

Homework #1

MEAM 502 Differential Equation Methods in Mechanical Engineering

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Noting that the functional given provides a nonlinear problem, it is better to find out a linearized problem that yields an iteration method to find a solution. Taking the first variation of the functional

$$F[v] = \frac{1}{2} \int_0^1 \sqrt{1 + \left(\frac{dv}{dx}\right)^2} dx - \int_0^1 \left(x - \frac{1}{2}\right) v dx - v \left(\frac{3}{4}\right)$$

we have

$$\begin{aligned} \delta F[v] &= \int_0^1 \delta \sqrt{1 + \left(\frac{dv}{dx}\right)^2} dx - \int_0^1 \left(x - \frac{1}{2}\right) \delta v dx - \delta v \left(\frac{3}{4}\right) \\ &= \int_0^1 \frac{1}{2} \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} 2 \frac{dv}{dx} \delta \frac{dv}{dx} dx - \int_0^1 \left(x - \frac{1}{2}\right) \delta v dx - \delta v \left(\frac{3}{4}\right) \\ &= \int_0^1 \frac{1}{2} \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} 2 \frac{dv}{dx} \frac{d\delta v}{dx} dx - \int_0^1 \left(x - \frac{1}{2}\right) \delta v dx - \delta v \left(\frac{3}{4}\right) \end{aligned}$$

After decomposing the interval (0,1) into two sub-domains (0,3/4) and (3/4,1), we apply the integration by parts rule :

$$\begin{aligned}
\delta F_{\text{v}} &= \int_0^{3/4} \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \frac{d\delta v}{dx} dx + \int_{3/4}^1 \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \frac{d\delta v}{dx} dx - \int_0^1 \left[x - \frac{1}{2} \right] \delta v dx - \delta v \left[\frac{3}{4} \right] \\
&= \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \delta v \Big|_0^{3/4} - \int_0^{3/4} \frac{d}{dx} \left[\frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right] \delta v dx \\
&\quad + \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \delta v \Big|_{3/4}^1 - \int_{3/4}^1 \frac{d}{dx} \left[\frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right] \delta v dx - \int_0^1 \left[x - \frac{1}{2} \right] \delta v dx - \delta v \left[\frac{3}{4} \right] \\
&= \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \delta v \Big|_0^{3/4} + \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \delta v \Big|_{3/4-}^{3/4+} - \frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \delta v \Big|_{3/4+}^1 - \delta v \left[\frac{3}{4} \right] \\
&\quad + \int_0^1 \left[-\frac{d}{dx} \left[\frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right] - \left[x - \frac{1}{2} \right] \right] \delta v dx
\end{aligned}$$

we have the Euler equation

$$-\frac{d}{dx} \left[\frac{1}{\sqrt{1 + \left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right] = \left[x - \frac{1}{2} \right] \quad \text{in} \quad \left[0, \frac{3}{4} \right] \& \left[\frac{3}{4}, 1 \right]$$

and

$$\left. \frac{1}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right|_{3/4-} - \left. \frac{1}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}} \frac{dv}{dx} \right|_{3/4+} - 1 = 0 \quad \text{at} \quad x = \frac{3}{4}$$

by using the constraint $v(0) = v(1) = 0$. The differential equation is clearly nonlinear.

If we assume

$$a(v, x) = \frac{1}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}}$$

the Euler equation can be written by

$$-\frac{d}{dx} \left[a(v, x) \frac{dv}{dx} \right] = \left[x - \frac{1}{2} \right] \quad \text{in} \quad \left[0, \frac{3}{4} \right] \& \left[\frac{3}{4}, 1 \right]$$

$$\left. a(v, x) \frac{dv}{dx} \right|_{3/4-} - \left. a(v, x) \frac{dv}{dx} \right|_{3/4+} = 1 \quad \text{at} \quad x = \frac{3}{4}$$

This suggests us a method to find a solution by an iteration method :

$$-\frac{d}{dx} \left[a(v^{(k)}, x) \frac{dv^{(k+1)}}{dx} \right] = \left[x - \frac{1}{2} \right] \quad \text{in} \quad \left[0, \frac{3}{4} \right] \& \left[\frac{3}{4}, 1 \right]$$

$$\left. a(v^{(k)}, x) \frac{dv^{(k+1)}}{dx} \right|_{3/4-} - \left. a(v^{(k)}, x) \frac{dv^{(k+1)}}{dx} \right|_{3/4+} = 1 \quad \text{at} \quad x = \frac{3}{4}$$

for $k = 1, 2, 3, \dots$, where we assume an appropriate initial guess $v^{(1)}$, say the solution of the linearized differential equation with the coefficient $a(0, x)$. If we can expect convergence of the sequence of solutions $v^{(1)}, v^{(2)}, v^{(3)}, \dots$, its limit will satisfy the original nonlinear problem. The corresponding functional to the linearized problem is then given by

$$F(u_{v^{b_{k+1}}}) = \frac{1}{2} \int_0^1 a(u_{v^{b_k}}, x) \left| \frac{dv^{b_{k+1}}}{dx} \right|^2 dx - \int_0^1 \left[x - \frac{1}{2} v^{b_{k+1}} \right] dx - v^{b_{k+1}} \left[\frac{3}{4} \right].$$

and the corresponding minimization problem is stated as

$$\min_{v^{b_{k+1}} \in K} F(u_{v^{b_{k+1}}}) \quad , \quad K = \{v \mid v|_0 = v|_1 = 0\}.$$

(1) Ritz Method

Assuming an approximated solution u_h in the following form

$$u_h = \sum_{i=1}^N c_i \phi_i(x) \quad , \quad \phi_i(x) = x(x-1)x^{i-1}$$

we shall solve a linearized problem

$$\min_{c_i} F(u_h^{b_{k+1}})$$

where

$$F(u_h^{b_{k+1}}) = \frac{1}{2} \int_0^1 a(u_h^{b_k}, x) \left| \frac{du_h^{b_{k+1}}}{dx} \right|^2 dx - \int_0^1 \left[x - \frac{1}{2} u_h^{b_{k+1}} \right] dx - u_h^{b_{k+1}} \left[\frac{3}{4} \right].$$

Substitution of the approximation into the linearized functional yields

$$\begin{aligned}
F(u_h^{(k+1)}) &= \frac{1}{2} \int_0^1 a(u_h^{(k)}, x) \left(\sum_{i=1}^N c_i^{(k+1)} \frac{d\phi_i}{dx} \right)^2 dx - \int_0^1 \left(x - \frac{1}{2} \sum_{i=1}^N c_i^{(k+1)} \phi_i \right) dx - \sum_{i=1}^N c_i^{(k+1)} \phi_i \left(\frac{3}{4} \right) \\
&= \sum_{i=1}^N \sum_{j=1}^N \frac{1}{2} c_i^{(k+1)} c_j^{(k+1)} \int_0^1 a(u_h^{(k)}, x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx - \sum_{i=1}^N c_i^{(k+1)} \int_0^1 \left(x - \frac{1}{2} \sum_{i=1}^N c_i^{(k+1)} \phi_i \right) dx - \sum_{i=1}^N c_i^{(k+1)} \phi_i \left(\frac{3}{4} \right) \\
&= \sum_{i=1}^N \sum_{j=1}^N \frac{1}{2} c_i^{(k+1)} c_j^{(k+1)} k_{ij} - \sum_{i=1}^N c_i^{(k+1)} b_i
\end{aligned}$$

where

$$\begin{aligned}
k_{ij} &= \int_0^1 a(u_h^{(k)}, x) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \\
b_i &= \int_0^1 \left(x - \frac{1}{2} \sum_{i=1}^N c_i^{(k+1)} \phi_i \right) dx + \phi_i \left(\frac{3}{4} \right)
\end{aligned}$$

Noting that

$$\min_{c_i} F(u_h^{(k+1)}) \Leftrightarrow \frac{\partial F(u_h^{(k+1)})}{\partial c_i} = 0 \quad , \quad i = 1, 2, \dots, N$$

the solution of the linearized problem must satisfy the following matrix equation :

$$\sum_{j=1}^N k_{ij} c_j^{(k+1)} = b_i \quad , \quad i = 1, 2, \dots, N \quad \Leftrightarrow \quad \mathbf{K} \mathbf{c}^{(k+1)} = \mathbf{b}$$

Solving this matrix equation, we obtain the $(k+1)th$ approximation of u_h .

A sample MATHEMATICA script program is given as follows :

```

imax=5;
bf=Table[x*(x-1)*x^(i-1),{i,1,imax}];
dbf=D[bf,x];
c=Table[0,{i,1,imax}];

```

```

tole=1;
iteration=0;
tolerance=10^(-3);
While[tole>tolerance,
vo=c.bf;
ao=Sqrt[1+(D[vo,x])^2];
b=Table[0,{i,1,imax}];
K=Table[0,{i,1,imax},{j,1,imax}];
Block[{i,j},
Do[bi=NIntegrate[(x-1/2)*bf[[i]],{x,0,1}]+
N[ bf[[i]]/.{x->0.75}];
b[[i]]=bi;
Do[kij=NIntegrate[ao*dbf[[i]]*dbf[[j]],{x,0,1}];
K[[i,j]]=kij;
K[[j,i]]=kij,{j,i,imax},{i,1,imax}]];
c=Inverse[K].b;
v1=c.bf;
iteration=iteration+1;
tole=Sqrt[NIntegrate[(v1-vo)^2,{x,0,1}]]/Sqrt[NIntegrate[v1^2,{x,0,1}]];
Print["iteration = ",iteration];
Print["relative error = ",tole];
Print["vh = ",v1];
Plot[v1,{x,0,1}];
vo=v1]

```

(2) Finite Element Method

Only the difference from the Ritz method described in above, we shall use the special basis function ϕ_i constructed by the finite element method :

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

for the finite element discretization

$$[0, 1] = \bigcup_{i=1}^E [x_i, x_{i+1}] \quad , \quad x_i = 0 \quad \text{and} \quad x_{E+1} = 1.$$

Because of the essential boundary condition at $x = 0$ and $x = 1$, we must satisfy

$$c_1 = c_{E+1} = 0.$$

The coefficient matrix, that is called the stiffness matrix in the finite element method, is then a tri-diagonal matrix because of the nature of the basis function.

(3) Finite Difference Method

We shall set up the finite difference scheme based on the Euler equation :

$$-\frac{a_{i+\frac{1}{2}} \left(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{x_{i+1} - x_i} \right) - a_{i-\frac{1}{2}} \left(\frac{u_i^{k+1} - u_{i-1}^{k+1}}{x_i - x_{i-1}} \right)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = f_i \left(x_i - \frac{1}{2} \right) \quad , \quad i \text{ except } x_i = \frac{3}{4}$$

that is

$$-a_{i+1} u_{i+1}^{k+1} + a_i u_i^{k+1} - a_{i-1} u_{i-1}^{k+1} = b_i$$

where

$$a_{i+1} = \frac{a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i+\frac{1}{2}}}{\mathbb{E} x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \mathbb{E} b x_{i+1} - x_i \mathbb{Q}} \quad , \quad a_{i-1} = \frac{a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i-\frac{1}{2}}}{\mathbb{E} x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \mathbb{E} b x_i - x_{i-1} \mathbb{Q}} \quad , \quad a_i = a_{i+1} + a_{i-1}$$

and

$$b_i = x_i - \frac{1}{2}.$$

At $x_i = 3/4$, instead of the second order differential equation, we must satisfy the “jump” condition :

$$a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x \mathbb{J} \frac{du^{\mathbb{D}_{k+1} \mathbb{Q}}}{dx} \Big|_{x=\frac{3}{4}-} - a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x \mathbb{J} \frac{du^{\mathbb{D}_{k+1} \mathbb{Q}}}{dx} \Big|_{x=\frac{3}{4}+} = 1$$

Approximating this by the finite difference method, we have

$$a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i-} \mathbb{J} \frac{u_i^{\mathbb{D}_{k+1} \mathbb{Q}} - u_{i-1}^{\mathbb{D}_{k+1} \mathbb{Q}}}{x_i - x_{i-1}} - a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i+} \mathbb{J} \frac{u_{i+1}^{\mathbb{D}_{k+1} \mathbb{Q}} - u_i^{\mathbb{D}_{k+1} \mathbb{Q}}}{x_{i+1} - x_i} = 1 \quad \text{at } i \text{ s.t. } x_i = \frac{3}{4}$$

that is

$$-a_{i+1} u_{i+1}^{\mathbb{D}_{k+1} \mathbb{Q}} + a_i u_i^{\mathbb{D}_{k+1} \mathbb{Q}} - a_{i-1} u_{i-1}^{\mathbb{D}_{k+1} \mathbb{Q}} = b_i$$

where

$$a_{i+1} = \frac{a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i+}}{x_{i+1} - x_i} \quad , \quad a_{i-1} = \frac{a \mathbb{E} u^{\mathbb{D}_k \mathbb{Q}}, x_{i-}}{x_i - x_{i-1}} \quad , \quad a_i = a_{i+1} + a_{i-1} \quad , \quad b_i = 1.$$

Thus, solving the tri-diagonal system, we can find the approximated solution. Here we must constrain $u^{(k+1)} = 0$ at $x = 0$ and $x = 1$.