On Stevenhagen's conjecture

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History of Pell's equation

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$$x^2 - dy^2 = 1 \text{ to be solved in } x, y \in \mathbb{Z}$$

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Lagrange was the first to give an algorithm with proof of correctness.

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Question: as we vary d, how often is the negative Pell equation soluble?

Solubility over the rationals

Define $\mathcal{D}$ to be the set of squarefree integers having as odd prime divisors only primes $p \equiv 1 \bmod 4$ and define $\mathcal{D}^-$ to be the set of squarefree integers for which negative Pell is soluble.

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Refined question: what is the density of $\mathcal{D}^-$ inside $\mathcal{D}$?

Conjectures on the negative Pell equation

Nagell (1930s) conjectured that

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}}$$

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Stevenhagen (1995) conjectured that

$$\lim_{X\to\infty} \frac{\#\{d \le X : d \in \mathcal{D}^-\}}{\#\{d \le X : d \in \mathcal{D}\}} = 1 - \alpha,$$

where

$$\alpha = \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1} \approx 0.41942.$$

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Fouvry and Klüners (2010) proved that

$$\frac{5\alpha}{4} \leq \liminf_{X\to\infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \limsup_{X\to\infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} \leq \frac{2}{3}.$$

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CKMP (2019) improved the lower bound to

$$\alpha \cdot \sum_{n=0}^{\infty} 2^{-n(n+3)/2} \approx \alpha \cdot 1.28325.$$

Stevenhagen's conjecture is true

Theorem 1 (K., Pagano (2021))

We have

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in accordance with Stevenhagen's conjecture.

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Corollary 2

We have

$$\#\{d \leq X : d \in \mathcal{D}^-\} \sim C \cdot (1 - \alpha) \cdot \frac{X}{\sqrt{\log X}}.$$

A criterion for solubility

Recall that the narrow class group $\mathrm{Cl}^+(K)$ is defined as the quotient of the ideal group I_K by the principal ideals P_K^+ admitting a totally positive generator, while the class group is the quotient by the principal ideals P_K.

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$$\begin{aligned} x^2 - dy^2 = -1 \text{ is soluble } &\iff \text{ fundamental unit } \epsilon \text{ has negative norm} \\ &\iff (\sqrt{d}) \text{ is trivial in } \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})). \end{aligned}$$

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There is a fundamental exact sequence

$$1 \to \frac{P_K}{P_K^+} \to \mathrm{Cl}^+(K) \to \mathrm{Cl}(K) \to 1$$

with $\#\frac{P_K}{P_K^+} \in \{1, 2\}$ and $\frac{P_K}{P_K^+}$ generated by $(\sqrt{d})$.

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Goal: study joint distribution of $(\mathrm{Cl}^+(K)[2^\infty], \mathrm{Cl}(K)[2^\infty])$.

Genus theory

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The description of $\mathrm{Cl}^+(K)[2]$ is due to Gauss and is known as genus theory. We have that

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and $\mathrm{Cl}^+(K)[2]$ is generated by the ramified prime ideals of $\mathcal{O}_K$.

If p divides the discriminant of $\mathbb{Q}(\sqrt{d})$, then p ramifies, so

$$\begin{array}{ccccc} \mathbb{Q}(\sqrt{d}) & \qquad & \mathfrak{p} & \qquad & \mathfrak{p}^2 = (p). \\ \Big| & & \Big| & & \\ \mathbb{Q} & & p & & \end{array}$$

There is precisely one relation between the ramified primes.

Cohen–Lenstra–Gerth

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More formally, Cohen and Lenstra conjectured that

$$\lim_{X \to \infty} \frac{\#\left\{K \text{ im. quadr.} : |D_K| < X \text{ and } \mathrm{Cl}(K)[p^\infty] \cong A\right\}}{\#\left\{K \text{ im. quadr.} : |D_K| < X\right\}} = \frac{\prod_{i=1}^{\infty}\left(1 - \frac{1}{p^i}\right)}{\#\mathrm{Aut}(A)}$$

for every finite, abelian p-group A.

Gerth's adaptation

Gerth adapted the conjecture of Cohen and Lenstra to $p = 2$

$$\lim_{X\to\infty} \frac{\#\left\{K \text{ im. quadr.} : |D_K| < X, 2\mathrm{Cl}(K)[2^\infty] \cong A\right\}}{\#\left\{K \text{ im. quadr.} : |D_K| < X\right\}} = \frac{\prod_{i=1}^{\infty}\left(1 - \frac{1}{2^i}\right)}{\#\mathrm{Aut}(A)}$$

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Two major difficulties: $\mathcal{D}$ has density 0 in the squarefree integers, and $\mathcal{D}$ naturally ends up in the error term in Smith's proof!

Strategy for Stevenhagen's conjecture

Example 1 (Definition of 2^k-rank)

Take

$$A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \quad .$$

Then $\mathrm{rk}_2 A = 3$, $\mathrm{rk}_4 A = \mathrm{rk}_8 A = 1$, $\mathrm{rk}_{2^k} A = 0$ *for* $k > 3$.

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Find for every integer $m \geq 1$, the density of $d \in \mathcal{D}$ for which

$$\mathrm{rk}_{2^k} \mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_{2^k} \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) > 0 \text{ for } 1 \leq k \leq m \text{ and}$$

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Duality of abelian groups

For a finite abelian group A, define

$$A^\vee := \mathrm{Hom}(A, \mathbb{C}^*).$$

There is a natural pairing

$$\mathrm{Art}_1 : A[2] \times A^\vee[2] \to \{\pm 1\}, \quad (a, \chi) \mapsto \chi(a).$$

Left kernel of Art_1 is $2A[4]$ and right kernel is $2A^\vee[4]$.

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So 4-rank is determined by the pairing Art_1. We start by describing $\mathrm{Cl}^{+,\vee}(K)[2]$.

The dual class group

Theorem 4 (Class field theory)

We have an isomorphism

$$\mathrm{Cl}^+(K) \cong \mathrm{Gal}(H^+(K)/K)$$

given by sending a prime ideal $\mathfrak{p}$ to $\mathrm{Art}(\mathfrak{p})$. Furthermore, if K is Galois, this isomorphism respects the natural Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ on both sides.

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If K is quadratic with odd discriminant, then $\mathrm{Cl}^{+,\vee}(K)[2]$ is generated by the quadratic characters χ_{p^*}, where p^* satisfies $|p^*| = |p|$ and $p^* \equiv 1 \bmod 4$.

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If K is quadratic with odd discriminant, then $\mathrm{Cl}^{+,\vee}(K)[2]$ is generated by the quadratic characters χ_{p^*}, where p^* satisfies $|p^*| = |p|$ and $p^* \equiv 1 \bmod 4$. In particular

$$d \in \mathcal{D} \iff \mathrm{rk}_2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d})) = \mathrm{rk}_2\mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \iff (\sqrt{d}) \in 2\mathrm{Cl}^+(\mathbb{Q}(\sqrt{d}))[4].$$

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Under the earlier identifications, we have that

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Let $p_1, \dots, p_t$ be the prime divisors of d. The Rédei matrix is

$$\begin{array}{cccccc} & \chi_{p_1^*} & \chi_{p_2^*} & \cdots & \chi_{p_t^*} \\ p_1 & * & \left(\frac{p_2^*}{p_1}\right) & \cdots & \left(\frac{p_t^*}{p_1}\right) \\ p_2 & \left(\frac{p_1^*}{p_2}\right) & * & \cdots & \left(\frac{p_t^*}{p_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_t & \left(\frac{p_1^*}{p_t}\right) & \left(\frac{p_2^*}{p_t}\right) & \cdots & * \end{array}.$$

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$$\begin{array}{ccccc} & \chi_{p_1^*} & \chi_{p_2^*} & \cdots & \chi_{p_t^*} \\ p_1 & * & \left(\frac{p_2^*}{p_1}\right) & \cdots & \left(\frac{p_t^*}{p_1}\right) \\ p_2 & \left(\frac{p_1^*}{p_2}\right) & * & \cdots & \left(\frac{p_t^*}{p_2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_t & \left(\frac{p_1^*}{p_t}\right) & \left(\frac{p_2^*}{p_t}\right) & \cdots & * \end{array}.$$

Left kernel surjects on $2\mathrm{Cl}^+(K)[4]$ with 1-dimensional kernel.

Interlude: Stevenhagen's conjecture

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Conjecture 1 (Stevenhagen's conjecture)

We have

$$\lim_{X \to \infty} \frac{\#\{d \leq X : d \in \mathcal{D}^-\}}{\#\{d \leq X : d \in \mathcal{D}\}} = \sum_{j=0}^{\infty} \frac{\mathbb{P}(4\text{-rank of } d \in \mathcal{D} \text{ equals } j)}{2^{j+1} - 1}.$$

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Furthermore,

$$\mathbb{P}(\text{4-rank of } d \in \mathcal{D} \text{ equals } j) = \lim_{t \to \infty} \mathbb{P}(t \times t \text{ sym. matrix has ker. of dim. } j).$$

The second Artin pairing

There is a natural pairing

$$\mathrm{Art}_2 : 2A[4] \times 2A^\vee[4] \to \{\pm 1\}, \quad (a, \chi) \mapsto \psi(a),\ 2\psi = \chi.$$

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Fact: a degree 4 unramified, abelian extension of $\mathbb{Q}(\sqrt{d})$ is Galois over $\mathbb{Q}$ with Galois group D_4.

Such extensions are of the shape $\mathbb{Q}(\sqrt{d}, \sqrt{a}, \sqrt{\alpha})$, where

$$x^2 = ay^2 + \frac{d}{a}z^2 \text{ with } x, y, z \in \mathbb{Z} \text{ and } \gcd(x, y, z) = 1, \quad \alpha := x + y\sqrt{a}.$$

Reflection principles

In the literature there are many known results that compare different class groups. For example, we have

$$\mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})) \leq \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{-3d})) \leq 1 + \mathrm{rk}_3\mathrm{Cl}(\mathbb{Q}(\sqrt{d})),$$

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Smith's idea is to look for situations where the compositum of various Hilbert class fields is in some sense *small*.

Intersections of Hilbert class fields

Take primes p_1, p_2, q_1, q_2. Now suppose that we have a degree 4 unramified, abelian extension of $\mathbb{Q}(\sqrt{dp_i q_j})$ each lifting the character χ_a.

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Recall that we then get $\alpha_{i,j} \in \mathbb{Q}(\sqrt{a})$ with

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$$\mathrm{Art}_{2,dp_1q_1}(b, \chi_a) + \mathrm{Art}_{2,dp_1q_2}(b, \chi_a) + \mathrm{Art}_{2,dp_2q_1}(b, \chi_a) + \mathrm{Art}_{2,dp_2q_2}(b, \chi_a) = 0$$

for $b \in 2\mathrm{Cl}(\mathbb{Q}(\sqrt{dp_i q_j}))[4]$ a fixed divisor of d.

Another reflection principle

With similar techniques, Smith proves another reflection principle

$$\begin{aligned}&\mathrm{Art}_{2,dp_1q_1}(b,\chi_{ap_1})+\mathrm{Art}_{2,dp_1q_2}(b,\chi_{ap_1})+\\&\qquad\mathrm{Art}_{2,dp_2q_1}(b,\chi_{ap_2})+\mathrm{Art}_{2,dp_2q_2}(b,\chi_{ap_2})=\sum_{r\mid b}\mathrm{Frob}_{K_{p_1p_2,q_1q_2}/\mathbb{Q}}(r).\end{aligned}$$

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The above reflection principle is useless in both cases.

We develop two new reflection principles. Unlike Smith's work, they make essential use of Hilbert reciprocity in multiquadratic fields.

Bonus slide: new reflection principles

For the Artin pairing with dp_iq_j we have (following Smith's ideas)

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For the pairing between a and χ_a we also develop a new reflection principle.

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Thank you for your attention!