

TEICHMÜLLER THEORY AND CRITICALLY FINITE ENDOMORPHISMS

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ABSTRACT. We present a systematic way to generate critically finite endomorphisms of $\mathbb{P}^n$. These maps arise in the context of Teichmüller theory, specifically in Thurston's topological characterization of rational maps. The dynamical objects for the endomorphisms correspond to central objects from Thurston's theorem. Our theorems build infinitely many of these endomorphisms; in fact, a large number of examples of critically finite endomorphisms of $\mathbb{P}^n$ found in the literature arise from this construction.

INTRODUCTION

Let S^2 be an oriented topological 2-sphere. We begin with an orientation-preserving branched cover $f : (S^2, P) \rightarrow (S^2, P)$, where the domain and range spheres are identified, and P is the postcritical set of f . If $|P| < \infty$, then f is called a Thurston map. Each Thurston map induces a holomorphic endomorphism $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ where $\mathcal{T}_P$ is the Teichmüller space of the pair (S^2, P) . Thurston's topological characterization of rational maps (Theorem 4.1) says that a Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ is *combinatorially equivalent* to a rational map $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ if and only if there are no *obstructing multicurves*. One proves Thurston's theorem by showing that the map f is equivalent to a rational map F if and only if the associated endomorphism $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ has a fixed point.

This map σ_f rarely descends to the moduli space $\mathcal{M}_P$. However, sometimes an “inverse” of σ_f descends to yield a *moduli space map* $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, such that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P \\ \pi_P \downarrow & & \downarrow \pi_P \\ \mathcal{M}_P & \xleftarrow{g_f} & \mathcal{M}_P \end{array}$$

where $\pi_P : \mathcal{T}_P \rightarrow \mathcal{M}_P$ is a holomorphic universal covering map. L. Bartholdi and V. Nekrashevych were the first to prove that moduli space maps exist by explicitly computing these maps for three examples in [BN]. We generalize this result and prove that for certain classes of Thurston maps, a moduli space map exists. The moduli space map g_f is naturally defined on $\pi_P(\sigma_f(\mathcal{T}_P)) \subseteq \mathcal{M}_P$ which may only be a proper subset of moduli space.

If a moduli space map exists, we compactify $\mathcal{M}_P$ and ask if the map extends to the compactification. If f is a *topological polynomial*, then we use the $\mathbb{P}^n$ compactification of the moduli space. Theorem 5.17 asserts that if the Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ is a

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topological polynomial with periodic critical points, then $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. Theorem 5.18 asserts that if the Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ is a unicritical topological polynomial, then $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$.

In general, one obtains a correspondence on the level of moduli space. Determining whether the graph of this correspondence is the graph of a map g_f is rather difficult. We prove that it is actually a nondynamical issue as it depends on the *Hurwitz class* of f . We define the *Hurwitz space* $\mathcal{W}_f$ associated to f ; we prove that it is a complex manifold and a finite cover of the moduli space $\mathcal{M}_P$ (Theorem 2.6).

Although the existence of a moduli space map is a nondynamical matter, we wish to exploit this map (when it exists) in a dynamical setting. As a general rule, the dynamical objects for $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ correspond to those for $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ and therefore also correspond to central objects in Thurston's theorem, Theorem 4.1. In this way, we can learn about $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ by examining the moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, and vice versa.

The critically finite endomorphisms $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ arising as moduli space maps have a variety of very nice properties. For example, the postcritical locus of each consists of $(n^2 + 3n + 2)/2$ hyperplanes (see Proposition 5.5). These hyperplanes comprise the boundary of $\mathcal{M}_P$ inside $\mathbb{P}^n$, so the complement of the postcritical locus of G_f in $\mathbb{P}^n$ is therefore Kobayashi hyperbolic (as a corollary of Royden's theorem [Roy]).

Critically finite endomorphisms of $\mathbb{P}^n$ were first studied by J.E. Fornæss and N. Sibony in [FS1]; part of their motivation was to provide a positive answer to a question asked by C. McMullen (see [B]). In addition to Fornæss and Sibony, several other mathematicians have studied critically finite endomorphisms of projective space. Among them are S. Crass, C. Dupont, M. Jonsson, F. Rong, and T. Ueda. Many of the existing examples of critically finite endomorphisms in the literature arise via our construction. For example, we can recover the original Fornæss and Sibony examples in [FS1] with our methods, in addition to the examples of Crass in [C]. There is one rather curious example that we cannot recover however; see Example 8.1 in Section 8.

In this article, we present a systematic way to generate these endomorphisms which arises from Teichmüller theory; Theorem 5.17 and Theorem 5.18 build infinitely many of them, and for all of them, the complement of the postcritical locus is Kobayashi hyperbolic.

0.1. Outline. We begin with some background material in Section 1: we discuss some Teichmüller theory and define the map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$. We then continue with a nondynamical discussion of the *Hurwitz space*, $\mathcal{W}_f$ in Section 2 (Theorem 2.6); we use this space to define the correspondence on the level of moduli space. We discuss moduli space maps in Section 3.

In Section 4, we shift to a dynamical setting, where we present Thurston's topological characterization of rational maps, and discuss the fixed points of the moduli space correspondence. In Section 5, we find sufficient conditions on the Thurston map $f : (S^2, P) \rightarrow (S^2, P)$, for which $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ exists in Proposition 5.1 and Proposition 5.3. We then discuss different compactifications of $\mathcal{M}_P$, and prove in Propositions 5.7 and 5.11 that these moduli

space maps extend holomorphically to the $\mathbb{P}^n$ compactification of $\mathcal{M}_P$, giving an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. We then prove that these endomorphisms are critically finite (Theorem 5.17 and Theorem 5.18).

In Section 6, we prove that the complement of the postcritical locus of each endomorphism is Kobayashi hyperbolic, and in Section 7, we compare the dynamical objects of $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ to those of $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$. In Section 8, we discuss the examples of critically finite endomorphisms in the literature that can be recovered by these methods, and one rather curious example that cannot. Lastly, we conclude the article with a few questions for further study in Section 9.

1. PRELIMINARIES

A Riemann surface is a connected oriented topological surface together with a complex structure: a maximal atlas of charts with holomorphic overlap maps. For a given oriented, compact topological surface X , we denote the set of all complex structures on X by $\mathcal{C}(X)$. An orientation-preserving branched covering map $f : X \rightarrow Y$ induces a map $f^* : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$; in particular, for any orientation-preserving homeomorphism $\psi : X \rightarrow X$, there is an induced map $\psi^* : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$.

1.1. The pullback map. Let $A \subset X$ be finite, and suppose $|A| \geq 3$. The *Teichmüller space* of (X, A) is

$$\mathcal{T}_{(X,A)} := \mathcal{C}(X) / \sim_A$$

where $c \sim_A c'$ if and only if $c = \psi^*(c')$ for some orientation-preserving homeomorphism $\psi : X \rightarrow X$ which is isotopic to the identity relative to A . It is well-known that $\mathcal{T}_{(X,A)}$ has a natural topology and a complex manifold structure (see [H1]).

By the homotopy-lifting property, if

- $B \subset Y$ is finite and contains the critical values of f , and
- $A \subseteq f^{-1}(B)$,

then $f^* : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$ descends to a well-defined map σ_f between the corresponding Teichmüller spaces:

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{f^*} & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathcal{T}_{(Y,B)} & \xrightarrow{\sigma_f} & \mathcal{T}_{(X,A)} \end{array}$$

This map is known as the *pullback map* induced by f . The map is holomorphic and contracting for the Teichmüller metric (see [DH]).

We will restrict our attention to the case where X is S^2 . It then follows from the Uniformization Theorem that the Teichmüller space $\mathcal{T}_{(S^2,A)}$ is the space of all orientation-preserving homeomorphisms $\phi : (S^2, A) \rightarrow (\mathbb{P}^1, \phi(A))$ where $\phi_1 \sim \phi_2$ if and only if there is a Möbius transformation μ such that

- $\phi_1 = \mu \circ \phi_2$ on the set A , and

- ϕ_1 is isotopic to $\mu \circ \phi_2$ relative to the set A .

The *moduli space* of (S^2, A) is the space of all injective maps $\psi : A \hookrightarrow \mathbb{P}^1$ modulo postcomposition with Möbius transformations. The moduli space $\mathcal{M}_{(S^2, A)}$ is an irreducible quasiprojective variety and a complex manifold. As we will continue to restrict our attention to S^2 , we will use the notation $\mathcal{T}_A$ to denote the Teichmüller space of (S^2, A) , and $\mathcal{M}_A$ to denote the moduli space of (S^2, A) .

If ϕ represents an element of the Teichmüller space $\mathcal{T}_A$, the restriction $\phi \mapsto \phi|_A$ induces a universal covering map $\pi_A : \mathcal{T}_A \rightarrow \mathcal{M}_A$ which is a local biholomorphism with respect to the complex structures on $\mathcal{T}_A$ and $\mathcal{M}_A$. The group of deck transformations of $\pi_A : \mathcal{T}_A \rightarrow \mathcal{M}_A$ is isomorphic to the pure mapping class group of (S^2, A) , PMod_A . By definition, PMod_A is the quotient of the group of orientation-preserving homeomorphisms $h : (S^2, A) \rightarrow (S^2, A)$ fixing A pointwise, by the subgroup of such maps isotopic to the identity relative to A ; it is generated by finitely many isotopy classes of Dehn twists about curves in $S^2 - A$.

1.1.1. *Conventions.* We establish some conventions for the remainder of the article.

- (1) We make the standing assumption that all subsequent maps $S^2 \rightarrow S^2$ are orientation-preserving.
- (2) Let $f : (S^2, A) \rightarrow (S^2, B)$ be a branched cover. If
 - the critical values of f are contained in B ,
 - $A \subseteq f^{-1}(B)$, and $2 < |A|, |B| < \infty$,

then $f : (S^2, A) \rightarrow (S^2, B)$ will be called an *admissible* branched cover.

Let $f : (S^2, A) \rightarrow (S^2, B)$ be an admissible branched cover. It follows from the discussion above that there is a pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$.

The Uniformization Theorem provides a more concrete description of the pullback map σ_f . Let $\phi : (S^2, B) \rightarrow (\mathbb{P}^1, \phi(B))$ be a homeomorphism fixing B pointwise. Pulling back this complex structure under the map f , we obtain a complex structure $\psi : (S^2, A) \rightarrow (\mathbb{P}^1, \psi(A))$, such that the composition

$$F_\phi := \phi \circ f \circ \psi^{-1} : (\mathbb{P}^1, \psi(A)) \rightarrow (\mathbb{P}^1, \phi(B))$$

is holomorphic; that is, the following diagram commutes, and F_ϕ is a rational function.

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ f \downarrow & & \downarrow F_\phi \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

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The homeomorphism ψ is not unique; if we pullback ϕ to another complex structure ψ' , then there is necessarily a Möbius transformation μ so that the following diagram commutes.

$$\begin{array}{ccc} & & (\mathbb{P}^1, \psi'(A)) \\ & \nearrow \psi' & \downarrow \mu \\ (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \end{array}$$

Notice that this implies that $[\psi] = [\psi']$ in $\mathcal{T}_A$, and $\sigma_f : [\phi] \mapsto [\psi]$.

2. THE HURWITZ SPACE

We continue our nondynamical discussion and define a very important space. To this end, we introduce the notion of *Hurwitz equivalence*. Let $f_1 : (S^2, A) \rightarrow (S^2, B)$ and $f_2 : (S^2, A) \rightarrow (S^2, B)$ be admissible branched covers. The pair of homeomorphisms

$$(h : (S^2, B) \rightarrow (S^2, B), \quad h' : (S^2, A) \rightarrow (S^2, A))$$

is a Hurwitz equivalence for the pair (f_1, f_2) if h fixes B pointwise, h' fixes A pointwise, and the following diagram commutes.

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{h'} & (S^2, A) \\ f_1 \downarrow & & \downarrow f_2 \\ (S^2, B) & \xrightarrow{h} & (S^2, B) \end{array}$$

We say that f_1 and f_2 are Hurwitz equivalent if there is a Hurwitz equivalence for (f_1, f_2) ; Hurwitz equivalence is evidently an equivalence relation on the set of all admissible branched covering maps $f : (S^2, A) \rightarrow (S^2, B)$.

Consider the subgroup of *liftable homeomorphisms*

$$\{h : (S^2, B) \rightarrow (S^2, B) \mid \exists h' : (S^2, A) \rightarrow (S^2, A) \text{ so that } (h, h') \text{ is a Hurwitz equivalence for } (f, f)\}.$$

If (h, h') and (h, h'') are Hurwitz equivalences for (f, f) , then $\alpha := h' \circ (h'')^{-1} : (S^2, A) \rightarrow (S^2, A)$ is a deck transformation for $f : (S^2, A) \rightarrow (S^2, B)$.

Lemma 2.1. *The deck transformation $\alpha : (S^2, A) \rightarrow (S^2, A)$ is isotopic to the identity, relative to A .*

Proof. Because α is a deck transformation, $f = f \circ \alpha$, and there are pullback maps

$$\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A \quad \text{and} \quad \sigma_{f \circ \alpha} : \mathcal{T}_B \rightarrow \mathcal{T}_A.$$

Note that $\sigma_{f \circ \alpha} = \sigma_\alpha \circ \sigma_f$ where $\sigma_\alpha : \mathcal{T}_A \rightarrow \mathcal{T}_A$ is the automorphism induced by the mapping class element represented by α . We have $\sigma_f = \sigma_\alpha \circ \sigma_f$, so every point of $\sigma_f(\mathcal{T}_B)$ is a fixed point of σ_α . But PMod_A acts freely on $\mathcal{T}_A$, so $\sigma_\alpha = \text{id}$, and the deck transformation $\alpha : (S^2, A) \rightarrow (S^2, A)$ is isotopic to the identity relative to A . $\square$

Corollary 2.2. *The subgroup of liftable homeomorphisms descends to a subgroup $H_f \leq \text{PMod}_B$ of liftable mapping classes.*

Proof. The result follows from Lemma 2.1 and the homotopy-lifting property. $\square$

Remark 2.3. The subgroup H_f has finite index in PMod_B as proved in Proposition 3.1 of [KPS].

2.1. The space $\mathcal{W}_f$. Recall the holomorphic map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$; it is rather mysterious and highly transcendental. We would like to record the algebraic data of σ_f . To this end, we associate the space $\mathcal{W}_f$ to the map $f : (S^2, A) \rightarrow (S^2, B)$; this space is a complex manifold, and it is an invariant of the Hurwitz class of f .

Let $\phi : (S^2, B) \rightarrow (\mathbb{P}^1, \phi(B))$ and $\psi : (S^2, A) \rightarrow (\mathbb{P}^1, \psi(A))$ be homeomorphisms such that this diagram commutes, and F_ϕ is a rational map.

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ f \downarrow & & \downarrow F_\phi \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

Recall that $\sigma_f : [\phi] \mapsto [\psi]$. We wish to record the data $(F_\phi, \phi|_B, \psi|_A)$, up to appropriate equivalence. This triple is an element of

$$R_{A,B} := \text{Rat}_d \times (\mathbb{P}^1)^B \times (\mathbb{P}^1)^A$$

where Rat_d denotes the space of rational maps of degree d , $(\mathbb{P}^1)^B$ denotes the space of all *injective* maps $B \hookrightarrow \mathbb{P}^1$, and $(\mathbb{P}^1)^A$ denotes the space of all *injective* maps $A \hookrightarrow \mathbb{P}^1$ (we are abusing notation a bit here). The space Rat_d is a smooth affine variety which we may identify with an open subset of $\mathbb{P}^{2d+1}$, and each of the spaces $(\mathbb{P}^1)^B$ and $(\mathbb{P}^1)^A$ can be identified with some $(\mathbb{P}^1)^N - \Delta$ where Δ is the closed set defined by the property that at least two coordinates coincide. Hence the space $R_{A,B}$ is a smooth affine variety.

The group $G := \text{PSL}(2, \mathbb{C}) \times \text{PSL}(2, \mathbb{C})$ acts on $R_{A,B}$ in the following way

$$(\mu, \nu) \bullet (F, i_B, j_A) \mapsto (\mu \circ F \circ \nu^{-1}, \mu \circ i_B, \nu \circ j_A)$$

This action is evidently free as the sets A and B each contain at least three points. We now appeal to geometric invariant theory to study the quotient $R_{A,B}/G$.

Recall that an algebraic group H over an algebraically closed field is *reductive* if the unipotent radical of H is trivial. The group $\text{PSL}(2, \mathbb{C})$ is a semi-simple Lie group and is therefore reductive. Since the product of two reductive groups is reductive, G is reductive as well. It then follows from [Mu], Proposition 0.8 and Amplification 1.3 that $R_{A,B}/G$ is a geometric quotient. As such, it is a smooth complex manifold and a quasiprojective variety of dimension $2d + 1 + |A| + |B| - 6$.

The map $\omega_f : \mathcal{T}_B \rightarrow R_{A,B}/G$, defined by $\omega_f : [\phi] \mapsto [(F_\phi, \phi|_B, \psi|_A)]$ is well-defined and holomorphic because $\sigma_f : [\phi] \mapsto [\psi]$ is well-defined and holomorphic.

Lemma 2.4. *The map $\omega_f : \mathcal{T}_B \rightarrow R_{A,B}/G$ is an immersion.*

Proof. There are maps $R_{A,B}/G \rightarrow \mathcal{M}_B$ and $R_{A,B}/G \rightarrow \mathcal{M}_A$, given by $\rho_B : [(F, i_B, j_A)] \mapsto [i_B]$ and $\delta_A : [(F, i_B, j_A)] \mapsto [j_A]$ respectively. It follows from the definition of $\omega_f : \mathcal{T}_B \rightarrow R_{A,B}/G$ that the following diagram commutes.

$$\begin{array}{ccccc}
\mathcal{T}_B & \xrightarrow{\sigma_f} & \mathcal{T}_A & & \\
\downarrow \pi_B & \searrow \omega_f & \downarrow \pi_A & & \\
& R_{A,B}/G & & & \\
& \swarrow \rho_B \quad \searrow \delta_A & & & \\
\mathcal{M}_B & & \mathcal{M}_A & &
\end{array}$$

Since the map $\pi_B : \mathcal{T}_B \rightarrow \mathcal{M}_B$ is an immersion, the map $\omega_f : \mathcal{T}_B \rightarrow R_{A,B}/G$ must be an immersion as well. $\square$

Lemma 2.5. *Let $\tau_1, \tau_2 \in \mathcal{T}_B$. Then $\omega_f(\tau_1) = \omega_f(\tau_2)$ if and only if there exists $m \in H_f$ such that $m \bullet \tau_1 = \tau_2$.*

Proof. Let $\phi_1 : (S^2, B) \rightarrow (\mathbb{P}^1, \phi_1(B))$, and $\phi_2 : S^2 \rightarrow (\mathbb{P}^1, \phi_2(B))$ be homeomorphisms representing τ_1 and τ_2 respectively. There exist homeomorphisms $\psi_1 : (S^2, A) \rightarrow (\mathbb{P}^1, \psi_1(A))$ and $\psi_2 : (S^2, A) \rightarrow (\mathbb{P}^1, \psi_2(A))$, and rational maps $F_1 : (\mathbb{P}^1, \psi_1(A)) \rightarrow (\mathbb{P}^1, \phi_1(B))$ and $F_2 : (\mathbb{P}^1, \psi_2(A)) \rightarrow (\mathbb{P}^1, \phi_2(B))$ so that the diagrams below commute.

$$\begin{array}{ccc}
(S^2, A) & \xrightarrow{\psi_1} & (\mathbb{P}^1, \psi_1(A)) \\
f \downarrow & & \downarrow F_1 \\
(S^2, B) & \xrightarrow{\phi_1} & (\mathbb{P}^1, \phi_1(B))
\end{array}
\qquad
\begin{array}{ccc}
(S^2, A) & \xrightarrow{\psi_2} & (\mathbb{P}^1, \psi_2(A)) \\
f \downarrow & & \downarrow F_2 \\
(S^2, B) & \xrightarrow{\phi_2} & (\mathbb{P}^1, \phi_2(B))
\end{array}$$

If $\omega_f(\tau_1) = \omega_f(\tau_2)$ then there are Möbius transformations μ and ν so that

$$F_1 \circ \nu = \mu \circ F_2, \quad \phi_1|_B = \mu \circ \phi_2|_B, \quad \text{and} \quad \psi_1|_A = \nu \circ \psi_2|_A.$$

Define the homeomorphisms

$$h := \phi_1^{-1} \circ \mu \circ \phi_2 : (S^2, B) \rightarrow (S^2, B), \quad \text{and} \quad h' := \psi_1^{-1} \circ \nu \circ \psi_2 : (S^2, A) \rightarrow (S^2, A).$$

It follows that h fixes B pointwise, and h' fixes A pointwise, so h represents an element of PMod_B , and h' represents an element of PMod_A .

In the diagram below, the outer and inner squares commute, the right panel commutes, and the top and bottom panels commute. The left panel therefore commutes as well.

$$\begin{array}{ccccc}
(S^2, A) & \xrightarrow{\psi_2} & & & (\mathbb{P}^1, \psi_2(A)) \\
& \searrow h' & & \swarrow \nu & \\
& (S^2, A) & \xrightarrow{\psi_1} & & (\mathbb{P}^1, \psi_1(A)) \\
& \downarrow f & & \downarrow F_1 & \\
& (S^2, B) & \xrightarrow{\phi_1} & & (\mathbb{P}^1, \phi_1(B)) \\
& \nearrow h & & \nwarrow \mu & \\
(S^2, B) & \xrightarrow{\phi_2} & & & (\mathbb{P}^1, \phi_2(B))
\end{array}$$

f (left vertical), F_2 (right vertical), F_1 (middle right vertical), ψ_1 (middle top horizontal), ϕ_1 (middle bottom horizontal), ψ_2 (top horizontal), ϕ_2 (bottom horizontal), h' (top left diagonal), h (bottom left diagonal), ν (top right diagonal), μ (bottom right diagonal).

Thus $[h] \in H_f$, and set $m := [h]$, so $m \bullet \tau_1 = \tau_2$.

The proof in the other direction is similar; we will require the commutative cube from above, which will be completed in a different order. If $\tau_2 = m \bullet \tau_1$ for some $m \in H_f$, then in particular, $\pi_B(\tau_1) = \pi_B(\tau_2)$, so there is a unique Möbius transformation μ so that $\phi_2|_B = \mu \circ \phi_1|_B$. Define the homeomorphism

$$h := \phi_1^{-1} \circ \mu \circ \phi_2 : (S^2, B) \rightarrow (S^2, B).$$

Clearly h fixes B pointwise, so $[h] \in \text{PMod}_B$. Moreover,

$$[h] \bullet \tau_1 = \tau_2 = m \bullet \tau_1$$

which implies that $m = [h]$ as PMod_B acts freely on $\mathcal{T}_B$. Therefore, the homeomorphism h is liftable, that is, there exists $h' : (S^2, A) \rightarrow (S^2, A)$ fixing A pointwise so that $h \circ f = f \circ h'$. Define the homeomorphism

$$\nu := \psi_1 \circ h' \circ \psi_2^{-1} : (\mathbb{P}^1, \psi_2(A)) \rightarrow (\mathbb{P}^1, \psi_1(A)).$$

Then in the diagram above, the left panel commutes, the top and bottom panels commute, and therefore the panel on the right commutes also. The homeomorphism ν is in fact a Möbius transformation since locally, ν can be expressed as a composition of analytic maps at all but finitely many points. Therefore $\omega_f(\tau_1) = \omega_f(\tau_2)$. $\square$

Theorem 2.6. *The image $\mathcal{W}_f := \omega_f(\mathcal{T}_B) \subseteq R_{A,B}/G$ is a complex manifold of dimension $|B| - 3$, and the map $\rho_B : \mathcal{W}_f \rightarrow \mathcal{M}_B$ is a finite covering map.*

Proof. We first prove that $\mathcal{W}_f$ is a complex manifold. It follows from Lemma 2.5 that there is a continuous injection $\alpha : \mathcal{T}_B/H_f \hookrightarrow R_{A,B}/G$ so that the following diagram commutes.

$$\begin{array}{ccc}
& \mathcal{T}_B & \\
\swarrow & & \searrow \omega_f \\
\mathcal{T}_B/H_f & \xrightarrow{\alpha} & R_{A,B}/G
\end{array}$$

Since the quotient map $\mathcal{T}_B \rightarrow \mathcal{T}_B/H_f$ is a local biholomorphism, and since $\omega_f : \mathcal{T}_B \rightarrow R_{A,B}/G$ is a holomorphic immersion, the map α is holomorphic as well. As a holomorphic injective map, $\alpha : \mathcal{T}_B/H_f \rightarrow R_{A,B}/G$ is a homeomorphism onto its image, $\omega_f(\mathcal{T}_B)$. Therefore $\omega_f(\mathcal{T}_B)$

is a topological manifold of dimension $|B| - 3$, which inherits a complex structure from the immersion ω_f . It follows that $\alpha : \mathcal{T}_B/H_f \rightarrow \mathcal{W}_f$ is a biholomorphism, and that $\omega_f : \mathcal{T}_B \rightarrow \mathcal{W}_f$ is a holomorphic universal covering map.

In the diagram below on the left, the outer square commutes since both of the smaller cells do. This fact, coupled with the fact that the diagram above commutes, imply that the diagram below on the right commutes as well.

$$\begin{array}{ccc}
 & \mathcal{T}_B & \\
 \swarrow & & \searrow \omega_f \\
 \mathcal{T}_B/H_f & \xrightarrow{\pi_B} & \mathcal{W}_f \\
 \searrow & & \swarrow \rho_B \\
 & \mathcal{M}_B &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{T}_B/H_f & \xrightarrow{\alpha} & \mathcal{W}_f \\
 & \searrow & \swarrow \rho_B \\
 & \mathcal{M}_B &
 \end{array}$$

It follows that $\rho_B : \mathcal{W}_f \rightarrow \mathcal{M}_B$ is a covering map. It is a finite cover because H_f has finite index in $\text{PMod}(S^2, B)$ (see Remark 2.3). $\square$

We define the *Hurwitz space* of the map $f : (S^2, A) \rightarrow (S^2, B)$ to be the complex manifold $\mathcal{W}_f$ defined in Theorem 2.6.

Corollary 2.7. *Let $f : (S^2, A) \rightarrow (S^2, B)$ and $g : (S^2, A) \rightarrow (S^2, B)$. Then*

- $\mathcal{W}_f = \mathcal{W}_g \iff f$ and g are Hurwitz equivalent, and
- $\mathcal{W}_f = \mathcal{W}_g$ or $\mathcal{W}_f \cap \mathcal{W}_g = \emptyset$.

Proof. This follows immediately from Lemma 2.5. $\square$

The Hurwitz space $\mathcal{W}_f$ is a very natural space to consider; it provides a porthole in which one can study $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$.

$$\begin{array}{ccc}
 \mathcal{T}_B & \xrightarrow{\sigma_f} & \mathcal{T}_A \\
 \pi_B \downarrow & \searrow \omega_f & \downarrow \pi_A \\
 & \mathcal{W}_f & \\
 \rho_B \swarrow & & \searrow \delta_A \\
 \mathcal{M}_B & & \mathcal{M}_A
 \end{array}$$

We immediately observe the following facts:

- (1) $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ is ramified if and only if $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ is ramified,
- (2) $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ has discrete fibers if and only if $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ has discrete fibers, and
- (3) the image of $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ contains an open subset of $\mathcal{M}_A$ if and only if the image of $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ contains an open subset of $\mathcal{T}_A$.

The map $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ is much more accessible than $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ and can be analyzed to understand the pullback map σ_f .

Proposition 2.8. *If $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ is a covering map (of its image), then*

- $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ is a (universal) covering map of its image
- the image of σ_f is

$$\sigma_f(\mathcal{T}_B) = \pi_A^{-1}(\pi_A(\sigma_f(\mathcal{T}_B))).$$

Proof. Define $\mathcal{M}'_A \subseteq \mathcal{M}_A$ to be the image of $\mathcal{T}_B$ under the composition $\pi_A \circ \sigma_f$; note that this coincides with the image of $\mathcal{T}_B$ under the map $\delta_A \circ \omega_f$ by construction. Since $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}'_A$ is a covering map, the map

$$\delta_A \circ \omega_f : \mathcal{T}_B \rightarrow \mathcal{M}'_A$$

is a covering map, and therefore a universal covering map since $\mathcal{T}_B$ is simply connected. The restriction

$$\pi_A : \pi_A^{-1}(\mathcal{M}'_A) \rightarrow \mathcal{M}'_A$$

is also a covering map. Because the map $\mathcal{T}_B \rightarrow \mathcal{M}'_A$ is a universal cover, there is a lift $\mathcal{T}_B \rightarrow \pi_A^{-1}(\mathcal{M}'_A)$;

$$\begin{array}{ccc} \mathcal{T}_B & \xrightarrow{\quad} & \pi_A^{-1}(\mathcal{M}'_A) \\ & \searrow & \downarrow \\ & & \mathcal{M}'_A \end{array} \quad \begin{array}{ccc} \mathcal{T}_B & \xrightarrow{\quad} & \pi_A^{-1}(\mathcal{M}'_A) \\ & \searrow & \downarrow \\ & & \mathcal{M}'_A \end{array}$$

choose the lift which maps some $\tau \in \mathcal{T}_B$ to $\sigma_f(\tau)$. Then this lift necessarily coincides with the map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$. In this case, σ_f is a cover (and hence a universal cover) of its image $\sigma_f(\mathcal{T}_B) = \pi_A^{-1}(\mathcal{M}'_A)$. $\square$

In [BEKP], the authors use the map δ_A to exhibit an example of a critically finite rational map for which the associated pullback map is constant, and an example of a critically finite rational map for which the associated pullback map is a surjective, ramified, Galois cover.

Remark 2.9. There is another space

$$\mathcal{V}_f := (\pi_B \times \pi_A)(\text{Graph}(\sigma_f)) \subseteq (\mathcal{M}_B \times \mathcal{M}_A)$$

which fits into the following diagram.

$$\begin{array}{ccc} & \mathcal{W}_f & \\ & \downarrow & \\ & \mathcal{V}_f & \\ \swarrow & & \searrow \\ \mathcal{M}_B & & \mathcal{M}_A \end{array}$$

In general $\mathcal{V}_f$ is not isomorphic to $\mathcal{W}_f$; in fact, $\mathcal{V}_f$ may not even be a manifold. Furthermore, if $f : (S^2, A) \rightarrow (S^2, B)$ is a ramified cover for which σ_f is constant, then $\mathcal{V}_f \rightarrow \mathcal{M}_B$ is an

isomorphism, and therefore $\mathcal{W}_f \rightarrow \mathcal{V}_f$ will be a covering map. In this case, if $\deg(f) > 1$, then the degree of this covering map $\mathcal{W}_f \rightarrow \mathcal{V}_f$ will be greater than 1.

2.2. Examples. Hurwitz equivalence captures “topological combinatorics” such as combinatorial data of covering maps, or function composition.

Example 2.10. Let $F_\alpha(t) = (1 - (1 - \alpha)t)^d$, where $\alpha \neq 1$ is a d th root of unity. Each F_α is a unicritical polynomial of degree d , with critical point $\omega = 1/(1 - \alpha)$, such that

$$F_\alpha(\omega) = 0, \quad F_\alpha(0) = 1, \quad \text{and} \quad F_\alpha(1) = 1.$$

Define the sets $A := \{0, 1, \infty\}$ and $B := \{0, 1, \infty\}$. The maps

$$F_\alpha : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B) \quad \text{and} \quad F_\beta : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$$

are Hurwitz equivalent if and only if $\alpha = \beta$. Indeed,

$$F_\alpha : (\mathbb{P}^1 - \{\omega, \infty\}) \rightarrow (\mathbb{P}^1 - \{0, \infty\})$$

is a covering map of degree d ; it is therefore cyclic with covering group isomorphic to $\mathbb{Z}/d\mathbb{Z}$. Any point p in the range cylinder $\mathbb{P}^1 - \{0, \infty\}$ has d inverse images in the domain cylinder $\mathbb{P}^1 - \{\omega, \infty\}$. Connect the d inverse images of p with a simple closed curve $\gamma \subset \mathbb{P}^1 - \{\omega, \infty\}$. This curve γ can be used to define a cyclic order of the d inverse images of p , invariant under any (orientation-preserving) homeomorphism of the domain cylinder. The point $1 \in B$ has d inverse images in the domain cylinder, exactly two of which are marked points in A . The d th root of unity parameter indicates the “circular distance” given by the ordering from γ . This quantity is preserved under any homeomorphism and is therefore a Hurwitz invariant.

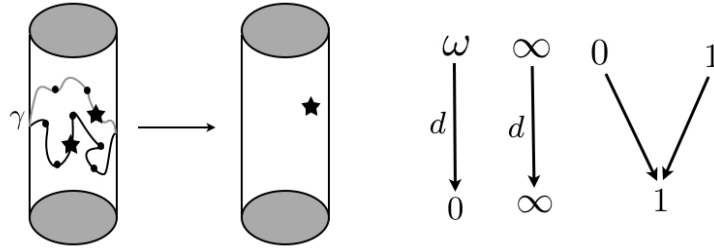


FIGURE 1. The maps F_α are cyclic covers of degree d . The star in the range indicates a marked point in B , and the points in the domain connected by γ are its inverse images. Two of these inverse images are marked points in A , indicated with stars. On the right, we see how F_α maps the points in A to the points in B .

The following example is presented in a dynamical context, but this is not necessary.

Example 2.11. Let $f : (S^2, P) \rightarrow (S^2, P)$ be a ramified cover of degree d , where the domain and range spheres are identified. Let P be the postcritical set of f , and suppose that P is finite. Consider the composition $f \circ f : (S^2, P) \rightarrow (S^2, P)$ which is a ramified cover of degree d^2 , with postcritical set P . Let $g : (S^2, P) \rightarrow (S^2, P)$ be a ramified cover of degree d^2 with postcritical set P which cannot be expressed as a second iterate of any map $(S^2, P) \rightarrow (S^2, P)$. Then $f \circ f$ and g are not Hurwitz equivalent.

3. MAPS ON MODULI SPACE

The only known examples for which the pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ descends to a map $\mathcal{M}_B \rightarrow \mathcal{M}_A$ are those for which σ_f is constant. For instance, the pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ associated to the rational function

$$f(z) = 2i \left(z^2 - \frac{1+i}{2} \right)^2, \quad A = B = \{0, -1, 1, \infty\}$$

is constant as proved in [BEKP].

As previously mentioned, the map $\rho_B : \mathcal{W}_f \rightarrow \mathcal{M}_B$ is not injective if $\deg(f) > 1$, however, for certain admissible $f : (S^2, A) \rightarrow (S^2, B)$, the map $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ is injective. Although the map σ_f does not descend to the level of moduli space, remarkably, an inverse of it does, that is, there exists a *moduli space map* $g_f : \mathcal{M}_A \dashrightarrow \mathcal{M}_B$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{T}_B & \xrightarrow{\sigma_f} & \mathcal{T}_A \\ \pi_B \downarrow & & \downarrow \pi_A \\ \mathcal{M}_B & \xleftarrow{g_f} & \mathcal{M}_A \end{array}$$

The map g_f is only defined on $\mathcal{M}'_A := \pi_A \circ \sigma_f(\mathcal{T}_B) \subseteq \mathcal{M}_A$.

3.1. Examples. We compute $\mathcal{W}_f$ for two examples.

Example 3.1. This example was first computed in [BN]. Let $R : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the “rabbit polynomial” given by $R(z) = z^2 + c_r$ where c_r is the root of the polynomial $(c^2 + c)^2 + c = 0$ with positive imaginary part. Let

$$A = B = \{0, c_r, c_r^2 + c_r, \infty\};$$

the polynomial R maps A to B in the following way, where 0 and ∞ are critical points of local degree 2.

$$\begin{array}{cccc} 0 & c_r & c_r^2 + c_r & \infty \\ \downarrow 2 & \downarrow & \downarrow & \downarrow 2 \\ c_r & c_r^2 + c_r & 0 & \infty \end{array}$$

We wish to compute $\mathcal{W}_R$; to this end, we first normalize. Let $\phi : (\mathbb{P}^1, B) \rightarrow (\mathbb{P}^1, \phi(B))$ be a homeomorphism so that

- $\phi(0) = 0, \phi(\infty) = \infty$, and $\phi(c_r) = 1$, and
- for notation, set $y := \phi(c_r^2 + c_r)$.

There is a unique $\psi : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, \psi(A))$, normalized so that $\psi(0) = 0, \psi(\infty) = \infty$, and $\psi(c_r) = 1$, such that

$$F_\phi := \phi \circ R \circ \psi^{-1} : (\mathbb{P}^1, \psi(A)) \rightarrow (\mathbb{P}^1, \phi(B))$$

is rational. For notation, set $x := \psi(c_r^2 + c_r)$. Implicit in this choice of normalization is an identification of $\mathcal{M}_B = \mathcal{M}_A$ with $\mathbb{P}^1 - \{0, 1, \infty\}$.

We have $\sigma_R([\phi]) = [\psi]$, $\pi_B([\phi]) = y$, and $\pi_A([\psi]) = x$. With respect to this normalization, a point in $\mathcal{W}_R$ is represented by a triple (F_ϕ, y, x) . Using the conventions above, we find a normal form for the rational function F_ϕ . The rational map F_ϕ has degree 2, and it maps $\psi(A)$ to $\phi(B)$ in the following way:

$$F_\phi : 0 \mapsto 1 \quad F_\phi : \infty \mapsto \infty \quad F_\phi : x \mapsto 0 \quad \text{and} \quad F_\phi : 1 \mapsto y,$$

where 0 and ∞ are critical points of local degree 2. A normal form for such a rational map is $F_\phi(t) = Ct^2 + 1$, where C is a parameter. Imposing the condition that $x \mapsto 0$ implies that $C = -1/x^2$, and imposing the condition that $F_\phi(1) = y$ yields the relation $1 - 1/x^2 = y$. In summary,

$$F_\phi(t) = 1 - t^2/x^2 \quad \text{and} \quad y = 1 - 1/x^2.$$

The triple

$$(1 - t^2/x^2, 1 - 1/x^2, x), \quad x \in \mathbb{P}^1 - \{0, 1, \infty, -1\},$$

represents a point in $\mathcal{W}_R$. The map $\delta_A : (1 - t^2/x^2, 1 - 1/x^2, x) \mapsto x$ is injective, so a moduli space map $g_R : x \mapsto y$ exists

$$g_R : x \mapsto 1 - 1/x^2.$$

The map g_R extends to a map $G_R : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The dynamics of this map provides insights into the dynamics of $\sigma_R : \mathcal{T}_B \rightarrow \mathcal{T}_A$ as discussed in Section 7.

No moduli space map exists in the following example; instead, $\mathcal{W}_f$ induces a *correspondence* on the level of moduli space.

Example 3.2. Let $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the rational function defined by

$$F(z) = \frac{3z^2}{2z^3 + 1},$$

and set $A = B = \{0, 1, \omega, \bar{\omega}\}$ where

$$\omega = -1/2 + i\sqrt{3}/2 \quad \text{and} \quad \bar{\omega} = -1/2 - i\sqrt{3}/2.$$

The set A consists of the four simple critical points of F , and the set B consists of the corresponding critical values. The map F sends A to B in the following way.

$$\begin{array}{cccc} 0 & 1 & \omega & \bar{\omega} \\ \downarrow 2 & \downarrow 2 & \downarrow 2 & \downarrow 2 \\ 0 & 1 & \bar{\omega} & \omega \end{array}$$

Computing as above, we find that $\delta_A : \mathcal{W}_F \rightarrow \mathcal{M}_A$ is a ramified double cover (of its image), and $\rho_B : \mathcal{W}_F \rightarrow \mathcal{M}_B$ is a covering map of degree 4. The explicit computation for this example can be found in [BEKP].

We will explore these moduli space maps further in Section 5.3, but in a dynamical setting. When the map exists, we will compactify moduli space and try to extend the moduli space map to the compactification. We will use the $\mathbb{P}^n$ compactification for our purposes, but a priori, there is no reason to prefer this compactification to others, like $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. In fact, there are good reasons to use the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$, as we discuss in the following sections. However, in the dynamical setting, we will restrict much

of our attention to topological polynomials where the $\mathbb{P}^n$ compactification is arguably more natural.

4. DYNAMICS

We now shift focus to the dynamical setting and identify the domain and range spheres for a ramified cover $f : S^2 \rightarrow S^2$. Let Ω_f be the critical set of f , and let P_f be the postcritical set of f ; that is,

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f).$$

We call f a *Thurston map* if P_f is a finite set. We will restrict our attention to Thurston maps which are admissible (see Section 1.1.1). Note that this excludes precisely those Thurston maps f for which $|P_f| = 2$.

A Thurston map f is a *topological polynomial* if there is a point $\omega \in \Omega_f$, so that $f^{-1}(\{\omega\}) = \{\omega\}$.

4.1. Thurston's Topological Characterization of Rational Maps. The pair of orientation-preserving homeomorphisms

$$(\phi_0 : (S^2, P_f) \rightarrow (S^2, P_g), \quad \phi_1 : (S^2, P_f) \rightarrow (S^2, P_g))$$

gives a *combinatorial equivalence* for the maps $f : (S^2, P_f) \rightarrow (S^2, P_f)$ and $g : (S^2, P_g) \rightarrow (S^2, P_g)$ if $\phi_0 \circ f = g \circ \phi_1$, and if ϕ_0 is isotopic to ϕ_1 relative to P_f . In particular, we have the following commutative diagram.

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi_1} & (S^2, P_g) \\ f \downarrow & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\phi_0} & (S^2, P_g) \end{array}$$

Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P . Thurston's theorem provides a purely topological criterion to test whether f is combinatorially equivalent to a rational map F . This criterion is stated in terms of curve systems on $S^2 - P$.

A simple closed curve γ is *nonperipheral* if each component of $S^2 - \gamma$ contains no fewer than two points of P . A *multicurve*, $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, is a set of simple, closed, disjoint, nonhomotopic, essential, nonperipheral curves in $S^2 - P$. (By essential, we mean not nullhomotopic). A multicurve Γ is *f-stable* if for all $\gamma \in \Gamma$, every nonperipheral component of $f^{-1}(\gamma)$ is homotopic in $S^2 - P$ to a curve in Γ . Following notation in [BFH], let γ_{ij}^α be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i rel P (where we index the components with α), and let d_{ij}^α be the degree of the map $f|_{\gamma_{ij}^\alpha} : \gamma_{ij}^\alpha \rightarrow \gamma_j$. The *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$ is

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{ij}^\alpha} \gamma_i.$$

The matrix of f_Γ has nonnegative entries, so there is a leading eigenvalue which is real and non-negative by the Perron-Frobenius theorem. Let $\lambda(f_\Gamma)$ denote this leading eigenvalue.

Theorem 4.1 (Thurston). *A Thurston map f with hyperbolic orbifold¹ is combinatorially equivalent to a rational function if and only if for any f -stable multicurve Γ , $\lambda(f_\Gamma) < 1$. In that case, the rational function is unique up to conjugation by a Möbius transformation.²*

An f -stable multicurve with $\lambda(f_\Gamma) \geq 1$ is called a *Thurston obstruction*. If the Thurston map f is not combinatorially equivalent to a rational map, then f is said to be *obstructed*.

Consider the pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ induced by $f : (S^2, P) \rightarrow (S^2, P)$. One proves Theorem 4.1 by showing that the multicurve criterion given in the theorem is equivalent to the existence of a fixed point of σ_f ; that is, the Thurston map f is combinatorially equivalent to a rational map if and only if the pullback map σ_f has a fixed point in $\mathcal{T}_P$.

4.2. Fixed points. Given a point $\tau_0 \in \mathcal{T}_P$, define the associated sequence of iterates $\tau_i := \sigma_f(\tau_{i-1})$ for $i \geq 1$. It follows from a theorem of Royden in [Roy] that σ_f is weakly contracting for the Teichmüller metric on $\mathcal{T}_P$. Since the space $\mathcal{T}_P$ is connected, the sequence of iterates τ_i either converges to the unique fixed point of σ_f , or it diverges and σ_f has no fixed point. As a consequence of Proposition 5.1 and Lemma 5.2 in [DH], τ_i converges in $\mathcal{T}_P$ if and only if the sequence $\omega_f(\tau_i) \in \mathcal{W}_f$ converges. In fact, the amount by which σ_f contracts at a point $\tau \in \mathcal{T}_P$ depends only on the point $\omega_f(\tau) \in \mathcal{W}_f$, see [DH].

As previously discussed, the Hurwitz space $\mathcal{W}_f$, and the maps

$$\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A \quad \text{and} \quad \rho_B : \mathcal{W}_f \rightarrow \mathcal{M}_B \quad \text{where} \quad A = B = P$$

induce a correspondence on the level of moduli space.

$$\begin{array}{ccc} & \mathcal{W}_f & \\ \rho_P \swarrow & & \searrow \delta_P \\ \mathcal{M}_P & & \mathcal{M}_P \end{array}$$

While the fixed point of $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ (if it exists) corresponds to a conjugacy class of rational maps which are combinatorially equivalent to f , the “fixed point(s)” of this correspondence induced by the maps ρ_P and δ_P correspond to conjugacy classes of rational maps which are combinatorially equivalent to a map $g : (S^2, P) \rightarrow (S^2, P)$, which is Hurwitz equivalent to f . To be more precise, consider the equalizer of the pair of maps (ρ_P, δ_P) . By definition, this is the locus

$$\text{Eq}(\rho_P, \delta_P) := \{w \in \mathcal{W}_f : \rho_P(w) = \delta_P(w)\}.$$

If f is combinatorially equivalent to a rational map, σ_f has a unique fixed point $\tau \in \mathcal{T}_P$ (provided f is not a *flexible Lattès map*), and clearly $\omega_f(\tau) \in \text{Eq}(\rho_P, \delta_P)$. However, if f is obstructed, $\text{Eq}(\rho_P, \delta_P)$ may be empty, see Example 4.5. The equalizer $\text{Eq}(\rho_P, \delta_P) \subseteq \mathcal{W}_f$ is evidently a Hurwitz invariant.

¹For more information about the orbifold associated to a Thurston map, see [DH].

²This is known as *Thurston rigidity*.

Remark 4.2. The notion of (P, P) -Hurwitz equivalence depends on what the points of P actually are, whereas combinatorial equivalence allows for the points in respective marked sets to be different. In particular, let $f : (S^2, P_f) \rightarrow (S^2, P_f)$ and $g : (S^2, P_g) \rightarrow (S^2, P_g)$ be Thurston maps, and suppose that f and g are combinatorially equivalent. Choose an orientation-preserving homeomorphism $h : (S^2, P_g) \rightarrow (S^2, P_f)$; then the maps $f : (S^2, P_f) \rightarrow (S^2, P_f)$ and $h \circ g \circ h^{-1} : (S^2, P_f) \rightarrow (S^2, P_f)$ are (P_f, P_f) -Hurwitz equivalent.

Proposition 4.3. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P , and let $[(F, i_P, j_P)] \in \text{Eq}(\rho_P, \delta_P)$. Then there is $\mu \in \text{Aut}(\mathbb{P}^1)$ so that $F \circ \mu$ is critically finite and combinatorially equivalent to a Thurston map $g : (S^2, P) \rightarrow (S^2, P)$, which is Hurwitz equivalent to f .*

Proof. Because $[(F, i_P, j_P)] \in \text{Eq}(\rho_P, \delta_P)$, $i_P : P \hookrightarrow \mathbb{P}^1$, and $j_P : P \hookrightarrow \mathbb{P}^1$ represent the same point in $\mathcal{M}_P$, therefore there exists a $\mu \in \text{Aut}(\mathbb{P}^1)$ so that $j_P = \mu \circ i_P$. The triple $(F, i_P, \mu \circ i_P)$ is equivalent to the triple $(F \circ \mu, i_P, i_P)$ in $\mathcal{W}_f$. The rational map

$$F \circ \mu : (\mathbb{P}^1, i_P(P)) \rightarrow (\mathbb{P}^1, i_P(P))$$

is critically finite, with postcritical set $i_P(P)$. Because $[(F \circ \mu, i_P, i_P)] \in \mathcal{W}_f$, there is a homeomorphism $\phi : (S^2, P) \rightarrow (\mathbb{P}^1, \phi(P))$ and a homeomorphism $\psi : (S^2, P) \rightarrow (\mathbb{P}^1, \psi(P))$ such that

- $\phi|_P = \psi|_P = i_P$, and
- $\phi \circ f \circ \psi^{-1} = F \circ \mu$.

Let $h : (\mathbb{P}^1, i_P(P)) \rightarrow (S^2, P)$ be a homeomorphism sending $i_P(P)$ to P . Then the map $g := h \circ (F \circ \mu) \circ h^{-1} : (S^2, P) \rightarrow (S^2, P)$ is a Thurston map with postcritical set P , which is combinatorially equivalent to $F \circ \mu$, and

$$f = (h \circ \phi)^{-1} \circ g \circ (h \circ \psi),$$

so f and g are Hurwitz equivalent. □

Conversely, we have the following proposition.

Proposition 4.4. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P . If the Thurston map $g : (S^2, P) \rightarrow (S^2, P)$ is Hurwitz equivalent to $f : (S^2, P) \rightarrow (S^2, P)$, and if g is combinatorially equivalent to a critically finite rational map $F : (\mathbb{P}^1, P_F) \rightarrow (\mathbb{P}^1, P_F)$, then there is a point in $[(F, i_P, j_P)] \in \text{Eq}(\rho_P, \delta_P)$ such that*

$$(i_P \circ f)|_P = (F \circ j_P)|_P \quad \text{and} \quad (i_P \circ f)(P) = (F \circ j_P)(P) = P_F.$$

Proof. If g is combinatorially equivalent to F , there are homeomorphisms (ϕ_0, ϕ_1) providing a combinatorial equivalence for (g, F) , and if f is Hurwitz equivalent to g , there are homeomorphisms (h, h') providing a Hurwitz equivalence for (f, g) ; that is, the following diagrams commute

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{h'} & (S^2, P) \\ f \downarrow & & \downarrow g \\ (S^2, P) & \xrightarrow{h} & (S^2, P) \end{array} \quad \begin{array}{ccc} (S^2, P) & \xrightarrow{\phi_1} & (\mathbb{P}^1, P_F) \\ g \downarrow & & \downarrow F \\ (S^2, P) & \xrightarrow{\phi_0} & (\mathbb{P}^1, P_F) \end{array}$$

and ϕ_0 and ϕ_1 are isotopic relative to P . As a consequence,

$$f = (\phi_0 \circ h)^{-1} \circ F \circ \phi_1 \circ h', \quad \text{thus} \quad [(F, \phi_0 \circ h|_P, \phi_1 \circ h'|_P)] \in \mathcal{W}_f.$$

Furthermore, $\phi_0 \circ h|_P = \phi_1 \circ h'|_P$ since ϕ_0 and ϕ_1 are isotopic relative to P , and h and h' fix P pointwise, moreover

$$\phi_0(h(P)) = \phi_1(h'(P)) = P_F,$$

and we have

$$[(F, \phi_0 \circ h|_P, \phi_1 \circ h'|_P)] \in \text{Eq}(\rho_P, \delta_P).$$

□

Example 4.5. Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map of degree 3 with postcritical set $P := \{a, b, c, d\}$. Suppose that f has the following ramification portrait;

$$a \curvearrowright_2 \quad b \curvearrowright_2 \quad c \curvearrowright_2 \quad d \curvearrowright_2$$

that is, f has four simple critical points, each of which is fixed. Note that such a ramified self-cover of the sphere exists, see [BBLPP]. If f were combinatorially equivalent to a rational map F , then F would be a degree 3 rational map with four fixed critical points. A rational map of degree 3 has exactly four fixed points, so if such a rational F exists, then all of its fixed points would be superattracting. It follows from the holomorphic index formula that this is impossible, so any Thurston map f with this ramification portrait is necessarily obstructed. Therefore the set $\text{Eq}(\rho_P, \delta_P) \subseteq \mathcal{W}_f$ is empty.

In [BN], L. Bartholdi and V. Nekrashevych solve the “twisted rabbit problem”: consider the rabbit polynomial R as discussed in Example 3.1. The postcritical set of R is the set $P_R := \{0, \infty, c_r, c_r^2 + c_r\}$. Consider a curve γ separating two of these postcritical points from the other two, and let the map $D_\gamma : (\mathbb{P}^1, P_R) \rightarrow (\mathbb{P}^1, P_R)$ be a Dehn twist about γ . For each $n \geq 0$, define the Thurston map $f_n := D_\gamma^n \circ R : (\mathbb{P}^1, P_R) \rightarrow (\mathbb{P}^1, P_R)$. By a theorem of S. Levy [Le], each of the f_n is combinatorially equivalent to a rational map, evidently a quadratic polynomial whose critical point is periodic of period 3. Up to affine conjugacy, there are exactly three such polynomials: the rabbit $R(z) = z^2 + c_r$, the corabbit $C(z) = z^2 + \bar{c}_r$, and the airplane, $A(z) = z^2 + c_a$, where c_a is the (nonzero) real root of $(c^2 + c)^2 + c = 0$. That is, there are three combinatorial equivalence classes of Thurston maps with the following ramification portrait.

$$\cdot \xrightarrow{2} \cdot \xrightarrow{\quad} \cdot \quad \cdot \curvearrowright_2$$

The twisted rabbit problem asks which polynomial f_n is combinatorially equivalent to, as a function of n . Clearly, f_n is Hurwitz equivalent to f_m . Let $P_A = \{0, \infty, c_a, c_a^2 + c_a\}$ be the postcritical set of the airplane polynomial and let $P_C = \{0, \infty, \bar{c}_r, \bar{c}_r^2 + \bar{c}_r\}$ be the postcritical set of the corabbit polynomial. Let $h_A : (\mathbb{P}^1, P_A) \rightarrow (\mathbb{P}^1, P_R)$ and $h_C : (\mathbb{P}^1, P_C) \rightarrow (\mathbb{P}^1, P_R)$ be orientation-preserving homeomorphisms, and define the maps

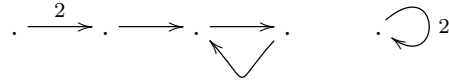
$$g_A := h_A \circ A \circ h_A^{-1} : (\mathbb{P}^1, P_R) \rightarrow (\mathbb{P}^1, P_R) \quad \text{and} \quad g_C := h_C \circ C \circ h_C^{-1} : (\mathbb{P}^1, P_R) \rightarrow (\mathbb{P}^1, P_R).$$

Then g_A is combinatorially equivalent to the airplane polynomial, and g_C is combinatorially equivalent to the corabbit polynomial, and both g_A and g_C are (P_R, P_R) -Hurwitz equivalent to the rabbit polynomial. Corollary 2.7 implies that

$$\mathcal{W}_R = \mathcal{W}_{g_A} = \mathcal{W}_{g_C}.$$

Moreover, $\text{Eq}(\rho_{P_R}, \delta_{P_R})$ consists of three points, one for each of the three combinatorial equivalence classes in this Hurwitz class, represented by the maps R, g_A, g_C .

In [BN], L. Bartholdi and V. Nekrashevych also prove that for polynomial $q : z \mapsto z^2 + i$ with postcritical set P , there exists infinitely many homeomorphisms $h : (S^2, P) \rightarrow (S^2, P)$ fixing P pointwise such that the Thurston map $f_h := h \circ q$ is obstructed. In fact, they prove that there are infinitely many combinatorial equivalence classes of maps with the following ramification portrait.



Exactly two of these combinatorial classes are unobstructed, and they are represented by $z \mapsto z^2 + i$ and $z \mapsto z^2 - i$. Moreover, if f and g are two Thurston maps with postcritical set P , realizing the ramification portrait above, then they are Hurwitz equivalent. Hence $\mathcal{W}_f = \mathcal{W}_g$, and $\text{Eq}(\rho_P, \delta_P)$ consists of exactly two points, one for each of the two unobstructed combinatorial equivalence classes in this Hurwitz class.

5. CRITICALLY FINITE ENDOMORPHISMS OF $\mathbb{P}^n$

In this section, we continue in the dynamical setting. We first provide sufficient conditions for which a moduli space map exists. We then discuss compactifications of $\mathcal{M}_P$, asking if our map extends holomorphically to different compactifications. In the case of the $\mathbb{P}^n$ compactification, the maps we consider do extend holomorphically, providing interesting dynamical systems in their own right.

5.0.1. Conventions. We now establish some conventions for the convenience of the exposition in the following sections. The Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ will be a topological polynomial with critical set Ω and postcritical set $P = \{p_0, \dots, p_{n+2}\}$. Suppose that $f^{-1}(p_{n+2}) = \{p_{n+2}\}$.

We say that the homeomorphism $\phi : (S^2, P) \rightarrow (\mathbb{P}^1, \phi(P))$ is *normalized* if

$$\phi(p_0) = 0, \quad \phi(p_1) = 1, \quad \text{and} \quad \phi(p_{n+2}) = \infty.$$

In this way, we identify the point $[\phi|_P] \in \mathcal{M}_P$ with $(z_1, \dots, z_n) \in \mathbb{C}^n$ where $z_i := \phi(p_{i+1})$, and $\mathcal{M}_P \approx \mathbb{C}^n - \Delta$ where Δ is the locus where at least two coordinates coincide.

5.1. Sufficient conditions: polynomials with periodic critical points. If $\Omega \subseteq P$, then all critical points of f are periodic, $f|_P : P \rightarrow P$ is a bijection, and a moduli space map exists.

Proposition 5.1. *If f is a topological polynomial such that $\Omega \subseteq P$, then $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is injective.*

Proof. We must compute $\mathcal{W}_f$ for such an f . Let $\mu : [0, n+1] \rightarrow [0, n+1]$ be the permutation defined by

$$p_{\mu(i)} = f(p_i),$$

which is induced by f , and define the inverse permutation $\nu := \mu^{-1}$. For $i \in [0, n+1]$, let m_i be the multiplicity of p_i as a critical point of f (if $p_i \notin \Omega$, then $m_i := 0$).

Let $\phi : (S^2, P) \rightarrow (\mathbb{P}^1, \phi(P))$, and $\psi : (S^2, P) \rightarrow (\mathbb{P}^1, \psi(P))$ be normalized homeomorphisms so that the following diagram commutes and F_ϕ is a rational map of degree $d := \deg(f)$.

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P)) \\ f \downarrow & & \downarrow F_\phi \\ (S^2, P) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P)) \end{array}$$

For notation, set $x_i := \psi(p_{i+1})$ and $y_i := \phi(p_{i+1})$ for $i \in [1, n]$. We are now in a position to compute $\mathcal{W}_f$ which consists of triples

$$(F_\phi, (y_1, \dots, y_n), (x_1, \dots, x_n)).$$

Note that F_ϕ is a polynomial of degree d whose critical points are contained in the set $\{0, 1, x_1, \dots, x_n\}$. The point $\psi(p_i) \in \{0, 1, x_1, \dots, x_n\}$ is critical if and only if $m_i > 0$.

By the conventions established above,

$$F_\phi(\psi(p_i)) = \phi(p_{\mu(i)}) \quad \text{or} \quad F_\phi(\psi(p_{\nu(i)})) = \phi(p_i), \quad \text{for } i \in [0, n+1],$$

so

$$F_\phi(\psi(p_{\nu(0)})) = 0 \quad \text{and} \quad F_\phi(\psi(p_{\nu(1)})) = 1.$$

A normal form for F_ϕ is

$$F_\phi(t) = A \cdot \int_{\psi(p_{\nu(0)})}^t \left(w^{m_0} \cdot (w-1)^{m_1} \cdot \prod_{i=1}^n (w-x_i)^{m_{i+1}} \right) dw,$$

where the parameter A is determined by the condition

$$F_\phi(\psi(p_{\nu(1)})) = 1.$$

Thus the polynomial F_ϕ is completely determined by $(x_1, \dots, x_n)$. Moreover for each $i \in [1, n]$ we have

$$y_i = \phi(p_i) = F_\phi(\psi(p_{\nu(i)})), \quad \text{where } \psi(p_{\nu(i)}) \in \{0, 1, x_1, \dots, x_n\},$$

thus $(y_1, \dots, y_n)$ is completely determined by $(x_1, \dots, x_n)$ as well. In the coordinates given by our normalization,

$$\delta_P : (F_\phi, (y_1, \dots, y_n), (x_1, \dots, x_n)) \mapsto (x_1, \dots, x_n),$$

which is injective, and hence a moduli space map

$$g_f : (x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$$

exists. This is well-defined because for each $i \in [1, n]$, there is a unique j such that $F_\phi(\psi(p_j)) = y_i$, namely $j = \nu(i)$. $\square$

The map $g_f : (x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$ is only defined on the subset in $\mathbb{C}^n$ corresponding to $\pi_P(\sigma_f(\mathcal{T}_P)) \subseteq \mathcal{M}_P$; we explicitly identify this locus in Corollary 5.16.

The only data about f that we used to compute $\mathcal{W}_f$ were:

- (1) the combinatorics of $f|_P : P \rightarrow P$ (the permutation μ), and

(2) the local degree information $\text{loc deg}(f)|_p$ for $p \in P$.

Any other Thurston map $g : (S^2, P) \rightarrow (S^2, P)$ which is a topological polynomial satisfying (1), such that

$$g|_P = f|_P \quad \text{and} \quad \forall p \in P, \text{loc deg}(g)|_p = \text{loc deg}(f)|_p$$

would yield an *identical* computation to that above; this fact coupled with Corollary 5.16 imply that $\mathcal{W}_f = \mathcal{W}_g$. Hence there is a unique Hurwitz class of Thurston maps f satisfying the three data points above.

Corollary 5.2. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P and critical set Ω such that $\Omega \subseteq P$, and suppose that there is $\omega \in \Omega$ with*

$$\text{loc deg } f|_\omega = \text{deg}(f).$$

Then $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is injective.

Proof. The map $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is a (nondynamical) Hurwitz invariant; such an f is Hurwitz equivalent to a topological polynomial $g : (S^2, P) \rightarrow (S^2, P)$ satisfying the hypotheses of Proposition 5.1. $\square$

5.2. Sufficient conditions: unicritical polynomials. Suppose that $f : (S^2, P) \rightarrow (S^2, P)$ is unicritical; that is the critical set Ω consists of exactly two points. Either $\Omega \subseteq P$, or there is exactly one point $\omega \in \Omega$ which is not in the set P . If $\Omega \subseteq P$, then a moduli space map exists by Proposition 5.1. Suppose on the contrary that there is a critical point $\omega \notin P$. The point ω is necessarily preperiodic for the map f : there is a smallest integer $m \geq 1$ such that $f^{\circ m}(\omega)$ is periodic of period l for some $l \geq 1$.

Proposition 5.3. *If f is a topological polynomial such that $|\Omega| = 2$, then $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is injective.*

Proof. Suppose that $\omega \notin P$, and let $m \geq 2$ be the preperiod of ω , and $l \geq 1$ be the period of $f^{\circ m}(\omega)$. Suppose also that

$$f(\omega) = p_0, \quad \text{and} \quad f(p_i) = p_{i+1} \quad \text{for} \quad i \in [0, n].$$

Given these conventions,

$$p_{m-1} = f^{\circ m}(\omega), \quad f(p_{n+1}) = p_{m-1}, \quad \text{and} \quad m + l - 2 = n + 1.$$

The point p_{m-1} is the unique postcritical point with two inverse images which are postcritical; that is $f^{-1}(\{p_{m-1}\}) \cap P = \{p_{m-2}, p_{n+1}\}$. We illustrate this schematically with the following ramification portrait, where the length of the periodic cycle is l .

$$\omega \xrightarrow{d} p_0 \longrightarrow \cdots \longrightarrow p_{m-2} \longrightarrow p_{m-1} \longrightarrow \cdots \longrightarrow p_{n+1} \xrightarrow{d} p_{n+2} \xrightarrow{d} \cdots \xrightarrow{d} p_{m-1}$$

Suppose that $\phi : (S^2, P) \rightarrow (\mathbb{P}^1, \phi(P))$, and $\psi : (S^2, P) \rightarrow (\mathbb{P}^1, \psi(P))$ are normalized homeomorphisms so that the following diagram commutes, and F_ϕ is a rational map of degree

$d := \deg(f)$.

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P)) \\ f \downarrow & & \downarrow F_\phi \\ (S^2, P) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P)) \end{array}$$

For notation, set $x_i := \psi(p_{i+1})$ and $y_i := \phi(p_{i+1})$ for $i \in [1, n]$. The map F_ϕ is a unicritical polynomial of degree d , whose critical point is $\alpha := \psi(\omega)$. By the conventions established above, $F_\phi(\alpha) = 0$, $F_\phi(0) = 1$, and a normal form for F_ϕ is

$$F_\phi(t) = \left(1 - \frac{t}{\alpha}\right)^d.$$

Furthermore,

$$F_\phi(\psi(p_i)) = \phi(p_{i+1}), \quad \text{and} \quad F_\phi(\psi(p_{m-2})) = F_\phi(\psi(p_{n+1})) = \phi(p_{m-1}).$$

Imposing these conditions, we eliminate the parameter α

$$\left(1 - \frac{\psi(p_{m-2})}{\alpha}\right)^d = \left(1 - \frac{\psi(p_{n+1})}{\alpha}\right)^d \implies \alpha = \frac{\psi(p_{m-2}) - \beta \cdot \psi(p_{n+1})}{1 - \beta}$$

where $\beta \neq 1$ is a d th root of unity determined by the Hurwitz class of f (see Example 2.10), and $\psi(p_{m-2}), \psi(p_{n+1}) \in \{0, 1, x_1, \dots, x_n\}$. We now compute $\mathcal{W}_f$ which consists of the triples

$$(F_\phi, (y_1, \dots, y_n), (x_1, \dots, x_n)).$$

A point $(x_1, \dots, x_n)$, together with a choice of the root of unity β , uniquely determine the rational map F_ϕ . Moreover $y_{i+1} = F_\phi(x_i)$ for $i \in [1, n]$, so $(y_1, \dots, y_n)$ is uniquely determined by F_ϕ and $(x_1, \dots, x_n)$. Given the root of unity β , the map

$$\delta_P : (F_\phi, (y_1, \dots, y_n), (x_1, \dots, x_n)) \mapsto (x_1, \dots, x_n)$$

is injective, so a moduli space map $g_f : (x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$ exists. $\square$

In the computation above, the data used to compute F_ϕ were

- (1) the combinatorics of $f|_P : P \rightarrow P$: the preperiod m and the period l , and
- (2) a choice of a d th root of unity $\beta \neq 1$ determined by the Hurwitz class of f .

Corollary 5.4. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P and critical set Ω . Suppose that f is bicritical, and that $\Omega \cap P \neq \emptyset$. Then the map $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is injective.*

Proof. The map $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is a (nondynamical) Hurwitz invariant; such an f is Hurwitz equivalent to a unicritical topological polynomial $g : (S^2, P) \rightarrow (S^2, P)$ satisfying the hypotheses of Proposition 5.3. $\square$

5.3. Compactifying $\mathcal{M}_P$. Recall that $P = \{p_0, \dots, p_{n+2}\}$, and let a, b, c be three distinct points of $\mathbb{P}^1$. Then any point of $\mathcal{M}_P$ is uniquely represented by $\varphi : P \hookrightarrow \mathbb{P}^1$ such that $\varphi(p_0) = a, \varphi(p_1) = b$, and $\varphi(p_{n+2}) = c$; the point $[\varphi] \in \mathcal{M}_P$ is uniquely determined by

$$(x_1, \dots, x_n) \in \prod_{j=1}^n (\mathbb{P}^1 - \{a, b, c\}), \quad \text{where } x_i := \varphi(p_{i+1}).$$

One possible compactification of $\mathcal{M}_P$ is clearly $\prod_{j=1}^n \mathbb{P}^1$. If either a, b , or c is equal to ∞ , then

$$(x_1, \dots, x_n) \in \prod_{j=1}^n \mathbb{C} - \{a, b\}.$$

In this case, another compactification is $\mathbb{P}^n$.

5.3.1. A specific normalization. Since we will be concerned with topological polynomials, we will normalize in this setting so that $\varphi(p_{n+2}) = \infty$, and $\varphi(p_0) = 0$. Thus $[\varphi] \in \mathcal{M}_P$ is determined (up to multiplication by a nonzero complex number) by $x_i := \varphi(p_i)$, for $i \in [1, n+1]$. In this way, we identify $[\varphi] \in \mathcal{M}_P$ with $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n - \Delta$, where Δ is the set of $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n$ such that at least two coordinates coincide, or at least one coordinate is equal to 0; we define $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ to be the identification induced by this particular normalization.

Proposition 5.5. *The locus $\Delta \subset \mathbb{P}^n$ consists of $(n^2 + 3n + 2)/2$ hyperplanes.*

Proof. The locus Δ consists of a hyperplane for each $x_i = 0$, and a hyperplane for each $x_i = x_j$; in total, we have

$$n + 1 + \binom{n+1}{2} = \frac{n^2 + 3n + 2}{2}.$$

□

Changing the normalization amounts to transforming this copy of $\mathbb{P}^n$ by the corresponding birational transformation.

Perhaps the most symmetric compactification of $\mathcal{M}_P$ is the Deligne-Mumford compactification; that is, the stable curves compactification $\overline{\mathcal{M}}_{0,n}$. This can be obtained as a “sequential blowup” of $\mathbb{P}^n$, see [Ki], [Ll].

Remark 5.6. In [Se], N. Selinger proved that the pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ extends to a continuous map

$$\widehat{\sigma}_f : \widehat{\mathcal{T}}_P \rightarrow \widehat{\mathcal{T}}_P$$

of the *augmented Teichmüller space* $\widehat{\mathcal{T}}_P$. The dynamics of $\widehat{\sigma}_f$ on the boundary strata of $\widehat{\mathcal{T}}_P$ give information about the map $f : (S^2, P) \rightarrow (S^2, P)$.

The quotient of $\widehat{\mathcal{T}}_P$ by the action of the mapping class group Mod_P is isomorphic to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$. This strongly suggests that we should use the Deligne-Mumford compactification of moduli space. We will see in Section 5.4 and Section 5.5 that the particular moduli space maps we consider extend holomorphically to the $\mathbb{P}^n$

compactification of moduli space. I know of no examples $f : (S^2, P) \rightarrow (S^2, P)$ such that $|P| > 4$, for which a moduli space map g_f (if it exists) extends holomorphically to the Deligne-Mumford compactification of moduli space.

5.4. Endomorphisms induced by polynomials with periodic critical points. We now prove that the maps obtained in Proposition 5.1 extend holomorphically to the $\mathbb{P}^n$ compactification of $\mathcal{M}_P$.

Proposition 5.7. *If f is a topological polynomial such that $\Omega \subseteq P$, then the moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$. The algebraic degree of G_f is equal to the topological degree of f .*

Proof. Recall the permutation $\mu : [0, n+1] \rightarrow [0, n+1]$ defined by $p_{\mu(i)} = f(p_i)$, with inverse $\nu := \mu^{-1}$, and for $i \in [0, n+1]$, m_i is the multiplicity of p_i as a critical point of f .

Set $x_0 := 0$, and let $Q \in \mathbb{C}[x_1, \dots, x_{n+1}, t]$ be the homogeneous polynomial of degree $d := \deg(f)$ defined by

$$Q(x_1, \dots, x_{n+1}, t) := \int_{x_{\nu(0)}}^t \left(d \cdot \prod_{i=0}^{n+1} (w - x_i)^{m_i} \right) dw.$$

Given $\mathbf{x} \in \mathbb{C}^{n+1}$, let $F_{\mathbf{x}} \in \mathbb{C}[t]$ be the monic polynomial defined by

$$F_{\mathbf{x}}(t) := Q(x_1, \dots, x_{n+1}, t).$$

The polynomial $F_{\mathbf{x}}$ is the unique monic polynomial of degree d which vanishes at $x_{\nu(0)}$ and whose critical points are exactly those points x_i for which $m_i > 0$.

Let $G_f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ be the homogeneous map of degree d defined by

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_{\nu(1)}) \\ \vdots \\ F_{\mathbf{x}}(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} Q(x_1, \dots, x_{n+1}, x_{\nu(1)}) \\ \vdots \\ Q(x_1, \dots, x_{n+1}, x_{\nu(n+1)}) \end{pmatrix}.$$

If $G_f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$, the map $G_f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ induces an endomorphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$. Indeed, if $G_f(\mathbf{x}) = \mathbf{0}$, then the monic polynomial $F_{\mathbf{x}}$ vanishes at $x_0, x_1, \dots, x_{n+1}$. The critical points of $F_{\mathbf{x}}$ are those points x_i for which $m_i > 0$. They are all mapped to 0 and thus, $F_{\mathbf{x}}$ has only one critical value in $\mathbb{C}$, namely 0.

Hence all preimages of this critical value must coincide and since $x_0 = 0$, they all coincide at 0: $x_0 = x_1 = \dots = x_{n+1} = 0$, so $G_f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$, and the induced map $\mathbb{P}^n \rightarrow \mathbb{P}^n$ is an endomorphism given in homogeneous coordinates as:

$$G_f : [x_1 : \dots : x_{n+1}] \mapsto [F_{\mathbf{x}}(x_{\nu(1)}) : \dots : F_{\mathbf{x}}(x_{\nu(n+1)})].$$

We must show that with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$, G_f coincides with the moduli space map g_f found in Proposition 5.1.

Let $\phi : (S^2, P) \rightarrow (S^2, P)$, and $\psi : (S^2, P) \rightarrow (S^2, P)$ be normalized homeomorphisms such that the following diagram commutes, where F is a rational map of degree d .

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P)) \\ f \downarrow & & \downarrow F \\ (S^2, P) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P)) \end{array}$$

The rational map F is a polynomial of degree d . Multiplying ψ by a nonzero complex number, we may assume that F is a monic polynomial. For all $i \in [0, n+1]$ set

$$x_i := \psi(p_i) \quad \text{and} \quad y_i := \phi(p_i).$$

We must show that $G_f(x_1, \dots, x_{n+1}) = (y_1, \dots, y_{n+1})$. For each $i \in [0, n+1]$, the following diagram commutes.

$$\begin{array}{ccc} p_{\nu(i)} & \xrightarrow{\psi} & x_{\nu(i)} \\ f \downarrow & & \downarrow F \\ p_i & \xrightarrow{\phi} & y_i \end{array}$$

Therefore $F(x_{\nu(i)}) = y_i$, and in particular $F(x_{\nu(0)}) = 0$. Furthermore, the critical points of F are exactly those critical points x_i for which $m_i > 0$, counted with multiplicity m_i . It follows that $F = F_{\mathbf{x}}$, and

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_{\nu(1)}) \\ \vdots \\ F_{\mathbf{x}}(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} F(x_{\nu(1)}) \\ \vdots \\ F(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

□

We now compute the critical loci of the maps $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ constructed in Proposition 5.7. To this end, we calculate the Jacobian $\text{Jac}(G_f)$. Note that since $G_f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is homogeneous, $\text{Jac}(G_f)$ is a homogeneous polynomial of degree $(n+1)(d-1)$ in the variables $x_1, \dots, x_{n+1}$.

Consider the polynomial $J \in \mathbb{C}[x_1, \dots, x_{n+1}]$ defined by

$$J(x_1, \dots, x_{n+1}) := \prod_{0 \leq i < j \leq n+1} (x_i - x_j)^{m_i + m_j}.$$

Proposition 5.8. *The Jacobian $\text{Jac}(G_f)$ is divisible by J .*

Proof. Set $G_0 := 0$. For $j \in [1, n+1]$, let G_j be the j th coordinate of $G_f(x_1, \dots, x_{n+1})$, that is

$$G_j := d \cdot \int_{x_{\nu(0)}}^{x_{\nu(j)}} \prod_{k=0}^{n+1} (w - x_k)^{m_k} dw.$$

For $0 \leq i < j \leq n+1$, note that setting $w = x_i + t(x_j - x_i)$, we have

$$\begin{aligned} G_{\mu(j)} - G_{\mu(i)} &= d \int_{x_i}^{x_j} \prod_{k=0}^{n+1} (w - x_k)^{m_k} dw \\ &= d \int_0^1 \prod_{k=0}^{n+1} (x_i + t(x_j - x_i) - x_k)^{m_k} \cdot (x_j - x_i) dt \\ &= (x_j - x_i)^{m_i + m_j + 1} \cdot H_{i,j} \end{aligned}$$

with

$$H_{i,j} := d \int_0^1 t^{m_i} (t-1)^{m_j} \cdot \prod_{\substack{k \in [0, n+1] \\ k \neq i, j}} (x_i - x_k + t(x_j - x_i))^{m_k} dt.$$

In particular, $G_{\mu(j)} - G_{\mu(i)}$ is divisible by $(x_j - x_i)^{m_i + m_j + 1}$.

For $k \in [0, n+1]$, let L_k be the row defined as:

$$L_k := \left[\frac{\partial G_k}{\partial x_1} \quad \cdots \quad \frac{\partial G_k}{\partial x_{n+1}} \right].$$

Note that L_0 is the zero row, and for $k \in [1, n+1]$, L_k is the k -th row of the Jacobian matrix of G_f . According to calculations above, the entries of $L_{\mu(j)} - L_{\mu(i)}$ are the partial derivatives of $(x_j - x_i)^{m_i + m_j + 1} \cdot H_{i,j}$. It follows that $L_{\mu(j)} - L_{\mu(i)}$ is divisible by $(x_j - x_i)^{m_i + m_j}$. Indeed, $L_{\mu(j)} - L_{\mu(i)}$ is either the difference of two rows of the Jacobian matrix of G_f , or such a row up to sign, when $\mu(i) = 0$ or $\mu(j) = 0$. Therefore, $\text{Jac}(G_f)$ is divisible by J . $\square$

Since $\sum m_j = d-1$, the lemma below shows that the degree of J is $(n+1) \cdot (d-1)$.

Lemma 5.9. *The degree of J is $(n+1) \cdot (d-1)$.*

Proof. The proof is just a simple calculation:

$$\begin{aligned} \sum_{0 \leq i < j \leq n+1} (m_i + m_j) &= \sum_{j=0}^{n+1} \sum_{i=0}^{j-1} m_j + \sum_{i=0}^{n+1} \sum_{j=i+1}^{n+1} m_i \\ &= \sum_{j=0}^{n+1} j m_j + \sum_{i=0}^{n+1} (n+1-i) m_i \\ &= \sum_{k=0}^{n+1} k m_k + \sum_{k=0}^{n+1} (n+1-k) m_k \\ &= (n+1) \sum_{k=0}^{n+1} m_k = (n+1) \cdot (d-1). \end{aligned}$$

$\square$

Since J and $\text{Jac}(G_f)$ are homogeneous polynomials of the same degree and since J divides $\text{Jac}(G_f)$, they are equal up to multiplication by a nonzero complex number. This shows that $\text{Jac}(G_f)$ vanishes exactly when J vanishes.

Corollary 5.10. *The critical locus of G_f is precisely*

$$\{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid x_i = x_j, \text{ and } m_i + m_j > 0 \text{ for } 0 \leq i < j \leq n+1\}.$$

5.5. Endomorphisms induced by unicritical polynomials.

Proposition 5.11. *If f is a topological polynomial such that $|\Omega| = 2$, then the moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$. The algebraic degree of G_f is equal to the topological degree of f .*

Proof. Recall that $\omega \notin P$, $m \geq 2$ is the preperiod of ω , $l \geq 1$ is the period of $f^{\circ m}(\omega)$, $f(\omega) = p_0$, and for $i \in [0, n]$, $f(p_i) = p_{i+1}$. Let $\beta_f \neq 1$ be a d th root of unity determined by the Hurwitz class of f (see Example 2.10). Also recall that $f(p_{m-2}) = f(p_{n+1})$.

Set $x_0 := 0$, and let $Q \in \mathbb{C}[x_{m-2}, x_{n+1}, t]$ be the homogeneous polynomial of degree $d := \deg(f)$ defined by

$$Q(x_{m-2}, x_{n+1}, t) := \left(t - \frac{\beta_f \cdot x_{n+1} - x_{m-2}}{\beta_f - 1} \right)^d.$$

Given $\mathbf{x} \in \mathbb{C}^{n+1}$, let $F_{\mathbf{x}} \in \mathbb{C}[t]$ be the polynomial defined by

$$F_{\mathbf{x}}(t) := Q(x_{m-2}, x_{n+1}, t).$$

Note that $F_{\mathbf{x}}$ is a monic polynomial of degree d which is unicritical, such that

$$F_{\mathbf{x}}(x_{m-2}) = F_{\mathbf{x}}(x_{n+1}),$$

and such a polynomial is unique up to the choice of a d th root of unity $\beta \neq 1$.

Let $G_f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ be the homogeneous map of degree d defined by

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_0) \\ \vdots \\ F_{\mathbf{x}}(x_n) \end{pmatrix} = \begin{pmatrix} Q(x_{m-2}, x_{n+1}, x_0) \\ \vdots \\ Q(x_{m-2}, x_{n+1}, x_n) \end{pmatrix}.$$

If $G_f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$, the map $G_f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ induces an endomorphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$. Indeed, if $G_f(\mathbf{x}) = \mathbf{0}$, then $F_{\mathbf{x}}$ vanishes at $x_0, \dots, x_n$. But $F_{\mathbf{x}}^{-1}(\{0\})$ consists of only one point since $F_{\mathbf{x}}$ is unicritical with critical value 0; therefore the points $x_0, \dots, x_n$ coincide, and since $x_0 = 0$, they must coincide at 0: $x_0 = x_1 = \dots = x_n = 0$. Note that if $F_{\mathbf{x}}(x_{m-2}) = 0$, then $F_{\mathbf{x}}(x_{n+1}) = 0$, so $F_{\mathbf{x}}$ also vanishes at x_{n+1} , therefore $x_{n+1} = 0$ as well.

The induced map $\mathbb{P}^n \rightarrow \mathbb{P}^n$ is an endomorphism given in homogeneous coordinates as:

$$G_f : [x_1 : \dots : x_{n+1}] \mapsto [F_{\mathbf{x}}(x_0) : \dots : F_{\mathbf{x}}(x_n)].$$

We must show that with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$, G_f coincides with the moduli space map g_f found in Proposition 5.3.

Let $\phi : (S^2, P) \rightarrow (S^2, P)$, and $\psi : (S^2, P) \rightarrow (S^2, P)$ be normalized homeomorphisms such that the following diagram commutes, where F is a rational map of degree d .

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P)) \\ f \downarrow & & \downarrow F \\ (S^2, P) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P)) \end{array}$$

The rational map F is a unicritical polynomial of degree d . Multiplying ψ by a nonzero complex number, we may assume that F is a monic polynomial. For all $i \in [0, n+1]$ set

$$x_i := \psi(p_i) \quad \text{and} \quad y_i := \phi(p_i).$$

We must show that $G_f(x_1, \dots, x_{n+1}) = (y_1, \dots, y_{n+1})$. For each $i \in [0, n]$, the diagram on the left commutes, and for $i = n+1$, the diagram on the right commutes.

$$\begin{array}{ccc} p_i & \xrightarrow{\psi} & x_i \\ f \downarrow & & \downarrow F \\ p_{i+1} & \xrightarrow{\phi} & y_{i+1} \end{array} \quad \begin{array}{ccc} p_{n+1} & \xrightarrow{\psi} & x_{n+1} \\ f \downarrow & & \downarrow F \\ p_{m-1} & \xrightarrow{\phi} & y_{m-1} \end{array}$$

So for $i \in [0, n]$, $F(x_i) = y_{i+1}$, and $F(x_{m-2}) = F(x_{n+1}) = y_{m-1}$. A monic, unicritical polynomial of degree which maps x_{m-2} and x_{n+1} to the same point, is uniquely determined by a d th root of unity $\beta \neq 1$. Since $\phi \circ f = F \circ \psi$, $\beta = \beta_f$; it follows that $F = F_{\mathbf{x}}$.

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_0) \\ \vdots \\ F_{\mathbf{x}}(x_n) \end{pmatrix} = \begin{pmatrix} F(x_0) \\ \vdots \\ F(x_n) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

□

We now compute the critical locus of the endomorphisms G_f constructed in Proposition 5.11. First we an important observation in the following lemma.

Lemma 5.12. *Let $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be the endomorphism of algebraic degree d constructed in Proposition 5.11. The map G_f can be decomposed as $G_f = s \circ \alpha$ where*

$$s : [x_1 : \dots : x_{n+1}] \mapsto [x_1^d : \dots : x_{n+1}^d]$$

and

$$\alpha : [x_1 : \dots : x_{n+1}] \mapsto [\kappa(x_0) : \dots : \kappa(x_n)] \quad \text{where}$$

$$\kappa(t) := \left(t - \frac{\beta_f \cdot x_{n+1} - x_{m-2}}{\beta_f - 1} \right).$$

Proof. This follows immediately from the proof of Proposition 5.11. □

Corollary 5.13. *The critical locus of G_f is*

$$\{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid \text{for some } x_i, \kappa(x_i) = 0\}.$$

Proof. Because $\alpha : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is an automorphism, the critical locus of G_f is $\alpha^{-1}(C_s)$, where C_s is the critical locus of $s : \mathbb{P}^n \rightarrow \mathbb{P}^n$. It is clear that

$$C_s = \{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid x_i = 0\}.$$

□

5.6. Critically finite endomorphisms. In this section we prove that the endomorphisms constructed in Proposition 5.7 and Proposition 5.11 are critically finite. Let $G : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be an endomorphism with critical locus C_G . The postcritical locus of G is defined as follows

$$V_G := \bigcup_{n \geq 0} G^{on}(C_g).$$

The endomorphism G is *critically finite* if V_G is algebraic.

We require the following topological lemma before we proceed.

Lemma 5.14. *Let X , Y and Z be path-connected, and locally path-connected topological spaces, with maps $i : X \hookrightarrow Y$ an inclusion, and surjective covering maps $f : Y \rightarrow Z$, and $g : X \rightarrow Z$ so that $g = f \circ i$. Then i is surjective.*

Proof. By hypothesis, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow g & \swarrow f \\ & Z & \end{array}$$

Suppose there is $y \in Y - i(X)$, and define $z := f(y)$. The fundamental group $\pi_1(Z, z)$ acts on the set $f^{-1}(z) \subset Y$ and the set $g^{-1}(z) \subset X$ where $i(g^{-1}(z)) \subset f^{-1}(z)$. Because X and Y are path connected, the action of $\pi_1(Z, z)$ on both sets $f^{-1}(z)$ and $g^{-1}(z)$ is transitive, so there is only one orbit, but this is impossible since y cannot be in the orbit of $i(g^{-1}(z))$. Therefore $Y - i(X) = \emptyset$, so i is surjective. □

Proposition 5.15. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P for which there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ that extends to an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to some normalization $i' : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ that identifies $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$. Then G_f is critically finite.*

Proof. Because there is a moduli space map, $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is injective, and the moduli space map $g_f : \mathcal{M}'_P \rightarrow \mathcal{M}_P$ is only defined on

$$\mathcal{M}'_P := \pi_P(\sigma_f(\mathcal{T}_P)) = \delta_P(\omega(\mathcal{T}_P)).$$

The map δ_P is an isomorphism $\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}'_P$, and we have the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P & & \\
\downarrow \pi_P & \searrow \omega_f & & \downarrow \pi_P & \\
& & \mathcal{W}_f & & \\
& \swarrow \rho_P & & \searrow \delta_P & \\
\mathcal{M}_P & \xleftarrow{g_f} & \mathcal{M}_P & & \\
\downarrow i' & & \downarrow i' & & \\
\mathbb{P}^n & \xleftarrow{G_f} & \mathbb{P}^n & &
\end{array}$$

Moreover, $g_f : \mathcal{M}'_P \rightarrow \mathcal{M}_P$ is a finite covering map because $\rho_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$ is a finite cover. Therefore, the critical value locus of the map $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ must be contained in Δ . We must now show that $G_f(\Delta) \subseteq \Delta$ to complete the proof.

Note that

$$\sigma_f(\mathcal{T}_P) \cap (G_f \circ i' \circ \pi_P)^{-1}(\Delta) = \emptyset.$$

Indeed, suppose that there is $\tau \in \mathcal{T}_P$ which lies in the above intersection. Then $\pi_P(\tau) \in \mathcal{M}'_P$, and therefore

$$G_f \circ i'(\pi_P(\tau)) = i' \circ g_f(\pi_P(\tau)).$$

But by assumption, $G_f \circ i'(\pi_P(\tau)) \in \Delta$, which is impossible since the image of $i' \circ g_f$ is equal to $\mathbb{P}^n - \Delta$. Therefore

$$\sigma_f(\mathcal{T}_P) \subseteq \mathcal{T}_P - (G_f \circ i' \circ \pi_P)^{-1}(\Delta) \implies i'(\mathcal{M}'_P) \subseteq \mathbb{P}^n - G_f^{-1}(\Delta).$$

Consider the following diagram

$$\begin{array}{ccc}
\mathcal{M}'_P & \xrightarrow{i'} & \mathbb{P}^n - G_f^{-1}(\Delta) \\
\searrow i \circ g_f & & \swarrow G_f \\
& \mathbb{P}^n - \Delta &
\end{array}$$

where $i' \circ g_f$ and G_f are surjective covering maps. The space $\mathcal{M}'_P$ is connected because it is the image of $\mathcal{T}_P$ (a connected space) by the continuous map $\pi_P \circ \sigma_f$. The spaces $\mathbb{P}^n - \Delta$ and $\mathbb{P}^n - G_f^{-1}(\Delta)$ are connected because Δ is a finite union of hyperplanes (see Proposition 5.5). By Lemma 5.14, the map

$$i' : \mathcal{M}'_P \hookrightarrow \mathbb{P}^n - G_f^{-1}(\Delta)$$

is surjective. This implies that $\Delta \subseteq G_f^{-1}(\Delta)$, therefore $G_f(\Delta) \subseteq \Delta$. $\square$

Corollary 5.16. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P for which there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ that extends to an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to some normalization $i' : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ that identifies $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$. Then the image of σ_f is*

$$\sigma_f(\mathcal{T}_P) = \mathcal{T}_P - (G_f \circ i' \circ \pi_P)^{-1}(\Delta).$$

Proof. This follows from Proposition 2.8 and the proof of Proposition 5.15. $\square$

We have ultimately proven the following theorems.

Theorem 5.17. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P , where $|P| = n + 3$, and let Ω be the critical set of f . If f is a topological polynomial such that $\Omega \subseteq P$, then there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ such that $g_f \circ \pi_P \circ \sigma_f = \pi_P$, and g_f extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$.*

Theorem 5.18. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P , where $|P| = n + 3$, and let Ω be the critical set of f . If f is a topological polynomial such that $|\Omega| = 2$, then there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ such that $g_f \circ \pi_P \circ \sigma_f = \pi_P$, and g_f extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$.*

6. KOBAYASHI HYPERBOLICITY

For dynamics in one complex variable, it is a very useful fact that if the rational map $G : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is critically finite, and the postcritical set of G contains at least 3 points, then the Poincaré metric is expanded by G on the complement of the postcritical set. In [B], C. McMullen asked about constructing analogous examples in higher dimensions. That is, are there nontrivial examples of endomorphisms $G : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that the complement of the postcritical locus is Kobayashi hyperbolic? In [FS1], Fornæss and Sibony analyzed the dynamics of two such maps. Each of the endomorphisms constructed by Theorems 5.17 and 5.18 has this property as we prove in this section.

The proof of Proposition 5.15 implies that the postcritical locus of the endomorphisms is contained in Δ . We will show that it is equal to Δ .

6.1. The postcritical locus: polynomials with periodic critical points.

Proposition 6.1. *Suppose that f is a topological polynomial such that $\Omega \subseteq P$, so that the moduli space map induces a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ which identifies $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$. Then the postcritical locus of G_f is equal to Δ .*

Proof. Set $x_0 := 0$, and recall the homogeneous polynomial Q

$$Q(x_1, \dots, x_{n+1}, t) := \int_{x_\nu(0)}^t \left(d \cdot \prod_{i=0}^{n+1} (w - x_i)^{m_i} \right) dw,$$

and the polynomial $F_{\mathbf{x}}(t) := Q(x_1, \dots, x_{n+1}, t)$. The endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is given by

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_{\nu(1)}) \\ \vdots \\ F_{\mathbf{x}}(x_{\nu(n+1)}) \end{pmatrix}.$$

Recall that in this setting, Δ is the locus of all $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n$ such that at least one coordinate is 0, or at least two coordinates coincide. Note that if $x_i = x_j$ for $i, j \in [0, n+1]$, $i \neq j$, then $y_{\mu(i)} = y_{\mu(j)}$. Identifying domain and range coordinates, we see that the component $x_i = x_j$ in Δ maps to the component $x_{\mu(i)} = x_{\mu(j)}$ in Δ .

Since $\Omega \subseteq P$, every postcritical point $p_k \in P$ is periodic. Let a_k be the length of the periodic cycle containing p_k . Then the hyperplane $x_i = x_j$ in Δ is periodic of period $a := \text{lcm}(a_i, a_j)$ since the map $G_f^{\circ a}$ maps the component $x_i = x_j$ in Δ to the component

$$x_{\mu^{\circ a}(i)} = x_{\mu^{\circ a}(j)}$$

which is just the component $x_i = x_j$. So every component of Δ is periodic. By Lemma 5.10, the critical locus of G_f is

$$\{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid x_i = x_j, \text{ and } m_i + m_j > 0 \text{ for } 0 \leq i < j \leq n+1\}.$$

Fix $i, j \in [0, n+1]$, $i \neq j$, and consider the component $x_i = x_j$ in Δ . The periodic cycle containing p_i must contain a critical point of f ; that is, there exists $b > 0$ so that $f^{\circ b}(p_i)$ is critical. Therefore the hyperplane

$$x_{\mu^{\circ b}(i)} = x_{\mu^{\circ b}(j)}$$

is critical, and $G_f^{\circ b}$ maps the hyperplane $x_i = x_j$ to the critical hyperplane $x_{\mu^{\circ b}(i)} = x_{\mu^{\circ b}(j)}$, and therefore the periodic cycle of hyperplanes in Δ containing the component $x_i = x_j$ also contains a critical component of G_f . Thus the postcritical locus of G_f is equal to Δ . $\square$

6.2. The postcritical locus: unicritical polynomials.

Proposition 6.2. *Suppose that f is a topological polynomial such that $|\Omega| = 2$, so the moduli space map induces an critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ which identifies $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$. Then the postcritical locus of G_f is equal to Δ .*

Proof. Set $x_0 := 0$, and recall the homogeneous polynomial Q

$$Q(x_{m-2}, x_{n+1}, t) := \left(t - \frac{\beta_f \cdot x_{n+1} - x_{m-2}}{\beta_f - 1} \right)^d.$$

and the polynomial $F_{\mathbf{x}}(t) := Q(x_{m-2}, x_{n+1}, t)$. Recall that $F_{\mathbf{x}}(x_{m-2}) = F_{\mathbf{x}}(x_{n+1})$, and the endomorphism G_f is given by

$$G_f : \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(x_0) \\ \vdots \\ F_{\mathbf{x}}(x_n) \end{pmatrix}.$$

Recall $G_f = s \circ \alpha : \mathbb{P}^n \rightarrow \mathbb{P}^n$ where $s : [x_1 : \dots : x_{n+1}] \mapsto [x_1^d : \dots : x_{n+1}^d]$, and $\alpha \in \text{Aut}(\mathbb{P}^n)$ is induced by

$$\kappa(t) := \left(t - \frac{\beta_f \cdot x_{n+1} - x_{m-2}}{\beta_f - 1} \right).$$

By Corollary 5.13, the critical locus of G_f is

$$\{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid \text{for some } x_i, \kappa(x_i) = 0\}.$$

It is clear that the union of the hyperplanes of the form $x_i = x_0$ comprise the critical value locus of the map G_f . Fix $i, j \in [0, n+1]$ where $i < j$. Then the component $x_{j-i} = x_0$ maps to the component $x_j = x_i$ in Δ . So every hyperplane $x_i = x_j$ is in the forward image of a critical value component, and therefore $x_i = x_j$ is postcritical. Thus the postcritical locus of G_f is equal to Δ . $\square$

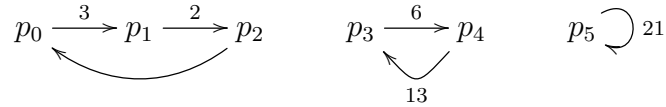
Corollary 6.3. *Let $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ be a critically finite endomorphism constructed in Proposition 5.7 or Proposition 5.11. Then the complement of the postcritical locus of G_f is Kobayashi hyperbolic.*

Proof. By Proposition 6.1 or Proposition 6.2, the complement of the postcritical locus of G_f is $\mathbb{P}^n - \Delta$. But $\mathbb{P}^n - \Delta$ is isomorphic to $\mathcal{M}_P$, which is Kobayashi hyperbolic, by Royden's Theorem [Roy]. $\square$

Remark 6.4. In [Gr], M. Green proved that the complement of $2n+1$ hyperplanes in $\mathbb{P}^n$ is Kobayashi hyperbolic if the hyperplanes are in general position. The locus Δ consists of $(n^2 + 3n + 2)/2$ hyperplanes. For $n > 1$, $(n^2 + 3n + 2)/2 > 2n + 1$, however the hyperplanes in Δ are not in general position, so this theorem cannot be directly applied to obtain the result in Corollary 6.3.

The following example illustrates an application of Theorem 5.17.

Example 6.5. Let $P = \{p_0, p_1, p_2, p_3, p_4, p_5\}$, and let $f : (S^2, P) \rightarrow (S^2, P)$ be any Thurston map with the following ramification portrait,



where the numbers over the arrows indicate the local degree by which f maps $p_i \mapsto p_j$. Using this data, Theorem 5.17 constructs a critically finite endomorphism $G_f : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ of algebraic degree 21, whose critical locus consists of periodic components, and whose postcritical locus is equal to Δ , which consists of 10 hyperplanes.

7. DYNAMICS: PULLBACK MAPS AND MODULI SPACE MAPS

Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P , and suppose there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ which extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ after identifying $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$ with respect to some normalization $i' : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$. In this section, we explore the extent to which dynamical objects for the moduli space map g_f (and G_f), relate to the dynamical objects for the pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$. We summarize the connections we have found so far between g_f and σ_f :

- (1) although it is a nondynamical result, Proposition 2.8 implies that if there is a moduli space map, then $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ is a (universal) cover of its image,
- (2) Propositions 4.3 and 4.4 provide a bijection between the fixed points of the moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ and conjugacy classes of rational maps which are each combinatorially equivalent to Thurston maps $g : (S^2, P) \rightarrow (S^2, P)$, which in turn are each Hurwitz equivalent to $f : (S^2, P) \rightarrow (S^2, P)$,

(3) and lastly, Corollary 5.16 identifies the image of σ_f , which is

$$\sigma(\mathcal{T}_P) = \mathcal{T}_P - (G_f \circ \pi_P)^{-1}(\Delta),$$

and therefore the domain of the moduli space map g_f is

$$\mathcal{M}'_P \approx \mathbb{P}^n - G_f^{-1}(\Delta).$$

There is more to say about this connection. For example, according to Proposition 4.2 of [FS2], an exceptional set in $\mathbb{P}^n$ can have at most $n + 1$ components (a variety $A \subseteq \mathbb{P}^n$ is *exceptional* for the map G_f if it is both forward and backward invariant). Thus, Δ cannot be exceptional as it contains $(n^2 + 3n + 2)/2$ components, so Δ is a proper subset of $G_f^{-1}(\Delta)$. As a consequence, $(G_f \circ \pi_P)^{-1}(\Delta) \subset \mathcal{T}_P$ is nonempty, so $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ cannot be surjective if there is a moduli space map g_f which extends to $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. If no moduli space map exists, it is perfectly possible for $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ to be surjective as demonstrated by way of example in [BEKP].

We have already discussed the fixed points of $g_f : \mathcal{M}'_P \rightarrow \mathcal{M}_P$ (and hence the fixed points of $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ which are contained in $\mathbb{P}^n - \Delta$). Of course, the periodic cycles of g_f are fixed points for some iterate of the map g_f ; hence the cycles of period $n > 1$ correspond to conjugacy classes of rational maps which are combinatorially equivalent to Thurston maps $g : (S^2, P) \rightarrow (S^2, P)$ which are in turn Hurwitz equivalent to $f^{\circ n} : (S^2, P) \rightarrow (S^2, P)$.

Proposition 7.1. *All periodic cycles of the map $g_f : \mathcal{M}'_P \rightarrow \mathcal{M}_P$ are repelling.*

Proof. This follows immediately from the fact that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P \\ \pi_P \downarrow & & \downarrow \pi_P \\ \mathcal{M}_P & \xleftarrow{g_f} & \mathcal{M}_P \end{array}$$

and from the fact that $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ is a contraction (see [DH]). □

Corollary 7.2. *All periodic cycles of $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ which are contained in $\mathbb{P}^n - \Delta$ are repelling.*

7.1. The Hurwitz semigroup. Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P , and let $h : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P that is Hurwitz equivalent to f . Then σ_f and σ_h both act on $\mathcal{T}_P$, and $\mathcal{W}_f = \mathcal{W}_h$ by Corollary 2.7; in particular, $g_f = g_h$ (if a moduli space map exists). We define the following semigroup.

$$\Theta_f := \{\sigma_h : \mathcal{T}_P \rightarrow \mathcal{T}_P \mid \text{the Thurston map } h : (S^2, P) \rightarrow (S^2, P)\}$$

is Hurwitz equivalent to $f : (S^2, P) \rightarrow (S^2, P)$.

This set is a semigroup with respect to composition. Given a point $\tau_0 \in \mathcal{T}_P$, we can look at the associated limit set

$$\Lambda(\tau_0) := \{\tau \in \mathcal{T}_P \mid \exists \text{ a sequence } \sigma_i \in \Theta_f \text{ so that}$$

$$\tau_i := \sigma_i \circ \cdots \circ \sigma_1(\tau_0) \text{ has a subsequence } \tau_{i_k} \rightarrow \tau \text{ as } k \rightarrow \infty\}.$$

Suppose that there is a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ which extends to an endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ with respect to some normalization $i' : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$ that identifies $\mathcal{M}_P \approx \mathbb{P}^n - \Delta$. Let $z_0 \in \mathbb{P}^n - \Delta$, and define $L(z)$ to be the accumulation set of the inverse images of z under G_f . In the one-dimensional case, $L(z)$ is equal to the Julia set of $G_f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, provided z is not an exceptional point for G_f .

Proposition 7.3. *Let $\tau_0 \in \mathcal{T}_P$, and define $z_0 := \pi_P(\tau_0) \in \mathbb{P}^n - \Delta$. Then*

$$\Lambda(\tau_0) = \pi_P^{-1}(L(z_0)).$$

Proof. This follows immediately from the fact that the following diagram commutes, and the fact that $i' \circ \delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P \hookrightarrow \mathbb{P}^n - G_f^{-1}(\Delta)$ is an isomorphism, by Corollary 5.16.

$$\begin{array}{ccc} \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P \\ \pi_P \downarrow & \searrow & \downarrow \pi_P \\ & \mathcal{W}_f & \\ \rho_P \swarrow & & \searrow \delta_P \\ \mathbb{P}^n - \Delta & \xleftarrow{G_f} & \mathbb{P}^n - G_f^{-1}(\Delta) \end{array}$$

□

We illustrate Proposition 7.3 by revisiting the rabbit example, see Example 3.1.

Example 7.4. Recall the rabbit polynomial $R : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ from Example 3.1. There is a moduli space map $g_R : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ which extends to a critically finite map $G_R : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with respect to the normalization used. In these coordinates, the map G_R is given by $z \mapsto 1 - 1/z^2$. The set Δ is $\{0, 1, \infty\}$, and $G_R^{-1}(\Delta) = \{-1, 0, 1, \infty\}$. The critical points of G_R are 0 and ∞ , and the map G_R has a superattracting cycle of length 3, displayed below.

$$\begin{array}{ccccc} 0 & \xrightarrow{2} & \infty & \xrightarrow{2} & 1 \\ & \searrow & & \swarrow & \\ & & & & \end{array}$$

By Proposition 5.16, the image of σ_R is

$$\sigma_R(\mathcal{T}_P) = \mathcal{T}_P - \pi_P^{-1}(\{-1\}).$$

Let τ_0 be the unique fixed point of σ_R .

The map G_R has three fixed points in $\mathbb{P}^1 - \{0, 1, \infty\}$ corresponding to the three conjugacy classes of quadratic polynomials whose critical point is periodic of period 3: the rabbit, the corabbit, and the airplane.

The space $\mathcal{W}_R$ is isomorphic to $\mathcal{M}'_P \approx \mathbb{P}^1 - \{-1, 0, 1, \infty\}$; for each point $x \in \mathcal{M}'_P$, there is a rational map $F_x := t \mapsto (1 - t^2/x^2)$, where

$$F_x : (\mathbb{P}^1, \{0, 1, \infty, x\}) \rightarrow (\mathbb{P}^1, \{0, 1, \infty, y\}) \quad y = 1 - 1/x^2.$$

In this way, $\mathcal{M}'_P$ is a *nondynamical* parameter space. Let $z_0 \in \mathcal{M}'_P$ be the fixed point of G_R corresponding to the rabbit polynomial; then $z_0 = \pi_P(\tau_0)$.

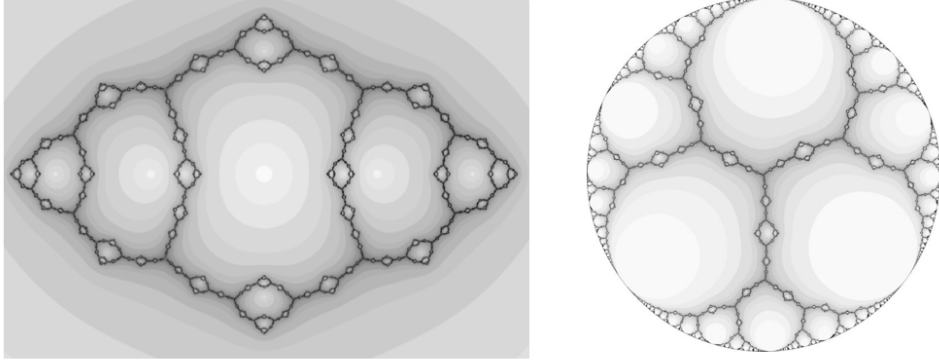


FIGURE 2. On the left, we see a copy of $\mathcal{M}'_P \approx \mathbb{P}^1 - \{-1, 0, 1, \infty\}$ drawn with the Julia set of the critically finite rational map $G_R : z \mapsto 1 - 1/z^2$, which is just $L(z_0)$. On the right, we see the limit set $\Lambda(\tau_0)$ in $\mathcal{T}_P \approx \mathbb{D}$, centered at τ_0 . Using the behavior of G_R near z_0 , we can determine the local behavior of σ_R near τ_0 : σ_R contracts lengths by about $3/4$, and rotates by an angle of about $2\pi/3$. The three largest components in the picture on the right form a 3-cycle for the map σ_R .

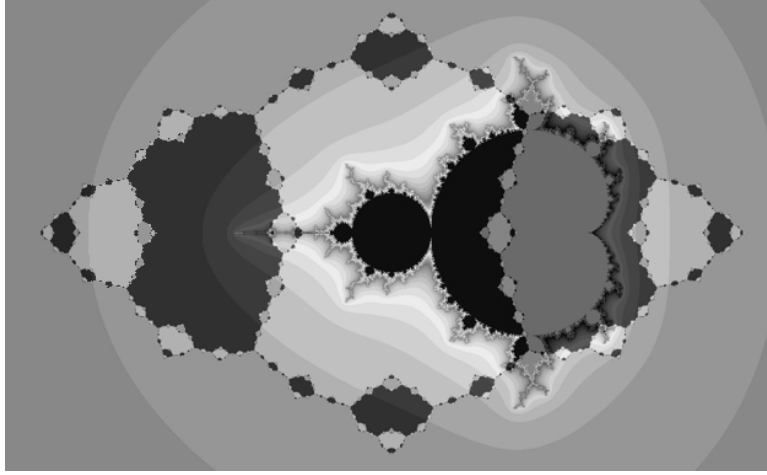


FIGURE 3. This is a superposition of two parameter spaces: the dynamical a -parameter plane for the quadratic family $t \mapsto (a - 1)t^2 + 1$ which contains the Mandelbrot set M , and the nondynamical parameter space $\mathcal{M}'_P \approx \mathbb{P}^1 - \{-1, 0, 1, \infty\}$ which has a dynamical system on it, namely $G_R : z \mapsto 1 - 1/z^2$. The Julia set of G_R is drawn in $\mathcal{M}'_P$. These two parameter spaces coincide at the three fixed points of G_R ; with respect to our normalization, the “rabbit” fixed point of G_R coincides with the “rabbit” parameter in M , the “corabbit” fixed point of G_R coincides with the “corabbit” parameter in M , and the “airplane” fixed point of G_R coincides with the “airplane” parameter in M .

7.2. Other connections. There are other dynamical objects of interest for the map $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. As proved in Proposition 7.1, all of the periodic cycles contained in $\mathbb{P}^n - G_f^{-1}(\Delta)$ are repelling. What about the periodic cycles of G_f which are contained in Δ ?

Fixed points of $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ in Δ correspond to invariant multicurves; that is, if $\mathbf{x} \in \Delta$ is a fixed point of G_f , then there is a Thurston map $g : (S^2, P) \rightarrow (S^2, P)$ which is Hurwitz equivalent to f , and there is a multicurve $\Gamma \subset S^2 - P$ which is a g -stable multicurve. If $\mathbf{x}$ is a repelling fixed point, then there is a g -stable multicurve which is an obstructing multicurve. For more details about this, see [Se].

8. RECOVERING EXAMPLES IN THE LITERATURE

Theorems 5.17 and 5.18 provide a new and systematic way for generating critically finite endomorphisms $G : \mathbb{P}^n \rightarrow \mathbb{P}^n$, whose postcritical locus is equal to Δ which consists of $(n^2 + 3n + 2)/2$ hyperplanes. A large number of critically finite maps found in the literature can actually be recovered as moduli space maps for some Thurston map $f : (S^2, P \cup A) \rightarrow (S^2, P \cup A)$ where P is the postcritical set of f , and $A \subset S^2$ is a finite union of periodic cycles of f . The two critically finite maps discussed in [FS1] can be recovered through these constructions, and the family of examples in [C] can also be recovered by these methods as we now discuss.

In [C] S. Crass constructs a family of critically finite endomorphisms on $G : \mathbb{P}^n \rightarrow \mathbb{P}^n$ using a reflection action of the symmetric group $\mathcal{S}_{n+2}$ on $\mathbb{P}^n$. Each of these maps can be recovered as a moduli space map associated to a Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ where $P = \{p_0, \dots, p_{n+2}\}$ is the postcritical set of f such that

- $f^{-1}(p_{n+2}) = \{p_{n+2}\}$,
- for all $i \in [0, n+1]$, $f(p_i) = p_i$, and p_i is a simple critical point of f .

There is a unique combinatorial class of Thurston maps with this property (see [BEK]), and the class is unobstructed.

It is clear that if a critically finite endomorphism $G : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is going to be recovered from these methods, the postcritical locus must necessarily be a union of $(n^2 + 3n + 2)/2$ hyperplanes. Thus far, I know of exactly one critically finite endomorphism in the literature with this property that cannot be recovered as a moduli space map. We present the example here; it can be found in [D].

Example 8.1 (Dupont). Let $G : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the endomorphism given by

$$G : [x : y : z] \mapsto [(x - y + z)^2 : (x - z + y)^2 : (y + z - x)^2].$$

The critical locus of this map consists of the hyperplanes

$$C = \{x - y + z = 0, x - z + y = 0, y + z - x = 0\},$$

and the following scheme illustrates the dynamics of G on its critical and postcritical locus.

$$\begin{array}{ccccccc}
x - y + z = 0 & & y + z - x = 0 & \longrightarrow & z = 0 & & \\
\downarrow & & & & \downarrow & & \\
x = 0 & \longrightarrow & x = y & \longrightarrow & x = z & \longrightarrow & y = z \longrightarrow x = y \\
& & & & \uparrow & & \\
& & & & x - z + y = 0 & \longrightarrow & y = 0
\end{array}$$

This map has algebraic degree 2, and it is critically finite with postcritical locus

$$\{x = 0, y = 0, z = 0, x = y, y = z, x = z\}.$$

Suppose that the Thurston map $f : (S^2, P) \rightarrow (S^2, P)$ induced this endomorphism on $\mathbb{P}^2$. Then the postcritical set P would have five points, and f would have degree 2. Since the critical locus is not periodic, the critical points of f would not all be contained in the postcritical set. Running through all finite combinatorial possibilities of ramification portraits of such a Thurston map, we note that the endomorphism G is not induced by any of them. It therefore does not arise as a moduli space map. It should be mentioned that C. Dupont proved that this map is a 2-dimensional Lattès example; see [D].

9. QUESTIONS FOR FURTHER STUDY

In the dynamical context, these moduli space maps provide a very intriguing connection between dynamics in several complex variables, and Thurston's topological characterization of rational maps, Theorem 4.1. In principle, dynamical objects of interest for $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ correspond to dynamical objects of interest for the moduli space map (and hence for the critically finite endomorphisms $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$).

Some of the endomorphisms $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ will have periodic cycles in Δ which are super-saddles, and as such, there is an unstable manifold associated to the repelling directions, and there is a superstable manifold corresponding to the superattracting directions. This structure can be lifted to the Teichmüller space $\mathcal{T}_P$ and can potentially be used to further understand the pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$, or the extended pullback map $\hat{\sigma}_f : \hat{\mathcal{T}}_P \rightarrow \hat{\mathcal{T}}_P$ (see [Se] for more information about the extended pullback map).

Similarly, some of the endomorphisms $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ will have fixed points (periodic cycles) in Δ which are superattracting; as such, we can ask about the existence of Böttcher coordinates; as proved in [BEK], some of these endomorphisms do admit Böttcher coordinates in higher dimensions. With a Böttcher coordinate, one can define higher dimensional analogues of external rays; again, this structure can be lifted to Teichmüller space and can be potentially used to understand the pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$. What information do these extra structures provide?

Theorem 5.17 and Theorem 5.18 provide sufficient conditions for which a moduli space map exists, and (with respect to the normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^n$), the map extends to a critically finite endomorphism $G_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. As the existence of a moduli space map is a nondynamical issue, Corollary 5.2 and Corollary 5.4 provide other sufficient conditions for

which a moduli space map exists, but this map may not extend holomorphically to $\mathbb{P}^n$. In fact, if f is not a topological polynomial, then the $\mathbb{P}^n$ compactification of moduli space may not be natural, so we ask, if a moduli space map exists, does it extend holomorphically to some compactification of moduli space?

Example 9.1. Let $F : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ be the polynomial

$$F(z) = (-0.187 - 0.421i)z^3 + (-2.332 - 0.239i)z^2 + (-0.285 + 2.879i)z + 1$$

with postcritical set $P = \{0, 1, -1.804 + 2.219i, -2.257 + 2.472i, \infty\}$. Here is the ramification portrait for F ,

$$0.121 + 0.721i \xrightarrow{2} 0 \longrightarrow 1 \longrightarrow -1.804 + 2.219i \xrightarrow{2} -2.257 + 2.472i \quad \infty \curvearrowright 3$$

and here is a picture of the filled Julia set of F .

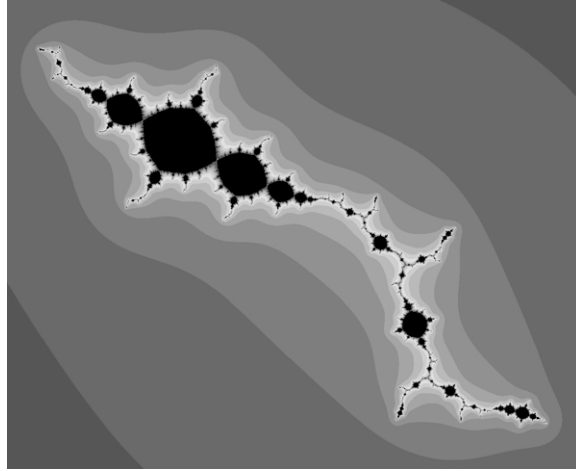


FIGURE 4. The filled Julia set of the critically finite polynomial F . There is no normalization for which the moduli space map $g_F : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends holomorphically to $\mathbb{P}^2$.

There is a moduli space map $g_F : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ for this example, but the map *does not* extend holomorphically to $\mathbb{P}^2$ with respect to any normalization $i : \mathcal{M}_P \hookrightarrow \mathbb{P}^2$. This is somewhat surprising; the map is a polynomial, so the $\mathbb{P}^2$ compactification is “natural” in some sense.

One of the biggest challenges is find necessary and sufficient conditions on the admissible branched cover $f : (S^2, A) \rightarrow (S^2, B)$ for which a moduli space map exists. Recall that a moduli space map exists if $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ is injective. As discussed in Section 2, this is a nondynamical matter which depends on the Hurwitz class of f .

In [Go], L. Goldberg counts the number of rational maps of degree d with a given set of $2d - 2$ critical points; she counts them up to postcomposition by Möbius transformations.

Theorem 9.2 (Goldberg). *A $(2d - 2)$ -tuple is the branch set of at most $\text{Cat}(d)$ classes in Rat_d . The maximum is attained by a Zariski open subset of $(2d - 2)$ -tuples.*

The number $\text{Cat}(d)$ is the Catalan number

$$\text{Cat}(d) = \frac{1}{d} \binom{2d-2}{d-1}.$$

Corollary 9.3. *Let $f : (S^2, A) \rightarrow (S^2, B)$ be an admissible branched cover of degree $d \geq 3$, and suppose that A contains $2d-2$ critical points of f , so that $|A| = 2d-2$. Suppose further that $|B| = 2d-2$, so $f|_A : A \rightarrow B$ is bijective. Then a moduli space map does not exist.*

Proof. The hypothesis $d \geq 3$ was required to ensure that $|A| = |B| \geq 3$. The map $\delta_A : \mathcal{W}_f \rightarrow \mathcal{M}_A$ cannot be injective as there is a Zariski open subset $Z \subseteq \mathcal{M}_A$ for which if $z \in Z$, $|\delta_A^{-1}(\{z\})| = \text{Cat}(d)$. The Catalan number $\text{Cat}(d) > 1$ if $d \geq 3$. $\square$

The following dynamical corollary is immediate.

Corollary 9.4. *Let $f : (S^2, P) \rightarrow (S^2, P)$ be a Thurston map with postcritical set P and critical set Ω . If $\Omega \subseteq P$ and $|\Omega| = 2d-2$, then a moduli space map does not exist.*

There is work by I. Scherbak which uses Schubert calculus to generalize Theorem 9.2 to the case where the critical points of f are not required to be simple (see [Sc]). What are the necessary and sufficient conditions for the Hurwitz class of $f : (S^2, A) \rightarrow (S^2, B)$ to ensure that a moduli space map exists?

As the existence of a moduli space map is a Hurwitz invariant, it is important to understand the Hurwitz class of a branched cover $f : (S^2, A) \rightarrow (S^2, B)$. For example, how many Hurwitz classes are there of ramified covering maps $g : (S^2, A) \rightarrow (S^2, B)$ for which

$$f|_A = g|_A \quad \text{and} \quad \forall a \in A, \quad \text{loc deg } f|_a = \text{loc deg } g|_a?$$

This is related to an old problem of Hurwitz (see [La], for example).

9.1. Correspondences. Even if there is no moduli space map, the Hurwitz space $\mathcal{W}_f$ and the pair of maps

$$\delta_P : \mathcal{W}_f \rightarrow \mathcal{M}_P \quad \text{and} \quad \rho_P : \mathcal{W}_f \rightarrow \mathcal{M}_P$$

induce a correspondence on the level of moduli space. This correspondence can be “iterated”. These correspondences are very interesting objects, and provide insights into the iteration of $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ on Teichmüller space.

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