Homework #1
MEAM 501 Analytical Methods in Mechanical Engineering

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1. Let X be the linear space C(0,T). Which, if any, of the following subsets of X are linear subspaces ? If they are linear subspaces, provide the proof.

$B_1 = \{ f \in C(0,T) \mid f(0)=f(T)$, i.e., functions are *periodic*. $\}$

Since, for $\alpha, \beta \in \boldsymbol{R}$ *and* $f_1, f_2 \in B_1$,

$$(\alpha f_1 + \beta f_2)(0) = \alpha f_1(0) + \beta f_2(0) = \alpha f_1(T) + \beta f_2(T) = (\alpha f_1 + \beta f_2)(T),$$

this is a linear subspace of X.

$B_2 = \{ f \in C(0,T) \mid f(0)=1 \}$

$(\alpha f_1 + \beta f_2)(0) = \alpha f_1(0) + \beta f_2(0) = \alpha + \beta \neq 1$, *that is, this is not a linear subspace.*

$B_3 = \{ f \in C(0,T) \mid \int_0^T f(x)\, dx = 1 \}$

$\int_0^T (\alpha f_1 + \beta f_2) dt = \alpha \int_0^T f_1 dt + \beta \int_0^T f_2 dt = \alpha + \beta \neq 1$, *that is, this is not a linear subspace.*

$B_4 = \{ f \in C(0,T) \mid | f(x_1) - f(x_2) | \le 10 | x_1 - x_2 |$ for all $x_1, x_2 \in (0,T) \}$.

$$\begin{aligned}
&\left|(\alpha f_1 + \beta f_2)(x_1) - (\alpha f_1 + \beta f_2)(x_2)\right| \\
&= \left|\alpha\left(f_1(x_1) - f_1(x_2)\right) + \beta\left(f_2(x_1) - f_2(x_2)\right)\right| \\
&\le \left|\alpha\left(f_1(x_1) - f_1(x_2)\right)\right| + \left|\beta\left(f_2(x_1) - f_2(x_2)\right)\right| \\
&= |\alpha|\left|f_1(x_1) - f_1(x_2)\right| + |\beta|\left|f_2(x_1) - f_2(x_2)\right| \\
&\le 10\left(|\alpha| + |\beta|\right)\left|x_1 - x_2\right|
\end{aligned}$$

This means that this is not a linear subspace.

2. Let $\boldsymbol{A}$ be a 4-by-4 matrix defined by

$$\boldsymbol{A}(t)=\begin{bmatrix} \exp\left(-\frac{t}{1+t^2}\right) & \sin(\cos(t)) & -\exp\left(-\frac{t}{1+t^2}\right) & 0 \\ \sin(\cos(t)) & 2 & -\cos(\cos(t)) & -1 \\ -\exp\left(-\frac{t}{1+t^2}\right) & -\cos(\cos(t)) & \exp\left(-\frac{t}{1+t^2}\right) & \sin^2(\cos(t)) \\ 0 & -1 & \sin^2(\cos(t)) & \frac{1}{2+\cos(t)} \end{bmatrix}$$

Using MATLAB (i.e. MAPLE) or MATHEMATICA, compute $\frac{d\boldsymbol{A}}{dt}$ and $\frac{d\boldsymbol{A}^{-1}}{dt}$ at t = 2.

Noting that

$$\frac{d\boldsymbol{A}^{-1}}{dt}=-\boldsymbol{A}^{-1}\frac{d\boldsymbol{A}}{dt}\boldsymbol{A}^{-1},$$

we can compute these using the following MATHEMATICA script program :

```
A={{Exp[-t/(1+t^2)],Sin[Cos[t]],-Exp[-t/(1+t^2)],0},
   {Sin[Cos[t]],2,-Cos[Cos[t]],-1},
   {-Exp[-t/(1+t^2)],-Cos[Cos[t]],Exp[-t/(1+t^2)],(Sin[Cos[t]])^2},
   {0,-1,(Sin[Cos[t]])^2,1/(2+Cos[t])}}
DA=N[D[A,t]/.{t->2}]
AI=N[Inverse[A/.{t->2}]]
DAI=-AI.DA.AI
```

Matrix A is defined by

```
Out[8]=
                2                              2
   -(t/(1 + t ))                  -(t/(1 + t ))
{{E              , Sin[Cos[t]], -E              , 0},

  {Sin[Cos[t]], 2, -Cos[Cos[t]], -1},

                  2                              2
     -(t/(1 + t ))                  -(t/(1 + t ))
  {-E              , -Cos[Cos[t]], E              ,

                 2                      2       1
   Sin[Cos[t]] }, {0, -1, Sin[Cos[t]] , ----------}}
                                        2 + Cos[t]
```

This is the first derivative of A at t = 2, i.e., $\frac{d\boldsymbol{A}}{dt}$ at t = 2 :

```
Out[9]=
 {{0.0804384, -0.831692, -0.0804384, 0},

  {-0.831692, 0, 0.367574, 0},
```

```
 {-0.0804384, 0.367574, 0.0804384, 0.672405},

 {0, 0, 0.672405, 0.362473}}
```

This is the inverse of the matrix A at t = 2 :

```
Out[10]=
 {{0.222604, -0.914699, -0.717608, -1.26302},

  {-0.914699, 0.0373926, -0.937248, 0.3018},

  {-0.717608, -0.937248, -0.152397, -1.44502},

  {-1.26302, 0.3018, -1.44502, 2.43585}}
```

This is the the first derivative $\frac{d\boldsymbol{A}^{-1}}{dt}$ of the inverse of A at t = 2 :

```
Out[11]=
 {{-2.68944, -0.116361, -1.37154, 1.66003},

  {-0.116361, 0.31621, -0.350741, 1.4165},

  {-1.37154, -0.350741, -0.0649722, 0.453188},

  {1.66003, 1.4165, 0.453188, 2.26674}}
```

It is clear that if the inverse of A is taken at first, and then the first derivative is computed, you can find a lot of extra work required.

3. Let a 5-by-4 matrix $\boldsymbol{A}$ be defined by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 1 \\ 2 & 4 & -2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

(1) What is the transpose of $\boldsymbol{A}$?

$$\boldsymbol{A}^T = \begin{bmatrix} -1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 4 & 0 & 1 \\ 1 & 1 & -2 & 3 & 2 \\ -1 & 1 & -2 & 1 & 1 \end{bmatrix}$$

(2) Find whether the column vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_4$ are linearly independent ?

$$\sum_{i=1}^{4}\beta_i\boldsymbol{a}_i=\beta_1\begin{Bmatrix}-1\\1\\2\\-1\\0\end{Bmatrix}+\beta_2\begin{Bmatrix}0\\2\\4\\0\\1\end{Bmatrix}+\beta_3\begin{Bmatrix}1\\1\\-2\\3\\2\end{Bmatrix}+\beta_4\begin{Bmatrix}-1\\1\\-2\\1\\1\end{Bmatrix}=0$$

that is

$$-\beta_1+\beta_3-\beta_4=0 \quad\Rightarrow\quad \beta_4=-\beta_1+\beta_3$$
$$\beta_1+2\beta_2+\beta_3+\beta_4=0 \quad\Rightarrow \beta_1+2\beta_2+\beta_3-\beta_1+\beta_3=0 \quad\Rightarrow\quad \beta_3=-\beta_2$$
$$\beta_1+2\beta_2-\beta_3-\beta_4=0 \quad\Rightarrow\quad \beta_1+2\beta_2-\beta_3+\beta_1-\beta_3=0 \quad\Rightarrow\quad \beta_2=-\frac{1}{2}\beta_1$$

$-\beta_1+3\beta_3+\beta_4=0$ is automatically satisfied.

$\beta_2+2\beta_3+\beta_4=0$ is automatically satisfied.

This means that

$$\beta_1=\beta_2=\beta_3=\beta_4=0$$

need not be required to be $\sum_{i=1}^{4}\beta_i\boldsymbol{a}_i=0$. *Therefore, these are not linearly independent.*

In order to answer to the following questions, we shall use MATLAB :

```
% HW#1-3 / 95F
% ________________________________________________________
% define the matrix A
A=[-1,0,1,-1;1,2,1,1;2,4,-2,-2;-1,0,3,1;0,1,2,1]
% set some column vectors
a2=A(:,2)
a3=A(:,3)
% 3-(3)
a2Ta3=a2'*a3
a2a3T=a2*a3'
% 3-(4)
Ar=rank(A)
[U,S,V]=svd(A)
diff=A-U*S*V'
% 3-(6)
A2=norm(A)
% 3-(7)
B=A(1:3,1:3)
BI=inv(B)
```

```
u=[-0.01;0;-0.01]
v=[-0.01;0;0.01]
BP=B+u*v'
BPISM=BI-BI*u*v'*BI/(1+v'*BI*u)
diff=BPISM-inv(BP)
% 3-(8)
detB=det(B)
```

(3) Find $\boldsymbol{a}_2{}^T\boldsymbol{a}_3$ and $\boldsymbol{a}_2\boldsymbol{a}_3{}^T$.

$\boldsymbol{a}_2{}^T\boldsymbol{a}_3$ = -4

$\boldsymbol{a}_2\boldsymbol{a}_3{}^T$ =

0	0	0	0	0
2	2	-4	6	4
4	4	-8	12	8
0	0	0	0	0
1	1	-2	3	2

(4) Find the singular value decomposition of ***A*** and define the $R(\boldsymbol{A})$ and $N(\boldsymbol{A})$ of the above matrix ***A***.

Rank of ***A*** = 3

Singular Value Decomposition of the matrix ***A***

U = $\begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 & \boldsymbol{u}_5\end{bmatrix}$ =

-0.1056	-0.0709	-0.8223	0.5498	0.0736
0.1241	-0.5858	0.4087	0.6040	-0.3311
0.8797	-0.2695	-0.2768	-0.2749	-0.0368
-0.4214	-0.5219	-0.2751	-0.4956	-0.4782
-0.1487	-0.5539	0.0668	-0.1083	0.8093

S =

5.8493	0	0	0
0	4.1279	0	0
0	0	1.9355	0
0	0	0	0.0000
0	0	0	0

V = $\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}$ =

0.4121	-0.1289	0.4921	-0.7559

0.6186	-0.6792	-0.1152	0.3780
-0.5646	-0.6762	-0.2850	-0.3780
-0.3590	-0.2548	0.8145	0.3780

Thus we have

$$R(\boldsymbol{A}) = \left\{ \boldsymbol{y} \in \boldsymbol{R}^5 : \quad \boldsymbol{y} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3, \forall \alpha_1, \alpha_2, \alpha_3 \in \boldsymbol{R} \right\}$$

and

$$N(\boldsymbol{A}) = \left\{ \boldsymbol{y} \in \boldsymbol{R}^5 : \quad \boldsymbol{y} = \beta \boldsymbol{v}_4 = \bar{\beta} \begin{Bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{Bmatrix}, \forall \beta \in \boldsymbol{R} \right\}$$

If the singular value decomposition is not applied to determine the range and null spaces of the matrix A, how should we do ? Note that $\boldsymbol{a}_4 = 2\boldsymbol{a}_1 - \boldsymbol{a}_2 + \boldsymbol{a}_3$, that is the fourth column vector is linearly dependent of the rest of three.

(5) Find $S^\perp$ for $S = \text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_3\}$.

$$S^\perp = \left\{ \boldsymbol{y} \quad : \quad \boldsymbol{y}^T \boldsymbol{a}_1 = 0 \quad and \quad \boldsymbol{y}^T \boldsymbol{a}_2 = 0 \right\}$$

that is

$$-y_1 + y_2 + 2y_3 - y_4 = 0 \Rightarrow y_4 = -y_1 + y_2 + 2y_3$$

and

$$y_1 + y_2 - 2y_3 + 3y_4 + 2y_5 = 0 \Rightarrow y_5 = -\frac{1}{2}(y_1 + y_2 - 2y_3 + 3y_4) \Rightarrow y_5 = y_1 - 2y_2 - 2y_3$$

Thus we chave

$$S^\perp = \left\{ \boldsymbol{y} \quad : \quad \boldsymbol{y} = \alpha \begin{Bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{Bmatrix} + \beta \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -2 \end{Bmatrix} + \gamma \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{Bmatrix}, \forall \alpha, \beta \in \boldsymbol{R} \right\}$$

Ideed, for such a $\boldsymbol{y}$

$$\boldsymbol{y}^T \boldsymbol{a}_1 = 0 \quad and \quad \boldsymbol{y}^T \boldsymbol{a}_2 = 0$$

(6) Compute $\|\boldsymbol{A}\|_2$.

$\|\boldsymbol{A}\|_2$ = 5.8493

(7) Consider a sub-matrix $\boldsymbol{B} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 4 & -2 \end{bmatrix}$ of A, and its perturbation

$$\boldsymbol{B} + \boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 4 & -2 \end{bmatrix} + \begin{Bmatrix} -0.01 \\ 0 \\ -0.01 \end{Bmatrix} \{-0.01 \quad 0 \quad 0.01\}.$$

Find the inverse of the perturbed matrix by the Sherman-Morison formula, and compare it with the one obtained by taking its inverse directly.

```
B =

    -1     0     1
     1     2     1
     2     4    -2
```

B^{-1} =

```
   -1.0000    0.5000   -0.2500
    0.5000         0    0.2500
         0    0.5000   -0.2500
```

```
u =

   -0.0100
         0
   -0.0100


v =

   -0.0100
         0
    0.0100
```

$B + u\ v^T$ =

```
   -0.9999         0    0.9999
    1.0000    2.0000    1.0000
    2.0001    4.0000   -2.0001
```

```
B^-1 + B^-1 u v^T B^-1/(1 + v^T B^-1 u) =

   -1.0001    0.5000   -0.2500
    0.5001         0    0.2500
   -0.0000    0.5000   -0.2500


diff =

   1.0e-16 *

         0         0         0
         0   -0.0333    0.2776
    0.6904         0         0
```

(8) Compute $\det \boldsymbol{B}$.

detB = 8