A Stabilized Mixed Quadrilateral Plate Bending Element for Reissner-Mindlin Type

by

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1. Introduction

We had developed some plate bending (PB) and plane stress (PS) elements.

1) 4-node quadrilateral PB element (1991)

2) 3-node PB element by using mixed formulation (1993) (Kirchhoff, Reissner-Mindlin)

3) Non-conforming PS element (1995)

4) Stabilized mixed quadrilateral PB element (1997)

Today, I would like to introduce 4-th one.

Objective

To improve the performance of the basic 4-node plate bending element.

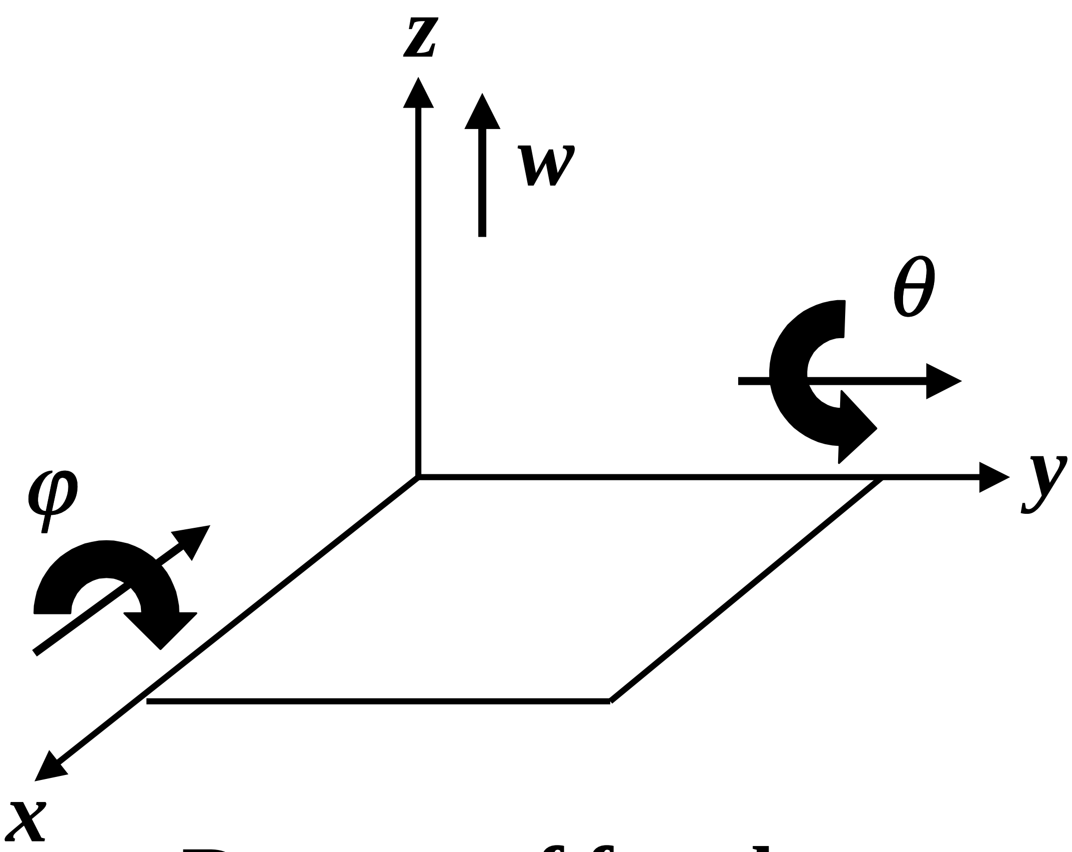
5 ideas are introduced to improve elements.

1. **Full isoparametric biquadratic shape functions were added for lateral deflections.**
2. **Introducing coupling between lateral deflections and 2 rotations, without increasing element d.o.f. with the interelement compatibility preserved.**
3. **Mesh dependent stabilization procedure was added to the original variational functional for controlling spurious (zero energy) modes.**

4. **Transverse shear forces are assumed ‘constant’ in an element.**

5. **Second order derivatives in variational functional are computed by using bilinear interpolation.**

2. Overview of the Plate bending theory



Degrees of freedom

Assumption in this presentation

Isotropic homogenious material and constant thickness are assumed.

The normal stress σ_z in *z*-direction is negligible.

Displacements are assumed as

$$u = z\theta(x, y), \quad v = z\varphi(x, y), \quad w = w(x, y)$$

Generalized strain-displacement relations

$$k_x = \frac{\partial \theta}{\partial x}, \quad k_y = \frac{\partial \varphi}{\partial y}, \quad k_{xy} = \frac{\partial \theta}{\partial y} + \frac{\partial \varphi}{\partial x},$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta + \frac{\partial w}{\partial x},$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \varphi + \frac{\partial w}{\partial y}$$

Generalized stress-strain relations

Moment-Curvature relations

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

Shear force-strain relations

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \kappa G t \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad \left(\begin{array}{ll} \kappa = 5/6 & : \text{Reissner} \\ \kappa = \pi^2/12 & : \text{Mindlin} \end{array} \right.$$

$$D = \frac{E\,t^3}{12\left(1-\nu^2\right)}, \quad G = \frac{E}{2\left(1+\nu\right)}$$

D : Bending stiffness, G : Shear stiffness

E : Young's modulus, ν : Poisson's ratio

t : Plate thickness

κ is the shear correction factor to account for non-uniform distribution of transverse shear strains and stresses in *z*-direction.

Basic variational principles

$$\Pi_1(w,\theta,\varphi)$$

$$= \iint_A \left\{ \frac{1}{2} \left(M_x k_x + M_y k_y + M_{xy} k_{xy} + Q_x \gamma_{xz} + Q_y \gamma_{yz} \right) - pw \right\} dxdy$$

$$= \iint_A \left\{ \frac{D}{2} \left({k_x}^2 + {k_y}^2 + 2\nu k_x k_y + \frac{1-\nu}{2} {k_{xy}}^2 \right) + \frac{\kappa G t}{2} ({\gamma_{xz}}^2 + {\gamma_{yz}}^2) - pw \right\} dxdy$$

When $t \to 0$ $\rightarrow$ $D \ll Gt$ $\rightarrow$ $\gamma_{xz}^2 + \gamma_{yz}^2 \to 0$

$$O(D) = O(t^3)\,,\quad O(G) = O(t)$$

<u>Kirchhoff assumption</u>

$$t \to 0 \;\rightarrow\; \gamma_{xz} \to 0,\ \gamma_{yz} \to 0$$

$$\theta \approx -\frac{\partial w}{\partial x},\quad \varphi \approx -\frac{\partial w}{\partial y}$$

By introducing Lagrange multipliers, the variational functional based on Reissner's variational principle is obtained as follows:

$$\Pi_2(w,\theta,\varphi,Q_x,Q_y)$$

$$= \iint_A \left\{ \frac{D}{2}\left(k_x^{\ 2} + k_y^{\ 2} + 2\nu k_x k_y + \frac{1-\nu}{2} k_{xy}^{\ 2} \right) \right.$$

$$+ Q_x\left(\theta + \frac{\partial w}{\partial x}\right) + Q_y\left(\varphi + \frac{\partial w}{\partial y}\right)$$

$$\left. - \frac{1}{2\kappa G t}\left(Q_x^{\ 2} + Q_y^{\ 2}\right) - pw \right\} dxdy = \sum_e \Pi_{2e}$$

$$\Pi'_2\left(w,\theta,\varphi,Q_x,Q_y,\lambda_1,\lambda_2\right)$$

$$=\iint_A\left\{\frac{D}{2}\left(k_x{}^2+k_y{}^2+2\nu k_x k_y+\frac{1-\nu}{2}k_{xy}{}^2\right)\right.$$

$$+\frac{1}{2\kappa Gt}\left(Q_x{}^2+Q_y{}^2\right)+\lambda_1\left(Q_x-\kappa Gt\left(\theta+\frac{\partial w}{\partial x}\right)\right)$$

$$\left.+\lambda_2\left(Q_y-\kappa Gt\left(\varphi+\frac{\partial w}{\partial y}\right)\right)-pw\right\}dxdy=\sum_e \Pi_{2e}$$

Stationary condition $\lambda_1=-\dfrac{Q_x}{\kappa Gt},\ \lambda_2=-\dfrac{Q_y}{\kappa Gt}$

If $\frac{1}{2\kappa Gt}\left(Q_x{}^2 + Q_y{}^2\right)$ **is omitted in** Π_2

Kirchhoff plate's variational functional will be obtained.

In this case, the Kirchhoff conditions are

$$\gamma_{xz} = \theta + \frac{\partial w}{\partial x} = 0\,, \quad \gamma_{yz} = \varphi + \frac{\partial w}{\partial y} = 0$$

For the Reissner-Mindlin plate, the relations between the shear forces and the transverse shear strains are

$$Q_x = \kappa G t \gamma_{xz} = \kappa G t \left(\theta + \frac{\partial w}{\partial x} \right)$$

$$Q_y = \kappa G t \gamma_{yz} = \kappa G t \left(\varphi + \frac{\partial w}{\partial y} \right)$$

By differentiating the equalities and using <u>the mean square approach with the in-plane isotropy taken into account</u>, we may obtain a new mixed variational functional with mesh dependent stabilization terms added to $\prod_2$. **<u>Idea-3</u>**

$$\frac{\partial Q_x}{\partial x} = \kappa G t\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \frac{\partial Q_x}{\partial y} = \kappa G t\left(\frac{\partial \theta}{\partial y} + \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$\frac{\partial Q_y}{\partial y} = \kappa G t\left(\frac{\partial \varphi}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right), \frac{\partial Q_y}{\partial x} = \kappa G t\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$
\begin{aligned}
&\Pi_3\left(w,\theta,\varphi,Q_x,Q_y\right)=\Pi_2\left(w,\theta,\varphi,Q_x,Q_y\right)\\
&+\sum_e \frac{\alpha_e}{2}\iint_e \left[\left\{\frac{1}{\kappa G t}\frac{\partial Q_x}{\partial x}-\left(\frac{\partial \theta}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right\}^2+\left\{\frac{1}{\kappa G t}\frac{\partial Q_y}{\partial y}-\left(\frac{\partial \varphi}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right\}^2\right.\\
&\qquad +2\mu_e\left\{\frac{1}{\kappa G t}\frac{\partial Q_x}{\partial x}-\left(\frac{\partial \theta}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right\}\left\{\frac{1}{\kappa G t}\frac{\partial Q_y}{\partial y}-\left(\frac{\partial \varphi}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right\}\\
&\qquad \left.+\frac{1-\mu_e}{2}\left\{\frac{1}{\kappa G t}\left(\frac{\partial Q_x}{\partial y}+\frac{\partial Q_y}{\partial x}\right)-\left(\frac{\partial \theta}{\partial y}+\frac{\partial \varphi}{\partial x}+2\frac{\partial^2 w}{\partial x \partial y}\right)\right\}^2\right]dxdy
\end{aligned}
$$

α_e is a positive parameter, which may depend on element *e* . μ_e is a parameter similar to Poisson's ratio and is hereafter taken as zero. Although μ_e can be chosen from the range $-1 < \mu_e < 1$.

Additional terms vanish for the exact plate solution .

3. Proposed element

The displacements in each element are assumed as: **Idea-2**

$$w(\xi,\eta)=\sum_{i=1}^{4}\left\{w_i N_i(\xi,\eta)+\theta_i N_{\theta_i}(\xi,\eta)+\varphi_i N_{\varphi_i}(\xi,\eta)\right\}$$

$$\theta(\xi,\eta)=\theta_L(\xi,\eta)+\theta_B(\xi,\eta);$$

$$\theta_L=\sum_{i=1}^{4}\theta_i N_i(\xi,\eta),\quad \theta_B=\alpha_1 N_9(\xi,\eta)$$

$$\varphi(\xi,\eta)=\varphi_L(\xi,\eta)+\varphi_B(\xi,\eta);$$

$$\varphi_L=\sum_{i=1}^{4}\varphi_i N_i(\xi,\eta),\quad \varphi_B=\alpha_2 N_9(\xi,\eta)$$

Shape functions Idea-1

$$N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta) \quad N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta) \quad N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

$$N_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)+\frac{D_4-D_1}{4(D_1+D_3)}N_9(\xi,\eta)$$

$$N_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1-\eta^2)+\frac{D_1-D_2}{4(D_2+D_4)}N_9(\xi,\eta)$$

$$N_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)+\frac{D_2-D_3}{4(D_3+D_1)}N_9(\xi,\eta)$$

$$N_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1-\eta^2)+\frac{D_3-D_4}{4(D_4+D_2)}N_9(\xi,\eta)$$

$$N_9(\xi,\eta)=(1-\xi^2)(1-\eta^2)$$

$$D_i = \begin{vmatrix} x_j - x_i & x_m - x_i \\ y_j - y_i & y_m - y_i \end{vmatrix} \quad i = 1,2,3,4$$

$$N_{\phi_i}(\xi,\eta) = \frac{1}{8}\left\{(y_i - y_j)N_{i+4}(\xi,\eta) + (y_i - y_m)N_{m+4}(\xi,\eta)\right\}$$
$$+(-1)^{i+1}\frac{1}{32}(y_2 + y_4 - y_1 - y_3)N_9(\xi,\eta)$$

$$N_{\theta_i}(\xi,\eta) = \frac{1}{8}\left\{(x_i - x_j)N_{i+4}(\xi,\eta) + (x_i - x_m)N_{m+4}(\xi,\eta)\right\}$$
$$+(-1)^{i+1}\frac{1}{32}(x_2 + x_4 - x_1 - x_3)N_9(\xi,\eta)$$
$$(i = 1,2,3,4)$$

See. Kikuchi & Okabe, Comp. Meth. Appl. Mech. Eng. 1999

N_9 is the so-called bubble function, whose elementwise magnitudes are designated by α_1 and α_2. Such a function may be effectively used in the mixed methods to satisfy the so-called inf-sup condition.

<u>The biquadratic terms in *w* are coupled with θ and ϕ so that *w* can represent constant curvature states and satisfy interelement continuity</u> without increasing number of unknown parameters.

We assume <u>the transverse shear forces Q_x and Q_y to be constant in each elememt</u>.

Idea-4

$$Q_x = \kappa G t \overline{\gamma_{xz}} \quad , \quad Q_y = \kappa G t \overline{\gamma_{yz}}$$

$$\overline{\gamma_{xz}} = \frac{1}{A^e} \iint_e \gamma_{xz} dxdy = \frac{1}{A^e} \iint_e \left(\theta + \frac{\partial w}{\partial x} \right) dxdy$$

$$\overline{\gamma_{yz}} = \frac{1}{A^e} \iint_e \gamma_{yz} dxdy = \frac{1}{A^e} \iint_e \left(\varphi + \frac{\partial w}{\partial y} \right) dxdy$$

Thus Π_3 for the present approximation may be simplified as:

$$\Pi_4(w,\theta,\varphi) = \sum_e \iint_e \left\{ \frac{1}{2}\{\varepsilon^e\}^T[D^e]\{\varepsilon^e\} - pw + \frac{\kappa Gt}{2}\left(\overline{\gamma_{xz}}^2 + \overline{\gamma_{yz}}^2\right) \right\} dxdy$$

The magnitudes α_1 and α_2 of the bubble functions can be eliminated by the static condensation procedure.

Although the mixed quadrilateral element is free from locking, it has a deleterious side effect called rank deficiency.

By zero-energy-mode analysis, we can see that there are two spurious zero-energy modes in excess of the three rigid-body modes in the case of rectangular elements.

The in-plane twist mode seldom causes difficulties in practical computations.

Π_3 may be simplified as :

$$\Pi_5(w,\theta,\varphi) = \Pi_4(w,\theta,\varphi)$$

$$+\sum_e \frac{\alpha_e}{2}\iint_e \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \varphi}{\partial y} \right)^2 \right.$$

$$\left. + \frac{1}{2}\left(\frac{\partial \theta}{\partial y} + \frac{\partial \varphi}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dxdy$$

Where $\alpha_e = \beta_e E t h_e^2$.

Here h_e **is the diameter of element *e* and** β_e **is the positive stabilization parameter for *e*. If** β_e **is too large, it may deteriorate the accuracy of numerical solutions, usually producing too stiff. If** β_e **is too small, it may cause modes quite close to spurious zero energy modes, and cannot stabilize the element effectively.**

Now β_e**=0.25.**

We introduce some further simplifications to the second order derivatives of *w* as well as to θ and ϕ in the stabilization terms.

We prepare the following two functions as alternatives to $\partial w / \partial x$ and $\partial w / \partial y$:

$$w_{h,x} = \sum_{i=1}^{4} N_i(\xi,\eta) \left.\frac{\partial w}{\partial x}\right|_{(x_i,y_i)}$$

$$w_{h,y} = \sum_{i=1}^{4} N_i(\xi,\eta) \left.\frac{\partial w}{\partial y}\right|_{(x_i,y_i)}$$

We use the bilinear interpolations of

$\partial w / \partial x$ **and** $\partial w / \partial y$.

Then we approximate $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$

and in the stabilization terms by

$$\frac{\partial^2 w}{\partial x^2} \cong \frac{\partial w_{h,x}}{\partial x}, \quad \frac{\partial^2 w}{\partial y^2} \cong \frac{\partial w_{h,y}}{\partial y},$$

Idea-5

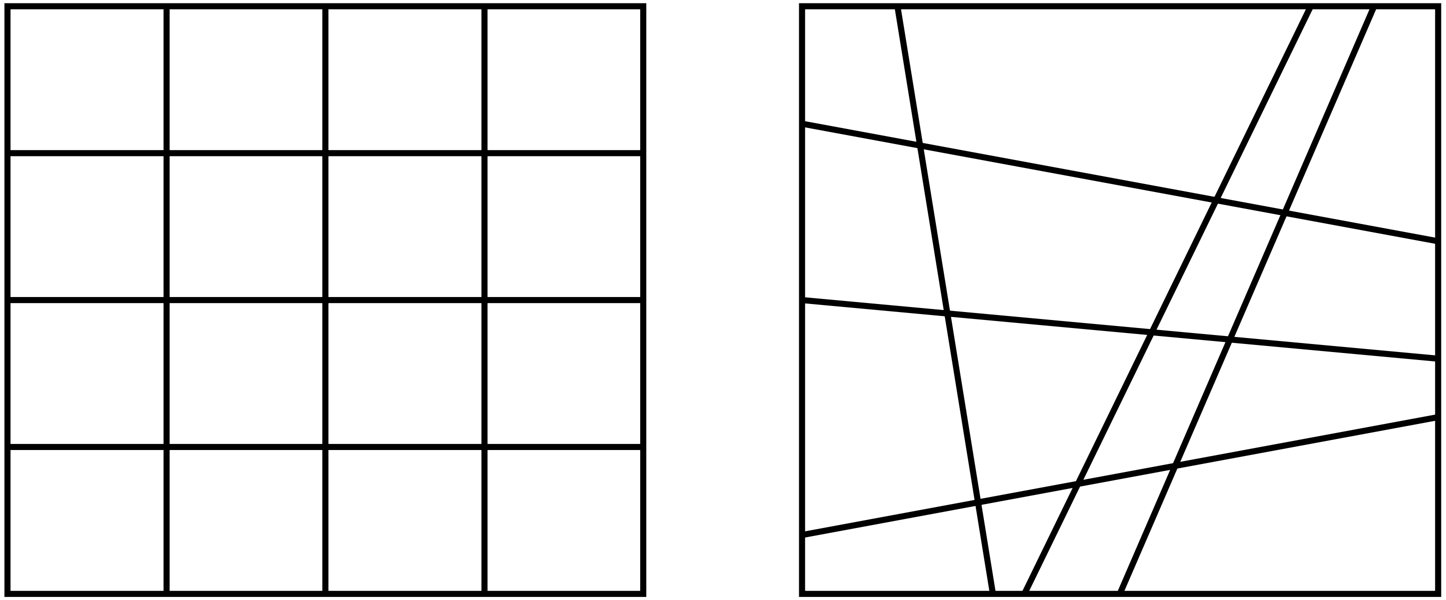
$$2\frac{\partial^2 w}{\partial x \partial y} \cong \frac{\partial w_{h,x}}{\partial y} + \frac{\partial w_{h,y}}{\partial x}$$

4. Numerical examples

SS : So-called “hard” simply supported boundary conditions,

CL : Clamped boundary conditions,

Nel : Number of elements.



Regular and Irregurar shape meshes

INT.	B.C.	Thickness / Side length			
		0.0001	0.001	0.01	0.1
2×2	SS	1.000	1.000	1.001	1.054
		1.000	1.000	1.001	1.044
	CL	0.988	0.988	0.990	1.189
		0.953	0.953	0.957	1.132
3×3	SS	0.999	0.999	1.000	1.053
		0.999	0.999	1.000	1.043
	CL	0.979	0.979	0.981	1.181
		0.951	0.951	0.955	1.128

Table 1. Central deflections normalized by the exact Kirchhoff solution in uniform mesh. Upper values for $\beta = 0.25$, lowers for $\beta = 0$.

INT.	B.C.	Thickness / Side length			
		0.0001	0.001	0.01	0.1
2×2	SS	0.994	0.994	0.994	1.049
		0.989	0.989	0.989	1.042
	CL	0.957	0.957	0.959	1.157
		0.943	0.943	0.946	1.119
3×3	SS	0.992	0.992	0.992	1.047
		0.985	0.985	0.986	1.041
	CL	0.945	0.945	0.948	1.147
		0.933	0.933	0.936	1.111

Irregular Mesh

Concluding remarks

We have proposed a stabilized mixed quadrilateral finite element for Reissner-Mindlin plates, which is free from shear locking and has fairly good accuracy in numerical results with stabilization effect.

However, more strict mathematical analysis such as convergence study of this new element appears to be necessary for establishing the validity of our approach.

Thank you very much for hearing my presentation.

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