely accepted as a standard practice, since it facilitates the effective exploration of a large design space where the close examinations of all design alternatives are not practical. It also naturally matches the thinking process of human designers, who tend to search for the embodiments of design specifications by progressively refining details.

For products whose primal functions are determined by their shape, conceptual design requires rapid generation of a large number of shape alternatives in a certain class of shapes, and their evaluations with respect to desired design specifications. It poses a unique challenge to designers since they must evaluate a shape alternative based on the incomplete descriptions of product geometry, which are being completed during this very evaluation process.

As a solution to an aspect of this challenge, this paper presents our first attempt towards a computer support for conceptual shape design of products with structural specifications, such as stress and displacement under certain loads. Considering that the complete details of product shapes are yet to be finalized during detail design, it is hypothesized that the accuracy of generated geometry is, if within a reasonable bound, is of less importance than the ease of generating alternative shapes that allows interactive concept exploration. Similarly, the accuracy of structural analysis, if within a reasonable bound, would be of less importance than the speed of analysis that allows interactive concept examination.

Based on this reasoning, the paper proposes a computational method for rapidly generating alternative product shapes and evaluating their structural responses during conceptual design. It integrates the methods of vague discrete interval modeling (VDIM) (Rusák, Z. and Horváth, I., 2005) to represent of a set of alternative product shapes, and the variational analysis (VA) technique (CADOE, 2002) as the method for quickly approximating structural responses based on the finite element analysis results of a nominal product shape. A simple case study on the two-dimensional cross section of a chair demonstrates that the method is capable of evaluating the structural responses of shape alternatives with a reasonable accuracy and negligible computational overhead.

The rest of the paper is organized as follows: First, relevant literatures are reviewed. Next, the proposed method is described, followed by the results of a case study. Finally, the conclusion and future work are given.

## 2. RELATED WORK

### 2.1. Shape alternative generation for conceptual design

One approach to creating large number of shape alternatives is called evolutionary design. It supports the designer by automatically generating a large number of shape mutations from an existing shape based on genetic algorithms. The designer is not required to understand the mathematical representation of the model, merely to use some fitness criterion at each generation and guide the evolution of shapes.

Smyth and Wallace (2000) applied evolutionary design approach to parametric surfaces, in which the structure is defined by a skeleton model. To generate shape alternatives, (a) first the designer chooses a handle type, i.e. skeleton, lattice or bounding box of the shape, then (b) enters generic operation settings e.g. the mutation rate, finally (c) the algorithm generates shape variances by alternating the parameters of the representation. Taura *et al*. (1998) combined evolutionary design and procedural modeling, in which shapes are represented by sets of rules instead of geometrical data. When these rules are to be applied to a given shape, an evolutionary approach is used to determine a cell division model that best fits to compute the results of shape manipulation. However, this cell division model is limited for well tessellated shapes that do not contain any holes or incomplete shape components.

When interactive approach is applied to generate shape concepts, communication between the designer and the modeling tool need to be amplified. To help the designer to easily express his ideas, several attempts have been made that focused on implementing shape grammars for product design. First efforts concentrated on only specific shape grammars that can be used to generate products, e.g. buildings (Heisserman, 1994), and coffeemakers (Cagan, 1998). By limiting the application field to specific products provided a better background for the researchers to develop highly sophisticated grammars. A general shape grammar based approach has been proposed by Hsiao and Chen (1997), which facilitates the designer to create shape variances in three steps. Although, they addressed the issue of mapping linguistic rules to shape manipulation commands, their solution was limited to a few rules.

Most of the above discussed methods are efficient to quickly generate alternative shapes, but their results cannot be directly used for physically based evaluation due to the mismatch in their geometric representation for shape conceptualization and for finite element analysis. To address this issue, Rusák and Horváth (2005) proposed a particles system representation as a part of vague discrete interval modeling (VDIM), which is summarized in the following section. This particle system representation inherently ensures that the same finite element mesh could be used to evaluate any shape instance that is derived from the same vague discrete interval model.

### 2.2. Shape alternative evaluation for conceptual design

Even with today's fast PCs, the finite element analysis of products with reasonable complexity can take several minutes to hours. It is undesirable, therefore, to conduct the finite element analyses of each shape alternative generated during the conceptual design. To facilitate interactive concept exploration, approximation methods can be applied to rapidly estimate the structural responses of an alternative shape based on the analysis results of one or several nominal shapes. Major classes of approximation methods for structural analyses are surrogate models and reanalysis.

Surrogate models estimate the structural responses of shape alternatives within certain ranges, by interpolating the analysis results of several nominal shapes sampled within the range. Prior to the evaluation of shape alternatives, the multiple sample shapes must be analyzed first, and then the parameters of the interpolation functions must be adjusted for the minimum estimation error of the sampled responses. Depending on the type of interpolation functions, Artificial Neural Network (ANN) (Cheng and Titterington, 1994), Polynomial Regression, Kriging Method

(Forsberg and Nilsson, 2005; Sakata, *et al*., 2003) and Radial Basis Function (Mechesheimer, 2001; Jin, *et al*., 2001) are popularly used. However, it can potentially take a long time to construct an accurate surrogate model, since the overall accuracy of the estimation is affected by the number of samples, Therefore, these methods may not be suitable for the use with concept exploration.

Reanalysis is a class of methods for approximating the structural responses of modified designs, based on the analysis results of a single original design. In contrast to surrogate models that require multiple analyses (multi-point approximation), reanalysis methods only utilize a single analysis result (single point approximation). Due to this character, reanalysis methods are more suitable for the concept exploration of shape alternatives within a neighborhood of a nominal design.

Of most notable in structural reanalysis is Combined Approximations (CA) method (Levy, 2000; Kirsch and Papalambros, 2001a; Kirsch and Papalambros, 2001b), which is based on the reduced-basis approximation of the nodal displacement vector in terms of a binomial series. Initially developed for static linear analyses, the method is later extended to nonlinear static analyses, eigenvalue analyses, and also applied to calculate design sensitivities (Kirsch, 2002; Kirsch, 2003). Another class of reanalysis methods is based on Taylor series approximation (Choi, and Chang, 1994; Yeh and Vance, 1998; CADOE, 2002), where the derivatives of desired structural responses with respect to design variables are calculated by using Adjoint Method (Choi, and Chang, 1994; Yeh and Vance, 1998) or Direct Method (CADOE, 2002). The variational analysis (VA) adopted in this paper is of this category.

## 3. PROPOSED METHOD

### 3.1. Overview

VDIM represents a class of non-parameterized geometry by a set of distribution trajectories along which the position of the points of an instance product geometry can be defined by the designer by applying linguistic rules. This facilitates fast and interactive modeling of variances of product geometries. VA technique approximates the displacements of a variant shape using the Taylor series expansion of the displacement vectors around the ones of a nominal shape computed by the conventional finite element analysis, The discrete nature of VDIM allows the seamless integration of the generation of alternative shapes and their rapid evaluation using VA, without the need of re-meshing product geometry. VA allows very quick estimation of structural performances of variant shapes without additional finite element analyses. The integration of these two techniques enables the interactive design modification and evaluation by the human designer, greatly facilitating the concept exploration for a class of products where both product geometry and structural responses are important design criteria.

### 3.2. Vague discrete interval models (VDIM)

For the representation of the variances of individual shapes as well as for modeling a cluster of shapes in one single geometric model, the theory and methodology of *vague discrete interval modeling* (VDIM) was developed. VDIM is vague in the sense that it models (i) a cluster of shapes with a single representation allowing the integration of a nominal shape with its domain of variance, (ii) describes the structural relations between shape components, which represent the shape either completely or incompletely, and (iii) supports manipulation of an evolving shape by means of dedicated modeling methods and tools. It is discrete since the representation of the geometry is composed from discrete entities. Finally, VDIM is an interval modeling technique, since it describes the domain of variance of the shape by a finite interval around a hypothesized nominal geometry.

The fundamental modeling entity of VDIM is the particle, $\pi$, or more precisely, coupled pairs of particles, c($\pi_1$, $\pi_2$). Coupling of particles makes it possible to introduce various physical relationships and constraints in order to provide the means for a physically based manipulation of shapes and for the investigation of their behavior as physical objects. In a general form, a particle is represented by its reference vector, $\underset{\sim}{\mathbf{r}}$, and its metric occurrence, E. Reference vectors provide positional information and metric occurrence and couplings supply morphological information for the geometric representation. Figure 1 shows the fundamental entities of VDIM. A vague discrete interval model is vague since the reference points of the particles are uncertainly specified. In practice it means, that a metric occurrence is defined as a finite distribution space and merged in the shape model represented as a particle system. Note that a particle system contains boundary and internal particles, but internal particles do not have metric occurrences. Internal particles are not used in the vague representation for the reason that they only play a role in physically based modeling.

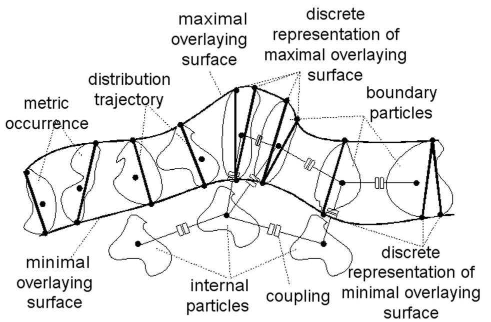


**Figure 1** Fundamental entities of VDIM

Having uncertain positions for particles, introduces weakness in the topological relationships between the neighboring elements, since they can have different neighbors depending on their actual position. In practice to use the same finite element mesh for each instance shape generated from the same VDIM model, the topological relationships of any neighboring particle has to be the same. To resolve this issue a characteristic discreteness, $\varepsilon$, is established in the vague discrete model, which is the distance between the neighboring particles. Discreteness of the model comes from the fact that the particles themselves never coincide. Nonetheless, the condition of discreteness does not assure topological robustness, since the actual discreteness depends on the actual positions of the particles so that a set of particles can be neighbors with a characteristic discreteness in a given position, but in another position they are not. Figure 2 illustrates the problem. To resolve this issue, the metric occurrence of the particles is generated in a way that avoids these situations. Having topologically robust representation assures that self-intersecting shapes are not instantiated from the vague model, and coupling relations of particles are the same for each instance.

The minimal and maximal overlaying surfaces are generated on the extremes of distribution specified by the boundary particles of the vague discrete model. The distribution interval is a subspace that is in between the minimal and maximal overlaying surfaces. Thus, it represents a cluster or a family of possible shapes rather than a single nominal shape. The images (actually, image points) of the particles on the two overlaying surfaces of the distribution interval are connected by so-called distribution trajectories, $\underset{\sim}{\tau}$. This is a sort of simplification of the representation, that enables to have 1D metric occurrences. Having a 1D metric occurrence reduces the required information over a vague shape as well as it regulates the discreteness of each shape that is represented in the vague interval model.

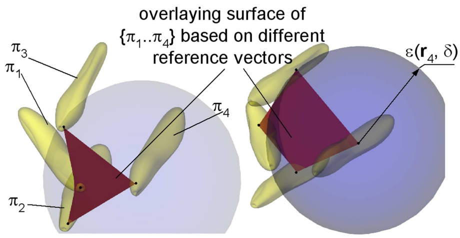


**Figure 2** Topologically weak vague discrete shape

To generate interval models with robust topological relationships, we have developed an algorithm that is shown in Algorithm 1. The input of this algorithm is two point sets, which define the minimal and maximal overlaying surfaces of a vague discrete interval model. In the first step of the algorithm, the closest points of the two point sets are looked for, and an initial distribution trajectory is generated between them. Then ascending order of the points is computed for each point set based on the shortest path from the end points of this initial distribution trajectory. Following this ascending order, distribution trajectories are generated one by one. In case a newly generated distribution trajectory causes topological anomaly in the structure of the interval model, then the next point is selected as a candidate for generating a distribution trajectory. The algorithm stops when each point in each point-set has been assigned to at least one distribution trajectory.

To identify topological anomalies in a particle cloud, $\Pi$, the geometric relation of the distribution trajectories is investigated between particles being closer to each other than $2\lfloor\underset{\sim}{\tau}\rfloor$. We can distinguish three relations of distribution trajectories, $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$, namely, that they: (a) are parallel, (b) intersect, and (c) are skewed. In case $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$ are parallel, the topological relation between each instance particle is the same. If they intersect, the topological relationship among the particles in the neighborhood of $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$, is influenced by the position of the selected instance particle(s), depending whether it is above or below the point of intersection. For skewed $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$, the topological relation to the neighboring particles is influenced by the position of the instance particle relative to the joining point of the normal traversal between $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$. The problem is illustrated in Figure 3. We have to investigate whether the intersecting point or the normal traversal of $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$ falls into the distribution interval. Topological anomalies are removed from the model, and new distribution trajectories are searched for those points that are not part of any other distribution trajectory.

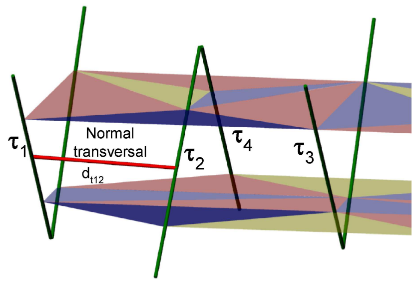


**Figure 3** Occurrence of topological anomalies (points connected by the dark triangle are in different neighboring relations below and above the normal traversal)

## 3.3. Variational Analysis (VA)

Variational Analysis (VA) (CADOE, 2002) in linear static analysis[1] is a type of reanalysis method based on Talylor series approximation of the displacement vector with respect to key design variables. The uniqueness of the VA over the other Taylor-series based reanalysis methods is the use of symbolic derivatives of the element stiffness matrices in computing the derivatives of the displacement vector. This allows scalable computational efficiency, approximation accuracy, and numerical stability, compared to the other methods that numerically evaluate derivatives such as finite differencing.

Let $\boldsymbol{u}(\boldsymbol{x})$ be the nodal displacement vector of the finite element model of a product geometry represented by design variable $\boldsymbol{x}$, and $\boldsymbol{x}_0$ be the value of design variable representing the nominal product geometry. VA estimates the nodal displacement vector $\boldsymbol{u}(\boldsymbol{x}_0 + \Delta \boldsymbol{x})$ of a modified geometry $\boldsymbol{x}_0 + \Delta \boldsymbol{x}$ using Taylor series

[1]While the description and case study in this paper assumes linear static structural analysis, VT can generalize to other type of analyses (CADOE, 2003).

Algorithm 1: Generating interval from two point-sets ($P_1$and $P_2$) based on topologically closest points

Match_Based_on_Topological_Relations($P_1$, $P_2$, $\Pi$)
If($\Pi$ empty) $\pi_c$=Find_Geometrically_Closest_Points($P_1$, $P_2$)
Else $\pi_c$=Find_Shortest_Distribution_Trajectory($\Pi$)
$P_{12}$= $P_1$+ $P_2$
Remove_Points_If_In_ParticleCloud($P_{12}$, $\Pi$)
Sort_Points_Ascending_By_Shortest_Path($P_{12}$, from $\pi_c$)
For p=each_point_of_$P_{12}$
$P_n$=CollectCloseNeighbors(p, $P_1$ (if p $\in$ $P_1$), $P_2$ (if p $\in$ $P_2$))
$\Pi_n$= CollectParticlesThatContainPoints($\Pi \mathfrak{sg}_n$)
$P_m$=CollectMatchedPointsOfCloseNeighbors(If $p_n$=$r_n$ return $p_m$=$r_n + \tau_n$; If $p_n$=$r_n + \tau_n$ return $p_m$=$r_n$)
$P_m$=CollectCloseNeighbors($p_m$, $P_1$ (if $p_m$ $\in P_1$), $P_2$ (if $p_m$ $\in P_2$))
Sort_Points_Ascending_By_Distance($P_m$, from p)
For $p_m$=each_point_of_$P_m$
$\pi_m$ =GenerateDistributionTrajectory ($p_m$, p)
add $\pi_m$ to $\Pi_n$
$\Pi_{anomalies}$ =Check_Topological_Anomalies($\Pi_n$)
If ($\Pi_{anomalies}$ $<>$0)
remove $\pi_m$ from $\Pi_n$
go_to_next_point_in $P_m$
Else
add $\pi_m$ to $\Pi$
go_to_next_point_in $P_{12}$
end for
end for

approximation:

$$u(x_0 + \Delta x) = u(x_0) + \sum_{i=1}^{n} \frac{1}{i!} u^{(i)}(x_0) \Delta x^i + O(\Delta x^{n+1}) \quad (1)$$

where $\boldsymbol{u}^{(i)}(\boldsymbol{x}_0)$ is the $i$-th order derivative of $\boldsymbol{u}(\boldsymbol{x})$ evaluated at $\boldsymbol{x} = \boldsymbol{x}_0$. The displacement vector $\boldsymbol{u}(\boldsymbol{x}_0)$ of the nominal design $\boldsymbol{x}_0$ is obtained by solving the static equilibrium equation:

$$\mathbf{K}(x_0) u(x_0) = f(x_0) \quad (2)$$

where $\mathrm{K}(\boldsymbol{x}_0)$ and $\boldsymbol{f}(\boldsymbol{x}_0)$ are the global stiffness matrix and the global load vector, respectively, of the nominal design $\boldsymbol{x}_0$. Note that both $\mathbf{K}$ and $\boldsymbol{f}$ are the functions of design variable $\boldsymbol{x}$.

To evaluate the right hand side of Equation (1), the derivative $\boldsymbol{u}^{(i)}(\boldsymbol{x}_0)$ needs be computed. In VA, this is done by exploiting the mathematical properties of the equilibrium equation, instead of numerically estimation. By differentiating the both sides of the equilibrium equation for arbitrary $\boldsymbol{x}$, one can obtain:

$$\begin{aligned} &\mathbf{K}^{(1)}(x)u(x) + \mathbf{K}(x)u^{(1)}(x) = f^{(1)}(x) \\ &\mathbf{K}^{(2)}(x)u(x) + 2\mathbf{K}^{(1)}(x)u^{(1)}(x) + \mathbf{K}(x)u^{(2)}(x) = f^{(2)}(x) \\ &\vdots \\ &\sum_{i=0}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} \mathbf{K}^{(n-i)}(x)u^{(i)}(x) + \mathbf{K}(x)u^{(n)}(x) = f^{(n)}(x) \end{aligned} \quad (3)$$

or equivalently:

$$
\begin{aligned}
&\mathbf{K}(x)u^{(1)}(x) = f^{(1)}(x) - \mathbf{K}^{(1)}(x)u(x) \\
&\mathbf{K}(x)u^{(2)}(x) = f^{(2)}(x) - \mathbf{K}^{(2)}(x)u(x) - 2\mathbf{K}^{(1)}(x)u^{(1)}(x) \\
&\vdots \\
&\mathbf{K}(x)u^{(n)}(x) = f^{(n)}(x) - \sum_{i=0}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} \mathbf{K}^{(n-i)}(x)u^{(i)}(x)
\end{aligned}
\tag{4}
$$

It can be seen from Equation (4) that:

1. The left hand sides of all equations are in the form **K*v*** (a vector ***v*** multiplied by the stiffness matrix), same as the original equilibrium equation (2).
2. The right hand side of the $i$-th equation (the equation with $\mathbf{K}(\boldsymbol{x})\boldsymbol{u}^{(i)}(\boldsymbol{x})$ in the left hand side) depends only on $\boldsymbol{f}^{(i)}(\boldsymbol{x})$, $\mathbf{K}^{(i)}(\boldsymbol{x})$, and $\boldsymbol{u}^{(j)}(\boldsymbol{x})$, $j$ =1,..., $i$-1.

The first observation leads to the efficient solution of Equations (4) by utilizing the decomposed stiffness matrix obtained while solving Equation (2). The second observation leads to the efficient evaluation of the right hand sides of Equations (4), by recursively substituting $\boldsymbol{u}^{(j)}(\boldsymbol{x}_0)$, $j$ = 1, 2, ..., $i$-1 to the $i$-th equation starting with $i$ = 1, once $\boldsymbol{f}^{(i)}(\boldsymbol{x}_0)$ and $\mathbf{K}^{(i)}(\boldsymbol{x}_0)$ are obtained.

The calculations of $\boldsymbol{f}^{(i)}(\boldsymbol{x}_0)$ and $\mathbf{K}^{(i)}(\boldsymbol{x}_0)$ are done by evaluating the analytic equations of the $i$-th derivatives of the boundary and loading conditions and the element stiffness matrix, respectively, with respect to design variable ***x***. As an illustration, let us consider a two-dimensional constant strain triangle (CST) element and suppose the design variable ***x*** is the nodal coordinates:

$$x = (x_1, y_1, x_2, y_2, x_3, y_3)^T \tag{4}$$

The element stiffness matrix of a CST element is given by (Chandrupatla and Belegundu, 1997):

$\mathbf{K}_e(\boldsymbol{x}) = tA(\boldsymbol{x})\ \mathbf{B}^T(\boldsymbol{x})\ \mathbf{D}\ \mathbf{B}(\boldsymbol{x})$ (5)

where $t$ and $A(\boldsymbol{x})$ are the thickness and area of the element, respectively, $\mathbf{B}(\boldsymbol{x})$ is the strain-displacement matrix, and **D** is the stress-strain matrix. They are given as:

$$A(x) = \frac{1}{2}|\Delta| \tag{5}$$

$$\mathbf{B}(x) = \frac{1}{\Delta}\begin{pmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{pmatrix} \tag{6}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix} \tag{7}$$

where:

$$
\begin{aligned}
&\Delta = x_{13}y_{23} - x_{23}y_{13} \\
&x_{ij} = x_i - x_j \\
&y_{ij} = y_i - y_j
\end{aligned}
\tag{8}
$$

Using Equations (5)-(8), the first derivative of the element stiffness matrix $\mathbf{K}_e(\boldsymbol{x})$ with respect to $x_1$, for example, is given as:

$$
\begin{aligned}
\frac{\partial \mathbf{K}_e(x)}{\partial x_1} &= t\frac{\partial A(x)}{\partial x_1}\mathbf{B}^T(x)\mathbf{D}\mathbf{B}(x) \\
&\quad + tA(x)\frac{\partial \mathbf{B}^T(x)}{\partial x_1}\mathbf{D}\mathbf{B}(x) \\
&\quad + tA(x)\mathbf{B}^T(x)\mathbf{D}\frac{\partial \mathbf{B}(x)}{\partial x_1}
\end{aligned}
\tag{9}
$$

where

$$\frac{\partial A(x)}{\partial x_1} = \begin{cases} y_{23} & \text{if } \Delta > 0 \\ -y_{23} & \text{otherwise} \end{cases} \tag{10}$$

$$\frac{\partial \mathbf{B}(x)}{\partial x_1} = \frac{1}{\Delta^2}\begin{pmatrix} -y_{23}y_{23} & 0 & -x_{32}y_{23} \\ 0 & -x_{32}y_{23} & -y_{23}y_{23} \\ -y_{31}y_{23} & 0 & \Delta - x_{13}y_{23} \\ 0 & \Delta - x_{13}y_{23} & -y_{31}y_{23} \\ -y_{12}y_{23} & 0 & -\Delta - x_{21}y_{23} \\ 0 & -\Delta - x_{21}y_{23} & -y_{12}y_{23} \end{pmatrix}^T \tag{11}$$

Basic steps for computing $\boldsymbol{u}(\boldsymbol{x}_0 + \Delta\boldsymbol{x})$ for given $\Delta\boldsymbol{x}$ can be summarized as follows:

1. Solve Equation (2) to obtain the displacement vector $\boldsymbol{u}(\boldsymbol{x}_0)$ for the nominal product shape.
2. For $i$ = 1, 2, ..., $n$, compute the $i$-th derivative $\partial^i\mathbf{K}_e(\boldsymbol{x})/\mathfrak{g}\boldsymbol{x}^i$of the element stiffness matrix and the $i$-th derivative $\partial^i\mathbf{f}(\boldsymbol{x})/\mathfrak{g}\boldsymbol{x}^i$ of the load vector, using analytical equations.
3. For $i$ = 1, 2, ..., $n$, assemble $\partial^i\mathbf{K}_e(\boldsymbol{x})/\mathfrak{g}\boldsymbol{x}^i$ to the $i$-th derivative $\partial^i\mathbf{K}(\boldsymbol{x})/\mathfrak{g}\boldsymbol{x}^i$of the global stiffness matrix.
4. From $i$ = 1 through $i = n$, recursively solve Equations (4) by using the decomposed global stiffness matrix $\mathbf{K}(\boldsymbol{x})$ obtained in step 1, and by substituting $\boldsymbol{u}^{(j)}(\boldsymbol{x}_0)$, $j$ = 1, 2, ..., $i$-1 to the right hand side of the $i$-th equation.
5. Compute $\boldsymbol{u}(\boldsymbol{x}_0 + \Delta\boldsymbol{x})$ by substituting $\boldsymbol{u}(\boldsymbol{x}_0)$ obtained in step 1 and $\boldsymbol{u}^{(i)}(\boldsymbol{x}_0)$, $i$ = 1, 2, ..., $n$, obtained in step 4, into Equation (1).

Since the global matrix is solved only once at step 1 and the subsequent steps are only assembling matrices and evaluating polynomials, $\boldsymbol{u}(\boldsymbol{x}_0 + \Delta\boldsymbol{x})$ can be obtained with far smaller computational cost than solving Equation (1) for $\boldsymbol{x} = \boldsymbol{x}_0 + \Delta\boldsymbol{x}$.

The choice of $n$ depends on the desired accuracy in Equation (1), which $n$ = 2 to 4 is reported as sufficient in most cases. Since the derivation of the analytical equations up to the $n$-th derivative can be extremely tedious and complex (especially for elements with large DOF), higher order derivatives can be approximated by a numerical method such as finite differencing.

## 4. EXAMPLES

VDIM and VA have been implemented as computer software, and a case study with a two-dimensional cross section of the seats of chairs is conducted to demonstrate the feasibility of the method. The finite element analysis and VA are implemented for CST elements with $n$ = 2. CST elements are chosen as a first attempt since they are relatively simple to implement VT due to the small DOF and compatible to the triangulated surfaces in the shape instances of VDIM.

### 4.1. VDIM

As an application of VDIM, three popular designs of the seats of chairs have been reproduced, and their 2D cross sections have been tested. The photos of Figure 5 illustrate the targeted seats. To produce the targeted shapes with VDIM, first a vague model was created as a composition of planar point-sets. The vague model had to be carefully chosen to be able to derive each targeted shape from it. This required generating a large interval. The left top image of Figure 5 presents the vague model of a seat. In the next step a number of regions have been selected by using a so called 3D cursor. Figure 4 shows the 3D cursor in action. Particles, that are in contact with the the 3D cursor during the selection procedure are assigned to a region. The selected regions represent various features of the targeted shapes. To derive the instance shapes, shape formation rules of curving, offsetting, and tilting were applied to the regions. Interested readers are referred to (Rusák and Horváth 2005) for details on shape instantiation based on shapes formation rules. The derived instance seats are presented on Figure 5.

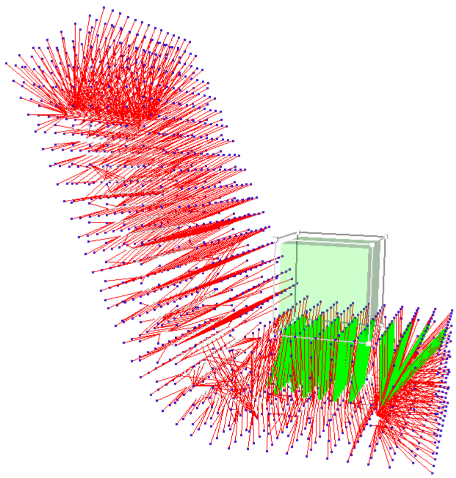

**Figure 4** Region selection with a 3D cursor

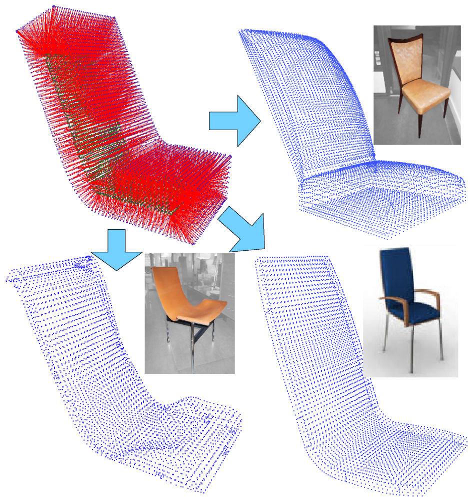

**Figure 5** Vague shape model of a seat and some instance shape variances

### 4.2. VA

The structural performances the shape alternatives instantiated from VDIM are examined by using VA. As an initial implementation, 2D cross sections of the seats are used for estimation and the results by VA are compared with the finite element analysis in terms of accuracy and speed.

Figure 6a shows the cross-section of the VDIM of seats in Figures 4 and 5, and its nominal shape. The dimension of the seats is modeled as approximately 700 mm in length and height, and 500 mm in width. As seen in Figure 6a, the shape intervals are defined only to the points on the seating surface, with the range between 15 mm and 30 mm. Figure 6b shows the boundary and loading conditions used for the evaluation of structural responses. These conditions intend to mimic the loads exerted by the person with 70 kg of weight sitting on the chair. For simplicity, the loads are assumed constant during the instantiation of different shape alternatives, *i.e.*, $\boldsymbol{f}^{(i)}(\boldsymbol{x}) \equiv 0$ in Equation (4).

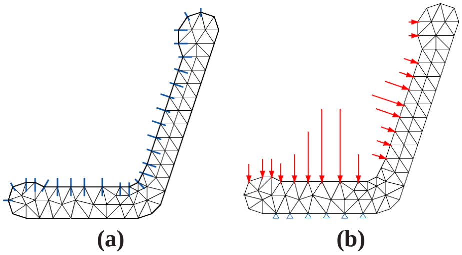


**Figure 6** (a) cross-section of the VDIM of seats and (b) boundary and loading conditions

Figure 7 shows a snapshot of the developed software implementing VA. The upper left window shows the displacement of the nominal shape obtained by the conventional finite element analysis, and the rest of the windows show the displacements of the shape alternatives derived from the VDIM in Figure 6a, estimated by VA utilizing up to the second-order terms in the Taylor series expansion. For clarification, displacements are magnified to 5 times. Although not shown in the figure, stress values estimated by VA can also be visualized. Using this user interface, a designer can quickly examine the structural performances of alternative shapes at virtually no computational overhead.

In order to statistically access the accuracy and speed of VA, samples of 100 shape variants are randomly instantiated from the VDIM in Figure 6a, and the estimation by VA are compared to the results of the finite element analyses. The comparisons are made by calculating the displacement errors $e_u$ and von Mises stress errors $e_\sigma$ averaged over all nodes in a sample shape variant:

$$\begin{aligned} e_u &= \frac{1}{6m}\sum_{i=1}^{6m}\frac{|u_i - v_i|}{v_i} \\ e_\sigma &= \frac{1}{m}\sum_{i=1}^{m}\frac{|\sigma_i - s_i|}{s_i} \end{aligned} \tag{12}$$

where $m$ is is the number of elements in the FE mesh, $u_i$and $v_i$ are the $i$-th element of the (global) nodal displacement vectors obtained by VA and FEM, respectively, and $\sigma_i$ and $s_i$ are von Mises stresses of the $i$-th CST element obtained by VA and FEM, respectively. Table 3.3 shows $e_u$ and $e_\sigma$ averaged over the 100 samples. It also shows the CPU time of the 100 runs for VA and FEA on a Windows laptop.

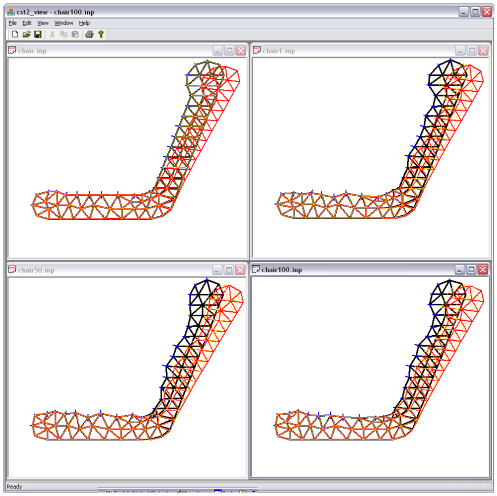


**Figure 7** Developed software visualizing the displacements of the nominal shape (upper left), and shape variants (rest).

The statistics show that von Mises stresses have 22% error on average, whereas the nodal displacements have error as large as 45%. While the error in von Mises stress is within an acceptable range (as standard FE is known to have an error as large as 10-20%), the displacement error is fairly large, with the notably large standard deviation (250%!).

Figures 8 and 9 show the displacement errors and von Mises stress errors of the 100 samples, plotted against the average and maximum coordinate deviations from the nominal shape. In these figures, the square and triangle points indicate the average coordinate deviation and the maximum coordinate deviation, respectively. The plots show no obvious correlation between the errors in structural responses and

**Table 1** Comparison of VA and FEA for 100 random samples

| quantities | avg of 100 samples | stddev of 100 samples |
|---|---|---|
| Average displacement error [%] | 45 | 250 |
| Average von Mises stress error [%] | 22 | 34 |
| CPU time for VT [ms] | 560 | 35 |
| CPU time for FEA [ms] | 3900 | 97 |

the deviations from the nominal shape. On the other hand, Figure 8 shows the several (6) samples with extremely large errors, with the rest of samples in one small cluster. These extreme samples are a likely cause of very large standard deviation (250%) of the displacement errors.

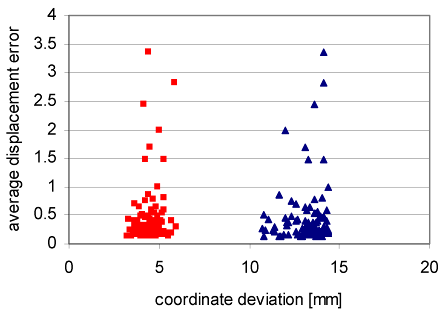


**Figure 8** Average displacement errors vs. average coordinate deviations (squares) and vs. maximum coordinate deviations (triangles) from the nominal shape

The close examination of each sample shape reveals that that the large errors in the displacements occur near the horizontal seat of the chair where the magnitude of displacements is almost zero compared to the size of the chair. For the sample shapes with relatively large shape variations at the seat, the displacement errors of the nodes on the seat can be very large since $v_i \approx 0$ in Equation (12). The errors in von Mises stresses in the element on the seat, on the other hand, are not as large, since the stress is constant in a

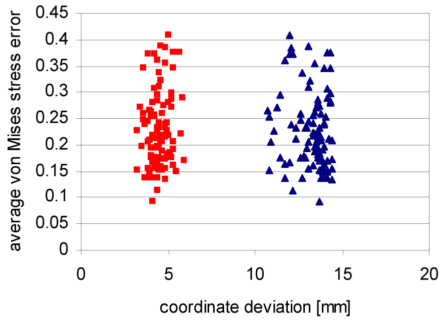


**Figure 9** Average von Mises stress errors vs. average coordinate deviations (squares) and vs. maximum coordinate deviations (triangles) from the nominal shape

CST element, which seems to average out the effects of errors in nodal displacements.

In terms of computational speed, VA clearly shows the advantage over FEA, with 7-fold improvement in the average CPU time. Since VA requires only one FE analysis of the nominal shape and the estimations for the shape variants are done by evaluating polynomials, this speed gain is expected to scale up well with the number of nodes in the finite element model.

## 5. CONCLUSION AND FUTURE WORK

The paper proposed the integration of the vague discrete interval models (VDIM) and the variational analysis (VA) technique, in order to realize a design environment for rapidly generating alternative shapes and evaluating their structural responses. While the accuracy of VA has some concern in the presented examples, it is likely due to the naïve implementation by the authors, considering the reported accuracy of the highly-tuned, commercial 3D implementation of the technique (CADOE, 2002). Since VDIM was originally developed for 3D (Rusák and Horváth, 2005), its integration with the 3D version of VA, possibly using a commercial code, will prove to be a very effective tool for concept exploration for practical designs.


## ACKNOWLEDGMENTS

The first author acknowledges the University of Michigan for partially supporting his sabbatical leave at the Delft University of Technology.

## REFERENCES


CADOE S. A., 2002, "Mechanical design synthesis: New processes for innovative product development," White Paper, September 2002. URL: http://www.cadoe.com/WhitePaper_CADOE.pdf

Cagan J. and Argawal M., "A blend of different tastes: the language of coffeemakers," *Environment and Planning B: Planning and Design*, 25():205-26.

Chandrupatla, T. R. and Belegundu, A. D., 1997, *Introduction to Finite Elements in Engineering*, $2^{nd}$ edition, Prentice Hall, New Jersey.

Cheng, B. and Titterington, D. M., 1994, "Neural networks: a review from a statistical perspective," Statistical Science, 9(1), pp. 2-54.

Choi, K. K. and Chang, K. H., 1994, "A study on design velocity filed computation for shape optimal design," Finite Elements in Analysis and Design, 15(4), pp. 317-341.

Forsberg, J. and Nilsson, L., 2005, "On polynomial response surfaces and Kriging for use in structural optimization of crashworthiness," Structural and Multidisciplinary Optimization, 29(3), pp. 232-243.

Heisserman J. and Woodbury R., 1994, "Geometric design with boundary solid grammars," *IFIP Transactions B: Computer Applications in Technology*, B-18:85-105.

Hsiao S. W. and Chen C. H., 1997, "A semantic and shape grammar based approach for product design," *Design Studies*, 18():275-95.

Jin, R., Chen, W., Simpson, T. W., 2001, "Comparative studies of metamodelling techniques under multiple modelling criteria," Structural and Multidisciplinary Optimization, 21(1), pp. 1-13.

Kirsch, U., 2002, Design-oriented analysis of structures, Kluwer Academic Publishers, Dordrecht.

Kirsch, U., 2003, "A unified reanalysis approach for structural analysis, design, and optimization," Structural and Multidisciplinary Optimization, 25(2), pp. 67 – 85.

Kirsch, U., and Papalambros, P. Y., 2001, "Structural reanalysis for topological modifications - A unified approach," Structural and Multidisciplinary Optimization, 21(5), pp. 333- 344.

Kirsch, U., and Papalambros, P. Y., 2001, "Exact and accurate reanalysis of structures for geometrical changes," Engineering with Computers, 17(4), pp. 363-372.

Levy, R., Kirsch, U., and Liu, S., 2000, "Reanalysis of trusses using modified initial designs," Structural and Multidisciplinary Optimization , 19(2), pp. 105-112.

Mechesheimer, 2001, "Metamodeling of combined discrete/continuous responses," AIAA Journal, 39 (9), pp. 1955-1959.

Rusák, Z. and Horváth, I., 2005, "Instantiation of shapes for products by shape formation rules based on Vague Discrete Interval Model," *Journal of Engineering Design*, 16(2),:135-155.

Sakata S, Ashida F, Zako M, 2003, "Strural optimization using Kriging approximation," Computer Methods in Applied Mechanics and Engineering, 192 (7-8), pp. 923-939.

Smyth S. N. and Wallace D. R., 2000, "Towards the synthesis of aesthetic product form," Proceedings of the ASME Design Engineering Technical Conferences, DETC-DTM-14554.

Taura T., Nagasaka I., and Yamagishi A., 1998, "Application of evolutionary programming to shape design," *Computer-Aided Design*, 30(1): 29-35.

Ullman, D. G. 2003, *The Mechanical Design Process*, $3^{rd}$ ed., McGraw-Hill, New York.

Yeh, T .P. and Vance, J. M., 1998, "Applying virtual reality techniques to sensitivity based structural shape design," Journal of Mechanical Design, 120(4), pp. 612–619.