On the equation $x^k - 1 = n!$ in function fields

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The diophantine equation

$$x^2 - 1 = n!$$

was first studied by Brocard [1], [2]. The only known solutions are $(x, n) \in \{(5, 4), (11, 5), (71, 7)\}$ and it is believed these are the only solutions. Erdős and Obláth [4] considered the more general equation

$$x^k - 1 = n! \tag{1}$$

They showed that there are no solutions for $k > 1$, except possibly for $k = 2$ and $k = 4$. Pollack and Shapiro [5] were able to deal with the case $k = 4$, but the case $k = 2$ is still open. An extensive history can be found in [6].

In this short note we consider an analogue of (1) over function fields. For this we will need an analogue of the factorial in the function field setting. Note that the usual factorial $n!$ vanishes in $\mathbb{F}_q(t)$ for n sufficiently large.

Let $K := \mathbb{F}_q(t)$ for q a power of a prime number. First we define for an integer $i \geq 0$

$$[i] := t^{q^i} - t.$$

Then we define the Carlitz factorial by $D_0 := 1$ and for an integer $i > 0$ by

$$D_i := \prod_{1 \leq j \leq i} [j]^{q^{i-j}}.$$

Carlitz [3] used the Carlitz factorial to define the Carlitz exponential

$$e_c(x) = \sum_{i=0}^{\infty} \frac{x^{q^i}}{D_i},$$

which shares many properties with the ordinary exponential function.

Theorem 1. *Let $k \in \mathbb{Z}_{\geq 2}$. If $f(t) \in \mathbb{F}_q(t)$ and $i \in \mathbb{Z}_{\geq 0}$ are such that*

$$f(t)^k - 1 = D_i, \tag{2}$$

then $i = 0$.

Proof. One way to proceed is to apply Mason's ABC Theorem, but in this case it is more convenient to directly attack the equation by differentiation. Suppose we have a solution to

$$f(t)^k - 1 = D_i = \prod_{1 \leq j \leq i} [j]^{q^{i-j}}$$

with $f(t) \in \mathbb{F}_q(t)$ and $i > 0$. Then differentiation with respect to t yields

$$kf'(t)f(t)^{k-1} = [i]' \prod_{1 \leq j \leq i-1} [j]^{q^{i-j}}.$$

But for $i > 0$ we have $[i]' = -1$. Hence we find that

$$f(t)^k - 1 = D_i = \prod_{1 \leq j \leq i} [j]^{q^{i-j}} = [i] \prod_{1 \leq j \leq i-1} [j]^{q^{i-j}} = -k[i]f'(t)f(t)^{k-1}. \quad (3)$$

Since $i > 0$, we see from (2) that $f(t) \notin \mathbb{F}_q$. Together with (3) this implies that $\gcd(k, p) = 1$. Choose a discrete valuation v on $\mathbb{F}_q(t)$ such that $v(f(t)) > 0$. It follows that

$$v(f'(t)) \geq v(f(t)) - 1 \geq 0 \quad (4)$$

and that

$$v(f(t)^k - 1) = 0. \quad (5)$$

Equation (3) gives after taking valuations and using (4) and (5)

$$0 = v([i]) + v(f'(t)) + (k-1)v(f(t)).$$

We conclude that $v([i]) < 0$, so v must be the infinite valuation, i.e., with $v(t) = -1$, and thus, $v([i]) = -q^i$. Since $v(f(t)) > 0$ implies that v is the infinite valuation, we conclude that $H_K(f(t)) = v(f(t))$. On the other hand we have from (2) that

$$kH_K(f(t)) = H_K(D_i) = iq^i. \quad (6)$$

Recall that $(k, p) = 1$, hence equation (6) tells us that $v(f(t)) = H_K(f(t)) > 1$ and therefore that $v(f'(t)) > 0$. In particular one finds that

$$(k-1)H_K(f(t)) = (k-1)v(f(t)) = -v([i]) - v(f'(t)) < q^i. \quad (7)$$

Combining (6) and (7) gives

$$iq^i = kH_K(f(t)) < \frac{k}{k-1}q^i.$$

Since $k \geq 2$, this implies $i = 1$. Then (6) becomes $kH_K(f(t)) = q$, which implies $k \mid q$. But this is impossible for $k \geq 2$ and $(k, p) = 1$. □

References

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