Analytical Solution of a Non-homogeneous PDE for Vibration of an Elastic String
Additional Note for Homework #2
February 13, 2000

We shall solve a vibration problem of an elastic string spanned on the interval $(0,1)$*:*

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=f(x,t) \quad , \quad (x,t)\in(0,1)\times(-\infty,+\infty)$$

with the homogeneous initial condition

$$u(x,0)=\frac{\partial u}{\partial t}(x,0)=0$$

and the homogeneous boundary condition

$$u(0,t)=u(1,t)=0.$$

To do this, we first assume that the non-homogeneous term $f(x,t)$ can be expressed by

$$f(x,t)=\sum_{k=1}^{\infty} f_k(t)\sin(k\pi x) \quad , \quad f_k(t)=2\int_0^1 f(\xi,t)\sin(k\pi\xi)\,d\xi$$

Then assuming the solution $u(x,t)$ in the following form

$$u(x,t)=\sum_{k=1}^{\infty} u_k(t)\sin(k\pi x)$$

we shall find appropriate coefficient functions $u_k(t)$. Substitution of this into the differential equation yields

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}&=\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\sum_{k=1}^{\infty} u_k(t)\sin(k\pi x)\\ &=\sum_{k=1}^{\infty}\left\{\frac{d^2u_k}{dt^2}(t)+(k\pi)^2 u_k(t)\right\}\sin(k\pi x)\\ &=f(x,t)=\sum_{k=1}^{\infty} f_k(t)\sin(k\pi x) \quad , \quad (x,t)\in(0,1)\times(-\infty,+\infty)\end{aligned}$$

and then

$$\frac{d^2u_k}{dt^2}(t)+(k\pi)^2 u_k(t)= f_k(t)$$

From the homogeneous initial condition, we have

$$u_k(0)=\frac{du_k}{dt}(0)=0$$

Assuming the solution $u_k(t)$ in the form

$$u_k(t)=a(t)\sin(k\pi t)+b(t)\cos(k\pi t)$$
$$\frac{du_k}{dt}=\frac{da}{dt}\sin(k\pi t)+ak\pi\cos(k\pi t)+\frac{db}{dt}\cos(k\pi t)-bk\pi\sin(k\pi t)$$
$$\frac{d^2u_k}{dt^2}=\frac{d^2a}{dt^2}\sin(k\pi t)+2\frac{da}{dt}k\pi\cos(k\pi t)-a(k\pi)^2\sin(k\pi t)$$
$$+\frac{d^2b}{dt^2}\cos(k\pi t)-2\frac{db}{dt}k\pi\sin(k\pi t)-b(k\pi)^2\cos(k\pi t)$$

that yields

$$\frac{d^2u_k}{dt^2}(t)+(k\pi)^2 u_k(t)$$
$$=\frac{d^2a}{dt^2}\sin(k\pi t)+2k\pi\frac{da}{dt}\cos(k\pi t)+\frac{d^2b}{dt^2}\cos(k\pi t)-2k\pi\frac{db}{dt}\sin(k\pi t)$$
$$= f_k(t)$$

Defining $\bar{a}=\frac{da}{dt}$ and $\bar{b}=\frac{db}{dt}$, we have

$$\frac{d\bar{a}}{dt}\sin(k\pi t)+2k\pi\bar{a}\cos(k\pi t)+\frac{d\bar{b}}{dt}\cos(k\pi t)-2k\pi\bar{b}\sin(k\pi t)= f_k(t)$$

Thus assuming

$$\bar{a}(t)=cf_k(t)\cos(k\pi t) \quad \& \quad \bar{b}(t)=-cf_k(t)\sin(k\pi t) \quad , \quad c\in \mathbf{R}$$

we have

$$\frac{d\bar{a}}{dt}\sin(k\pi t)+2k\pi\bar{a}\cos(k\pi t)+\frac{d\bar{b}}{dt}\cos(k\pi t)-2k\pi\bar{b}\sin(k\pi t)$$
$$=c\frac{df_k}{dt}\sin(k\pi t)\cos(k\pi t)-ck\pi f_k\sin^2(k\pi t)+2ck\pi f_k\cos^2(k\pi t)$$
$$-c\frac{df_k}{dt}\cos(k\pi t)\sin(k\pi t)-ck\pi f_k\cos^2(k\pi t)+2ck\pi f_k\sin^2(k\pi t)$$
$$=ck\pi f_k = f_k(t)$$

That is, $c=\dfrac{1}{k\pi}$ and

$$\bar{a}(t)=\frac{1}{k\pi}f_k(t)\cos(k\pi t) \quad \& \quad \bar{b}(t)=-\frac{1}{k\pi}f_k(t)\sin(k\pi t)$$

Integrating these in time, we have

$$a(t)=\frac{1}{k\pi}\int_0^t f_k(\tau)\cos(k\pi\tau)\,d\tau \quad \& \quad b(t)=-\frac{1}{k\pi}\int_0^t f_k(\tau)\sin(k\pi\tau)\,d\tau$$

that is

$$u_k(t)=\frac{1}{k\pi}\sin(k\pi t)\int_0^t f_k(\tau)\cos(k\pi\tau)\,d\tau-\frac{1}{k\pi}\cos(k\pi t)\int_0^t f_k(\tau)\sin(k\pi\tau)\,d\tau$$
$$=\frac{1}{k\pi}\int_0^t f_k(\tau)\sin\big(k\pi(t-\tau)\big)\,d\tau$$

It is also noted that this satisfies the homogeneous initial condition $u_k(0)=\dfrac{du_k}{dt}(0)=0$. Therefore, the solution of the differential equation becomes

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin(k\pi x)=\frac{1}{k\pi}\sum_{k=1}^{\infty}\sin(k\pi x)\int_0^t f_k(\tau)\sin\big(k\pi(t-\tau)\big)\,d\tau$$

<u>Exercise 1</u> Solve the same initial-boundary value problem by changing the boundary condition

$$u(0,t)=\frac{\partial u}{\partial x}(1,t)=0 \qquad ///$$

<u>Exercise 2</u> Plot the solution $u(x,t)$ of the non-homogeneous initial-boundary value problem with

homogeneous boundary and initial condition, when $f(x,t)$ is given by

$$f(x,t)=\begin{cases} t & \text{if} \quad (x,t)\in(0.48,0.52)\times(0,2) \\ 0 & \text{if otherwise} \end{cases} \qquad ///$$

<u>Exercise 3</u> Solve the problem with the homogeneous boundary and initial condition for the PDE

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=f(x,t) \quad , \quad (x,t)\in(0,1)\times(-\infty,+\infty)$$

for a given constant *c*. ///