

# Counting nilpotent extensions

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# Malle's conjecture

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Let  $G$  be a finite, non-trivial group. Then there exist numbers  $a(G) \in \mathbb{Q}_{>0}$ ,  $b(G) \in \mathbb{Z}_{\geq 0}$  and  $c(G) > 0$  such that

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Sometimes  $c(G)$  is an Euler product. This is expected to be true for  $S_n$  (Malle–Bhargava principle).

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- ▶ direct products  $S_n \times A$  for  $n \in \{3, 4, 5\}$  and  $A$  abelian by Wang (with  $\#A$  coprime to some values) and later by Masri–Thorne–Tsai–Wang.



# An exercise about hyperbolas

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$$\begin{aligned}\sum_{ab^2 \leq X} 1 &= \sum_{b \leq \sqrt{X}} \sum_{a \leq X/b^2} 1 = \sum_{b \leq \sqrt{X}} \left( \frac{X}{b^2} + O(1) \right) \\ &= X \sum_{b=1}^{\infty} \frac{1}{b^2} + O(\sqrt{X}).\end{aligned}$$

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Both observations fail now.

# Ramification theory

Let  $K/\mathbb{Q}$  be a Galois extension and suppose that  $p$  does not divide  $[K : \mathbb{Q}]$ . Then

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Moral: inertia subgroups tend to “typically” be as small as possible when counting by discriminant.

# An example

## Example (Non-Galois quartic $D_4$ )

If  $L/\mathbb{Q}$  is quartic  $D_4$  with quadratic subfield  $K$ , then for all  $p \neq 2$

$$v_p(D_L) = \begin{cases} 3 & \text{if } p \text{ is totally ramified} \\ 2 & \text{if } p \text{ is in all other cases} \\ 1 & \text{if } p \text{ is unramified in } K/\mathbb{Q} \text{ but ramifies in the biquadratic} \\ 0 & \text{if } p \text{ is unramified.} \end{cases}$$

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- ▶ main contribution comes from quadratic fields  $K$  with  $D_K < \log \log \log \log X$ ;
- ▶ a positive proportion of the quartic  $D_4$ -extensions have a given quadratic field  $K$  as their subfield.

# Difficulties with discriminant counting

Group theoretic properties greatly influence how difficult it is to count by discriminant, heavily exploited in previous works.

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*Note that  $D_{2^n} = \mathbb{Z}/2^n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ . The elements of minimal order are  $(k, 1)$  (reflections) and  $(2^{n-1}k, 0)$  (rotations with order dividing 2).*

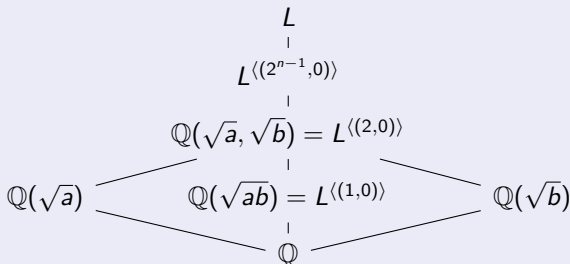


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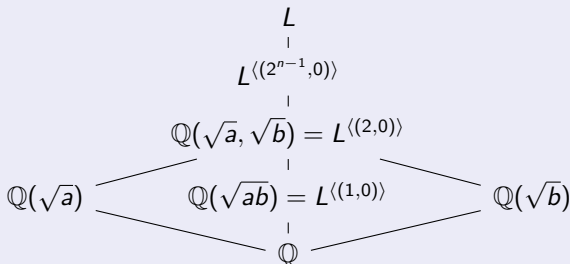


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Positive proportion of extensions have  $L^{\langle(2^{n-1}, 0)\rangle} / \mathbb{Q}(\sqrt{ab})$  unramified. So at least as hard as getting distribution of  $\text{Cl}(\mathbb{Q}(\sqrt{d}))[2^\infty]$ .

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Altug–Shankar–Varma–Wilson (2017): Malle’s conjecture for  $D_4$  by Artin conductor.



# Main result

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## Theorem (K.–Pagano)

Assume GRH. Let  $G$  be a nilpotent group with  $\#G$  odd. Then

$$\liminf_{X \rightarrow \infty} \frac{\# \left\{ K/\mathbb{Q} : \prod_{p: I_p \neq \{\text{id}\}} p \leq X, \text{Gal}(K/\mathbb{Q}) \cong G \right\}}{c'(G)X(\log X)^{b'(G)}} \geq 1,$$

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Surprisingly, the corresponding asymptotic

$$\lim_{X \rightarrow \infty} \frac{\#\left\{K/\mathbb{Q} : \prod_{p: I_p \neq \{\text{id}\}} p \leq X, \text{Gal}(K/\mathbb{Q}) \cong G\right\}}{c'(G)X(\log X)^{b'(G)}} = 1$$

is not true in general. Counterexamples exist for nilpotency class 2.

## Theorem (K.-Pagano)

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# Applications

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For every  $g_1 \in G_1 - \{\text{id}\}$ , every  $h_1 \in G_1$  and every  $\alpha$  coprime to  $p$  satisfying  $h_1 g_1 h_1^{-1} = g_1^\alpha$ , we have

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A non-trivial example is  $(G_1, G_2, G_3) = (\mathbb{F}_p^n, U(n+1, p)/Z, U(n+1, p))$ .

This is known as the Massey vanishing conjecture (recently proven by Harpaz–Wittenberg for all  $p$  and all number fields).

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For 2-groups, the situation is much more involved. The only known proof is a famous result of Shafarevich (inverse Galois for solvable groups).

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Let  $H$  be a  $p$ -group and let  $G$  be a central  $\mathbb{F}_p$ -extension of  $H$ , i.e.

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It is well-known that we have a local-to-global for the above diagram, which means that we have to control  $\pi(\text{Frob}_v)$  for all  $v$ .

## Scholz–Reichardt sketch II

Note that  $H$  also fits in an exact sequence

$$1 \rightarrow \mathbb{F}_p \rightarrow H \rightarrow H' \rightarrow 1.$$

Therefore we may twist our  $H$ -extension  $\pi : G_{\mathbb{Q}} \rightarrow H$  by  $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{F}_p$  to get  $\pi + \chi : G_{\mathbb{Q}} \rightarrow H$ .

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Here we use that  $p$  is odd in an essential way:  $\chi_{\ell}(\text{Frob}_q)$  and  $\chi_q(\text{Frob}_{\ell})$  are independent.

## Step 1a: parametrization

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$$0 \rightarrow \mathbb{F}_2 \rightarrow D_4 \xrightarrow{q} \mathbb{F}_2^2 \rightarrow 0$$

and a bijection

$$\text{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2^2) \leftrightarrow \{(a, b) \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^2 : a, b \text{ lin. ind.}\}.$$

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Given  $\pi \in \text{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2^2)$ , this leads to the *central embedding problem*

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It is well-known that a  $\mathbb{F}_2^2$ -extension  $\mathbb{Q}(\sqrt{a}, \sqrt{b})$  of  $\mathbb{Q}$  is contained in a  $D_4$ -extension if and only if  $x^2 = ay^2 + bz^2$  has a non-trivial point.

## Step 1b: parametrization

If  $\rho \in \text{Epi}(G_{\mathbb{Q}}, D_4)$  is a lift of  $\pi \in \text{Epi}(G_{\mathbb{Q}}, \mathbb{F}_2)$  and  $q : D_4 \rightarrow \mathbb{F}_2^2$ , then

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It turns out to be more convenient to work with seven variables  $\alpha_S$  for  $\emptyset \subset S \subseteq \{a, b, c\}$ , where  $\alpha_S$  is the product over all primes  $p$  dividing the variables in  $S$  and not dividing the variables in  $\{a, b, c\} - S$ .

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The variables  $\alpha_S$  are squarefree and pairwise coprime, and we have  $\text{rad}(|abc|) = \prod_{\emptyset \subset S \subseteq \{a, b, c\}} |\alpha_S|$ .

## Step 2: character sums

Define  $T(a)$  to be the subsets of  $\{a, b, c\}$  containing  $a$ . Then we have

$$a = \prod_{S \in T(a)} \alpha_S$$

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Now rewrite the above sum as a sum over Legendre symbols involving the variables  $\alpha_S$ .

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- ▶ Proof can most likely be made unconditional with a suitably strong large sieve for nilpotent extensions.



Thank you for your attention!