

Homework #5
MEAM 501 Analytical Methods in Mechanical Engineering

1997 Fall

1. Suppose that the following is the coordinates of the eleven control points of the Bezier spline curve C in the three dimensional space \mathbf{R}^3 .

{0,0,0}
{0.1, 0.870501, 0.938693}
{0.2, -0.968673, 1.13874}
{0.3, -1.71508, 0.323311}
{0.4, -0.202442, -0.385068}
{0.5, 1.25, 0}
{0.6, -0.278835, 0.530376}
{0.7, -3.25672, -0.613928}
{0.8, -2.53949, -2.98535}
{0.9, 3.15378, -3.40084}
{1., 7.5, 0}

For $n+1$ control points, the Bezier spline curve is defined by

$$\mathbf{r} = \sum_{i=1}^{n+1} \mathbf{r}_i B_i^n(s) \quad \text{i.e.} \quad \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{n+1} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} B_i^n(s)$$

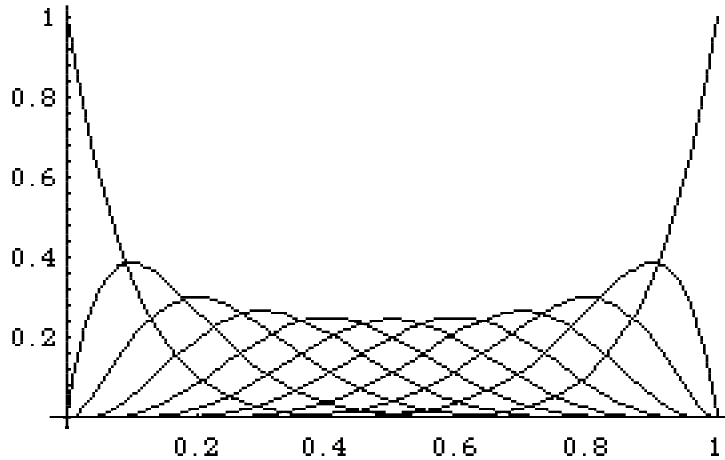
with

$$B_i^n(s) = \frac{n!}{(i-1)!(n-i+1)!} s^{i-1} (1-s)^{n-i+1}$$

for the parameter $s \in (0,1)$.

We first define the basis functions of the Bezier spline :

```
n=10;  
Bezier=Table[(n!/((i-1)! (n-i+1)!))*s^(i-1) (1-s)^(n-i+1),  
             {i,1,n+1}];  
Plot[Release[Bezier],{s,0,1}, PlotRange->All]
```

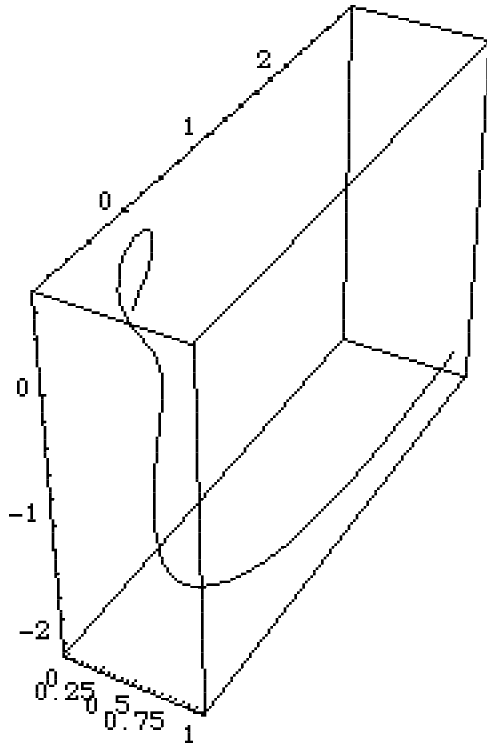


(1) Compute the total length L of the curve C by applying an appropriate quadrature.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} ds$$

Using the MATHEMATICA script program, we first compute the total length of the curve :

```
r={{0,0,0},
{0.1, 0.870501, 0.938693},
{0.2, -0.968673, 1.13874},
{0.3, -1.71508, 0.323311},
{0.4, -0.202442, -0.385068},
{0.5, 1.25, 0},
{0.6, -0.278835, 0.530376},
{0.7, -3.25672, -0.613928},
{0.8, -2.53949, -2.98535},
{0.9, 3.15378, -3.40084},
{1., 7.5, 0}};
xb=Sum[r[[i,1]]*Bezier[[i]],{i,1,n+1}];
yb=Sum[r[[i,2]]*Bezier[[i]],{i,1,n+1}];
zb=Sum[r[[i,3]]*Bezier[[i]],{i,1,n+1}];
ParametricPlot3D[{xb,yb,zb},{s,0,1}]
dxds=D[xb,s];
dyds=D[yb,s];
dzds=D[zb,s];
ds=Sqrt[dxds^2+dyds^2+dzds^2];
Length=NIntegrate[ds,{s,0,1}]
```



Total Length of the Curve =

12.3246

(2) Obtain the tangent, normal, and bi-normal vectors at the point P on the curve, where P is the 20% point of the total length L of the curve C. To identify the location of the 20% point of the total length, you may use either the bisection or Newton method.

Noting that the Newton iteration method to solve a nonlinear equation is given by

$$s^{(k+1)} = s^{(k)} - \frac{f(s^{(k)})}{f'(s^{(k)})}, \quad k = 1, 2, \dots$$

where

$$f(s) = \int_0^s \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} ds - L_s = 0$$

$$f'(s) = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

and L_s corresponds to the 20% length of the total one, we can develop the following MATHEMATICA script programs :

Newton's method for the initial assumption

```
sk=0.2;  
fk=NIntegrate[ds,{s,0,sk}]-0.2*L0;  
dfk=N[ds/.{s->sk}];  
dsk=-fk/dfk
```

Newton's method for general iteration (this must be repeatedly executed)

```
sk=sk+dsk  
fk=NIntegrate[ds,{s,0,sk}]-0.2*L0;  
dfk=N[ds/.{s->sk}];  
dsk=-fk/dfk
```

We can compute the value of s that yields the 20% length as follows :

sk =

0.583753

dsk (whose absolute number is the error of the Newton method) =

-9
-7.00638 10

Now we shall compute the tangent, normal, and bi-norma vectors :

```
cross[v1_,v2_] := {Det[{{1,0,0},v1,v2}],  
                  Det[{{0,1,0},v1,v2}],  
                  Det[{{0,0,1},v1,v2}]}  
gs={dxds,dyds,dzds}/.{s->0.583753};  
ds=Sqrt[dxds^2+dyds^2+dzds^2];  
ns={D[dxds/ds,s],D[dyds/ds,s],D[dzds/ds,s]}/.{s->0.583753};  
tv=gs/Sqrt[gs.gs]  
nv=ns/Sqrt[ns.ns]  
bv=cross[tv,nv]  
tv.nv  
tv.bv  
nv.bv
```

tv (unit tangent vector) =
{0.170689, -0.532384, -0.829115}

nv (unit normal vector) =
{-0.171329, 0.812609, -0.557057}

bv (unit bi-normal vector) =
{0.970315, 0.237135, 0.0474909}

tv.nv =
-17
5.55112 10

1.38778 10⁻¹⁷

nv.bv

-2.08167 10⁻¹⁷

2. Approximate the function

$$f(x) = \begin{cases} -x^2 + 1 & , x \in (-1,0) \\ 1-x & , x \in (0,1) \end{cases}$$

defined on the interval (-1,1) by using

(1) Legendre polynomials and the least squares method,

Noting that the Legendre polynomials $P_n(x)$, LegendreP[n,x] in MATHEMATICA, $n = 0,1,2,\dots$, is defined with the orthogonality relation

$$\int_{-1}^1 P_m(x)P_n(x)dx = 0 \quad , \quad m \neq n,$$

we define the orthonormal basis functions by the Legendre polynomials :

$$\phi_i(x) = \frac{P_i}{\|P_i\|}(x) = \frac{1}{\sqrt{\int_{-1}^1 P_i(x)^2 dx}} P_i(x) \quad , \quad i = 0,1,\dots$$

Then the least squares approximation of a given function f is given by

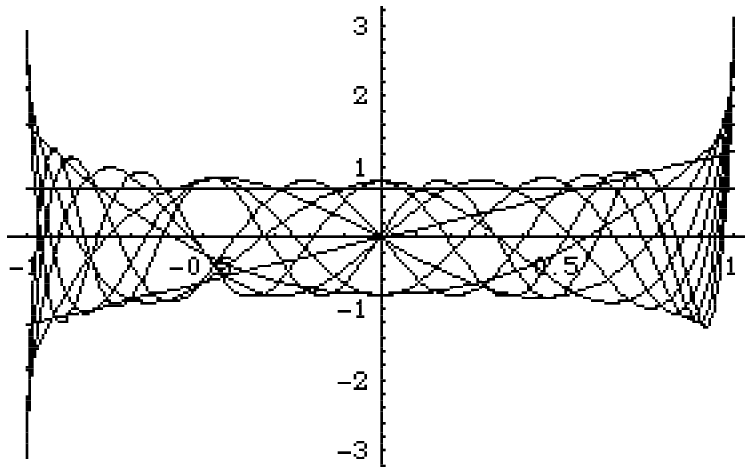
$$f(x) \approx f_n(x) = \sum_{i=0}^n c_i \phi_i(x) = \sum_{i=0}^n \left(\int_{-1}^1 f(x) \phi_i(x) dx \right) \phi_i(x).$$

Therefore, using the MATHEMATICA script program :

```
n=9;
LP=Table[LegendreP[i,x],{i,0,n}];
NLP=Table[Sqrt[NIntegrate[LP[[i]]^2,{x,-1,1}]],{i,1,n+1}];
LP=Table[LP[[i]]/NLP[[i]],{i,1,n+1}];
Plot[Release[LP],{x,-1,1}]
f1=-x^2+1;
f2=1-x;
f=If[x<0,f1,f2];
Ck=Table[NIntegrate[f1*LP[[i]],{x,-1,0}]+
          NIntegrate[f2*LP[[i]],{x,0,1}],{i,1,n+1}];
fn=Ck.LP;
Plot[{f,fn},{x,-1,1}]
```

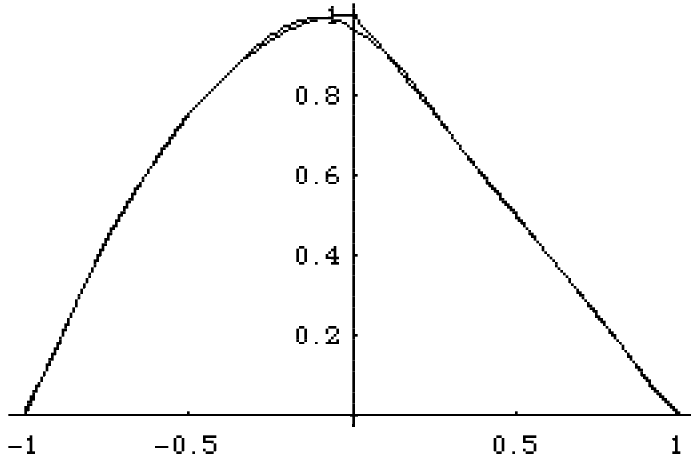
$\{\phi_i(x)\} =$

$\{0.707107, 1.22474 x, 0.790569 (-1 + 3 x),$
 $0.935414 (-3 x + 5 x^3), 0.265165 (3 - 30 x^2 + 35 x^4),$
 $0.293151 (15 x^3 - 70 x^5 + 63 x^5),$
 $0.159344 (-5 + 105 x^2 - 315 x^4 + 231 x^6),$
 $0.171163 (-35 x^3 + 315 x^5 - 693 x^7 + 429 x^7),$
 $0.0227772 (35 - 1260 x^2 + 6930 x^4 - 12012 x^6 + 6435 x^8),$
 $0.0240797 (315 x^3 - 4620 x^5 + 18018 x^7 - 25740 x^7 +$
 $12155 x^9) \}$



$\{c_i\} =$

$\{0.824958, -0.102062, -0.408461, 0.0779512, 0.0441942,$
 $-0.0122146, -0.019918, 0.00427908, 0.0113886, -$
 $0.00200665\}$



(2) trigonometric functions (i.e. Fourier series approximation),

Using the orthonormal basis functions by the trigonometric functions :

$$\{\phi_i(x)\} = \left\{ \frac{1}{\sqrt{2}}, \cos(n\pi x), \sin(n\pi x) \right\}, i = 1, 2, \dots, 2N + 1, n = 1, 2, \dots, N$$

we can approximate the function f as

$$f(x) \approx f_n(x) = \sum_{i=0}^n c_i \phi_i(x) = \sum_{i=0}^n \left(\int_{-1}^1 f(x) \phi_i(x) dx \right) \phi_i(x).$$

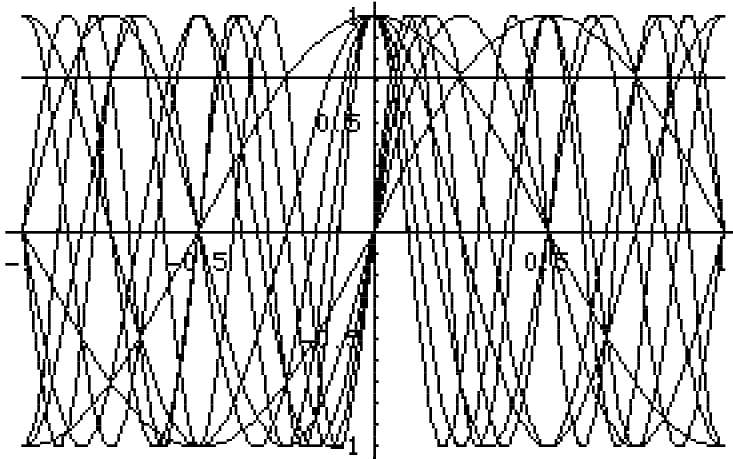
Using the following MATHEMATICA script program :

```
n=5;
LP1=Table[Cos[i*Pi*x], {i, 0, n}];
LP2=Table[Sin[i*Pi*x], {i, 1, n}];
LP=Flatten[{LP1, LP2}];
LP[[1]]=LP[[1]]/Sqrt[2];
LP
Plot[Release[LP], {x, -1, 1}]
f1=-x^2+1;
f2=1-x;
f=If[x<0, f1, f2];
Ck=Table[NIntegrate[f1*LP[[i]], {x, -1, 0}]+
          NIntegrate[f2*LP[[i]], {x, 0, 1}], {i, 1, 2*n+1}]
fn=Ck.LP;
Plot[{f, fn}, {x, -1, 1}]
```

$$\{\phi_i(x)\} =$$

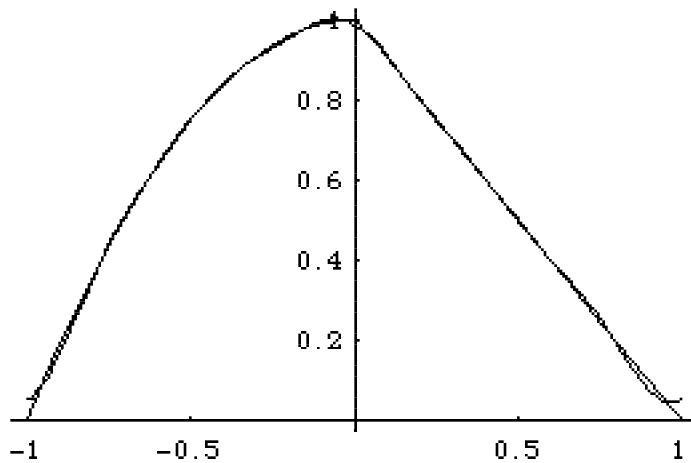
$$\left\{ \frac{1}{\sqrt{2}}, \cos[\pi x], \cos[2 \pi x], \cos[3 \pi x], \cos[4 \pi x], \right. \\ \left. \cos[5 \pi x], \sin[\pi x], \sin[2 \pi x], \sin[3 \pi x], \right.$$

$\{\sin[4 \text{ Pi } x], \sin[5 \text{ Pi } x]\}$



$\{c_i\} =$

$\{0.824958, 0.405285, -0.0506606, 0.0450316, -0.0126651,$
 $0.0162114, -0.129006, -5.55112 \cdot 10^{-17}, -0.00477801,$
 $5.55112 \cdot 10^{-17}, -0.00103205\}$



and

(3) the Lagrange interpolation. Plot the approximation and the original function, and also plot the amount of approximation error with respect to the number of terms used in approximation.

Noting that the Lagrange interpolation basis functions are defined by

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$

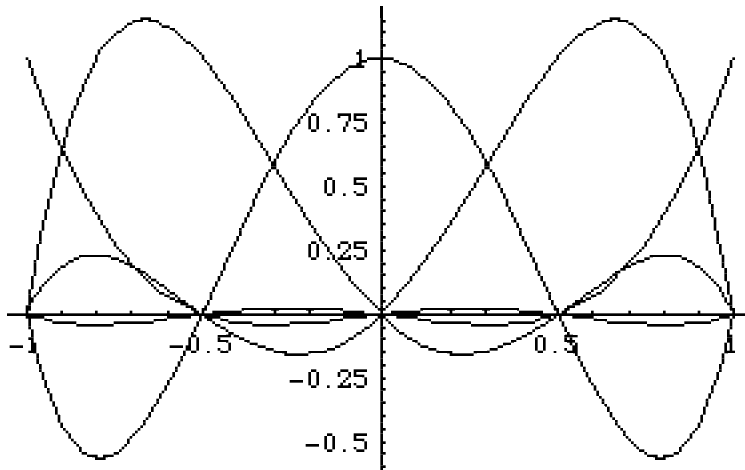
where

$$x_i = -1 + 2(i-1)/n, \quad i = 1, 2, \dots, n+1$$

we can develop a MATHEMATICA script program :

```
n=4;
xk=Table[-1+2*(i-1)/n,{i,1,n+1}]
LP=Table[1,{i,1,n+1}];
Block[{i,j},
  Do[If[i!=j,LP[[i]]=LP[[i]]*(x-xk[[j]])/(xk[[i]]-xk[[j]]),
    LP[[i]]=LP[[i]]*1],
    {j,1,n+1},{i,1,n+1}]
]
Plot[Release[LP],{x,-1,1},PlotRange->All]
f=If[x<0,-x^2+1,1-x];
fk=Table[f/.{x->xk[[i]]},{i,1,n+1}]
fn=fk.LP
Plot[{f,fn},{x,-1,1}]
```

The profile of the Lagrange interpolation basis functions for n=4 are given as follows :



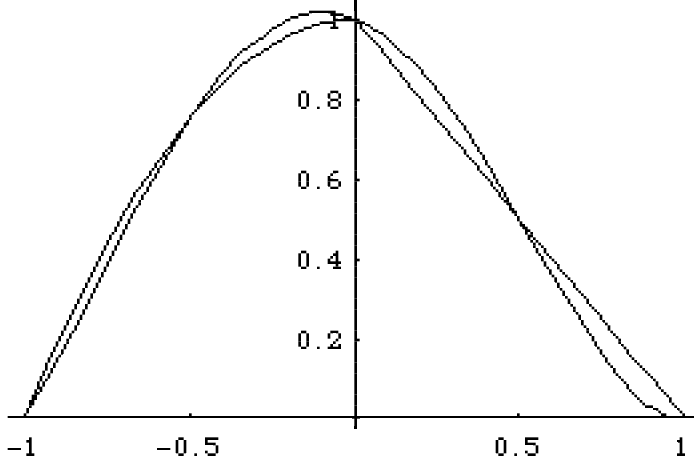
$$f(x) \approx f_n(x) = \sum_{i=1}^{n+1} f(x_i) L_i(x) =$$

$$-2 \frac{(-1+x) \left(-\frac{1}{2}+x\right) x (1+x)}{2} +$$

$$4 \frac{(-1+x) \left(-\frac{1}{2}+x\right) \left(\frac{1}{2}+x\right) (1+x)}{2} -$$

$$4 \frac{(-1+x) x \left(\frac{1}{2}+x\right) (1+x)}{2}$$

The following figure expresses the original function f and its Lagrange interpolation :



3. Consider the minimization of a functional

$$J(w) = \frac{1}{2} \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right)^2 + \cos\left(\frac{\pi}{2}x\right) w^2 \right\} dx - \int_0^1 \sin(\pi x) w(x) dx - w(1)$$

on the admissible space

$$K = \{w : w(0) = 0, w \text{ is piecewise continuously differentiable functions in } (0,1)\}.$$

(1) Find the first variation of the functional J , and obtain the necessary condition of the minimum of the functional J on K .

$$\begin{aligned} \delta J(w) &= \frac{1}{2} \delta \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right)^2 + \cos\left(\frac{\pi}{2}x\right) w^2 \right\} dx - \delta \int_0^1 \sin(\pi x) w(x) dx - \delta w(1) \\ &= \frac{1}{2} \int_0^1 \left\{ (1 + \exp(-x)) \delta \left(\frac{dw}{dx} \right)^2 + \cos\left(\frac{\pi}{2}x\right) \delta w^2 \right\} dx - \int_0^1 \sin(\pi x) \delta w(x) dx - \delta w(1) \\ &= \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right) \delta \left(\frac{dw}{dx} \right) + \cos\left(\frac{\pi}{2}x\right) w \delta w \right\} dx - \int_0^1 \sin(\pi x) \delta w(x) dx - \delta w(1) \\ &= \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right) \left(\frac{d\delta w}{dx} \right) + \cos\left(\frac{\pi}{2}x\right) w \delta w \right\} dx - \int_0^1 \sin(\pi x) \delta w(x) dx - \delta w(1) \geq 0 \\ &\forall \delta w \text{ s.t. } \delta w(0) = 0 \end{aligned}$$

that is

$$\begin{aligned} \delta J(w) &= \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right) \left(\frac{d\delta w}{dx} \right) + \cos\left(\frac{\pi}{2}x\right) w \delta w \right\} dx - \int_0^1 \sin(\pi x) \delta w(x) dx - \delta w(1) = 0 \\ &\forall \delta w \text{ s.t. } \delta w(0) = 0 \end{aligned}$$

(2) Find Euler's equation of the functional J , that is, find out the corresponding boundary value problem to the above minimization problem.

Applying the integration by parts to the necessary condition :

$$\begin{aligned}\delta J(w) &= \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw}{dx} \right) \left(\frac{d\delta w}{dx} \right) + \cos\left(\frac{\pi}{2}x\right) w \delta w \right\} dx - \int_0^1 \sin(\pi x) \delta w(x) dx - \delta w(1) \\ &= \left[(1 + \exp(-x)) \frac{dw}{dx} \delta w \right]_{x=0}^{x=1} + \int_0^1 \left\{ -\frac{d}{dx} \left[(1 + \exp(-x)) \left(\frac{dw}{dx} \right) \right] + \cos\left(\frac{\pi}{2}x\right) w - \sin(\pi x) \right\} \delta w dx - \delta w(1) \\ &= \left\{ (1 + \exp(-x)) \frac{dw}{dx} \Big|_{x=1} - 1 \right\} \delta w(1) + \int_0^1 \left\{ -\frac{d}{dx} \left[(1 + \exp(-x)) \left(\frac{dw}{dx} \right) \right] + \cos\left(\frac{\pi}{2}x\right) w - \sin(\pi x) \right\} \delta w dx = 0 \\ &\forall \delta w \text{ s.t. } \delta w(0) = 0\end{aligned}$$

we can find Euler's equation and the natural boundary condition :

$$\begin{aligned}-\frac{d}{dx} \left[(1 + \exp(-x)) \left(\frac{dw}{dx} \right) \right] + \cos\left(\frac{\pi}{2}x\right) w &= \sin(\pi x) \quad \text{in } (0,1) \\ (1 + \exp(-x)) \frac{dw}{dx} \Big|_{x=1} &= 1 \quad \text{at } x = 1\end{aligned}$$

while the constraint yields the essential boundary condition

$$w(0) = 0$$

(3) Solve the minimization problem by using the Ritz method. You may use polynomials or trigonometric functions as the basis functions.

In the Ritz method, we approximate the minimizer w and its arbitrary variation δw by

$$w \approx w_n = \sum_{i=1}^n c_i \phi_i(x) \quad \text{s.t. } \phi_i(0) = 0$$

and

$$\delta w \approx \delta w_n = \sum_{i=1}^n \delta c_i \phi_i(x) \quad \text{s.t. } \phi_i(0) = 0,$$

and we shall substitute these into the necessary condition (the first variation must be zero at the minimizer w) :

$$\begin{aligned}\delta J(w) &\approx \int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{dw_n}{dx} \right) \left(\frac{d\delta w_n}{dx} \right) + \cos\left(\frac{\pi}{2}x\right) w_n \delta w_n \right\} dx - \int_0^1 \sin(\pi x) \delta w_n(x) dx - \delta w_n(1) = 0 \\ &\forall \delta w_n \text{ s.t. } \delta w_n(0) = 0\end{aligned}$$

that is

$$\int_0^1 \left\{ (1 + \exp(-x)) \left(\frac{d}{dx} \sum_{j=1}^n c_j \phi_j(x) \right) \left(\frac{d}{dx} \sum_{i=1}^n \delta c_i \phi_i(x) \right) + \cos\left(\frac{\pi}{2}x\right) \sum_{j=1}^n c_j \phi_j(x) \delta \sum_{i=1}^n \delta c_i \phi_i(x) \right\} dx$$

$$- \int_0^1 \sin(\pi x) \delta \sum_{i=1}^n \delta c_i \phi_i(x) dx - \delta \sum_{i=1}^n \delta c_i \phi_i(1) = 0 \quad \forall \delta c_i, \quad i = 1, 2, \dots, n$$

Defining

$$k_{ij} = \int_0^1 \left\{ (1 + \exp(-x)) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + \cos\left(\frac{\pi}{2}x\right) \phi_i \phi_j \right\} dx$$

and

$$f_i = \int_0^1 \sin(\pi x) \phi_i dx + \phi_i(1)$$

we have

$$\sum_{i=1}^n \sum_{j=1}^n \delta c_i k_{ij} c_j - \sum_{i=1}^n \delta c_i f_i = 0, \quad \delta c_i, \quad i = 1, 2, \dots, n$$

that is

$$\sum_{i=1}^n \delta c_i \left(\sum_{j=1}^n k_{ij} c_j - f_i \right) = 0, \quad \delta c_i, \quad i = 1, 2, \dots, n.$$

Noting that

$$ax = 0, \quad \forall x \Leftrightarrow a = 0$$

we have

$$\sum_{j=1}^n k_{ij} c_j - f_i = 0, \quad i = 1, 2, \dots, n.$$

Solving this yields the unknown coefficients

$$c_j, \quad j = 1, 2, \dots, n.$$

Once the coefficients are obtained, substitution of these into

$$w \approx w_n = \sum_{i=1}^n c_i \phi_i(x)$$

we can find an approximation of the minimizer w .

Using MATHEMATICA script programs :

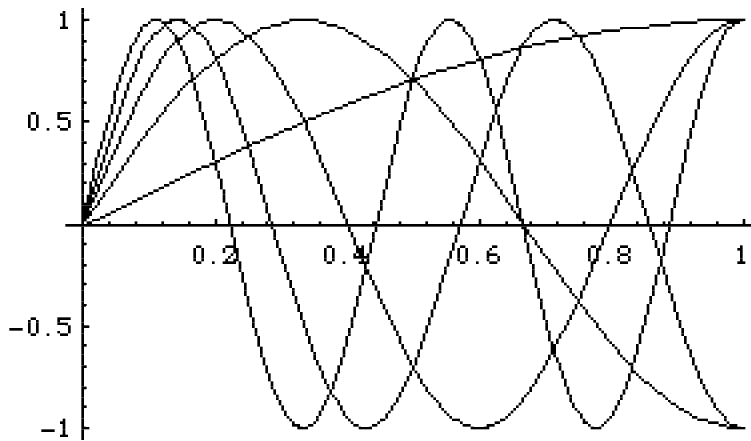
(Trigonometric Basis Functions : $\phi_i(x) = \sin\left((2i-1)\frac{\pi}{2}x\right), i = 1, 2, \dots, n$)

```

n=5;
LP=Table[Sin[(2*i-1)*Pi*x/2],{i,1,n}];
Plot[Release[LP],{x,0,1}]
DLP=D[LP,x];
fi=Table[0,{i,1,n}];
kij=Table[0,{i,1,n},{j,1,n}];
a=1+Exp[-x];
k0=Cos[Pi*x/2];
f=Sin[Pi*x];
Block[{i,j},
  Do[fi[[i]]=NIntegrate[f*LP[[i]],{x,0,1}]+LP[[i]]/.{x->1};
    Do[kij[[i,j]]=NIntegrate[a*DLP[[i]]*DLP[[j]]+
      k0*LP[[i]]*LP[[j]],{x,0,1}];
      kij[[j,i]]=kij[[i,j]],{j,i,n}],{i,1,n}]]
cj=Inverse[kij].fi
wn=cj.LP;
Plot[wn,{x,0,1}]

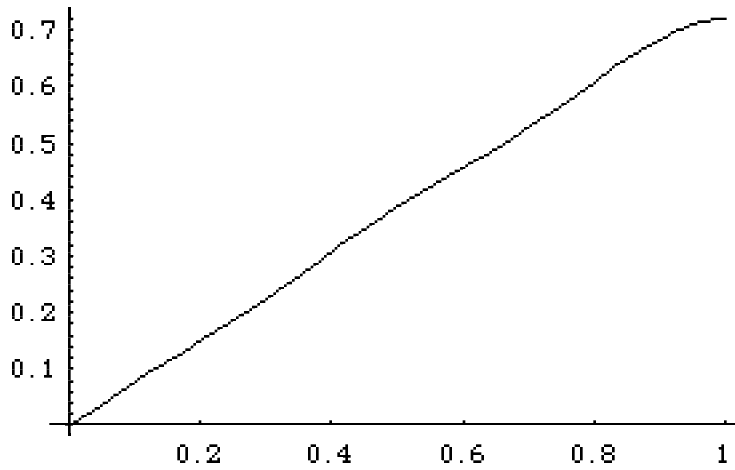
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Profile of the Basis Functions



$\{c_j\} =$

$\{0.615077, -0.0642824, 0.0209713, -0.0124196, 0.00650177\}$



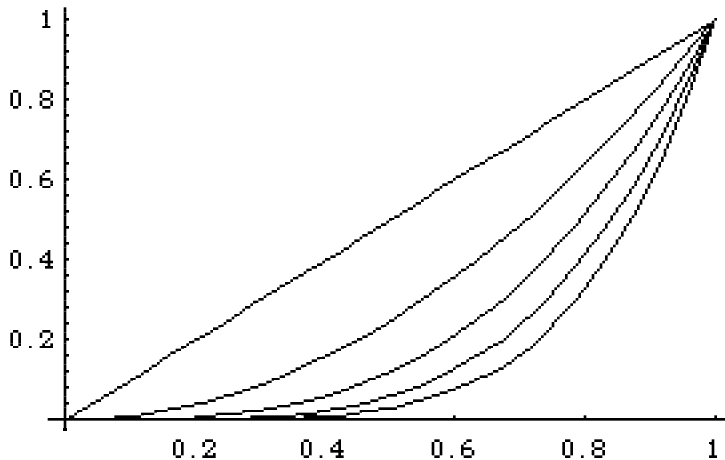
(Polynomial Basis Functions : $\phi_i(x) = x^i, i=1,2,\dots,n$)

```

n=5;
LP=Table[x^i,{i,1,n}];
Plot[Release[LP],{x,0,1}]
DLP=D[LP,x];
fi=Table[0,{i,1,n}];
kij=Table[0,{i,1,n},{j,1,n}];
a=1+Exp[-x];
k0=Cos[Pi*x/2];
f=Sin[Pi*x];
Block[{i,j},
  Do[fi[[i]]=NIntegrate[f*LP[[i]],{x,0,1}]+LP[[i]].{x->1};
    Do[kij[[i,j]]=NIntegrate[a*DLP[[i]]*DLP[[j]]+
      k0*LP[[i]]*LP[[j]],{x,0,1}];
      kij[[j,i]]=kij[[i,j]],{j,i,n}],{i,1,n}]]
cj=Inverse[kij].fi
wn=cj.LP;
Plot[wn,{x,0,1}]

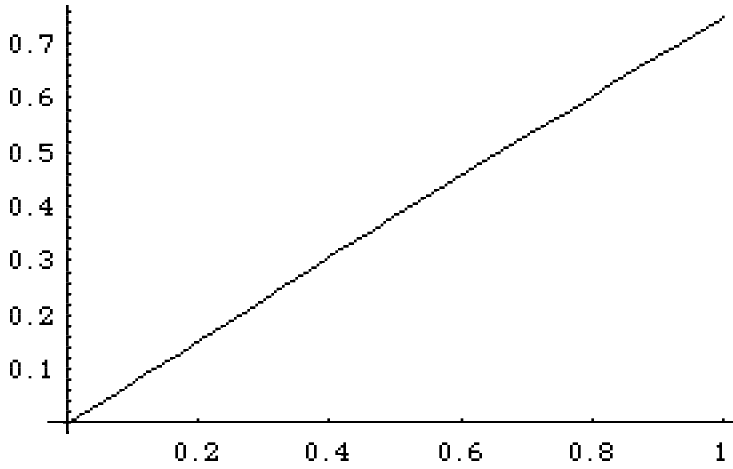
```

Profile of the Basis Functions



$\{c_j\} =$

(.....)



(4) Solve the minimization problem by the finite difference or finite element method.

Using Euler's equation

$$-\frac{d}{dx} \left[(1 + \exp(-x)) \left(\frac{dw}{dx} \right) \right] + \cos\left(\frac{\pi}{2}x\right)w = \sin(\pi x)$$

we apply the central difference approximation

$$\begin{aligned} \frac{d}{dx} \left[(1 + \exp(-x)) \left(\frac{dw}{dx} \right) \right] &\approx \frac{1}{\Delta x} \left((1 + \exp(-x)) \left(\frac{dw}{dx} \right) \Big|_{i+\frac{1}{2}} - (1 + \exp(-x)) \left(\frac{dw}{dx} \right) \Big|_{i-\frac{1}{2}} \right) \\ &\approx \frac{1}{\Delta x} \left(\left(1 + \exp\left(-x_{i+\frac{1}{2}}\right) \right) \frac{1}{\Delta x} (w_{i+1} - w_i) - \left(1 + \exp\left(-x_{i-\frac{1}{2}}\right) \right) \frac{1}{\Delta x} (w_i - w_{i-1}) \right) \\ &= \frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right)}{\Delta x^2} w_{i+1} - \frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right) + 1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2} w_i + \frac{1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2} w_{i-1} \end{aligned}$$

we have

$$-\frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right)}{\Delta x^2} w_{i+1} + \frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right) + 1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2} w_i - \frac{1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2} w_{i-1} + \cos\left(\frac{\pi}{2}x_i\right)w_i = \sin(\pi x_i)$$

that is

$$-c_{i+1}w_{i+1} + c_i w_i - c_{i-1}w_{i-1} = f_i$$

where

$$c_{i+1} = \frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right)}{\Delta x^2}$$

$$c_i = \frac{1 + \exp\left(-x_{i+\frac{1}{2}}\right) + 1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2} + \cos\left(\frac{\pi}{2}x_i\right)$$

$$c_{i-1} = \frac{1 + \exp\left(-x_{i-\frac{1}{2}}\right)}{\Delta x^2}$$

and

$$f_i = \sin(\pi x_i).$$

Using the backward difference approximation

$$(1 + \exp(-x)) \frac{dw}{dx} \Big|_{x=1} \approx \left(1 + \exp\left(-x_{n-\frac{1}{2}}\right)\right) \frac{w_n - w_{n-1}}{\Delta x}$$

to the natural boundary condition at the end point $x_n = 1$, we have

$$b(w_n - w_{n-1}) = 1 \Leftrightarrow w_n = w_{n-1} + \frac{1}{b}$$

where

$$b = \frac{1 + \exp\left(-x_{n-\frac{1}{2}}\right)}{\Delta x}$$

If the left end point is assumed to be $x_0 = 0$, then the constraint $w(0) = 0$ yields

$$w_0 = 0.$$

Therefore, for $i = 1, 2, \dots, n-1$, we have

$$-c_{i+1}w_{i+1} + c_i w_i - c_{i-1}w_{i-1} = f_i$$

and

$$w_n = w_{n-1} + \frac{1}{b} \quad \text{with} \quad w_0 = 0,$$

that is

$$-c_2 w_2 + c_1 w_1 = f_1,$$

$$-c_{i+1} w_{i+1} + c_i w_i - c_{j-1} w_{j-1} = f_i, \quad i = 2, \dots, n-2$$

$$(c_{n-1} - c_n) w_{n-1} - c_{n-2} w_{n-2} = f_{n-1} + \frac{c_n}{b}$$

and

$$w_n = w_{n-1} + \frac{1}{b}.$$

```
% HW#5_3_3 : Finite Difference Approximation
n=10;
dx=1/n;
A=zeros(n-1);
f=zeros(n-1,1);
for i=2:n-2
    xim=(i-0.5)*dx;
    xip=(i+0.5)*dx;
    xi=i*dx;
    A(i,i-1)=-(1+exp(-xim))/dx^2;
    A(i,i+1)=-(1+exp(-xip))/dx^2;
    A(i,i)=(1+exp(-xim)+1+exp(-xip))/dx^2 + cos(pi*xi/2);
    f(i)=sin(pi*xi);
end
% at the node n = 1
xim=0.5*dx;
xip=1.5*dx;
xi=dx;
A(1,1)=(1+exp(-xim)+1+exp(-xip))/dx^2 + cos(pi*xi/2);
A(1,2)=-(1+exp(-xip))/dx^2;
f(1)=sin(pi*xi);
% at the node n-1
xim=(n-1.5)*dx;
xip=(n-0.5)*dx;
xi=(n-1)*dx;
b=(1+exp(-xip))/dx;
A(n-1,n-1)=(1+exp(-xim)+1+exp(-xip))/dx^2 + cos(pi*xi/2);
A(n-1,n-1)=A(n-1,n-1)-(1+exp(-xip))/dx^2;
A(n-1,n-2)=-(1+exp(-xim))/dx^2;
f(n-1)=sin(pi*xi)+(1+exp(-xip))/(dx^2*b);
%
A
f
c=A\f
c(n)=c(n-1)+1/b;
w(1)=0;
w(2:n+1)=c;
w;
x=0:1/n:1;
plot(x,w)
xlabel('x')
```

title('Solution Profile')

