Homework #2, Winter 2000
ME 501 Analytical Methods in Mechanics and Mechanical Engineering

Consider transverse vibration of an elastic pipe submerged in water that is mathematically modeled by the following homogeneous initial-boundary value problem:

$$\rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial w}{\partial x} \right) + kw = 0 \ , \ (x,t) \in (0,L) \times (0,T)$$

with the homogeneous boundary condition

$$w(0,t) = 0 \quad , \quad EA \frac{\partial w}{\partial x}(L,t) = 0 \quad , \quad t \in (0,T)$$

and the non-homogeneous initial condition

$$w(x,0) = u_0(x) \ , \ \frac{\partial w}{\partial t}(x,0) = v_0(x) \ , \ x \in (0,L).$$

That is, vibration is induced by the initial displacement u_0 and velocity v_0.

(1) Assuming the solution $w(x,t)$ in the form

$$w(x,t) = \sum_{k=1}^{\infty} c_k \exp(i\omega_k t) u_k(x)$$

show that the following differential equation

$$-\frac{d}{dx}\left(EA \frac{du_k}{dx} \right) + ku_k = \rho \omega_k{}^2 u_k \quad , \quad x \in (0,L)$$

must be satisfied for all *k* for any complex numbers $c_k \in \mathbf{C}, k = 1, 2, ..., \infty$.

(2) Suppose that the initial displacement and velocity are expanded by

$$u_0(x) = \sum_{k=1}^{\infty} a_k u_k(x) \quad , \quad v_0(x) = \sum_{k=1}^{\infty} b_k u_k(x)$$

Show that

$$w(x,t) = \mathrm{Re}\left[\sum_{k=1}^{\infty}\left(a_k - i\frac{b_k}{\omega_k}\right)\exp(i\omega_k t)u_k(x)\right]$$

satisfies the non-homogeneous initial condition.

(3) We shall call the pair (λ_k, u_k), $k = 1, 2, \ldots.,$ $\lambda_k = \rho\omega_k{}^2$, is the eigenvalue and eigenfunction, and it satisfies the boundary value problem:

$$-\frac{d}{dx}\left(EA\frac{du_k}{dx}\right) + ku_k = \rho\omega_k{}^2 u_k \quad , \quad x \in (0, L)$$

and the boundary condition

$$u_k(0) = 0 \quad , \quad EA\frac{du_k}{dx}(L) = 0 \,.$$

Assuming that *EA*, *k* and ρ are constant, and assuming the solution as

$$u_k(x) = \sin\left(\frac{(2k-1)\pi x}{2L}\right) \quad , \quad k = 1, 2, \ldots$$

find λ_k, while it satisfies the boundary condition.

(4) Suppose that $v(x,t;\tau)$ is the solution of the homogeneous initial boundary value problem

$$\rho\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial v}{\partial x}\right) + kv = 0$$

with the homogeneous boundary condition

$$v(0,t) = 0 \ , \ EA\frac{\partial v}{\partial x}(L,t) = 0 \ , \ t \in (0,T)$$

and the specially designed non-homogeneous "initial" condition defined at time τ

$$v(x,\tau) = 0 \ , \ \rho\frac{\partial w}{\partial t}(x,\tau) = f(x,\tau) \ , \ x \in (0,L)$$

Then show that

$$w_f(x,t) = \int_0^t v(x,t;\tau)\, d\tau$$

is a solution of the non-homogeneous initial-boundary value problem

$$\rho \frac{\partial^2 w_f}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial w_f}{\partial x} \right) + k w_f = f$$

with homogeneous boundary and initial condition:

$$w_f(0,t) = 0 \ , \ EA \frac{\partial w_f}{\partial x}(L,t) = 0 \ , \ t \in (0,T)$$

$$w_f(x,0) = 0 \ , \ \frac{\partial w_f}{\partial t}(x,0) = 0 \ , \ x \in (0,L).$$

(5) Show that

$$u(x,t) = w_f(x,t) + w(x,t)$$

satisfies the non-homogeneous initial-boundary value problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) + k u = f \ , \ (x,t) \in (0,L) \times (0,T)$$

with the homogeneous boundary condition

$$u(0,t) = 0 \ , \ EA \frac{\partial u}{\partial x}(L,t) = 0 \ , \ t \in (0,T)$$

and the non-homogeneous initial condition

$$u(x,0) = u_0(x) \ , \ \frac{\partial u}{\partial t}(x,0) = v_0(x) \ , \ x \in (0,L).$$

It is noted that a solution of the non-homogeneous initial-boundary value problem can be obtained by adding the homogeneous solution w and a particular solution w_f obtained by using the homogeneous solution w.

Up to this point, we have only looked at mathematics. Now we shall input some engineering.

(A) Optimal Design: In (3), we pass through the eigenvalue problem for a constant EA. Suppose that EA can be a function of x, say

$$EA(x) = d_1 + d_2 \sin\left(\frac{\pi x}{2L}\right) + d_3 \sin\left(\frac{3\pi x}{2L}\right)$$

while the stiffness constraint

$$\int_0^L EA(x)\,dx \le EW_{\max}$$

is satisfied, where d_i are "design variables." Describe how you can determine the design variables d_i so as to maximize the minimum eigenvalue $\lambda_1 = \rho\omega_1^2$.

(B) Optimal Control: Suppose that the initial velocity $v_0(x) = \sin\left(\frac{\pi x}{2L}\right)$ is specified with zero initial displacement $u_0(x) = 0$. Now describe your idea how can we design the input force by the actuator that is set up at the middle point of the pipe so as to minimize the L2 norm of the displacement at the free end of the pipe, i.e., minimize $\left\| u|_{x=L} \right\| = \sqrt{\int_0^T |u(L,t)|^2\, dt}$.

Give me you idea how to approach to these optimal design and optimal control problems by knowing some mathematics described in above. Not that I do not ask you to solve the problems mathematically at this time. I wish to hear from you how you would approach to these problems. They are very typical problems we can see in mechanical engineering now a day.