Solution of Homework #1, Fall 1998

MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering

Consider the static equilibrium equation of an elastic string supported by distributed springs, whose spring constant is k per unit length, and spanned by a tensile force T with a possibly distributed force f:

$$-T\frac{d^2u}{dx^2}+ku=f \qquad in \qquad (0,L)$$

(1) Applying the boundary condition

$$u(0)=u(L)=0$$

derive a discrete system by using the weighted residual method together with the functions

$$\phi_j(x)=\sin\left(j\pi\frac{x}{L}\right) \quad , \quad j=1,\ldots\ldots,n$$
$$\psi_i(x)=x^{i-1} \quad , \quad i=1,\ldots\ldots,m$$

which approximate the trial function space U and the test function space Ω*, respectively. Here, it is noted that any linear combination of the functions* ϕ_i *satisfies the boundary condition, that is, a candidate solution*

$$u=\sum_{j=1}^{n}u_j\phi_j(x)$$

satisfies the boundary condition $u(0)=u(L)=0$ *a priori.*

Multiplying arbitrary basis functions $\psi_i(x)$ for a weighting function (or of the test function space W), we have

$$\int_0^L \psi_i \left(-T\frac{d^2u}{dx^2} + ku - f \right) dx = 0 \quad , \quad i = 1,2,\dots,m$$

$$\sum_{j=1}^{n} \left\{ \int_0^L \psi_i \left(-T\frac{d^2\phi_j}{dx^2} + k\phi_j \right) dx \right\} u_j = \int_0^L \psi_i f dx \quad , \quad i = 1,2,\dots,m$$

Putting

$$k_{ij} = \int_0^L \psi_i \left(-T\frac{d^2\phi_j}{dx^2} + k\phi_j \right) dx \quad , \quad f_i = \int_0^L \psi_i f dx$$

we have a system of linear equations (a discrete problem/system) :

$$\sum_{j=1}^{n} k_{ij} u_j = f_i \quad , \quad i = 1,\dots,m \,.$$

(2) Obtain the stiffness matrix ***K*** *and the generalized force vector* ***f****:*

$$k_{ij} = \left\langle -T\frac{d^2\phi_j}{dx^2} + k\phi_j \,, \psi_i \right\rangle = \int_0^L \left(-T\frac{d^2\phi_j}{dx^2} + k\phi_j \right) \psi_i dx$$

$$f_i = \langle f \,, \psi_i \rangle = \int_0^L f\psi_i dx$$

for

$$L = T = 1 \quad , \quad k = e^{-x} \quad , \quad and \quad f = 1 + \sin\left(3\pi \frac{x}{L} \right).$$

Using the MATLAB program

```
L=1;
T=1;
m=10;
n=7;
```

```
KG=zeros(m,n);
fG=zeros(m,1);
dx=L/100;
x=0:dx:L;
in=size(x,2);
for i=1:m
  for j=1:n
    kij=(T*(j*pi/L)^2+exp(-x)).*x.^(i-1).*sin(j*pi*x/L);
    fi=(1+sin(3*pi*x/L)).*x.^(i-1);
    KG(i,j)=0.01*sum(kij)-0.005*(kij(1)+kij(in));
    fG(i)=0.01*sum(fi)-0.005*(fi(1)+fi(in));
  end
end
KG
fG
```

for m = 10, n = 7, we have the following matrix **K** and vector **f** :

KG =

6.6780	0.0981	18.9790	0.0499	31.4379	0.0333	43.8667
3.3204	-6.3334	9.4596	-12.5783	15.6996	-18.8129	21.9193
1.9676	-6.3482	9.0306	-12.5802	15.4440	-18.8135	21.7371
1.2939	-5.3855	8.8194	-12.1015	15.3169	-18.4948	21.6462
0.9121	-4.4202	8.2389	-11.6231	14.9470	-18.1763	21.3778
0.6759	-3.6217	7.4732	-11.0459	14.4557	-17.7695	21.0206
0.5200	-2.9904	6.6761	-10.3697	13.8788	-17.2743	20.5879
0.4119	-2.4953	5.9245	-9.6389	13.2340	-16.7068	20.0863
0.3341	-2.1051	5.2481	-8.8982	12.5430	-16.0825	19.5254
0.2762	-1.7947	4.6536	-8.1791	11.8292	-15.4172	18.9160

fG =

1.2120
0.6060

0.4346

0.3489

0.2924

0.2505

0.2177

0.1914

0.1700

0.1522

(3) Solve the matrix equation

$$\mathbf{Ku} = \mathbf{f}$$

by using, for example, MATLAB with the command

pinv(K)

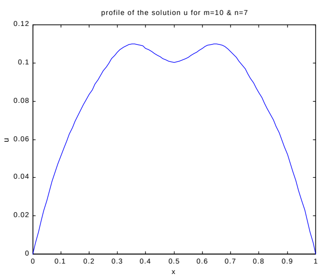
that takes the pseudo-inverse (or generalize inverse) of the matrix K. In MATHEMATICA the pseudo-inverse is taken by PseudoInverse[K]. That is, solve the matrix equation by

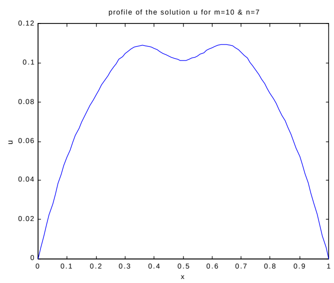
$$\mathbf{u} = \mathbf{K}^{+}\mathbf{f}$$

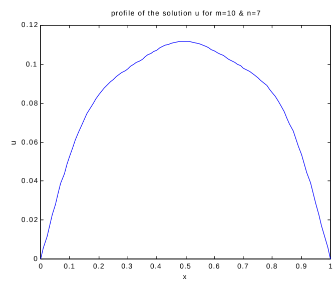
where $\mathbf{K}^{+}$ *is the pseudo-inverse of* ***K*** *which is a generalized inverse of a standard square matrix to a rectangular matrix. At this moment, please not make any question on the pseudo-inverse. We shall discuss this more details later when the singular value decomposition of a rectangular matrix will be studied. Make comparison the solutions for*

m	n
10	7
7	7
5	7

j	m=10	m=7	m=5
1	0.1214	0.1214	0.1214
2	-0.0003	-0.0004	-0.0003
3	0.0163	0.0162	0.0141
4	-0.0002	0.0000	-0.0003
5	-0.0015	-0.0010	0.0044
6	0.0001	0.0000	0.0001
n=7	0.0031	0.0029	-0.0001







(4) *If we assume* $\phi_j(x) = \sin\left(j\pi \frac{x}{L}\right)$, *then* $L(\phi_j)$, $L = -T\frac{d^2}{dx^2} + k$, *are infinitely many times continuously differentiable on the interval (0,L). Thus, we may take the functions for the test function space:*

$$\psi_i(x) = \delta_{x_i}(x)$$

where δ_{x_i} *is the Dirac delta function at a point* x_i. *In this case, the weighted residual method yields the collocation method. Repeat (3) for this choice by setting up the sampling points* x_i *from the interval (0,L) with equal distance of adjacent sampling points.*

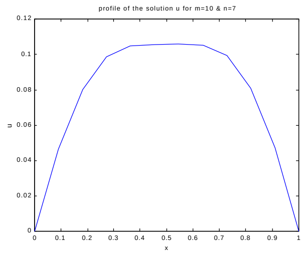
Using the following MATLAB program

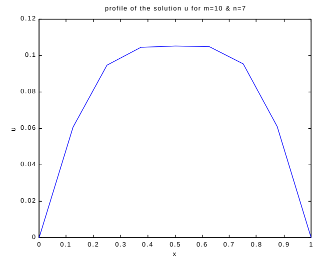
```
L=1;
T=1;
```

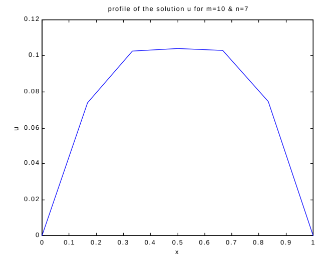
```
m=10;
n=7;
KG=zeros(m+2,n);
fG=zeros(m+2,1);
dx=L/(m+1);
x=0:dx:L;
in=size(x,2);
for j=1:n
   KG(:,j)=((T*(j*pi/L)^2+exp(-x)).*sin(j*pi*x/L))';
end
fG=(1+sin(3*pi*x/L))';
KG
fG
uG=pinv(KG)*fG
fj=zeros(in,n);
for j=1:n
   fj(:,j)=sin(j*pi*x/L)';
end
u=fj*uG;
plot(x,u)
title('profile of the solution u for m=10 & n=7')
xlabel('x')
ylabel('u')
```

we have

i	m=10	m=7	m=5
1	0.1206	0.1198	0.1186
2	-0.0004	-0.0004	-0.0004
3	0.0156	0.0153	0.0149
4	0.0000	0.0000	0.0000
5	0.0008	0.0007	0.0001
6	0.0000	0.0000	0.0000
7=n	0.0002	0.0001	-0.0001







(5)   If we change the boundary condition to

$$u(0)=0 \qquad and \qquad T\frac{du}{dx}(L)=0$$

that is, the elastic string is supported at x = 0, but the symmetry condition is applied at x = L, i.e., the slope of the string becomes zero at x = L. In this case, the choice of sine function as in (1) is not appropriate, because of the symmetry boundary condition. If we choose

$$\phi_j(x)=x^j \qquad , \qquad j=1,.....,n$$
$$\psi_i(x)=x^{i-1} \qquad , \qquad i=1,.....,m$$

derive the discrete problem. It is noted that the degrees of freedom, that is called the generalized displacement, must satisfy the boundary condition:

$$\sum_{j=1}^{n} u_j \frac{d\phi_j}{dx}=\sum_{j=1}^{n} ju_j x^{j-1}=0 \qquad at \qquad x=L \qquad , that\ is \qquad \sum_{j=1}^{n} ju_j L^{j-1}=0$$

Thus, this becomes a constraint while we solve the matrix equation $\mathbf{Ku}=\mathbf{f}$ *. To avoid this constraint, we may make modification to the weighted residual form:*

$$\langle L(u)-f\ ,w\rangle=\int_0^L\left(-T\frac{d^2u}{dx^2}+ku-f\right)wdx=-\left[T\frac{du}{dx}w\right]_{x=0}^{x=L}+\int_0^L\left(T\frac{du}{dx}\frac{dw}{dx}+kuw-fw\right)dx$$

by applying the integration by parts. Then the boundary condition, especially, the symmetry boundary condition is applied to the first term, that is,

$$\langle L(u)-f\ ,w\rangle=\int_0^L\left(-T\frac{d^2u}{dx^2}+ku-f\right)wdx=T\frac{du}{dx}(0)w(0)+\int_0^L\left(T\frac{du}{dx}\frac{dw}{dx}+kuw-fw\right)dx$$

and then we need not introduce an additional constraint to satisfy the symmetry condition. Using this framework, repeat (3).

In order to deal with the boundary condition involving the first derivative of the solution u, we prefer the form after applying the integration by parts (or more generally the divergence theorem for multiple space dimensions).

In order to simplify more, we would assume

$$w(0)=0 \quad i.e. \quad \psi_i(x)=x^i \quad , \quad i=1,2,....,m$$

then we have

$$\langle L(u)-f\ ,w\rangle=\int_0^L\left(T\frac{du}{dx}\frac{dw}{dx}+kuw-fw\right)dx\,.$$

Using the MATLAB program

```
L=1;
T=1;
m=5;
n=7;
KG=zeros(m,n);
fG=zeros(m,1);
dx=L/100;
x=0:dx:L;
in=size(x,2);
```

```
for i=1:m
   for j=1:n
      kij=T*i*j*x.^(i-1).*x.^(j-1)+exp(-x).*x.^i.*x.^j;
      fi=(1+sin(3*pi*x/L)).*x.^i;
      KG(i,j)=0.01*sum(kij)-0.005*(kij(1)+kij(in));
      fG(i)=0.01*sum(fi)-0.005*(fi(1)+fi(in));
   end
end
KG
fG
uG=pinv(KG)*fG
fj=zeros(in,n);
for j=1:n
   fj(:,j)=(x.^j)';
end
u=fj*uG;
plot(x,u)
title('profile of the solution u for m=10 & n=7')
xlabel('x')
ylabel('u')
```

we have

i	M=10	M=7	M=5
1	0.9974	1.0063	1.1025
2	0.3221	0.1486	-1.3431
3	-7.3551	-6.2620	1.0096
4	19.04841	15.9451	0.3846
5	-20.8524	-16.3045	-0.4784
6	9.8063	6.5644	-0.5301
7=n	-1.4969	-0.5922	0.3604