Factoring in number rings

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This Week's Discoveries

Leiden, Nederland, May 2019

Basic arithmetic

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A ring is a set where one can add, subtract and multiply the elements.

Some example of rings are: the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and

$$\left\{ \frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b \right\}.$$

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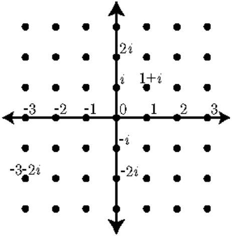

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$$(1 + i) + (-3 - 2i) = -2 - i$$

and multiply Gaussian integers by expanding brackets and using the rule $i^2 = -1$

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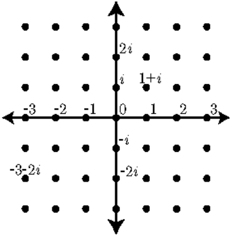

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We will now study factorization properties of the Gaussian integers.

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If we view $\mathbb{Z}[i]$ as a subset of the complex numbers $\mathbb{C}$ we have

$$N(a + bi) = |a + bi|^2,$$

where $| \cdot |$ is the absolute value on $\mathbb{C}$. We have the fundamental property

$$N((a + bi) \cdot (c + di)) = N(a + bi) \cdot N(c + di).$$

Computing the unit group

Now suppose that $a + bi \in \mathbb{Z}[i]$ is a unit. Then, by definition, there is $c + di \in \mathbb{Z}[i]$ such that

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Then we deduce $N(a + bi) = \pm 1$. But $N(a + bi) = a^2 + b^2$, so

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We conclude that the units of $\mathbb{Z}[i]$ are $\{1, -1, i, -i\}$.

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Example 2

For the integers, prime and irreducible are the same notion. The prime elements are exactly $\pm p$, where p is a prime number. This property is the key behind unique factorization!

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Example 3

We have that

$$-3 + 9i = i \cdot 3 \cdot (1 + i) \cdot (2 - i) = 3i \cdot (-1 + i) \cdot (-1 - 2i),$$

where 3, $1 + i$ and $2 - i$ are all irreducible and prime. In fact: irreducible and prime are the same notion in $\mathbb{Z}[i]$.

Failure of unique factorization

In the ring $\mathbb{Z}[\sqrt{-6}] := \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$ we have

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With a similar computation as for $\mathbb{Z}[i]$, it follows that the units of $\mathbb{Z}[\sqrt{-6}]$ are ± 1. We will show on the next slide that 2, 3, $\sqrt{-6}$ are irreducible. So we have two different factorizations of 6 in equation (1).

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This shows that 2 is irreducible, and similarly for 3 and $\sqrt{-6}$.

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Attached to an integer d, we can define an abelian group $\mathrm{Cl}(d)$ that measures the failure of unique factorization in $\mathbb{Z}[\sqrt{d}]$. We have

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Given d there is an algorithm that computes $\mathrm{Cl}(d)$.

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Together with Djordjo Milovic and Carlo Pagano I have been able to answer some of these questions.

Questions