Smith explained part II

Peter Koymans
Max Planck Institute for Mathematics

MAX-PLANCK-GESELLSCHAFT

Informal Seminar

22 October 2020

The Cohen-Lenstra heuristics

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More formally, Cohen and Lenstra conjectured that

$$\lim_{X\to\infty} \frac{|\{K \text{ im. quadr.} : |D_K| < X \text{ and } \mathrm{Cl}(K)[p^\infty] \cong A\}|}{|\{K \text{ im. quadr.} : |D_K| < X\}|} = \frac{\prod_{i=1}^{\infty}\left(1 - \frac{1}{p^i}\right)}{|\mathrm{Aut}(A)|}$$

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where $\mathrm{Cl}(K)[p^\infty]$ is now the quotient of a random abelian group.

Genus theory

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and $\mathrm{Cl}(K)[2]$ is generated by the ramified prime ideals of $\mathcal{O}_K$.

Indeed, if p divides the discriminant of $\mathbb{Q}(\sqrt{d})$, then p ramifies, so

$$\begin{array}{ccccc} \mathbb{Q}(\sqrt{d}) & \quad & \mathfrak{p} & \quad & \mathfrak{p}^2 = (p). \\ | & & | & & \\ \mathbb{Q} & & p & & \end{array}$$

There is precisely one relation between the ramified primes.

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Theorem 1 (Smith, 2017)

Gerth's conjecture is true.

The dual class group

Theorem 2 (Class field theory)

We have an isomorphism

$$\mathrm{Cl}(K) \cong \mathrm{Gal}(H(K)/K)$$

given by sending a prime ideal $\mathfrak{p}$ *to* $\mathrm{Art}(\mathfrak{p})$. *Furthermore, if* K *is Galois, this isomorphism respects the natural Galois action of* $\mathrm{Gal}(K/\mathbb{Q})$.

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Indeed,

$$\mathrm{Cl}^{\vee}(K)[2] = \mathrm{Hom}(\mathrm{Cl}(K), \mathbb{C}^*)[2] \cong \mathrm{Hom}(\mathrm{Gal}(H(K)/K), \{\pm 1\}).$$

Given $\chi \in \mathrm{Hom}(\mathrm{Gal}(H(K)/K), \{\pm 1\})$, look at $H(K)^{\ker(\chi)}$. The quadratic unramified characters are generated by χ_p with p dividing d.

The Artin pairing

Let A be a finite abelian 2-group. We have a natural pairing

$$\mathrm{Art}_m : 2^{m-1}A[2^m] \times 2^{m-1}A^\vee[2^m] \to \mathbb{F}_2$$

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$$\mathrm{Art}_{m,K} : (\mathfrak{p}, \chi) \mapsto \psi(\mathrm{Frob}_{\mathfrak{p}}).$$

Smith essentially proves that the Artin pairing is random. This implies Cohen–Lenstra.

Random Artin pairings

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Take an integer d and let $p_1, \ldots, p_r$ be its prime divisors ordered by size. Then we have natural surjective maps

$$\mathbb{F}_2^r \to \mathrm{Cl}(\mathbb{Q}(\sqrt{d}))[2], \quad \mathbb{F}_2^r \to \mathrm{Cl}^\vee(\mathbb{Q}(\sqrt{d}))[2].$$

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Real quadratic: random $N + 1$ by N matrices.
Imaginary quadratic: random N by N matrices.

The first Artin pairing

In matrix form Art_1 becomes

$$\begin{array}{ccccc} & \chi_{p_1} & \chi_{p_2} & \cdots & \chi_{p_r} \\ p_1 & * & \left(\frac{p_2}{p_1}\right) & \cdots & \left(\frac{p_r}{p_1}\right) \\ p_2 & \left(\frac{p_1}{p_2}\right) & * & \cdots & \left(\frac{p_r}{p_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_r & \left(\frac{p_1}{p_r}\right) & \left(\frac{p_2}{p_r}\right) & \cdots & * \end{array}.$$

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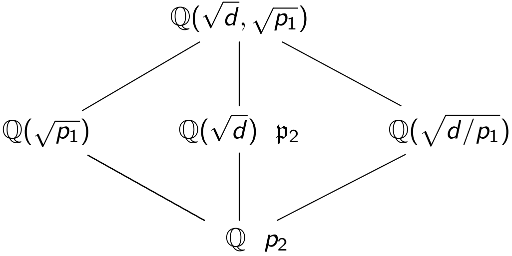
Indeed,



Prime divisors part I

This is an entirely analytic problem. To tackle this problem, our first aim is to gain a deeper understanding of the typical structure of the prime divisors of an integer.

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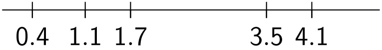
A good heuristic model is that $\log\log p_i$ is roughly equal to i.



Prime divisors part II

Hence to prove equidistribution of Art_1, restrict to integers n with $\omega(n) = r$, where

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We can cover the set of squarefree integers up to N with r prime divisors with product sets of the shape

$$X := X_1 \times \cdots \times X_r$$

where the X_i are suitable, disjoint intervals of primes. We view an element $x \in (x_1, \ldots, x_r)$ as a squarefree integer by multiplying out its coordinates.

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For this to work out, we need that most integers n satisfy

$$\log p_{i+1} - \log p_i \geq 1 \text{ for all } i.$$

We also need to shrink the intervals at the end.

Prime divisors part III

These boxes X are extremely useful. It will be the most natural way to set up our algebraic results later for the higher Artin pairings, while it also helps with analytic questions (allowing for inductive arguments).

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Smith shows that a typical integer is regularly spaced, i.e.

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for all $i \leq r/3$.

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Smith also shows that there is typically at least one big gap, i.e.

$$\log p_i > \log\log p_i \cdot \left(\sum_{j=1}^{i-1} \log p_j\right)$$

for some $i \in (0.5r^{1/4}, 0.5r^{1/2})$. It is then easy to show that this is also true for boxes (except for a negligible amount).

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	χ_{p_1}	χ_{p_2}	χ_{p_3}
p_1	?	Cheb	Cheb
p_2	Cheb	LarSie	LarSie
p_3	Cheb	LarSie	LarSie

This information is enough to recover for example the rank distribution as r goes to infinity, since there is only a ? in at most the top $0.5\sqrt{r}$ part of the matrix.

Recap: the Artin pairing

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Recap II: comparing Artin pairings

By genus theory we have a natural surjective map $\mathbb{F}_2^r \to \mathrm{Cl}(K)[2]$ and $\mathbb{F}_2^r \to \mathrm{Cl}^\vee(K)[2]$, where $r = \omega(\Delta_K)$.

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Pulling back $\mathrm{Art}_{m,K}$ then induces a pairing $\mathbb{F}_2^r \times \mathbb{F}_2^r \to \mathbb{F}_2$. Concretely, if $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the ramified prime ideals ordered by norm, then we are keeping track of the set of $(e_1, \dots, e_r) \in \mathbb{F}_2^r$ such that

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for all $m \geq 1$.

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Similarly, if $\chi_{p_1}, \dots, \chi_{p_r}$ are the unramified characters, then we bookkeep the set of $(e_1, \dots, e_r) \in \mathbb{F}_2^r$ with

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Higher Artin pairings

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$$\dim_{\mathbb{F}_3} \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \leq \dim_{\mathbb{F}_3} \mathrm{Cl}(\mathbb{Q}(\sqrt{-3d})) \leq 1 + \dim_{\mathbb{F}_3} \mathrm{Cl}(\mathbb{Q}(\sqrt{d})),$$

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The main algebraic result in Smith's work is in fact a reflection principle that compares the 2^m-torsion of 2^m quadratic fields.

How can we find such reflection principles?

Reflection principles for the second Artin pairing

Suppose that we have four fields

$$\{p, p'\} \times \{q, q'\} \times \{d\}$$

such that χ_a is a double in the dual class group (with $a \mid d$), i.e. in the right kernel of the various Art_1.

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Inspecting Art_1, we see that χ_a is a double in $\mathrm{Cl}^\vee(\mathbb{Q}(\sqrt{m}))$ if and only if

$$x^2 = ay^2 + \frac{m}{a}z^2$$

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This is a Galois extension of $\mathbb{Q}$ (in fact a D_4).

A small compositum

But from the equations

$$x^2 - ay^2 = \frac{dpq}{a}z^2, \quad x^2 - ay^2 = \frac{dpq'}{a}z^2, \quad x^2 - ay^2 = \frac{dp'q}{a}z^2$$

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Concretely, a part of the Hilbert class field of $\mathbb{Q}(\sqrt{dp'q'})$ is already inside the compositum of the Hilbert class fields of $\mathbb{Q}(\sqrt{dpq})$, $\mathbb{Q}(\sqrt{dpq'})$ and $\mathbb{Q}(\sqrt{dp'q})$.

A small compositum

But from the equations

$$x^2 - ay^2 = \frac{dpq}{a} z^2, \quad x^2 - ay^2 = \frac{dpq'}{a} z^2, \quad x^2 - ay^2 = \frac{dp'q}{a} z^2$$

we get a solution to

$$x^2 - ay^2 = \frac{dp'q'}{a} z^2.$$

Concretely, a part of the Hilbert class field of $\mathbb{Q}(\sqrt{dp'q'})$ is already inside the compositum of the Hilbert class fields of $\mathbb{Q}(\sqrt{dpq})$, $\mathbb{Q}(\sqrt{dpq'})$ and $\mathbb{Q}(\sqrt{dp'q})$.

This implies for $b \mid d$ a common 4-rank ideal

$$\mathrm{Art}_{2,dpq}(\chi_a, b) + \mathrm{Art}_{2,dpq'}(\chi_a, b) + \mathrm{Art}_{2,dp'q}(\chi_a, b) + \mathrm{Art}_{2,dp'q'}(\chi_a, b) = 0.$$

Rephrasing in terms of cocycles

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We define $N = \mathbb{Q}_2/\mathbb{Z}_2$ with trivial $G_\mathbb{Q}$ action and the discrete topology.

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We have a split exact sequence

$$0 \to \mathrm{Cocy}(\mathrm{Gal}(K/\mathbb{Q}), N(\chi))[2^k] \to \mathrm{Cocy}(\mathrm{Cl}(K) \rtimes \mathrm{Gal}(K/\mathbb{Q}), N(\chi))[2^k] \to \mathrm{Cl}(K)^\vee[2^k] \to 0,$$

where χ is the character corresponding to $\mathrm{Gal}(K/\mathbb{Q})$. Also note that $\mathrm{Cl}(K) \rtimes \mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Gal}(H(K)/\mathbb{Q})$.

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In simple words, we can lift dual class group elements to cocycles of $\mathrm{Gal}(H(K)/\mathbb{Q})$ valued in $N(\chi)$ (with an easily described kernel).

Cocycles surject to class group

On this slide we will prove that

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We claim that we can send σ to any element of $N(\chi)[2^k]$ and this uniquely defines our cocycle lift $\widetilde{\psi}$. The cocycle rule forces

$$0 = \widetilde{\psi}(\sigma^2) \overset{\text{cocycle rule}}{=} \sigma * \widetilde{\psi}(\sigma) + \widetilde{\psi}(\sigma) = -\widetilde{\psi}(\sigma) + \widetilde{\psi}(\sigma) = 0,$$

since $\chi(\sigma) = -1$, so no conditions as claimed. Now check that this extends to a cocycle.

A common space

But now we can view our cocycles as elements in

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Look at

$$\begin{aligned} d\psi_{dpq}(\sigma, \tau) &:= \psi_{dpq}(\sigma\tau) - \psi_{dpq}(\sigma) - \psi_{dpq}(\tau) \\ &= \chi_{dpq}(\sigma) * \psi_{dpq}(\tau) - \psi_{dpq}(\tau) \\ &= (\chi_{dpq}(\sigma) - 1) \cdot \psi_{dpq}(\tau) \\ &= \iota(\chi_{dpq}(\sigma)) \cdot \chi_a(\tau), \end{aligned}$$

where $\iota : \{\pm 1\} \to \mathbb{F}_2$.

A small compositum: a cocycle perspective

We have

$$d\left(\psi_{dpq}+\psi_{dp'q}+\psi_{dpq'}+\psi_{dp'q'}\right)(\sigma,\tau)= \iota(\chi_{dpq}(\sigma)\cdot\chi_{dp'q}(\sigma)\cdot\chi_{dpq'}(\sigma)\cdot\chi_{dp'q'}(\sigma))\cdot\chi_a=0,$$

which recovers our previous computation.