**Solutions of Review Problems for the Final Examination**
**MEAM 501 Analytical Methods in Mechanical Engineering**

1997 Fall

*1. State the following definitions, properties, and/or concept :*

*(1) What is the Lagrange interpolation of a function f defined on an interval (a,b).*

Within a given interval (a,b), we place n+1 points, say, $x_1, x_2, \ldots\ldots, x_{n+1}$, and using the values $f_i = f(x_i)$ of the given function $f(x)$ at these points, we approximate the function $f$ by a $n$ degree polynomial $f_n$ :

$$f_n(x) = \sum_{i=1}^{n+1} f_i L_i(x) \quad , \quad L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$

*(2) What is the Legendre polynomilas defined on the interval (-1,1) ?*

Legendre polynomials $P_n(x)$, $n = 0, 1, 2, \ldots\ldots$, are obtained by the orthogonalization of the polynomial basis functions $\{1, x, x^2, \ldots\ldots\}$ with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x) g(x) dx$$

together with an appropriate normalization. Using the generalized Rodriguez formula, they can be written by

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[ \left(1 - x^2\right)^n \right]$$

and they satisfy the differential equation

$$-\frac{d}{dx} \left[ \left(1 - x^2\right) \frac{dP_n}{dx} \right] = n(n+1) P_n$$

That is, the Legendre polynomials are the eigenfuctions of the differential operator

$$w \to -\frac{d}{dx} \left[ \left(1 - x^2\right) \frac{dw}{dx} \right]$$

associated with the eigenvalues $n(n+1)$.

Legendre Polynomials

```
In[1]:=
LP=Table[(-1)^n/(2^n*n!)D[(1-x^2)^n,{x,n}],{n,0,10}];
LP=Simplify[LP]
Out[2]=
              2             2           2       4
      -1 + 3 x   x (-3 + 5 x )  3 - 30 x  + 35 x
{1, x, ---------, -------------, -----------------,
          2            2                 8
```

$$\frac{x\,(15 - 70 x^2 + 63 x^4)}{8}, \quad \frac{-5 + 105 x^2 - 315 x^4 + 231 x^6}{16},$$

$$\frac{x\,(-35 + 315 x^2 - 693 x^4 + 429 x^6)}{16},$$

$$\frac{35 - 1260 x^2 + 6930 x^4 - 12012 x^6 + 6435 x^8}{128},$$

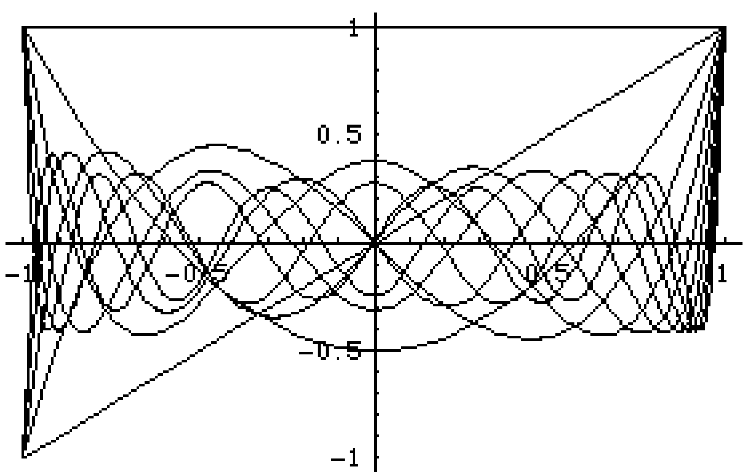
$$\frac{x\,(315 - 4620 x^2 + 18018 x^4 - 25740 x^6 + 12155 x^8)}{128},$$

$$(-63 + 3465 x^2 - 30030 x^4 + 90090 x^6 - 109395 x^8 +$$

$$46189 x^{10}) / 256\}$$

In[3]:=

Plot[Release[LP],{x,-1,1}]




Out[3]=

-Graphics-

In[4]:=

LegendreP[2,x]

Out[4]=

$$\frac{-1 + 3 x^2}{2}$$

*(3) What is the Chebyshev polynomilas defined on the interval (-1,1) ?*

Legendre polynomials $P_n(x)$, $n = 0,1,2,.....$, are obtained by the orthogonalization of the polynomial basis functions $\{1, x, x^2, ......\}$ with respect to the inner product

$$(f,g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)g(x)dx$$

together with an appropriate normalization. Using the generalized Rodriguez formula, they can be written by

$$P_n(x) = \frac{(-1)^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n-\frac{1}{2}}\right]$$

and they satisfy the differential equation

$$\left(1-x^2\right)\frac{d^2P_n}{dx^2} - x\frac{dP_n}{dx} + n^2 P_n = 0.$$

*(4) What is the Hermite polynomilas defined on the interval* $(-\infty,\infty)$ *?*

Legendre polynomials $P_n(x)$, $n = 0,1,2,.....$, are obtained by the orthogonalization of the polynomial basis functions $\{1, x, x^2, ......\}$ with respect to the inner product

$$(f,g) = \int_{-\infty}^{\infty} \exp\left(-x^2\right) f(x)g(x)dx$$

together with an appropriate normalization. Using the generalized Rodriguez formula, they can be written by

$$P_n(x) = (-1)^n \exp\left(x^2\right)\frac{d^n}{dx^n}\left[\exp\left(-x^2\right)\right]$$

and they satisfy the differential equation

$$\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + 2nP_n = 0.$$

*(5) What is the Lagerre polynomilas defined on the interval* $(0,\infty)$ *?*

Legendre polynomials $P_n(x)$, $n = 0,1,2,.....$, are obtained by the orthogonalization of the polynomial basis functions $\{1, x, x^2, ......\}$ with respect to the inner product

$$(f,g) = \int_{0}^{\infty} \exp(-x) f(x)g(x)dx$$

together with an appropriate normalization. Using the generalized Rodriguez formula, they can be written by

$$P_n(x) = \frac{1}{n!}\exp(x)\frac{d^n}{dx^n}\left[x^n \exp(-x)\right]$$

and they satisfy the differential equation

$$x\frac{d^2 P_n}{dx^2} + (1-x)\frac{dP_n}{dx} + nP_n = 0$$

*(6) Obtain the 2 point Gauss-Legendre quadrature to integrate a function defined on the interval (-1,1), that is, obtain the quadrature points and the associate weights.*

The quadrature points are identified with the roots of the second degree Legendre polynomial :

$$P_2(x) = \frac{(-1)^2}{2^2 2!}\frac{d^2}{dx^2}\left[\left(1-x^2\right)^2\right] = \frac{1}{2}\left(-1+3x^2\right)$$

that is

$$P_2(x) = \frac{1}{2}\left(-1+3x^2\right) = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$$

i.e.

$$x_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad x_2 = +\frac{1}{\sqrt{3}}.$$

Noting that the quadrature is defined by

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{2} w_i f\left(x_i\right) = w_1 f\left(x_1\right) + w_2 f\left(x_2\right)$$

and noting that this quadrature can integrate the polynomial 1 and $x$ exactly, we have the following relation :

$$\int_{-1}^{1} 1dx = w_1 + w_2 = 2$$

$$\int_{-1}^{1} xdx = w_1\left(-\frac{1}{\sqrt{3}}\right) + w_2\left(+\frac{1}{\sqrt{3}}\right) = 0 \quad \Leftrightarrow \quad w_1 = w_2$$

Therefore, the weights are obtained as

$$w_1 = w_2 = 1.$$

*(7) Obtain the 3 point Gauss-Lagerre quadrature to integrate a function defined on the interval* $(0,\infty)$*, that is, find the 3 point quadrature points and associated weights.*

We extablish the 3 point Gauss-Lagerre quadrature to integrate

$$\int_0^{+\infty} \exp(-x) f(x) dx \approx w_1 \exp(-x_1) f(x_1) + w_2 \exp(-x_2) f(x_2) + w_3 \exp(-x_3) f(x_3)$$

by using the roots of the third degree Lagerre polynomial

$$P_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

as the quadrature points $x_1, x_2$, and $x_3$, that is

```
In[11]:=
NSolve[LaguerreL[3,0,x]==0,x]
Out[11]=
{{x -> 0.415775}, {x -> 2.29428}, {x -> 6.28995}}
```

This means that the three quadrature points are given by

$$x_1 = 0.415775, \quad x_2 = 2.29428, \quad \text{and} \quad x_3 = 6.28995.$$

Then the weights would be determined so as to

$$\int_0^{+\infty} \exp(-x) 1 dx = w_1 \exp(-x_1) + w_2 \exp(-x_2) + w_3 \exp(-x_3)$$

$$\int_0^{+\infty} \exp(-x) x dx = w_1 \exp(-x_1) x_1 + w_2 \exp(-x_2) x_2 + w_3 \exp(-x_3) x_3$$

$$\int_0^{+\infty} \exp(-x) x^2 dx = w_1 \exp(-x_1) x_1^2 + w_2 \exp(-x_2) x_2^2 + w_3 \exp(-x_3) x_3^2$$

Using MATHEMATICA

```
In[12]:=
x1=0.415775;
x2=2.29428;
x3=6.28995;
P1=Integrate[Exp[-x],{x,0,Infinity}];
P2=Integrate[Exp[-x] x,{x,0,Infinity}];
P3=Integrate[Exp[-x] x^2,{x,0,Infinity}];
NSolve[{P1==w1*Exp[-x1]+w2*Exp[-x2]+w3*Exp[-x3],
     P2==w1*Exp[-x1]*x1+w2*Exp[-x2]*x2+w3*Exp[-x3]*x3,
     P3==w1*Exp[-x1]*x1^2+w2*Exp[-x2]*x2^2+w3*Exp[-x3]*x3^2},
     {w1,w2,w3}]
Out[18]=
{{w1 -> 1.07769, w2 -> 2.76214, w3 -> 5.60113}}
```

we can obtain the weights :

$$w_1 = 1.07769, \quad w_2 = 2.76214, \quad \text{and} \quad w_3 = 5.60113.$$

Therefore

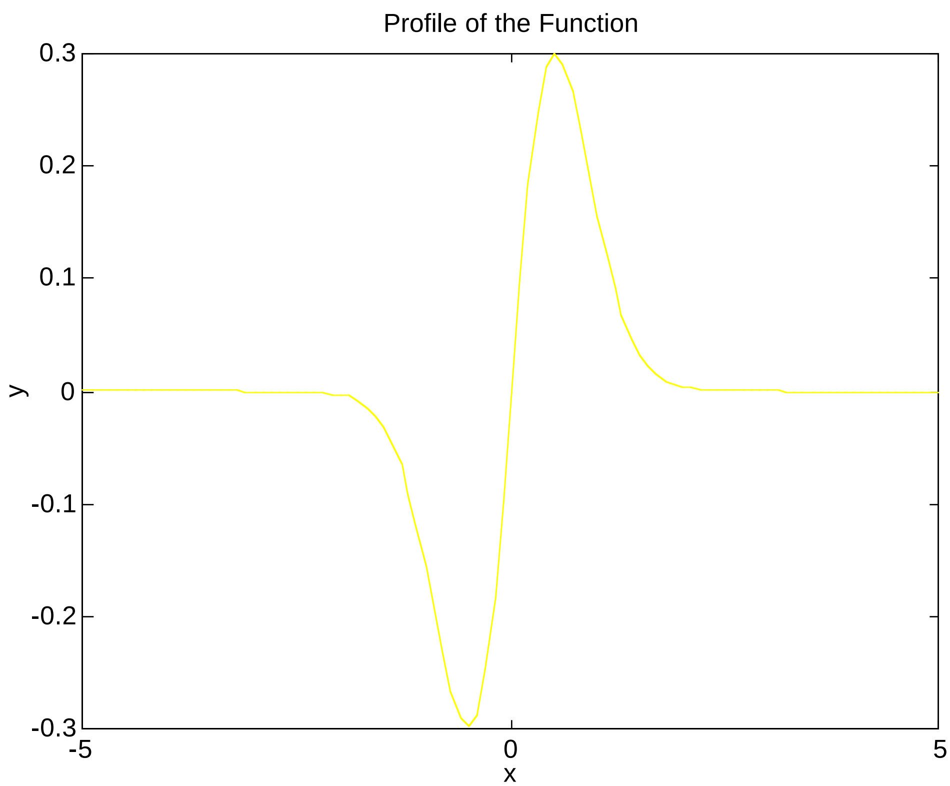
$$x_1 = 0.415775, \quad x_2 = 2.29428, \quad \text{and} \quad x_3 = 6.28995$$
$$w_1 = 1.07769, \quad w_2 = 2.76214, \quad \text{and} \quad w_3 = 5.60113$$

*(8) Integrate by using a numerical method to integrate*

$$\int_{-\infty}^{+\infty} \frac{\exp(-x^2)\sin x}{1+x^2}\,dx$$

*with the accuracy of* $10^{-6}$.

```
n=100;
xmin=-5;
xmax=5;
h=(xmax-xmin)/n;
x=xmin:h:xmax;
f=(exp(-x.^2).*sin(x))./(1+x.^2);
plot(x,f)
title('Profile of the Function')
xlabel('x')
ylabel('y')
I=h*(sum(f)-(f(1)+f(length(f)))/2)
```



Since the given function is anti-symmetric at $x$=0, the value of interal must be zero. Applying the extended trapezoid rule, we have

I =

4.4563e-17

*(9) What is the cubic Hermite interpolation of a function f defined on the interval (0,1) ?*

The cubic Hermite interpolation is defined by a cubic polynomial :

$$f_h(t) = \begin{Bmatrix} 1 & t & t^2 & t^3 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix},$$

where $a_i$ are coefficient of the basis $\{1, t, t^2, t^3\}$ of cubic polynomials. Let the degrees of freedom be given by $\left\{ f^{(1)}, \left(\frac{df}{dt}\right)^{(1)}, f^{(2)}, \left(\frac{df}{dt}\right)^{(2)} \right\}$, where $f^{(i)}$ is the value of at node $i$. Then the coefficient vector a is related to the degrees of freedom as follows :

$$\begin{Bmatrix} f^{(1)} \\ \left(\frac{df}{dt}\right)^{(1)} \\ f^{(2)} \\ \left(\frac{df}{dt}\right)^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad i.e. \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}^{-1} \begin{Bmatrix} f^{(1)} \\ \left(\frac{df}{dt}\right)^{(1)} \\ f^{(2)} \\ \left(\frac{df}{dt}\right)^{(2)} \end{Bmatrix}.$$

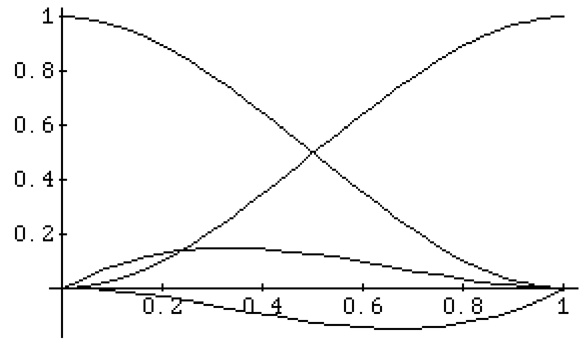
Therefore, the basis ( i.e. blending ) functions { $H_i(t)$ }of the cubic Hermite interpolation are obtained as

$$f_h(t) = \begin{Bmatrix} 1 & t & t^2 & t^3 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$$= \underbrace{\begin{Bmatrix} 1 & t & t^2 & t^3 \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}^{-1}}_{=\{H_1(t) \quad H_2(t) \quad H_3(t) \quad H_4(t)\}} \begin{Bmatrix} f^{(1)} \\ \left(\frac{df}{dt}\right)^{(1)} \\ f^{(2)} \\ \left(\frac{df}{dt}\right)^{(2)} \end{Bmatrix},$$

where

$$H_1(t) = 1 - 3t^2 + 2t^3$$
$$H_2(t) = t(1-t)^2$$
$$H_3(t) = 3t^2 - 2t^3$$
$$H_4(t) = t^2(-1+t)$$




*(10)   What is the Bezier spline approximation of a curve ?*

The Bezier curve characterized by $n$+1 control points of a characteristic polygon is defined by

$$\boldsymbol{r}(t) = \sum_{i=1}^{n+1} \boldsymbol{r}^i B_i^n(t) \qquad with \qquad B_i^n(t) = \frac{n!}{(i-1)!(n-i+1)!} t^{i-1}(1-t)^{n-i+1}$$

where $\{\boldsymbol{r}^i\}$ are the coordinates of the control points and the parameter $t \in [\,0, 1\,]$.

*(11)   What is the B-spline approximation of a curve ?*

The B-spline is introduced to confine the effect of change of the location of a control point in the localized neighborhood, while other curves defined by e.g. Bezier splines are globally affected by a local change of control points.   To do this, a curve is generated by

$$\boldsymbol{r}(t) = \sum_{i=1}^{n+1} \boldsymbol{r}^i N_i^k(t)$$

where $N_i^k(t)$ are the blending functions for the B-spline curve defined by the following recursive formula :

$$N_i^1(t) = \begin{cases} 1 & if \quad t_i \le t < t_{i+1} \\ 0 & otherwise \end{cases}$$

$$N_i^k(t) = \frac{(t - t_i)N_i^{k-1}(t)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - t)N_{i+1}^{k-1}(t)}{t_{i+k} - t_{i+1}} \qquad k = 1, .....$$

$$t \in [0, n - k + 2]$$

$$i = 1, ....., n + k + 1$$

$$t_i = \begin{cases} 0 & if \;\; i - 1 < k \\ i - k & if \;\; k \le i - 1 \le n + 1 \\ n - k + 2 & if \;\; n + 1 < i - 1 \end{cases}$$

Here we have used the convention $\frac{0}{0} = 1$.

*(12) What is the minimum principle ?*

Equilibrium is attained at the minimum point of the corresponding quadratic functional. For example, the equilirium represented by a system of linear equations ( or matrix equation ) :

$$\boldsymbol{Ax} = \boldsymbol{b}$$

for a symmetric matrix $\boldsymbol{A}$, we can define the minimization problem equivalent to this equilirium :

$$\min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Ax} - \boldsymbol{x}^T\boldsymbol{b}$$

If the matrix $\boldsymbol{A}$ is nonnegative in the sense that

$$\boldsymbol{x}^T\boldsymbol{Ax} \ge 0 \quad , \quad \forall \boldsymbol{x},$$

both are equivalent. That is, the solution of the system of linear equation is the minimizer of the functional, and also the minimizer of the functional is a solution of the system of linear equations.

*(13) What is the trapezoidal rule ?*

Integration of a function $f$ over an interval $(a,b)$ is approximated by the area of a quatrilateral formed by $a$, $b$, $f(b)$, and $f(a)$ :

$$I = \int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$

Integration error is approximately evaluated as

$$E \approx -\frac{1}{12}(b-a)^3 f''(\xi) \quad , \quad \xi \in (a,b).$$

*(14) What is the Simpson rule ?*

$$I = \int_a^b f(x)dx \approx \frac{b-a}{6}\left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

and the error is approximated by

$$E \approx -\frac{1}{2^5 90}(b-a)^5 f''''(\xi) \quad , \quad \xi \in (a,b)$$

Idea of the Simpson rule is that a given function f is approximated by a quadratic polynomial in the interval (a,b) by using the values $f(a)$, $f\left(\frac{a+b}{2}\right)$, and $f(b)$. More precisely,

$$f(x) \approx f(a)\frac{1}{2}s(s-1) + f\left(\frac{a+b}{2}\right)\left(1-s^2\right) + f(b)\frac{1}{2}s(s+1)$$

with

$$x = \frac{a+b}{2} + \frac{b-a}{2}s$$

and then

$$I = \int_a^b f(x)dx \approx \int_a^b \left\{ f(a)\frac{1}{2}s(s-1) + f\left(\frac{a+b}{2}\right)\left(1-s^2\right) + f(b)\frac{1}{2}s(s+1)\right\}dx$$

$$= \int_{-1}^1 \left\{ f(a)\frac{1}{2}s(s-1) + f\left(\frac{a+b}{2}\right)\left(1-s^2\right) + f(b)\frac{1}{2}s(s+1)\right\}\frac{b-a}{2}ds$$

$$= \frac{b-a}{6}\left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

(15) What is the exponential transformation for quadrature ?

*2. Obtain the first variation of the following functionals, the necessary conditions, and Euler's equations on the admissible set K :*

*(1)* $$J(v) = \frac{1}{2}\int_0^1 \left\{ (v')^2 + xv^2 \right\}dx - \int_0^1 fv dx$$

$$K = \left\{ v \in V : v(0) = v(1) = 0 \right\}$$

$$V = \left\{ v : \text{piecewise continuously differentiable functions on } (0,1) \right\}$$

$$
\begin{aligned}
\delta J(v) &= \frac{1}{2}\delta\int_0^1\left\{(v')^2 + xv^2\right\}dx - \delta\int_0^1 fv\,dx \\
&= \frac{1}{2}\int_0^1\left\{\delta(v')^2 + x\delta v^2\right\}dx - \int_0^1 f\delta v\,dx \\
&= \frac{1}{2}\int_0^1\{2v'\,\delta v' + x2v\delta v\}dx - \int_0^1 f\delta v\,dx \\
&= \int_0^1\{v'\,\delta v' + xv\delta v\}dx - \int_0^1 f\delta v\,dx \\
&= [v'\,\delta v]_{x=0}^{x=1} + \int_0^1\{-v''\delta v + xv\delta v\}dx - \int_0^1 f\delta v\,dx \\
&= [v'\,\delta v]_{x=0}^{x=1} + \int_0^1(-v'' + xv - f)\delta v\,dx = 0 \quad , \quad \forall\delta v \quad s.t. \quad \delta v(0) = \delta v(1) = 0
\end{aligned}
$$

that is

$$\int_0^1(-v'' + xv - f)\delta v\,dx = 0 \quad , \quad \forall\delta v \quad s.t. \quad \delta v(0) = \delta v(1) = 0$$

From this we can derive Euler's equation

$$-v'' + xv - f = 0 \quad \text{i.e.} \quad -v'' + xv = f \quad \text{in} \quad (0,1)$$

while we do not have the natural boundary condition, since we have constraints at the both end points.

*(2)* $$J(v) = \frac{1}{2}\int_0^1\left\{(v')^2 + xv^2\right\}dx + \frac{1}{2}k_0 v(0)^2 - \int_0^1 fv\,dx - Pv(0)$$

$$K = V$$

$$
\begin{aligned}
\delta J(v) &= \frac{1}{2}\delta\int_0^1\left\{(v')^2 + xv^2\right\}dx + \frac{1}{2}k_0\delta v(0)^2 - \delta\int_0^1 fv\,dx - P\delta v(0) \\
&= \frac{1}{2}\int_0^1\left\{\delta(v')^2 + x\delta v^2\right\}dx + \frac{1}{2}k_0\delta v(0)^2 - \int_0^1 f\delta v\,dx - P\delta v(0) \\
&= \int_0^1\{v'\,\delta v' + xv\delta v\}dx + k_0 v(0)\delta v(0) - \int_0^1 f\delta v\,dx - P\delta v(0) \\
&= [v'\,\delta v]_{x=0}^{x=1} + \int_0^1\{-v''\delta v + xv\delta v\}dx + k_0 v(0)\delta v(0) - \int_0^1 f\delta v\,dx - P\delta v(0) \\
&= v'(1)\delta v(1) - v'(0)\delta v(0) + \int_0^1\{-v''\delta v + xv\delta v\}dx + k_0 v(0)\delta v(0) - \int_0^1 f\delta v\,dx - P\delta v(0) \\
&= v'(1)\delta v(1) + [-v'(0) + k_0 v(0) - P]\delta v(0) + \int_0^1\{-v'' + xv - f\}\delta v\,dx = 0 \quad , \quad \forall\delta v
\end{aligned}
$$

Thus, we have the following Euler equation and two natural boundary conditions at the both end points :

$$-v''+xv = f \quad \text{in} \quad (0,1)$$
$$v'(1) = 0$$
$$v'(0) = k_0 v(0) - P$$

(3) $$J(v) = \frac{1}{2}\int_0^1 \left\{ \left(EI(x)v''\right)^2 - P(v')^2 + k(x)v^2 \right\} dx + \frac{1}{2}k_0 v(0)^2 + \frac{1}{2}k_1 v'(0)^2$$
$$-\int_0^1 fv dx - Fv(0) - Tv'(0)$$

$$K = V$$

$V = \{v$ : piecewise *twice* continuously differentiable functions on $(0,1)\}$

$$\delta J(v) = \int_0^1 \left\{ \left(EI(x)v''\right)\delta v'' - P(v')\delta v' + k(x)v\delta v \right\} dx + k_0 v(0)\delta v(0) + k_1 v'(0)\delta v'(0)$$
$$-\int_0^1 f\delta v dx - F\delta v(0) - T\delta v'(0)$$
$$= \left[\left(EI(x)v''\right)\delta v'\right]_{x=0}^{x=1} + \left[-\left(EI(x)v''\right)'\delta v\right]_{x=0}^{x=1} - \left[P(v')\delta v\right]_{x=0}^{x=1}$$
$$+\int_0^1 \left\{ \left(EI(x)v''\right)'' + (Pv') + kv - f \right\} \delta v dx$$
$$+k_0 v(0)\delta v(0) + k_1 v'(0)\delta v'(0) - F\delta v(0) - T\delta v'(0)$$
$$= \left(EI(1)v''(1)\right)\delta v'(1) + \left\{ -\left(EI(0)v''(0)\right) + k_1 v'(0) - T \right\} \delta v'(0)$$
$$-\left\{ (EIv'')'(1) + Pv'(1) \right\} \delta v(1) + \left\{ (EIv'')'(0) + Pv'(0) + k_0 v(0) - F \right\} \delta v(0)$$
$$+\int_0^1 \left\{ \left(EI(x)v''\right)'' + (Pv') + kv - f \right\} \delta v dx = 0 \quad , \quad \forall \delta v$$

Thus, we can get the following Euler equation and the natural boundary conditions :

$$\left(EI(x)v''\right)'' + (Pv') + kv = f \quad \text{in} \quad (0,1)$$

$EI(1)v''(1) = 0$ : moment is zero at $x = 1$

$(EIv'')'(1) + Pv'(1) = 0$ : force balance at $x = 1$

$EI(0)v''(0) = +k_1 v'(0) - T$ : moment balance at $x = 0$

$$(EIv'')'(0) + Pv'(0) = -k_0 v(0) + F \quad : \quad \text{force balance at } x = 0$$

*3 Solve the minimization problem by the Ritz method :*

$$\min_{\substack{v \\ \text{such that} \\ v(0)=0}} \quad J(v)$$

*where*

$$J(v) = \frac{1}{2}\int_0^1 \left\{ (v')^2 + xv^2 \right\} dx - \int_0^1 2v dx \cdot$$

In the Ritz method, we approximate the minimizer $w$ and its arbitrary variation $\delta w$ by

$$w \approx w_n = \sum_{i=1}^{n} c_i \phi_i(x) \quad \text{s.t.} \quad \phi_i(0) = 0$$

and

$$\delta w \approx \delta w_n = \sum_{i=1}^{n} \delta c_i \phi_i(x) \quad \text{s.t.} \quad \phi_i(0) = 0,$$

and we shall substitute these into the necessary condition ( the first variation must be zero at the minimizer $w$ ) :

$$\delta J(w) \approx \int_0^1 \left\{ \left( \frac{dw_n}{dx} \right) \left( \frac{d\delta w_n}{dx} \right) + x w_n \delta w_n \right\} dx - \int_0^1 2\delta w_n(x) dx = 0 \quad , \quad \forall \delta w_n \ \text{ s.t. } \ \delta w_n(0) = 0$$

that is

$$\int_0^1 \left[ \left( \frac{d}{dx} \sum_{j=1}^{n} c_j \phi_j(x) \right) \left( \frac{d}{dx} \sum_{i=1}^{n} \delta c_i \phi_i(x) \right) + x \sum_{j=1}^{n} c_j \phi_j(x) \delta \sum_{i=1}^{n} \delta c_i \phi_i(x) \right] dx$$

$$- \int_0^1 2\delta \sum_{i=1}^{n} \delta c_i \phi_i(x) dx = 0 \quad \forall \delta c_i \quad , \quad i = 1, 2, \ldots., n$$

Defining

$$k_{ij} = \int_0^1 \left\{ \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + x \phi_i \phi_j \right\} dx$$

and

$$f_i = \int^1 2\phi_i dx$$

we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\delta c_i k_{ij} c_j - \sum_{i=1}^{n}\delta c_i f_i = 0 \quad , \quad \delta c_i \; , i = 1,2,\ldots,n$$

that is

$$\sum_{i=1}^{n}\delta c_i\left(\sum_{j=1}^{n} k_{ij} c_j - f_i\right) = 0 \quad , \quad \delta c_i \; , i = 1,2,\ldots,n .$$

Noting that

$$ax = 0 \quad , \quad \forall x \quad \Leftrightarrow \quad a = 0$$

we have

$$\sum_{j=1}^{n} k_{ij} c_j - f_i = 0 \quad , \quad i = 1,2,\ldots,n .$$

Soving this yields the unknown coefficients

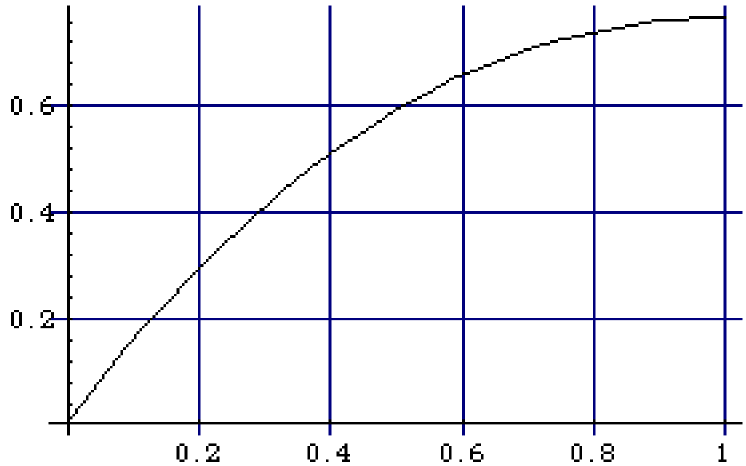
$$c_j \; , i = 1,2,\ldots,n .$$

Once the coefficients are obtained, substitution of these into

$$w \approx w_n = \sum_{i=1}^{n} c_i \phi_i(x)$$

we can find an approximation of the minimizer $w$.

```
n=7;
LP=Table[x^i,{i,1,n}];
Plot[Release[LP],{x,0,1},PlotRange->All]
DLP=D[LP,x];
fi=Table[0,{i,1,n}];
kij=Table[0,{i,1,n},{j,1,n}];
Block[{i,j},
  Do[fi[[i]]=NIntegrate[2*LP[[i]],{x,0,1}];
     Do[kij[[i,j]]=NIntegrate[DLP[[i]]*DLP[[j]]+
     x*LP[[i]]*LP[[j]],{x,0,1}];
     kij[[j,i]]=kij[[i,j]],{j,i,n}],{i,1,n}]]
cj=Inverse[kij].fi
wn=cj.LP;
Plot[wn,{x,0,1},GridLines->Automatic]
```




```
Out[101]=
{1.67519, -0.999956, -0.000397617, 0.141285, -0.0538021,

  0.00474059, 0.00019126}
```

*4. Consider a curve defined by*

$$\begin{cases} x = \cos\left(\theta + \theta^2\right) \\ y = \sin(\theta) + \sin\left(\theta^2\right) \\ z = \theta/2\pi \end{cases}$$

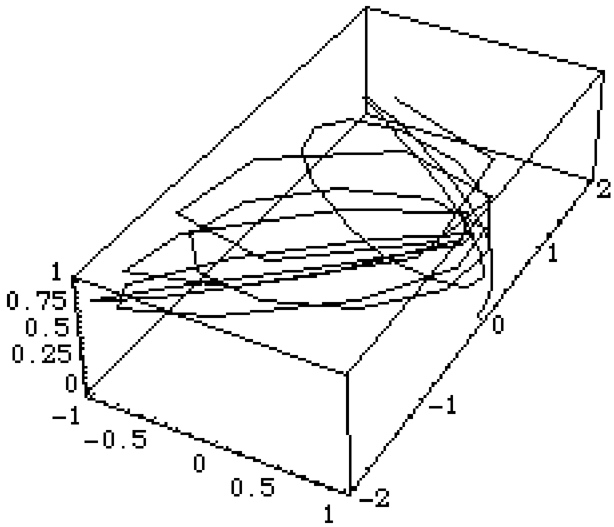
*using a parameter* $\theta$ *such that* $\theta \in (0, 2\pi)$.

*(1) Obtain the expression of the tangent vector* $\boldsymbol{t}$.

*(2) Obtain the normal and bi-normal vectors* $\boldsymbol{n}$ *and* $\boldsymbol{b}$*, respectively.*

*(3) What is the total length of this curve ?*

```
In[40]:=
x=Cos[s+s^2];
y=Sin[s]+Sin[s^2];
z=s/(2*Pi);
ParametricPlot3D[{x,y,z},{s,0,2*Pi}]
dx=D[x,s];
dy=D[y,s];
dz=D[z,s];
ds=Sqrt[dx^2+dy^2+dz^2];
L0=NIntegrate[ds,{s,0,2*Pi}]
tv={dx/ds,dy/ds,dz/ds}
n0=D[tv,s];
n1=Sqrt[n0[[1]]^2+n0[[2]]^2+n0[[3]]^2];
nv=n0/n1
```




```
Out[43]=
-Graphics3D-
NIntegrate::ncvb:
   NIntegrate failed to converge to prescribed accuracy
     after 7 recursive bisections in s near s = 4.83511.
Out[48]=
41.7097    .......  Total Length of the Curve
Out[49]=
                          2
{-(((1 + 2 s) Sin[s + s ]) /

              1                          2   2
     Sqrt[----- + (Cos[s] + 2 s Cos[s ])  +
                2
          4 Pi

              2           2 2
       (1 + 2 s)  Sin[s + s ] ]),

                    2
  (Cos[s] + 2 s Cos[s ]) /

            1                          2   2
   Sqrt[----- + (Cos[s] + 2 s Cos[s ])  +
              2
        4 Pi

            2           2 2
     (1 + 2 s)  Sin[s + s ] ],

                     1                          2   2
  1 / (2 Pi Sqrt[----- + (Cos[s] + 2 s Cos[s ])  +
                       2
                 4 Pi

              2           2 2
       (1 + 2 s)  Sin[s + s ] ])}
```

**Output of the normal and binormal vectors are more or less non-sense. Thus, I will not put them.**