Midterm #4 Practice Exam Questions - Solutions

Topics in relativity:

- Michelson-Morley and ether drift
- Postulates of special relativity
- Simultaneity
- Lorentz transformations
- Length contraction
- Relativistic velocity addition
- Relativistic Doppler effect
- Spacetime intervals and causality
- Energy momentum invariant
- Relativistic momentum
- Relativistic total energy
- Rest energy
- Massless particles

1: Two events occur in the stationary frame S with coordinates:

1: $x = x_0 \quad t = x_0/2c$

2: $x = 4x_0 \quad t = x_0/c$

Assume that the origins of S and S' are the same at t=t'=0. What velocity must a reference frame S' be travelling for these two events to occur at the same time in S'?

The easiest way is to use the Lorentz transformations:

$$\Delta x' = \gamma (\Delta x - v\Delta t)$$
$$\Delta t' = \gamma (\Delta t - v\Delta x/c^2)$$

I want to make $\Delta t' = 0$, so:

$$\gamma (\Delta t - v\Delta x/c^2) = 0 \qquad \text{or} \qquad \Delta t = v\Delta x/c^2$$

and

$$v = c^2\Delta t/\Delta x = c^2 (x_0/2c)/(3x_0) = c / 6$$

2: A pair of atomic clocks are synchronized on the ground. Clock A is then placed in an airplane which flies in circles above clock B at constant velocity v=300m/s for 1 week, stopping only briefly to refuel. Estimate the time difference between the two clocks.

It is reasonable, for these low velocities and accelerations, to treat this airplane as an inertial reference frame. What sets the limit? Well it is reasonable to consider the Earth, for most purposes, as an inertial frame, even though it is spinning, orbiting the sun at 10km/s and plowing through the galaxy at 200km/s. So here we have time dilation, with $\beta = 300\text{m/s} / 3\text{x}10^8\text{m/s} = 10^{-6}$:

$$\Delta t' = \gamma\Delta t = (1 - \beta^2)^{-1/2}\Delta t$$

with numbers this small it pays to use the binomial expansion:

$$(1+x)^n \approx 1 + nx + (1/2)n(n-1)x^2 + \ldots.$$

Which here implies:

$$\Delta t' = (1 - \beta^2)^{-1/2}\Delta t \approx (1 + 1/2\beta^2)\Delta t = (1 + 0.5*10^{-12})\Delta t$$

here Δt = 1 week = 7days*24hours/day*3600sec/hour = $6.05\text{x}10^5$s

So $\Delta t' - \Delta t = -0.5*10^{-12}*6.05\text{x}10^5\text{s} = -3.0\text{x}10^{-7}\text{s}$

Or about 0.3µs.

3: Two events are simultaneous, but separated by a distance x_0 in one reference frame S. What velocity must reference frame S' be moving with to cause the two events to occur in the same place? In other words, what velocity is needed to make $\Delta x' = 0$.

These two events are separated by a spacetime interval:

$$s^2 = c^2\Delta t^2 - \Delta x^2 = 0 - x_0^2$$

which is a negative number. This means that the interval is "spacelike". A spacelike interval is intrinsically spatial, it cannot be made a pure time interval. It is impossible to move in such a way that you can make $\Delta x = 0$. Remember that this interval must be the same in every frame. If you want to find a frame in which $\Delta x'=0$, you would measure an interval:

$$s^2 = c^2\Delta t^2$$

which must be positive. So there's no way to find a reference frame in which these events occur at the same place.

You can also see this from $\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x$. To make $\Delta x' = 0$ requires $\gamma = 0$, which cannot occur. γ can only equal zero if v^2 is negative and much larger than c^2. Note that it is possible to make $\Delta t > 0$, but only by making Δx larger, not smaller.

4: People sometimes imagine travelling to nearby stars by accelerating to something near the speed of light gradually and then coasting. Imagine that our spacecraft reaches a speed of 0.95c relative to the galaxy as a whole. As it travels through interstellar space it collides with hydrogen atoms which are at rest relative to the galaxy. The rest mass of hydrogen is about 0.94 GeV.
What is the energy of each hydrogen atom in the reference frame of the spacecraft?

The gamma factor for this motion is:

$$\gamma = (1 - 0.95^2)^{-1/2} = 3.2$$

so the energy of the hydrogen atoms observed by the ship is:

$$E = \gamma m_0c^2 = 3.2(0.94\text{GeV}) = 3.0\text{GeV}$$

5: What fractional error does one make when calculating the kinetic energy using the Newtonian equation $KE=1/2mv^2$ instead of the relativistic equation for an object travelling with a speed of 10^4m/s?

Compare $1/2mv^2$ to $(\gamma-1)mc^2$

Take the difference over the correct answer:

$$[1/2mv^2 - (\gamma-1)mc^2] / (\gamma-1)mc^2 = 1/2mv^2/(\gamma-1)mc^2 - 1$$

here $\gamma = 1.0000000005$, so

$$\text{Error} = 1/2(10^4)^2 / 5.555\text{x}10^{-10}(3\text{x}10^8)^2 - 1 = 5\text{x}10^7/4.999995\text{x}10^7 - 1 = 0.000001$$

The exact value would require a more accurate calculator. The most important point is that it is less than one ten thousanth of one percent.

6: Two events occur at the same time in reference frame S and are separated by 1km along the x axis. What is the time difference between these two events when measured in a reference frame S' moving with constant velocity along x if the spatial separation in S' is 2km?

We know here that $\Delta x = 1$km, $\Delta x'=2$km, and $\Delta t = 0$, so

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x$$

so

$$\gamma = 2 \text{ which implies } v = 0.75^{1/2}c$$

given this we can find $\Delta t'$ from

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = 2(-0.75^{1/2}c(1\text{km})/c^2) = -2*0.75^{1/2}*(1\text{km}/3\text{x}10^8\text{m/s}) = -5.8\text{x}10^{-6}\text{ s}$$

7: Consider two reference frame S, and S' moving with velocity v along the x axis of S. If a particle travels in reference frame S with u_x=c, what is its velocity u_x' in reference frame S'?

The speed of light is the same in every reference frame, so $u_x' = c$.

8: A set of π^0 mesons is produced with total energies ranging from 6.0×10^9 to 18.0×10^9eV. The rest energy of the π^0 is 135.1×10^6eV and its lifetime measured in its rest frame is 2×10^{-16}s.

a) What is the range of speeds of these pions?
b) What is the range of distances through which they travel (on average) before decaying?

a) $E = \gamma mc^2$, so the range of γ factors is:

$\gamma_{low} = 6 \times 10^9 / 135 \times 10^6 = 44.4$

$\gamma_{high} = 18 \times 10^9 / 135 \times 10^6 = 133.3$

Now we can use $\beta = (1 - \gamma^{-2})^{1/2}$ to get velocities:

$\beta_{low} = 0.99975$

$\beta_{high} = 0.999972$

b) In the second part, just figure out the time in the pion frame:

Lab lifetime = γ (pion rest lifetime)

And

Distance = $\beta c \gamma$(pion lifetime)

$D_{short} = 0.99975 * 3 \times 10^8 m/s * 44.4 * 2 \times 10^{-16} s = 2.66 \times 10^{-6} m$

$D_{long} = 0.999972 * 3 \times 10^8 m/s * 133.3 * 2 \times 10^{-16} s = 8.0 \times 10^{-6} m$

9: Is the classical concept of an incompressible fluid valid in relativity?

An incompressible fluid is one who's density cannot be altered. Since relativity allows the possibility of length contraction, the density of a fluid is higher when measured in a moving reference frame, and changes with the velocity of the observer. So there are no incompressible fluids in relativity.

10: Take the rest radius of the Earth to be 6400km, and its orbital speed around the sun to be 30km/s. By how much is the Earth's radius foreshortened when seen by an observer on the Sun?

The motion is perpendicular to the line of sight, so there is no length contraction.