Midterm Examination 1997 Winter
MEAM 305 Introduction to Finite Element Methods

March 14, 1997

1. Answer to the following quetions :

a) In which yaer was the finite element method introduced ? Who are the authors of the first paper ?

The finite element method was first introduced by Turner, Clough, Martin, and Topp in 1956 using the displacement method to derive the stiffness of plane elements, and their paper is regarded as the first paper on the finite element method in engineering, while mathematicians believe that R. Courant introduced the method in 1943 under the name of the generalized finite difference method which is the approximation by 3 node trangular elements.

b) Who is the person named the finite element method ? When ?

In 1960, Clough named the method as the finite element method.

c) Whai is the isoparametric element ?

Isoparametric finite elements imples the expression of the element geometry (that is the transformation of the parametric and local coordinates) and the displacement approximation using the same shape functions and the nodal coordinates and displacements, respectively. A typical mathematical expression of this isoparametric element is as follows :

(geometry)

$$x = \sum_{i=1}^{N_e} x_i N_i(\boldsymbol{x},\eta,\zeta)\,,\; y = \sum_{i=1}^{N_e} y_i N_i(\boldsymbol{x},\eta,\zeta)\,,\; z = \sum_{i=1}^{N_e} z_i N_i(\boldsymbol{x},\eta,\zeta)$$

(displacement)

$$u_x = \sum_{i=1}^{N_e} u_{xi} N_i(\boldsymbol{x},\eta,\zeta)\,,\; u_y = \sum_{i=1}^{N_e} u_{y_i} N_i(\boldsymbol{x},\eta,\zeta)\,,\; u_z = \sum_{i=1}^{N_e} u_{zi} N_i(\boldsymbol{x},\eta,\zeta)$$

Here $(x_i \quad y_i \quad z_i)$ *and* $(u_{xi} \quad u_{y_i} \quad u_{zi})$ *are the nodal coordinates and displacements, respectively, and* N_e *is the number of node in an element. Functions* $N_i(\boldsymbol{x},\eta,\zeta)$ *are the shape functions which are polynomials of the parametric coordinates.*

d) State the principle of minimum potential energy. Explain this principle using a spring model as shown in Fig. 1.

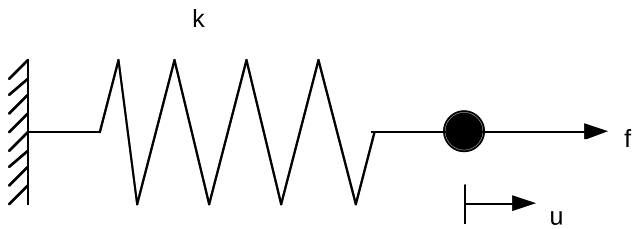


Figure 1. A Spring Model with A Single Degree of Freedom

Here k is the spring constant (stiffness), f is an applied load, and u is the amount of elongation or contraction.

The equilibrium state provides the minimum value of the total potential energy of a mechanical system. In the one degree spring mass system shown in above yields the total potential energy

$$\Pi = \frac{1}{2} k u^2 - f u .$$

Since it is convex (i.e. quadratic polynomial in u), the minimum of the total potential energy is characterized by vanishing of the first derivative of the total potential energy (i.e. zero gradient / slope) :

$$\frac{\partial \Pi}{\partial u} = ku - f = 0 \quad \Leftrightarrow \quad ku = f$$

e) Derive the stiffness of the bar element for axial loading and deformation. Here we assume that Young's modulus of the bar is E, cross sectional area is A, and the length of the bar is L, see Fig. 2.

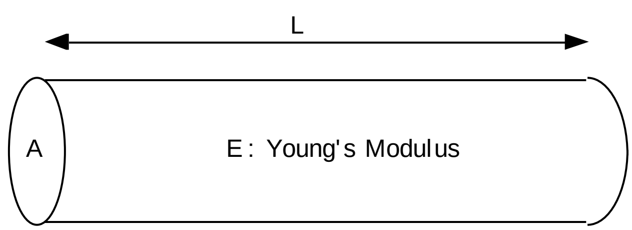


Figure 2. A Bar Element for Axial Loadings

Noting that

$$\sigma = E\varepsilon \quad \Rightarrow \quad \frac{P}{A} = E\frac{u}{L} \quad \Rightarrow \quad P = \frac{EA}{L} u$$

we can find the stiffness of the axially loaded bar :

$$k = \frac{EA}{L}.$$

2. Consider an 8 node hexagonal element shown in Fig. 3.

a) Define the shape functions $N_i(\xi, \eta, \zeta),\ i = 1, 2, \ldots, 8$ in terms of the parametric coordinates $\xi, \eta,$ and ζ.

Shape functions of HEXA8 are given by

$$N_i(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)$$

where (ξ_i, η_i, ζ_i) *are the nodal coordinates of the corner nodes in the parametric coordinate system. More precisely,*

$$N_1(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$$

$$N_2(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$$

$$N_3(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$

$$N_4(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$

$$N_5(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$$

$$N_6(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$$

$$N_7(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$$

$$N_8(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$

b) Evaluate N_4 at the cetroid of the element, at the second node, and at the fourth node.

$$N_4(\xi,\eta,\zeta) = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta) = \begin{cases} 1/8 & \text{at centroid} \\ 0 & \text{at node 2} \\ 1 & \text{at node 4} \end{cases}$$

c) Verify the property $\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)=1$.

$$
\begin{aligned}
&\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)\\
&=\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)+\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)\\
&+\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)+\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)\\
&+\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)+\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)\\
&+\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)+\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)
\end{aligned}
$$

$$
=\underline{\frac{1}{4}(1-\eta)(1-\zeta)+\frac{1}{4}(1+\eta)(1-\zeta)}+\underline{\frac{1}{4}(1-\eta)(1+\zeta)+\frac{1}{4}(1+\eta)(1+\zeta)}
$$

$$
=\frac{1}{2}(1-\zeta)+\frac{1}{2}(1+\zeta)=1
$$

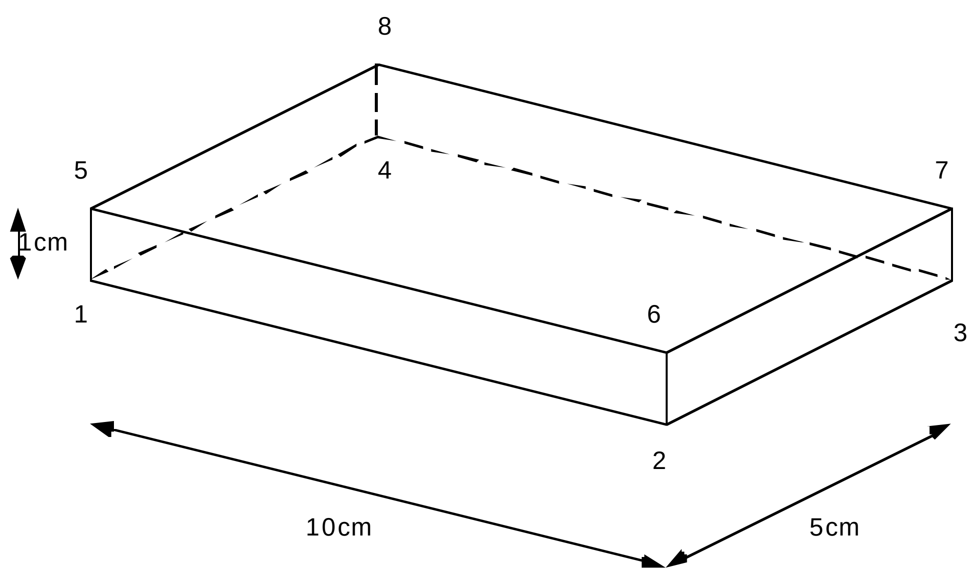
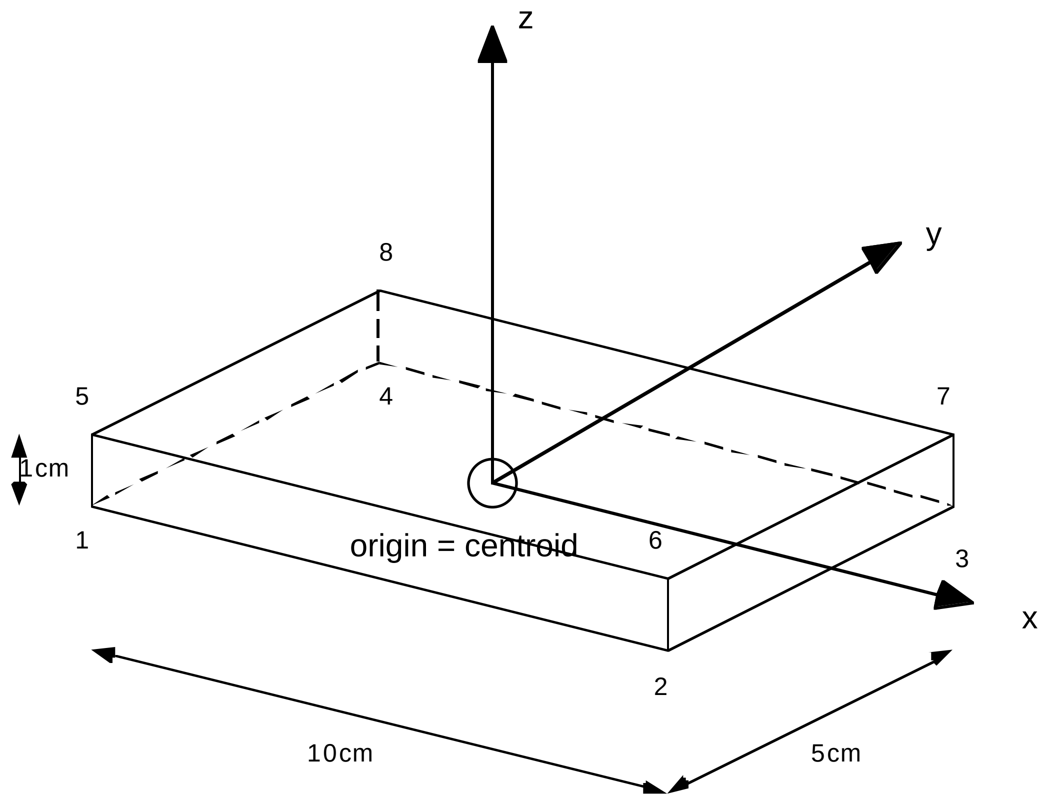


Figure 3. A 8 Node Hexagonal Element with Physical Dimensions

d) Sketch the local coordinate system (x,y,z). Where is the origin ?



e) Using the differential relation to an arbitrary function g

$$\begin{Bmatrix} \frac{\partial g}{\partial \xi} \\ \frac{\partial g}{\partial \eta} \\ \frac{\partial g}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{Bmatrix}$$

compute

e1) the three displacement components $\{u_x \quad u_y \quad u_z\}$ in the local coordinate system

e2) the normal strain $\varepsilon_x = \frac{\partial u_x}{\partial x}$, and the shear strain $\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$

when

$$\boldsymbol{d}^T = \{0 \quad 0 \quad 0 \quad 0.01 \quad 0 \quad 0 \quad 0.01 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.01 \quad 0 \quad 0 \quad 0.01 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$$

f) Find the values of ε_x at the centroid and at node 7.

Noting that the displacemeny field given by the degrees of freedom given in above is chracterized by

$$u_{x1} = u_{x4} = u_{x5} = u_{x8} = 0$$
$$u_{x2} = u_{x3} = u_{x6} = u_{x7} = 0.01$$
$$u_y = u_z = 0 \quad \text{at every node}$$

that is, only non-zero component is the one in the x direction, they implies that

1) *zero axial displacement at the left face 1-4-8-5 (i.e. plane* $\xi = -1$ *)*

2) *0.01 axial displacement at the right face 2-3-7-6 (i.e plane* $\xi = +1$ *)*

in the right direction. Therefore, the displacement components are given by

$$u_x = \sum_{i=1}^{8} u_{xi} N_i(\xi,\eta,\zeta)$$

$$= 0.01\left\{\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) + \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)\right.$$

$$\left. + \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) + \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)\right\}$$

$$= 0.01\left\{\frac{1}{4}(1+\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1+\eta)\right\}$$

$$= 0.01 \times \frac{1}{2}(1+\xi)$$

and

$$u_y = u_z = 0.$$

Furthermore, the element is rectanular parallelpipe, we have

$$x = -5 \times \frac{1}{2}(1-\xi) + 5 \times \frac{1}{2}(1+\xi) = 5\xi$$
$$y = -2.5 \times \frac{1}{2}(1-\eta) + 0.25 \times \frac{1}{2}(1+\eta) = 2.5\eta$$
$$z = -0.5 \times \frac{1}{2}(1-\zeta) + 0.5 \times \frac{1}{2}(1+\zeta) = 0.5\zeta$$

Thus, the normal strain ε_x *and shear strain* γ_{xy} *become*

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u_x}{\partial \xi} = \frac{1}{5}\frac{0.01}{2} = 0.001$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0$$

respectively.

That is, the normal strain ε_x *is constant in the element, i.e. 0.001 at the centroid and node 7.*