Final Examination
A Typical Solution

MEAM305 Introduction to Finite Element Methods

April 21, 98W

1. Knowing the parametric coordinates of the node of the HEXA8 element as follows (20 points) :

node	ξ	η	ζ
1	-1	-1	-1
2	+1	-1	-1
3	+1	+1	-1
4	-1	+1	-1
5	-1	-1	+1
6	+1	-1	+1
7	+1	+1	+1
8	-1	+1	+1

(a) define the shape function $N_3(\xi,\eta,\zeta)$ of node 3, and (b) evaluate it at node 2, at the centroid, and at a point $(\xi,\eta,\zeta)=\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{3}}\right)$. (c) Sketch the function profile of the shape function N_3 on the surface defined by $\eta = 0$ and $\zeta = 0$. It should become a function of ξ in the interval (-1,1).

(a) The shape function becomes

$$N_3(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$

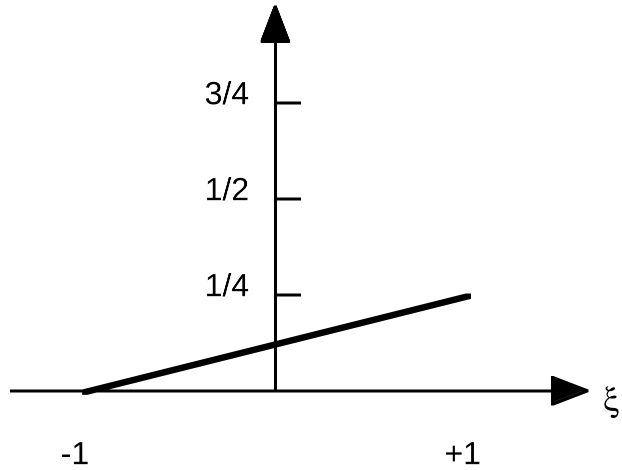
at node 3,

(b)

$$N_3=\begin{cases} 0 & \text{at node 2} \\ \frac{1}{8} & \text{at the centroid } (0,0,0) \\ \frac{1}{8}\left(1-\frac{1}{\sqrt{3}}\right)^3 & \text{at } (\xi,\eta,\zeta)=\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},+\frac{1}{\sqrt{3}}\right) \end{cases}$$

(c)

$$N_3(\xi,0,0) = \frac{1}{8}(1+\xi)$$



2. When the HEXA8 element is used to approximate the geometry and displacement, and when the strain assumed approximation is considered, explain why the component ε_x of the strain is approximated by a polynomial

$$\varepsilon_x = a_0 + a_1\eta + a_2\zeta$$

where a_0, a_1, and a_2 are unknown coefficients. Similarly, the shear strain γ_{xy} should be approximated by

$$\gamma_{xy} = b_0 + b_1\zeta$$

in the strain assumed element. Explain why this approximation makes sense. Here we have assumed that the local coordinates (x,y,z) are almost parallel to the parametric coordinates (ξ,η,ζ). (20 points)

When the normal strain is assumed to be

$$\varepsilon_x = a_0 + a_1\eta + a_2\zeta$$

we have

$$\Rightarrow \quad u_x = \int_x \varepsilon_x dx = \int_x \frac{\partial u_x}{\partial x} dx \approx \int_\xi (a_0 + a_1\eta + a_2\zeta) d\xi$$

$$= c_0(\eta,\zeta) + (a_0 + a_1\eta + a_2\zeta)\xi$$

that is, the displacement component u_x becomes a trilinear function of the parametric coordinates which is the same with the displacement approximation in HEXA8.

On the other hand if it is a linaer function of ξ, then the diplacement becomes quadratic in ξ. However, this is not the case of HEXA8 element.

Similarly, for the shear

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = b_0 + b_1\zeta$$

$$\Rightarrow \quad \int_y \int_x \gamma_{xy} dxdy \approx \int_\eta \int_\xi \gamma_{xy} d\xi d\eta = e_0(\xi,\zeta) + \{d_0(\zeta) + (b_0 + b_1\zeta)\xi\}\eta$$

and this is compartible with the displacement approximation in HEXA8 element.

3. Suppose that a given function

$$g(\xi) = 1 - \xi^2$$

in the interval $(-1,1)$, must be approximated by a constant function

$$g_{approximation}(\xi) = a\,,$$

where the coefficient a is an appropriate number. Find a by using the least squares method. (20 points)

The least squares problem becomes

$$\min_a \frac{1}{2}\int_{-1}^{1}\left(a - \left(1-\xi^2\right)\right)^2 d\xi = \min_a \int_{-1}^{1} F(a) d\xi$$

where $F(a) = \frac{1}{2}\left(a - \left(1-\xi^2\right)\right)^2$, *and then*

$$\frac{\partial}{\partial a}\int_{-1}^{1} F(a) d\xi = \int_{-1}^{1} \frac{\partial F}{\partial a} d\xi = \int_{-1}^{1} \left\{a - \left(1-\xi^2\right)\right\} d\xi$$

$$= 2a - \left[\xi - \frac{1}{3}\xi^3\right]_{-1}^{1} = 2a - 2\frac{2}{3} = 0$$

$$\Rightarrow \quad a = \frac{2}{3}$$

4. For the finite element model consisting of two bar elements axially loaded as shown in Fig. 1. Let Young's modulus be E1 and E2, let the length of the elements be L1 and L2, and let the cross sectional area of the elements be A1 and A2, respectively. (20 points)

(1) Find the sensitivity of the displacement at the loading point, that is, at node 3, if design variables is the cross sectional area A1.

Noting that the finite element equation of the problem becomes

$$\begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

we have

$$\frac{\partial}{\partial A_1}\left\{ \begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \right\} = \frac{\partial}{\partial A_1}\begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

$$\begin{bmatrix} \frac{E_1}{L_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} + \begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_2}{\partial A_1} \\ \frac{\partial u_3}{\partial A_1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_2}{\partial A_1} \\ \frac{\partial u_3}{\partial A_1} \end{Bmatrix} = -\begin{Bmatrix} \frac{E_1}{L_1}u_2 \\ 0 \end{Bmatrix}$$

and then

$$\begin{Bmatrix} \frac{\partial u_2}{\partial A_1} \\ \frac{\partial u_3}{\partial A_1} \end{Bmatrix} = -\begin{bmatrix} \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{E_1}{L_1}u_2 \\ 0 \end{Bmatrix}$$

$$= -\frac{1}{\left(\frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2}\right)\frac{E_2A_2}{L_2} - \left(\frac{E_2A_2}{L_2}\right)^2} \begin{bmatrix} \frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \\ \frac{E_2A_2}{L_2} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} \frac{E_1}{L_1}u_2 \\ 0 \end{Bmatrix}$$

This yields

$$\frac{\partial u_3}{\partial A_1} = -\frac{\frac{E_2 A_2}{L_2}\frac{E_1}{L_1}u_2}{\left(\frac{E_1 A_1}{L_1}+\frac{E_2 A_2}{L_2}\right)\frac{E_2 A_2}{L_2}-\left(\frac{E_2 A_2}{L_2}\right)^2} = -\frac{\frac{E_1}{L_1}u_2}{\frac{E_1 A_1}{L_1}} = -\frac{u_2}{A_1}$$

(2) Find the sensitivity of the axial stress of the first bar element when a design variable is Young's modulus E1 of the first bar element.

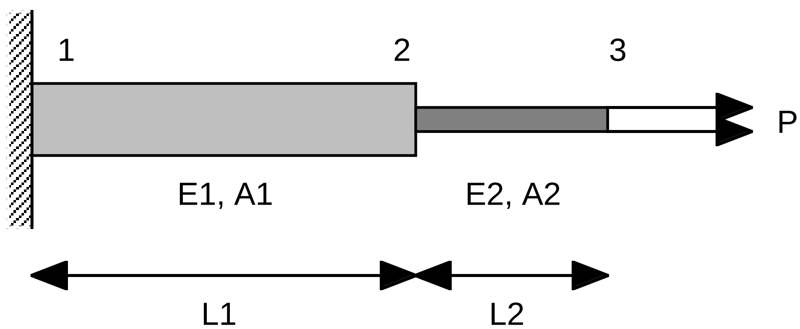


Figure 1 Two Bar Element Model (Axially Loaded)

Similarly

$$\frac{\partial}{\partial E_1}\left\{\begin{bmatrix} \frac{E_1 A_1}{L_1}+\frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}\right\} = \frac{\partial}{\partial E_1}\begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

and then

$$\begin{bmatrix} \frac{A_1}{L_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} + \begin{bmatrix} \frac{E_1 A_1}{L_1}+\frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}\begin{Bmatrix} \frac{\partial u_2}{\partial E_1} \\ \frac{\partial u_3}{\partial E_1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u_2}{\partial E_1} \\ \frac{\partial u_3}{\partial E_1} \end{Bmatrix} = -\begin{bmatrix} \frac{E_1 A_1}{L_1}+\frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}^{-1}\begin{Bmatrix} \frac{A_1}{L_1}u_2 \\ 0 \end{Bmatrix}$$

$$= -\frac{1}{\left(\frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2}\right)\frac{E_2A_2}{L_2}-\left(\frac{E_2A_2}{L_2}\right)^2}\begin{bmatrix}\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2}\\ \frac{E_2A_2}{L_2} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2}\end{bmatrix}\begin{Bmatrix}\frac{A_1}{L_1}u_2\\ 0\end{Bmatrix}$$

which yields

$$\frac{\partial u_2}{\partial E_1} = -\frac{\frac{E_2A_2}{L_2}\frac{A_1}{L_1}u_2}{\left(\frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2}\right)\frac{E_2A_2}{L_2}-\left(\frac{E_2A_2}{L_2}\right)^2} = -\frac{u_2}{E_1}$$

Therefore, the sensitivity of the axial stress of the first element is obtained by

$$\frac{\partial \sigma_1}{\partial E_1} = \frac{\partial}{\partial E_1}\left(E_1\frac{u_2}{L_1}\right) = \frac{u_2}{L_1}+\frac{E_1}{L_1}\frac{\partial u_2}{\partial E_1} = \frac{u_2}{L_1}-\frac{E_1}{L_1}\frac{u_2}{E_1} = 0$$

since

$$\sigma_1 = E_1\varepsilon_1 = E_1\frac{u_2}{L_1}.$$

5. How is the stress gradient related to the finite element approximation error ? Similarly how is the size of the finite element related to the finite element approximation error ? (10 points)

Finite element approximation error is proportional to the magnitude of the stress gradient, and it is also proprtional to the size of the finite element if no stress singularity exist in the element.

6. Explain the h-element and p-element in the adaptive finite element method. (10 points)

h-adaptive method is based on the local refinement of the initial element without changing the shape functions of the element. Typical h-method is refinement of a QUAD element into 4 QUAD elements, a HEXA element into 8 HEXA elements, at a time of adaptation.

p-adaptation is based on increasing the degrees of polynomials of the shape functions, while the original finite element mesh is kept. In this case, the original element connectivity of finite elements and the nodal coordinates are not changed, but the involved shape functions are modified by using higher order polynomials.