**Homework #3**
**MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering**

Kikuchi, Fall 1997

## 1. Problem (8.3) in the textbook

For the function representation

$$y = \alpha \exp(\beta x)$$

taking log in the both sides yields

$$\log y = \log \alpha + \log \exp(\beta x) = \log \alpha + \beta x .$$

Thus, by using polyfit[x,logy,1], we have

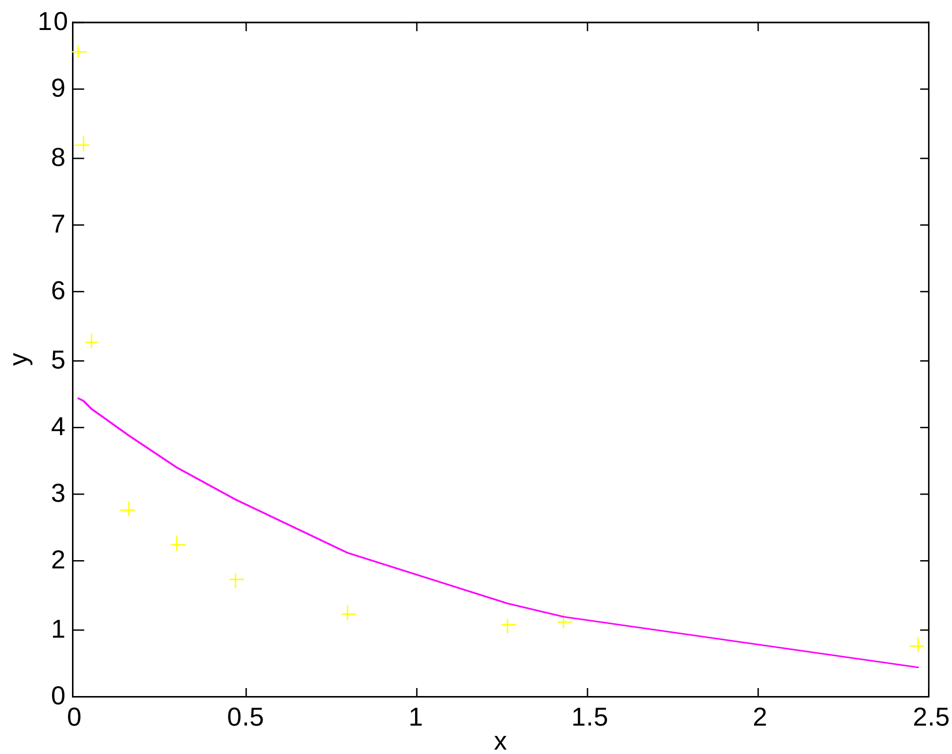
$$\beta = c_1 \quad \text{and} \quad \log \alpha = c_2 \quad \Rightarrow \quad \alpha = \exp(c_2)$$

For the data

x=[0.0129,0.0247,0.0530,0.1550,0.3010,0.4710,0.8020,1.2700,1.4300,2.4600];
y=[9.5600,8.1845,5.2616,2.7917,2.2611,1.7340,1.2370,1.0674,1.1171,0.7620];

we have

beta = -0.9238 and alfa = 4.4717

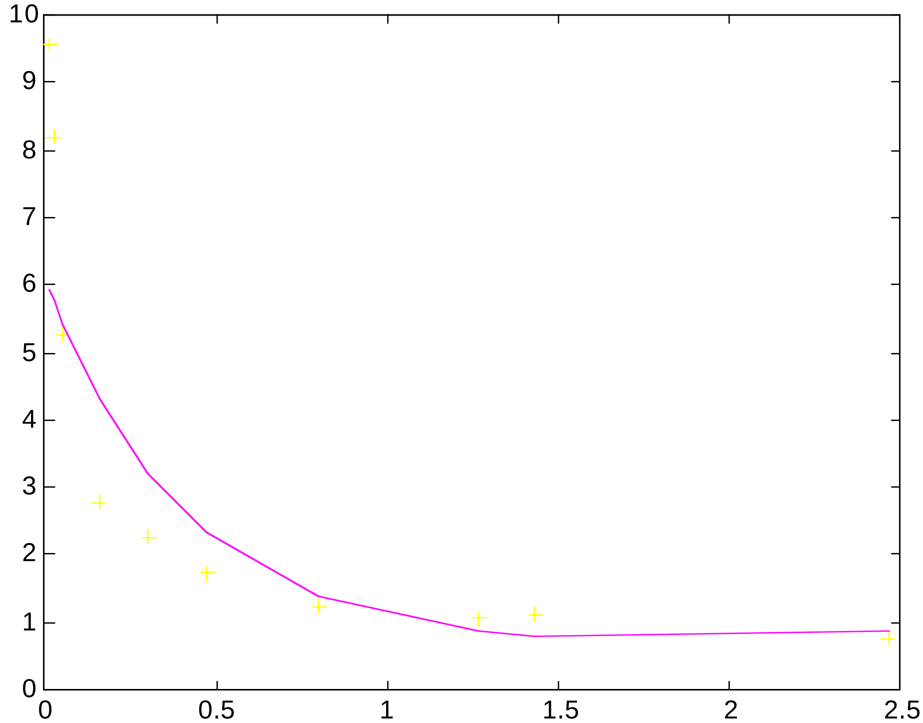


using the MATLAB script program :

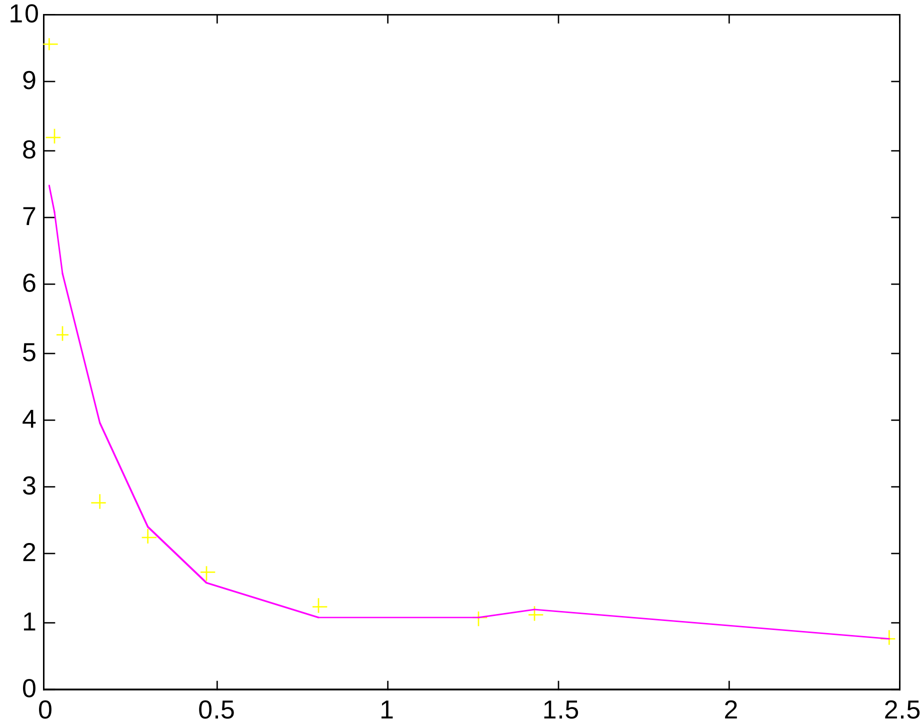
```
% Problem 8.3
x=[0.0129,0.0247,0.0530,0.1550,0.3010,0.4710,0.8020,1.2700,1.4300,2.4600];
y=[9.5600,8.1845,5.2616,2.7917,2.2611,1.7340,1.2370,1.0674,1.1171,0.7620];
plot(x,y)
c=polyfit(x,log(y),1)
beta=c(1)
alfa=exp(c(2))
g=alfa*exp(beta*x);
plot(x,y,'+',x,g)
xlabel('x')
ylabel('y')
pause
% using one more term
c=polyfit(x,log(y),2)
beta1=c(1);
beta2=c(2);
alfa=exp(c(3));
g=alfa*exp(beta1*x.^2+beta2*x);
plot(x,y,'+',x,g)
pause
% using two more terms
c=polyfit(x,log(y),3)
beta1=c(1);
beta2=c(2);
beta3=c(3);
alfa=exp(c(4));
g=alfa*exp(beta1*x.^3+beta2*x.^2+beta3*x);
plot(x,y,'+',x,g)
```

Since the linear approximation is not quite accurate, I try two more cases : quadratic and cubic approximations :

$$y = \alpha \exp\left(\beta_2 x + \beta_1 x^2\right) \quad \text{and} \quad y = \alpha \exp\left(\beta_3 x + \beta_2 x^2 + \beta_1 x^3\right)$$




and



## 2. Problem (8.9) in the textbook

Using the curve

$$g(x) = c_1 + c_2 x + c_3 \sin(\pi x) + c_4 \sin(2\pi x)$$

we shall fit the data

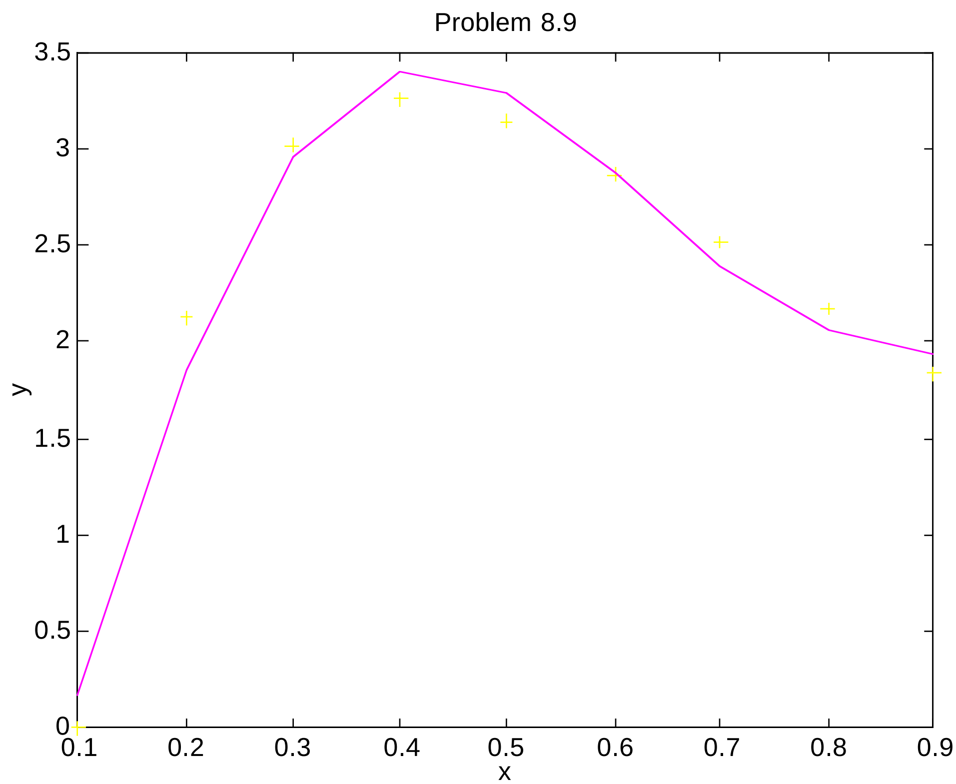
x=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9]
y=[0.0000,2.1220,3.0244,3.2568,3.1399,2.8579,2.5140,2.1639,1.8358]

Using the MATLAB script program :

```
% Problem 8.9
x=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9];
y=[0.0000,2.1220,3.0244,3.2568,3.1399,2.8579,2.5140,2.1639,1.8358];
A(:,1)=ones(9,1);
A(:,2)=x';
A(:,3)=(sin(pi*x))';
A(:,4)=(sin(2*pi*x))';
c=A\y'
g=c(1)+c(2)*x+c(3)*sin(pi*x)+c(4)*sin(2*pi*x);
plot(x,y,'+',x,g)
title('Problem 8.9')
xlabel('x')
ylabel('y')
```

```
c =

   -1.8576
    3.8144
    3.2419
    1.0941
```

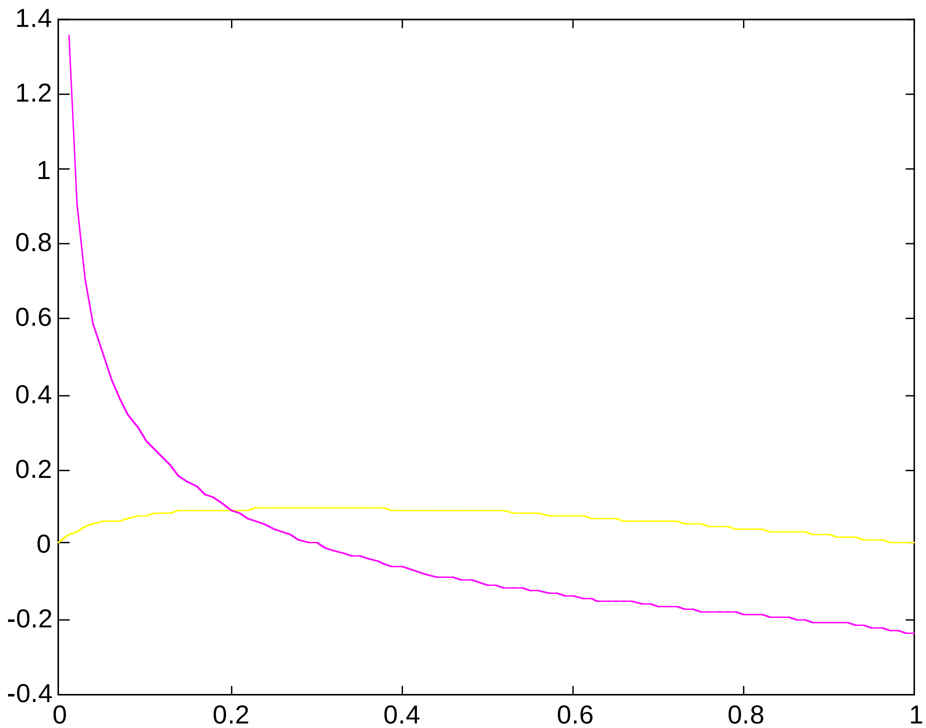



**3. Problem (7.5) in the textbook**

$$y = 0.2969\sqrt{x} - 0.126x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4$$

and

$$y' = \frac{0.2969}{2\sqrt{x}} - 0.126 - 2 * 0.3516x + 3 * 0.2843x^2 - 4 * 0.1015x^3$$




(a) We shall find x that gives the maximum thickness, i.e., the solution of the nonlinear equation that is the first derivative of the airfoil profile function

$$f(x) = \frac{0.2969}{2\sqrt{x}} - 0.126 - 2*0.3516x + 3*0.2843x^2 - 4*0.1015x^3 = 0.$$

by the bi-section method.

(b) Noting that we have two roots for this solution. To find them we shall solve the nonlinear equation that is the profile equation

$$g(x) = 0.2969\sqrt{x} - 0.126x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4 - 0.05 = 0$$

by the bi-section method.

```
% Problem 7.5
x=0:0.01:1;
y=0.2969*sqrt(x)-0.126*x-0.3516*x.^2+0.2843*x.^3-0.1015*x.^4;
dy=(0.2969/2)*x.^(-1/2)-0.126-2*0.3516*x+3*0.2843*x.^2-4*0.1015*x.^3;
plot(x,y,x,dy)
% (a)
tolerance=10^(-5);
error=1;
i=0;
x1=0;
x2=1;
while error>tolerance
x3=(x2+x1)/2;
f1=(0.2969/2)*x1^(-1/2)-0.126-2*0.3516*x1+3*0.2843*x1^2-4*0.1015*x1^3;
f2=(0.2969/2)*x2^(-1/2)-0.126-2*0.3516*x2+3*0.2843*x2^2-4*0.1015*x2^3;
f3=(0.2969/2)*x3^(-1/2)-0.126-2*0.3516*x3+3*0.2843*x3^2-4*0.1015*x3^3;
```

```
if f1*f3<0
        x1=x1;
        x2=x3;
end
if f3*f2<0
        x1=x3;
        x2=x2;
end
i=i+1;
error=abs(f3);
[i,error,x1,x2,x3]
end
xmax=x3
% (b)-1   First Root
tolerance=10^(-5);
error=1;
i=0;
x1=0;
x2=xmax;
while error>tolerance
x3=(x2+x1)/2;
g1=0.2969*sqrt(x1)-0.126*x1-0.3516*x1^2+0.2843*x1^3-0.1015*x1^4-0.05;
g2=0.2969*sqrt(x2)-0.126*x2-0.3516*x2^2+0.2843*x2^3-0.1015*x2^4-0.05;
g3=0.2969*sqrt(x3)-0.126*x3-0.3516*x3^2+0.2843*x3^3-0.1015*x3^4-0.05;
if g1*g3<0
        x1=x1;
        x2=x3;
end
if g3*g2<0
        x1=x3;
        x2=x2;
end
i=i+1;
error=abs(g3);
[i,error,x1,x2,x3]
end
x3
% (b)-2   Second Root
tolerance=10^(-5);
error=1;
i=0;
x1=xmax;
x2=1;
while error>tolerance
x3=(x2+x1)/2;
g1=0.2969*sqrt(x1)-0.126*x1-0.3516*x1^2+0.2843*x1^3-0.1015*x1^4-0.05;
g2=0.2969*sqrt(x2)-0.126*x2-0.3516*x2^2+0.2843*x2^3-0.1015*x2^4-0.05;
g3=0.2969*sqrt(x3)-0.126*x3-0.3516*x3^2+0.2843*x3^3-0.1015*x3^4-0.05;
if g1*g3<0
        x1=x1;
        x2=x3;
end
if g3*g2<0
        x1=x3;
        x2=x2;
end
i=i+1;
error=abs(g3);
```

```
[i,error,x1,x2,x3]
end
x3
```

(a)

| | | | | |
|---|---|---|---|---|
| 1.0000 | 0.1052 | 0 | 0.5000 | 0.5000 |
| 2.0000 | 0.0421 | 0.2500 | 0.5000 | 0.2500 |
| 3.0000 | 0.0488 | 0.2500 | 0.3750 | 0.3750 |
| 4.0000 | 0.0093 | 0.2500 | 0.3125 | 0.3125 |
| 5.0000 | 0.0146 | 0.2812 | 0.3125 | 0.2812 |
| 6.0000 | 0.0022 | 0.2969 | 0.3125 | 0.2969 |
| 7.0000 | 0.0036 | 0.2969 | 0.3047 | 0.3047 |
| 8.0000 | 0.0007 | 0.2969 | 0.3008 | 0.3008 |
| 9.0000 | 0.0008 | 0.2988 | 0.3008 | 0.2988 |
| 10.0000 | 0.0000 | 0.2998 | 0.3008 | 0.2998 |
| 11.0000 | 0.0004 | 0.2998 | 0.3003 | 0.3003 |
| 12.0000 | 0.0002 | 0.2998 | 0.3000 | 0.3000 |
| 13.0000 | 0.0001 | 0.2998 | 0.2999 | 0.2999 |
| 14.0000 | 0.0000 | 0.2998 | 0.2999 | 0.2999 |
| 15.0000 | 0.0000 | 0.2998 | 0.2998 | 0.2998 |

xmax = 0.2998

(b-1)

| | | | | |
|---|---|---|---|---|
| 1.0000 | 0.0391 | 0 | 0.1499 | 0.1499 |
| 2.0000 | 0.0200 | 0 | 0.0750 | 0.0750 |
| 3.0000 | 0.0023 | 0 | 0.0375 | 0.0375 |
| 4.0000 | 0.0118 | 0.0187 | 0.0375 | 0.0187 |
| 5.0000 | 0.0040 | 0.0281 | 0.0375 | 0.0281 |
| 6.0000 | 0.0007 | 0.0328 | 0.0375 | 0.0328 |
| 7.0000 | 0.0008 | 0.0328 | 0.0351 | 0.0351 |
| 8.0000 | 0.0000 | 0.0328 | 0.0340 | 0.0340 |
| 9.0000 | 0.0003 | 0.0334 | 0.0340 | 0.0334 |
| 10.0000 | 0.0001 | 0.0337 | 0.0340 | 0.0337 |
| 11.0000 | 0.0001 | 0.0338 | 0.0340 | 0.0338 |
| 12.0000 | 0.0000 | 0.0339 | 0.0340 | 0.0339 |

x3 = 0.0339

(b-2)

| | | | | |
|---|---|---|---|---|
| 1.0000 | 0.0189 | 0.6499 | 1.0000 | 0.6499 |
| 2.0000 | 0.0110 | 0.6499 | 0.8250 | 0.8250 |

|  |  |  |  |  |
|---|---|---|---|---|
| 3.0000 | 0.0048 | 0.7374 | 0.8250 | 0.7374 |
| 4.0000 | 0.0028 | 0.7374 | 0.7812 | 0.7812 |
| 5.0000 | 0.0010 | 0.7593 | 0.7812 | 0.7593 |
| 6.0000 | 0.0009 | 0.7593 | 0.7703 | 0.7703 |
| 7.0000 | 0.0001 | 0.7648 | 0.7703 | 0.7648 |
| 8.0000 | 0.0004 | 0.7648 | 0.7675 | 0.7675 |
| 9.0000 | 0.0002 | 0.7648 | 0.7662 | 0.7662 |
| 10.0000 | 0.0000 | 0.7648 | 0.7655 | 0.7655 |
| 11.0000 | 0.0000 | 0.7651 | 0.7655 | 0.7651 |
| 12.0000 | 0.0000 | 0.7651 | 0.7653 | 0.7653 |

```
x3 = 0.7653
```

**4. Problem (7.10) in the textbook**

Two ellipses have zero to four intersections. The following equations represent two ellipses. Find the coordinates of intersections by a graphic method first and then by Newton iteration.

$$(x-2)^2 + (y-3+2x)^2 = 5$$

and

$$2(x-2)^2 + \left(\frac{y}{3}\right)^2 = 4.$$

We shall first make a parametric representation of the two ellipes. For the first ellipse decribed by

$$(x-2)^2 + (y-3+2x)^2 = 5$$

we shall use the parametric representation :

$$x-2 = \sqrt{5}\cos\theta \quad \Rightarrow \quad x = 2+\sqrt{5}\cos\theta$$

$$y-3+2x = \sqrt{5}\sin\theta \quad \Rightarrow \quad y = 3+\sqrt{5}\sin\theta - 2\left(2+\sqrt{5}\cos\theta\right) = -1+\sqrt{5}\sin\theta - 2\sqrt{5}\cos\theta$$

Similarly, for the second ellipse

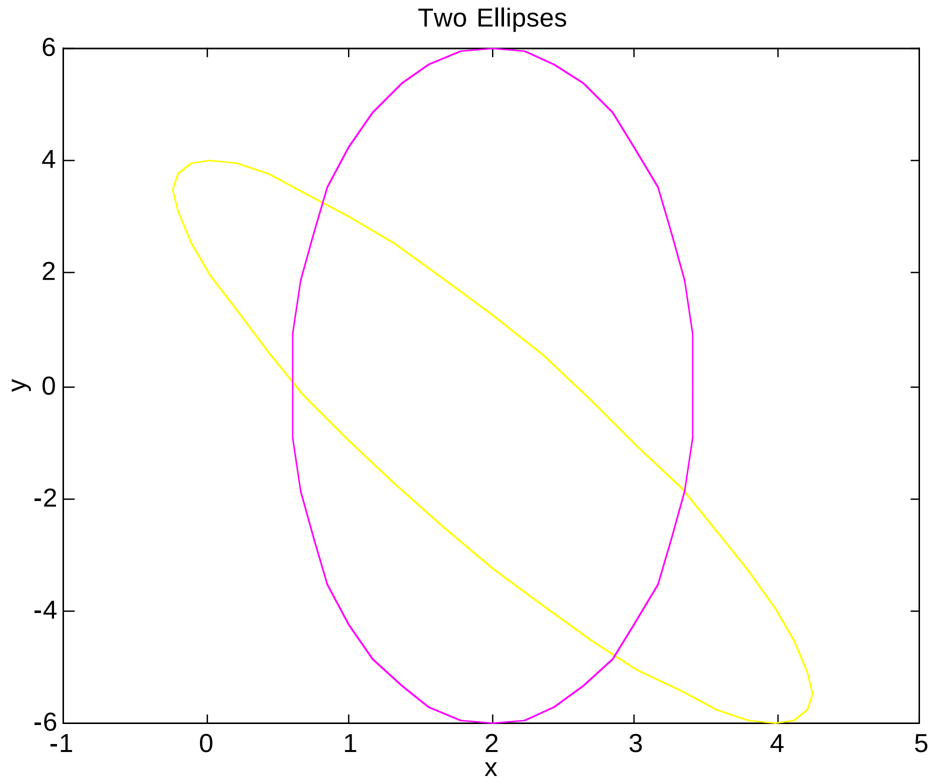
$$2(x-2)^2 + \left(\frac{y}{3}\right)^2 = 4$$

we shall use the parametric representation

$$x-2 = \sqrt{2}\cos\phi \quad \Rightarrow \quad x = 2+\sqrt{2}\cos\phi$$

$$y = 6\sin\phi$$

The profiles of the two ellipses are as follows :




Thus, the intersections can be obtained by solving the equations

$$2+\sqrt{5}\cos\theta = 2+\sqrt{2}\cos\phi \quad \Rightarrow \cos\theta - \sqrt{\frac{2}{5}}\cos\phi = 0$$

$$-1+\sqrt{5}\sin\theta - 2\sqrt{5}\cos\theta = 6\sin\phi \quad \Rightarrow \quad \sin\theta - 2\cos\theta - \frac{6}{\sqrt{5}}\sin\phi - \frac{1}{\sqrt{5}} = 0$$

that is

$$\begin{cases} f_1\left(\begin{Bmatrix}\theta \\ \phi\end{Bmatrix}\right) = \cos\theta - \sqrt{\frac{2}{5}}\cos\phi = 0 \\ f_2\left(\begin{Bmatrix}\theta \\ \phi\end{Bmatrix}\right) = \sin\theta - 2\cos\theta - \frac{6}{\sqrt{5}}\sin\phi - \frac{1}{\sqrt{5}} = 0 \end{cases}$$

If we shall use Newton's method, we need the first derivatives of the two equations :

$$\boldsymbol{J}\left(\begin{Bmatrix}\theta \\ \phi\end{Bmatrix}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial\theta} & \frac{\partial f_1}{\partial\phi} \\ \frac{\partial f_2}{\partial\theta} & \frac{\partial f_2}{\partial\phi} \end{bmatrix} = \begin{bmatrix} -\sin\theta & \sqrt{\frac{2}{5}}\sin\phi \\ \cos\theta + 2\sin\theta & -\frac{6}{\sqrt{5}}\cos\phi \end{bmatrix}$$

Using this, Newton's method is decribed by the following iteration scheme :

$$\begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(k+1)} = \begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(k)} - \boldsymbol{J}\left(\begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(k)}\right)^{-1}\begin{Bmatrix} f_1\left(\begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(k)}\right)\\ f_2\left(\begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(k)}\right)\end{Bmatrix}$$

```
% Problem 7-10
% ______________________________________________________________
% Graphical Representation
q=0:2*pi/40:2*pi;
x1=2+sqrt(5)*cos(q);
y1=-1+sqrt(5)*(sin(q)-2*cos(q));
x2=2+sqrt(2)*cos(q);
y2=6*sin(q);
plot(x1,y1,x2,y2)
xlabel('x')
ylabel('y')
title('Two Ellipses')
% Newton's Method
it=0;
tolerance=10^(-5);
error=1;
q1=1.5;
q2=1;
while error>tolerance
        it=it+1;
        f1=cos(q1)-sqrt(2/5)*cos(q2);
        f2=sin(q1)-2*cos(q1)-6*sin(q2)/sqrt(5)-1/sqrt(5);
        f=[f1,f2]';
        JM=[-sin(q1),sqrt(2/5)*sin(q2);cos(q1)+2*sin(q1),-6*cos(q2)/sqrt(5)];
        dq=-JM\f;
        error=norm(dq);
        q1=q1+dq(1);
        q2=q2+dq(2);
        [it,error,q1,q2]
end
% Plot the Intersections
xi=2+sqrt(5)*cos(q1);
yi=-1+sqrt(5)*(sin(q1)-2*cos(q1));
plot(x1,y1,x2,y2,xi,yi,'*')
xlabel('x')
ylabel('y')
title('Two Ellipses')
```

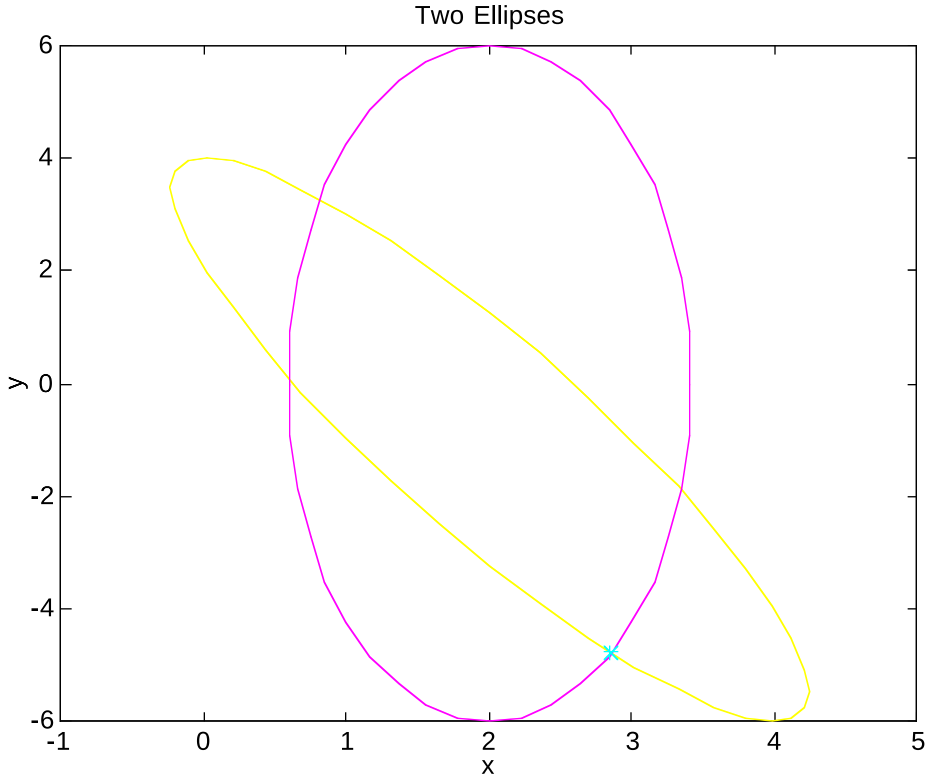
( Intersection #1 ) Using the initial value

$$\begin{Bmatrix}\theta\\ \phi\end{Bmatrix}_{(0)} = \begin{Bmatrix}-0.5\\ -0.3\end{Bmatrix}$$

we have the first intersection

```
1.0000    1.1052   -1.3470   -1.0100
2.0000    0.1853   -1.1826   -0.9244
```

| | | | |
|---|---|---|---|
| 3.0000 | 0.0057 | -1.1781 | -0.9210 |
| 4.0000 | 0.0000 | -1.1781 | -0.9210 |

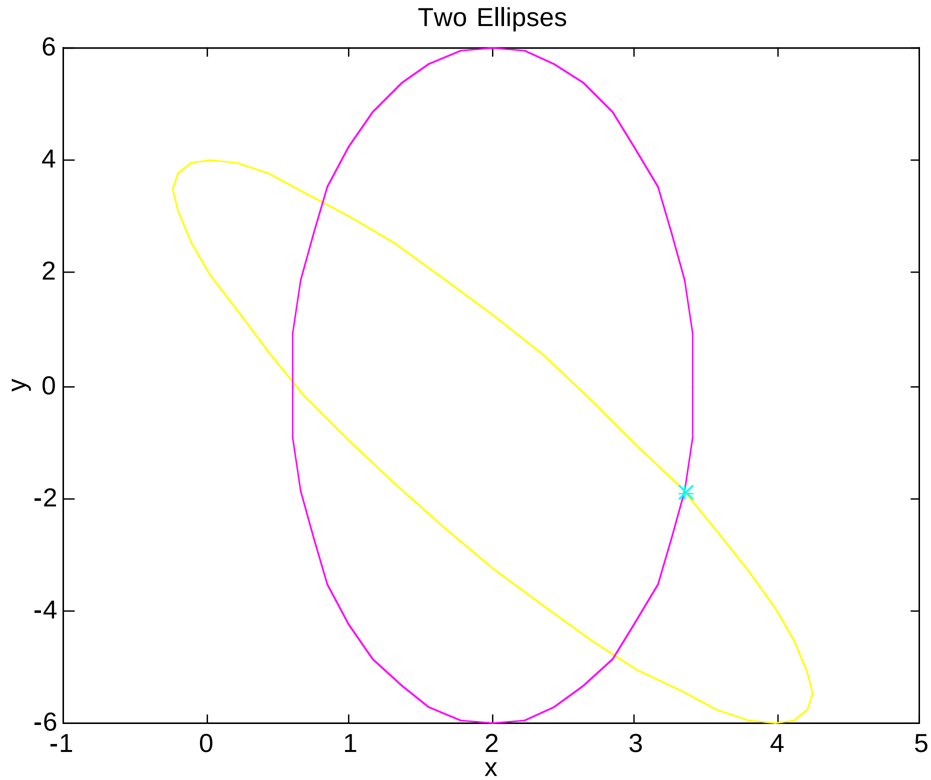



( Intersection #2 ) Starting from the initial value

$$\begin{Bmatrix} \theta \\ \phi \end{Bmatrix}_{(0)} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix}$$

we can find the 2nd intersection :

| | | | |
|---|---|---|---|
| 1.0000 | 1.5103 | 0.2327 | -0.9865 |
| 2.0000 | 1.1574 | 1.0380 | -0.1553 |
| 3.0000 | 0.2050 | 0.9214 | -0.3238 |
| 4.0000 | 0.0063 | 0.9272 | -0.3213 |
| 5.0000 | 0.0000 | 0.9272 | -0.3213 |
| 6.0000 | 0.0000 | 0.9272 | -0.3213 |

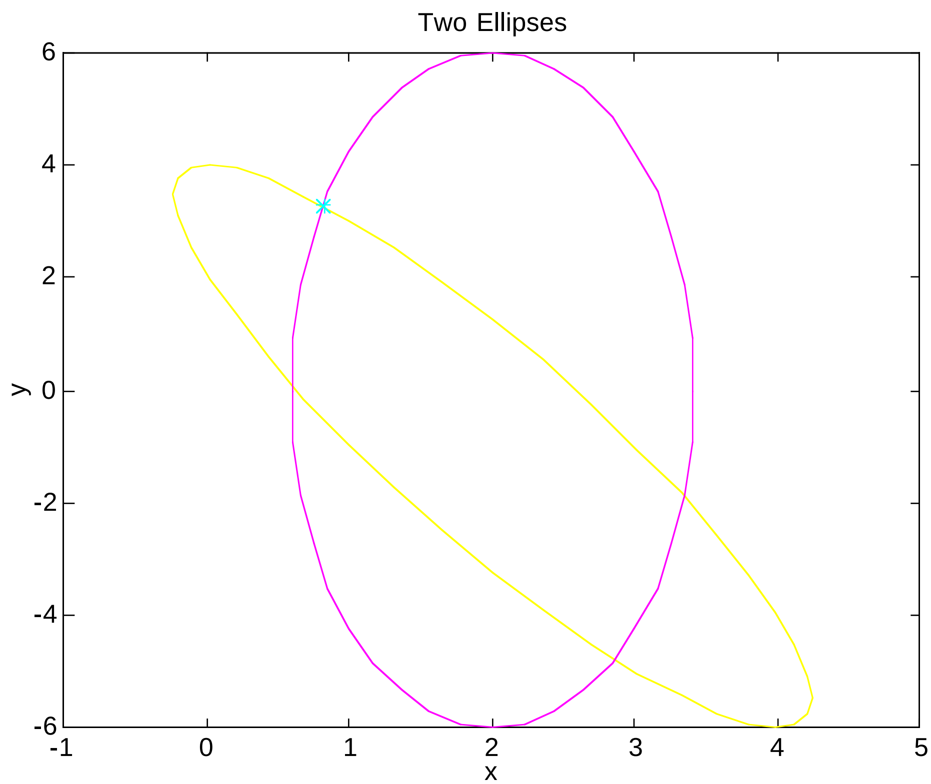



( Intersection #3 )    Starting from the initial value

$$\begin{Bmatrix}\theta \\ \phi\end{Bmatrix}_{(0)} = \begin{Bmatrix}2 \\ 1.8\end{Bmatrix}$$

we have the third intersection :

| | | | |
|---|---|---|---|
| 1.0000 | 1.2045 | 2.4556 | 2.9150 |
| 2.0000 | 0.4969 | 2.1239 | 2.5450 |
| 3.0000 | 0.0216 | 2.1300 | 2.5657 |
| 4.0000 | 0.0002 | 2.1299 | 2.5657 |
| 5.0000 | 0.0000 | 2.1299 | 2.5657 |

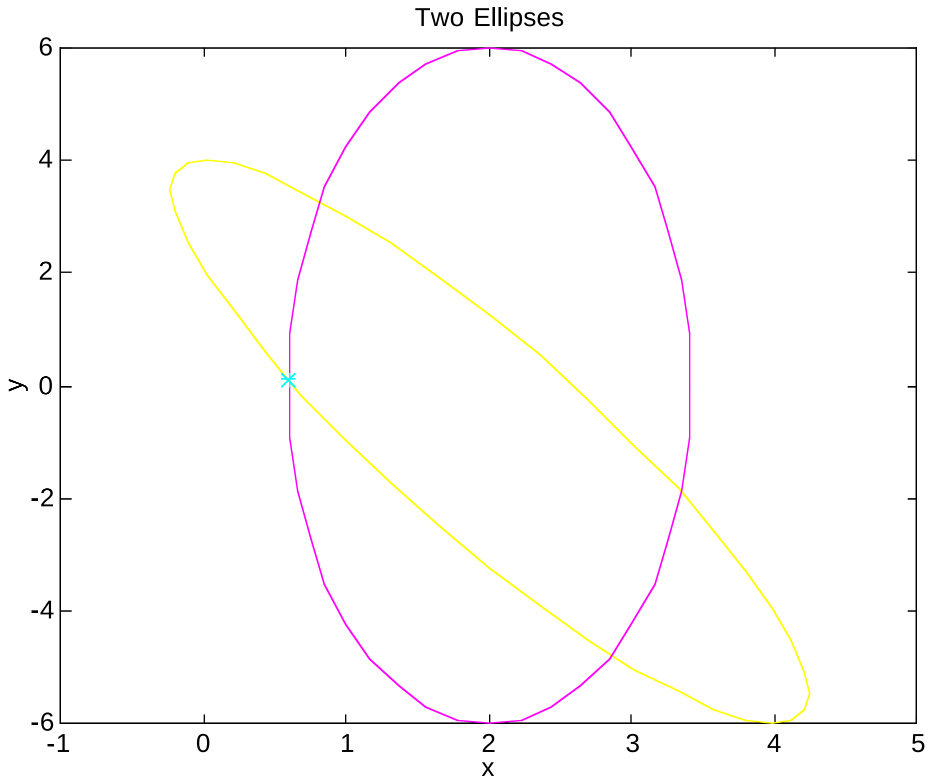



( Intersection #4 )   Starting from the initial value

$$\begin{Bmatrix} \theta \\ \phi \end{Bmatrix}_{(0)} = \begin{Bmatrix} 1.5 \\ 1 \end{Bmatrix}$$

we have

| | | | |
|---|---|---|---|
| 1.0000 | 7.9894 | -2.4706 | -5.9329 |
| 2.0000 | 4.1634 | 0.6902 | -8.6427 |
| 3.0000 | 12.7068 | 9.2072 | -18.0727 |
| 4.0000 | 4.4285 | 4.9265 | -16.9378 |
| 5.0000 | 2.4196 | 3.3652 | -15.0894 |
| 6.0000 | 1.1878 | 4.3563 | -15.7441 |
| 7.0000 | 0.3062 | 4.0529 | -15.7032 |
| 8.0000 | 0.0325 | 4.0278 | -15.7239 |
| 9.0000 | 0.0001 | 4.0278 | -15.7239 |
| 10.0000 | 0.0000 | 4.0278 | -15.7239 |




**5. Problem (7.11) in the textbook**

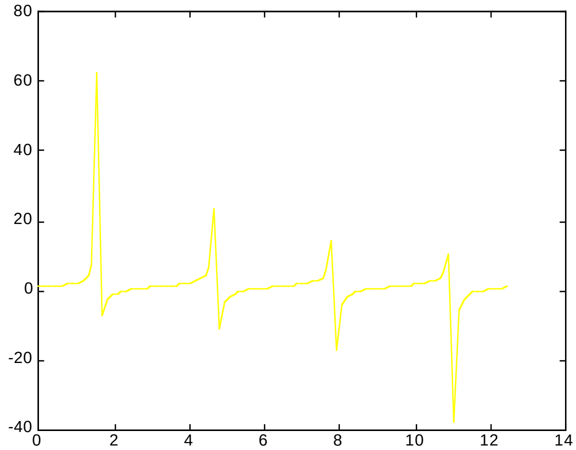
The natural frequencies of vibration of a uniform beam clamped at one end and free at the other end are solutions of

$$\tan(\beta L)\tanh(\beta L)+1=0$$

where

$$\beta=\rho\omega^2/EI$$
$$L=1\,(\text{length of the beam}), m$$
$$\omega=\text{frequency}, s^{-1}$$
$$EI=\text{flexural rigidity}, Nm^2$$
$$\rho=\text{density of the beam material}, kg/m^3$$

Investigate the $\beta$ values that satisfy the equation first by the graphic method and then determine the lowest three values of $\beta$ satisfying the equation by Newton iteration.




Noting that the first derivative of the function

$$f(\beta) = \tan(\beta)\tanh(\beta) + 1 = 0$$

becomes

$$f'(\beta) = \text{sech}(\beta)^2 \tan(\beta) + \sec(\beta)^2 \tanh(\beta)$$

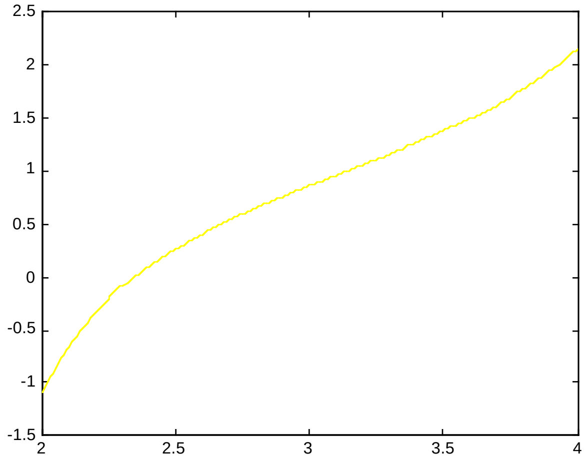
we can set up the Newton iteration scheme as follows :

```
% Problem 7-11
b=8.5;
it=0;
tolerance=10^(-5);
error=1;
while error>tolerance
f=tan(b)*tanh(b)+1;
df=tan(b)*sech(b)^2+tanh(b)*sec(b)^2;
db=-f/df;
error=abs(db);
it=it+1;
b=b+db;
[it,error,b,f]
end
b
```

( #1 : The Lowest Frequency )

| | | | |
|---|---|---|---|
| 1.0000 | 0.2044 | 2.2044 | -1.1064 |
| 2.0000 | 0.1208 | 2.3252 | -0.3284 |
| 3.0000 | 0.0214 | 2.3466 | -0.0439 |
| 4.0000 | 0.0005 | 2.3470 | -0.0009 |
| 5.0000 | 0.0000 | 2.3470 | -0.0000 |

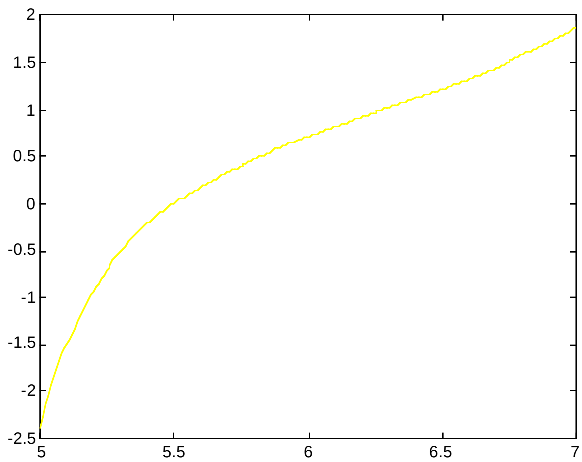
$\beta$ = 2.3470




( #2 : Second Frequency )

| | | | |
|---|---|---|---|
| 1.0000 | 0.1915 | 5.1915 | -2.3802 |
| 2.0000 | 0.1966 | 5.3881 | -0.9246 |
| 3.0000 | 0.0968 | 5.4849 | -0.2474 |
| 4.0000 | 0.0127 | 5.4976 | -0.0260 |
| 5.0000 | 0.0002 | 5.4978 | -0.0003 |
| 6.0000 | 0.0000 | 5.4978 | -0.0000 |

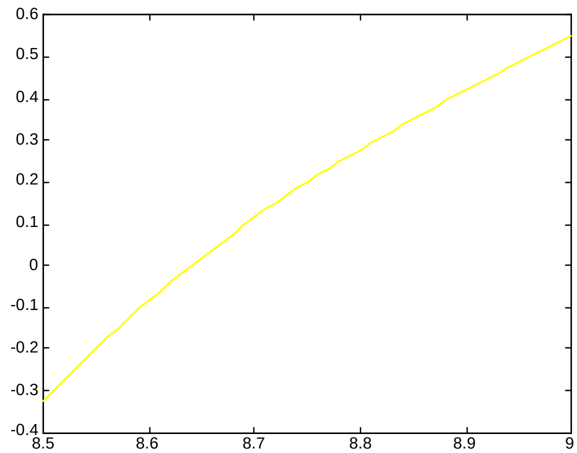
$\beta$ = 5.4978




( #3 : Third Frequency )

| | | | |
|---|---|---|---|
| 1.0000 | 0.1183 | 8.6183 | -0.3264 |
| 2.0000 | 0.0206 | 8.6389 | -0.0431 |
| 3.0000 | 0.0005 | 8.6394 | -0.0009 |
| 4.0000 | 0.0000 | 8.6394 | -0.0000 |

$\beta$ = 8.6394


0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
8.5
8.6
8.7
8.8
8.9
9