Final Examination
MEAM 501 Analytical Methods in Mechanical Engineering

1997 Fall

1. State the following definitions, properties, and/or concept (40 points) :

(1) Define the Lagrange interpolation of a function *f* defined on an interval (*a*,*b*) by using *n*+1 points $x_1,....,x_{n+1} \in [a,b]$. State clearly major properties of the basis functions of the Lagrange interpolation.

Within a given interval (a,b), we place n+1 points, say, $x_1, x_2,.....,x_{n+1}$, and using the values $f_i = f(x_i)$ of the given function *f*(*x*) at these points, we approximate the function *f* by a *n* degree polynomial f_n :

$$f_n(x) = \sum_{i=1}^{n+1} f_i L_i(x) \quad , \quad L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$

The basis functions $L_i(x)$ have the following properties :

1) they are *n* degree polynomials

and

$$\text{2)} \; L_i(x_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

(2) State the property of the Legendre polynomilas defined on the interval (-1,1) ?

They are orthogonal with respect to the inner product $(f,g) = \int_{-1}^{1} f(x)g(x)dx$, and they are obtained from the set of polynomial basis functions $\{1, x, x^2, x^3,......, x^n,.....\}$ by Gram-Schmidt orthogonalization process .

(3) Are the two functions $f_1(x) = x + 1$ and $f_2(x) = \sin(\pi x)$ orthogonal in an interval (0,1) ? If not, orthogonalize them by adding an appropriate constant or polynomial to the function f_1 . From these make up two orthonormal basis functions.

$$(f_1, f_2) = \int_0^1 (x+1)\sin(\pi x)dx = \int_0^1 (x+1)\left\{-\frac{1}{\pi}\frac{d}{dx}\cos(\pi x)\right\}dx$$

$$= \left[-\frac{1}{\pi}(x+1)\cos(\pi x)\right]_{x=0}^{x=1} + \int_0^1 (x+1)' \frac{1}{\pi}\cos(\pi x)dx$$

$$= \frac{3}{\pi} + \left[\frac{1}{\pi^2}\sin(\pi x)\right]_{x=0}^{x=1} = \frac{3}{\pi}$$

Thus, they are not orthogonal. We shall now consider

$$\bar{f}_1 = f_1 + k = x + 1 + k$$

to orthogonalize. In deed,

$$\left(\bar{f}_1, f_2\right) = \left(f_1 + k, f_2\right) = \frac{3}{\pi} + k\int_0^1 \sin(\pi x)dx = \frac{3}{\pi} + k\left[-\frac{1}{\pi}\cos(\pi x)\right]_{x=0}^{x=1} = \frac{3}{\pi} + \frac{2k}{\pi} = 0$$

that is

$$k = -\frac{3}{2}.$$

Therefore, $\bar{f}_1 = f + k = x - \frac{1}{2}$ and $f_2 = \sin(\pi x)$ are ortogonal.

(4) What is the trazoidal rule of numerical integration of a function *f* on an interval (*a*,*b*) ? What is the order of quadrature error in terms of the length of the interval $h = b - a$.

$$\int_{x=a}^{x=b} f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$

The Taylor expansion

$$f(a) = f(x) + f'(x)(a-x) + \frac{1}{2}f''(x)(a-x)^2 + ...$$

$$f(b) = f(x) + f'(x)(b-x) + \frac{1}{2}f''(x)(b-x)^2 + ...$$

yields

$$\begin{aligned}
&f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} \\
&= \left(f(x) + f'(x)(a-x) + \frac{1}{2}f''(x)(a-x)^2 + ...\right)\frac{b-x}{b-a} \\
&+ \left(f(x) + f'(x)(b-x) + \frac{1}{2}f''(x)(b-x)^2 + ...\right)\frac{x-a}{b-a} \\
&\approx f(x) + \frac{1}{2}f''(x)\left\{(a-x)^2\frac{b-x}{b-a} + (b-x)^2\frac{x-a}{b-a}\right\} \\
&= f(x) - \frac{1}{2}f''(x)(a-x)(b-x)
\end{aligned}$$

Integration of this over the interval implies

$$\frac{b-a}{2}\left(f(a) + f(b)\right) \approx \int_a^b f(x)dx - \frac{1}{2}\int_a^b f''(x)(a-x)(b-x)dx$$

Thus, the quadrature error is propotional to the cubic power of the length of the interval and the norm of the second derivative of the integrand

(5) What is the two point Gauss-Legendre quadrature of a function *f* on an interval (-1,1) ?

Using the two quadrature points $\pm\frac{1}{\sqrt{3}}$ which are the two roots of the second degree Legendre polynomial, and the associated weights 1, we form the quadrature :

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

This rule can integrate upto cubic (2*n* - 1 degree) polynomials exactly, where *n* is the number of quadrature points.

(6) What is the Bezier spline of a curve using *n*+1 control points $x_1,....,x_{n+1} \in [a,b]$?

The Bezier curve characterized by *n*+1 control points of a characteristic polygon is defined by

$$x(t) = \sum_{i=1}^{n+1} x_i B_i^n(t) \qquad \text{with} \qquad B_i^n(t) = \frac{n!}{(i-1)!(n-i+1)!} t^{i-1}(1-t)^{n-i+1}$$

where x_i are the coordinates of the control points and the parameter $t \in$ [0, 1].

(7) State the minimum principle ?

Equilibrium is attained at the minimum point of the corresponding quadratic functional. For example, the equilirium represented by a system of linear equations (or matrix equation) :

$$\boldsymbol{Ax} = \boldsymbol{b}$$

for a symmetric matrix $\boldsymbol{A}$, we can define the minimization problem equivalent to this equilirium :

$$\min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Ax} - \boldsymbol{x}^T\boldsymbol{b}$$

If the matrix $\boldsymbol{A}$ is nonnegative in the sense that

$$\boldsymbol{x}^T\boldsymbol{Ax} \geq 0 \quad , \quad \forall \boldsymbol{x},$$

both are equivalent. That is, the solution of the system of linear equation is the minimizer of the functional, and also the minimizer of the functional is a solution of the system of linear equations.

2. Obtain the first variation of the following functional, the necessary conditions, Euler's equation, and the natural boundary condition (20 points) :

$$J(v) = \frac{1}{2}\int_0^1 \left\{ \left(a_0 + a_1 \sin(n\pi x)\right)(v')^2 + xv^2 \right\} dx + \frac{1}{2} k_1 v(1)^2 - \int_0^1 fv dx - P_0 v(0) - P_1 v(1)$$

$$K = V$$

$$V = \left\{ v : \text{piecewise continuously differentiable functions on } (0,1) \right\}$$

The necessary condition can be obtained as follows :

$$\delta J(v) = \frac{1}{2}\delta\int_0^1 \left\{ \left(a_0 + a_1 \sin(n\pi x)\right)(v')^2 + xv^2 \right\} dx + \frac{1}{2} k_1 \delta v(1)^2 - \delta\int_0^1 fv dx - P_0 \delta v(0) - P_1 \delta v(1)$$

$$= \frac{1}{2}\int_0^1 \left\{ \left(a_0 + a_1 \sin(n\pi x)\right)\delta(v')^2 + x\delta v^2 \right\} dx + k_1 v(1)\delta v(1) - \int_0^1 f\delta v dx - P_0 \delta v(0) - P_1 \delta v(1)$$

$$= \int_0^1 \left\{ \left(a_0 + a_1 \sin(n\pi x)\right)v' \,\delta v' + xv\delta v \right\} dx + k_1 v(1)\delta v(1) - \int_0^1 f\delta v dx - P_0 \delta v(0) - P_1 \delta v(1)$$

$$= \left[\left(a_0 + a_1 \sin(n\pi x)\right)v' \,\delta v \right]_{x=0}^{x=1} + \int_0^1 \left\{ -\frac{d}{dx}\left(\left(a_0 + a_1 \sin(n\pi x)\right)\frac{dv}{dx} \right) + xv - f \right\} \delta v dx$$

$$+ k_1 v(1)\delta v(1) - P_0 \delta v(0) - P_1 \delta v(1)$$

$$= \left(\left(a_0 + a_1 \sin(n\pi)\right)\frac{dv}{dx}(1) + k_1 v(1) - P_1 \right)\delta v(1) - \left(a_0 \frac{dv}{dx}(0) + P_0 \right)\delta v(0)$$

$$+ \int_0^1 \left\{ -\frac{d}{dx}\left(\left(a_0 + a_1 \sin(n\pi x)\right)\frac{dv}{dx} \right) + xv - f \right\} \delta v dx \geq 0 \quad , \quad \forall \delta v$$

Thus, Euler's equation and the natural boundary conditions are obtained as follows :

$$-\frac{d}{dx}\left(\left(a_0 + a_1 \sin(n\pi x)\right)\frac{dv}{dx} \right) + xv = f \quad \text{in} \quad (0,1)$$

$$\left(a_0 + a_1 \sin(n\pi)\right)\frac{dv}{dx}(1) + k_1 v(1) - P_1 = 0$$

$$a_0 \frac{dv}{dx}(0) + P_0 = 0$$

3 Solve the minimization problem by the Ritz method with a single term (20 points) :

$$\min_{\substack{v \\ \text{such that} \\ v(0)=v(1)=0}} \quad J(v)$$

where

$$J(v) = \frac{1}{2}\int_0^1 (v')^2\,dx - \int_0^1 2v\,dx\,.$$

Noting that $\phi(x) = x(x-1)$ satisfies the essential boundary conditions (constraints) at x = 0 and 1, we approximate the solution by the Ritz metod

$$v(x) \approx c\phi(x) = cx(x-1)$$

where c is determined to be minimizing the functional

$$J(v) \approx J_1(c) = \frac{1}{2}\int_0^1 \left(\{cx(x-1)\}'\right)^2 dx - \int_0^1 2cx(x-1)\,dx\,.$$

Noting that

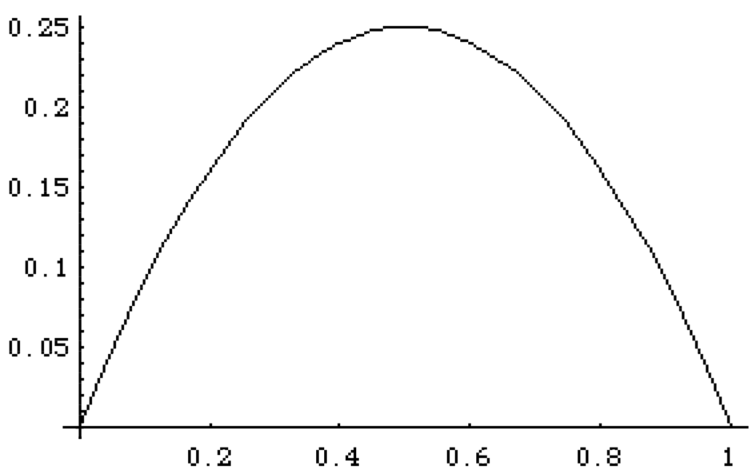
$$J_1(c) = \frac{1}{2}c^2\int_0^1 (2x-1)^2\,dx - 2c\int_0^1 x(x-1)\,dx$$

and the necessary condition of the minimization

$$\frac{\partial J_1}{\partial c} = c\int_0^1 (2x-1)^2\,dx - 2\int_0^1 x(x-1)\,dx = 0$$

we have

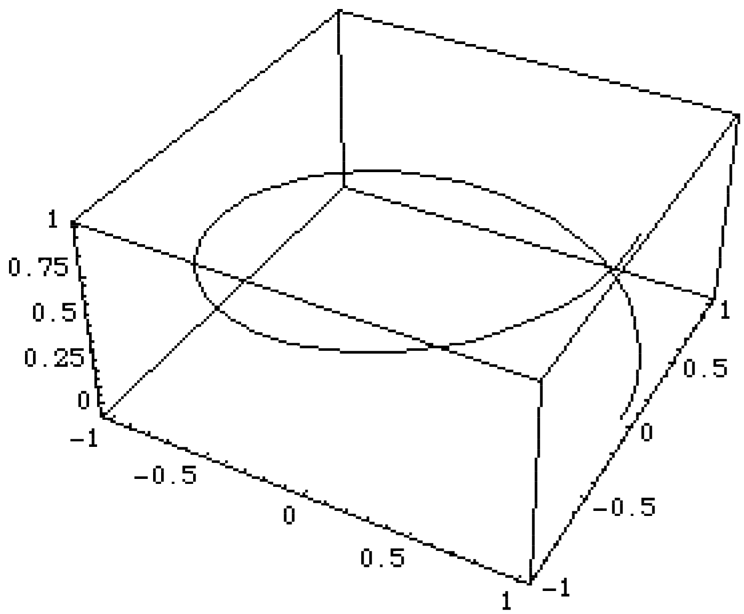
$$c = \frac{2\int_0^1 x(x-1)\,dx}{\int_0^1 (2x-1)^2\,dx} = -1$$



4. Consider a curve defined by

$$\begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \\ z = \theta/2\pi \end{cases}$$

using a parameter θ such that $\theta \in (0, 2\pi)$ whose profile is shown in the following figure. (20 points)



(1) Obtain the expression of the tangent vector $\boldsymbol{t}$.

$$\boldsymbol{t} = \left\{(-\sin\theta)\vec{e}_x + (\cos\theta)\vec{e}_y + \vec{e}_z\right\} \Big/ \sqrt{1+\left(\frac{1}{2\pi}\right)^2}$$

(2) Obtain the normal and bi-normal vectors $\boldsymbol{n}$ and $\boldsymbol{b}$, respectively.

$$\boldsymbol{n} = \frac{\dfrac{\partial \boldsymbol{t}}{\partial\theta}}{\left\|\dfrac{\partial \boldsymbol{t}}{\partial\theta}\right\|} = \left\{(-\cos\theta)\vec{e}_x + (-\sin\theta)\vec{e}_y\right\}$$

$$\boldsymbol{b} = \boldsymbol{t}\times\boldsymbol{n} = \frac{1}{\sqrt{1+\left(\frac{1}{2\pi}\right)^2}} \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -\sin\theta & \cos\theta & 1 \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= \left\{(\sin\theta)\vec{e}_x + (-\cos\theta)\vec{e}_y + \vec{e}_z\right\} \Big/ \sqrt{1+\left(\frac{1}{2\pi}\right)^2}$$

(3) Obtain the total length of this curve ?

$$L = \int_0^{2\pi} ds = 2\pi\sqrt{1+\left(\frac{1}{2\pi}\right)^2}$$