Homework #1, MEAM 501 Fall 1998

September 19, 1998 Due Date October 1

Consider the static equilibrium equation of an elastic string supported by distributed springs, whose spring constant is k per unit length, and spanned by a tensile force T with a possibly distributed force f:

$$-T\frac{d^2u}{dx^2}+ku=f \qquad in \qquad (0,L)$$

(1) Applying the boundary condition

$$u(0)=u(L)=0$$

derive a discrete system by using the weighted residual method together with the functions

$$\phi_j(x)=\sin\left(j\pi\frac{x}{L}\right) \quad , \quad j=1,\ldots..,n$$
$$\psi_i(x)=x^{i-1} \quad , \quad i=1,\ldots..,m$$

which approximate the trial function space U and the test function space Ω, respectively. Here, it is noted that any linear combination of the functions ϕ_i satisfies the boundary condition, that is, a candidate solution

$$u=\sum_{j=1}^{n}u_j\phi_j(x)$$

satisfies the boundary condition $u(0)=u(L)=0$ a priori.

(2) Obtain the stiffness matrix **K** and the generalized force vector **f**:

$$k_{ij}=\left\langle -T\frac{d^2\phi_j}{dx^2}+k\phi_j\,,\psi_i\right\rangle=\int_0^L\left(-T\frac{d^2\phi_j}{dx^2}+k\phi_j\right)\psi_i\,dx$$
$$f_i=\langle f\,,\psi_i\rangle=\int_0^L f\psi_i\,dx$$

for

$$L = T = 1 \quad , \quad k = e^{-x} \quad , \quad and \quad f = 1 + \sin\left(3\pi \frac{x}{L}\right).$$

(3) Solve the matrix equation

$$\mathbf{Ku} = \mathbf{f}$$

by using, for example, MATLAB with the command

pinv(K)

that takes the pseudo-inverse (or generalize inverse) of the matrix K. In MATHEMATICA the pseudo-inverse is taken by PseudoInverse[K]. That is, solve the matrix equation by

$$\mathbf{u} = \mathbf{K}^{+}\mathbf{f}$$

where $\mathbf{K}^{+}$ is the pseudo-inverse of $\mathbf{K}$ which is a generalized inverse of a standard square matrix to a rectangular matrix. At this moment, please not make any question on the pseudo-inverse. We shall discuss this more details later when the singular value decomposition of a rectangular matrix will be studied. Make comparison the solutions for

m	n
10	7
7	7
5	7

(4) If we assume $\phi_j(x) = \sin\left(j\pi \frac{x}{L}\right)$, then $L(\phi_j)$, $L = -T\frac{d^2}{dx^2} + k$, are infinitely many times continuously differentiable on the interval (0,L). Thus, we may take the functions for the test function space:

$$\psi_i(x) = \delta_{x_i}(x)$$

where δ_{x_i} is the Dirac delta function at a point x_i. In this case, the weighted residual method yields the *collocation method*. Repeat (3) for this choice by setting up the sampling points x_i from the interval (0,L) with equal distance of adjacent sampling points.

(5) If we change the boundary condition to

$$u(0) = 0 \qquad and \qquad T\frac{du}{dx}(L) = 0$$

that is, the elastic string is supported at x = 0, but the symmetry condition is applied at x = L, i.e., the slope of the string becomes zero at x = L. In this case, the choice of sine function as in (1) is not appropriate, because of the symmetry boundary condition. If we choose

$$\phi_j(x) = x^j \qquad , \qquad j = 1,.....,n$$
$$\psi_i(x) = x^{i-1} \qquad , \qquad i = 1,.....,m$$

derive the discrete problem. It is noted that the degrees of freedom, that is called the generalized displacement, must satisfy the boundary condition:

$$\sum_{j=1}^{n} u_j \frac{d\phi_j}{dx} = \sum_{j=1}^{n} j u_j x^{j-1} = 0 \qquad at \qquad x = L \qquad , that\ is \qquad \sum_{j=1}^{n} j u_j L^{j-1} = 0$$

Thus, this becomes a constraint while we solve the matrix equation $\mathbf{Ku} = \mathbf{f}$. To avoid this constraint, we may make modification to the weighted residual form:

$$\langle L(u) - f\ , w \rangle = \int_0^L \left(-T\frac{d^2u}{dx^2} + ku - f \right) w dx = -\left[T\frac{du}{dx} w \right]_{x=0}^{x=L} + \int_0^L \left(T\frac{du}{dx}\frac{dw}{dx} + kuw - fw \right) dx$$

by applying the integration by parts. Then the boundary condition, especially, the symmetry boundary condition is applied to the first term, that is,

$$\langle L(u) - f\ , w \rangle = \int_0^L \left(-T \frac{d^2u}{dx^2} + ku - f \right) w dx = T \frac{du}{dx}(0) w(0) + \int_0^L \left(T \frac{du}{dx} \frac{dw}{dx} + kuw - fw \right) dx$$

and then we need not introduce an additional constraint to satisfy the symmetry condition. Using this framework, repeat (3).

In order to deal with the boundary condition involving the first derivative of the solution u, we prefer the form after applying the integration by parts (or more generally the divergence theorem for multiple space dimensions).