Hilbert schemes of points

Devlin Mallory

February 10, 2019

Example. We begin with an example. The Hilbert scheme of points of a variety X will parametrize all 0-dimensional closed subschemes of X; it will decompose into connected components corresponding to the length of the subscheme, i.e., the number of points. Let's begin by calculating what this looks like for $X = \operatorname{Spec} k[x, y]$ for k algebraically closed (you may as well think of $k = \mathbb{C}$). Denote by n the length we're considering. A length-n subscheme corresponds to a quotient $k[x, y]/I$ of length n.

$n = 1$: In this case, a length-1 subscheme is a quotient of $k[x, y]$ of length 1, i.e., a field, so the set of length-1 subschemes is in bijection with the points of the variety. Thus, the Hilbert scheme of length-1 subschemes is X itself.

$n = 2$: Say $k[x, y]/I$ has length 2. Then we must have a linear dependence

$$a + bx + cy = 0$$

in $k[x, y]/I$. By a linear change of coordinates we may replace this by the equation $y = 0$; that is, $y \in I$ in $k[x, y]$. Now, consider $x^2 \in k[x, y]/I$; this must be a linear combination of 1 and x, i.e.,

$$x^2 + dx + e \in I.$$

Factoring this, we obtain that

$$I = (y, (x - \alpha)(x - \beta))$$

for some α, β. There are then two cases: $\alpha \neq \beta$, in which case we have that

$$I = (y, x - \alpha) \cap (y, x - \beta),$$

i.e., the subscheme consists of two distinct points. The other case is $\alpha = \beta$; by a change of coordinates we may assume $\alpha = 0$, so that $I = (y, x^2)$. Here, we have supp $k[x, y]/I = (x, y)$, but we have a nilpotent, the element x. This should be interpreted as a tangent vector at the point (x, y), pointing along the x-axis.

Geometrically, the idea is as follows: points of the second kind are obtained by letting two distinct points collide; the tangent vector recovers the direction of collision. Note that this suggests what the Hilbert scheme of length-2 subschemes is: take the singular variety $\mathbb{A}^2 \times \mathbb{A}^2/S_2$ and blow up along the singular locus.

$n = 3$: The only new case of interest is where $\operatorname{supp} k[x, y]/I = (x, y)$ (without loss of generality); otherwise we obtain I either as the intersection of three maximal ideals (i.e., the closed subscheme of three distinct points) or as the intersection of a maximal ideal with an ideal of the form (y, x^2) (i.e., the union of a length-2 subscheme at a point and another distinct point).

So, suppose $\operatorname{supp} k[x, y]/I = (x, y)$ (so I is (x, y)-primary).

(1) Case 1: I doesn't contain any linear forms. Note then that $I \supset (x, y)^2$, since it's contained in some maximal power of (x, y), which by assumption isn't (x, y) itself. But then we have a surjection

$$k[x, y]/I \to k[x, y]/(x, y)^2;$$

both have length 3, so this is actually an isomorphism and $I = (x, y)^2$.

(2) Case 2: I contains a linear form, say y. We must have a linear dependence $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = 0$ in $k[x, y]/I$, i.e.,
$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \in I.$$
Note that α_3 cannot be 0, or else $1, x, x^2$ fail to be linearly independent and the length must be less than 3. But now it's clear that
$$\operatorname{length}\bigl(k[x, y]/(y, \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\bigr) = 3,$$
so that $I = (y, \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)$. But note that we can factor $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$; the only way for $\operatorname{supp} k[x, y]/I$ to be (x, y) is if this factorization is just x^3. That is, in suitable coordinates, we have $k[x, y]/(y, x^3)$.

So, what's the intuition behind these two kinds of "length-3 fat points"? Note that $k[x, y]/(x, y)^2$ has two linearly independent square-0 elements x and y, while $k[x, y](x, y^3)$ has a cube-0 nilpotent x. The intuition here is that the former is a point together with two distinct tangent directions and the latter a point with first- and second-order tangent vectors.

Exercise. Consider the family of ideals $I_t = (x, y) \cap (x - t, y) \cap (x, y - t)$. Show that if set $t = 0$ then we have $I = (x, y)^2$ (note that setting $t = 0$ does not commute with intersecting the ideals!). Thus we can view I as arising from letting three points collide along distinct directions of approach. In contrast, let $J_t = (x, y) \cap (x - t, y - (x - t)^2) \cap (x + t, y - (x + t)^2)$. Show that setting $t = 0$ we get (y, x^3). Thus, this ideal is obtained by letting three points collide along the smooth curve $y = x^2$.

Exercise. Perhaps more surprisingly, show that you obtain the same result for any curve $f(x, y)$ smooth at the origin and parametrizations of points colliding along $f(x, y)$ (e.g., take $f(x, y) = y - x^5$ and consider $J_t = (x, y) \cap (x - t, y - (x - t)^5) \cap (x - t^2, y - (x - t^2)^5)$ and show that setting $t = 0$ we still get (y, x^3)!).

Preliminaries

Throughout we work over a field k (though essentially everything in the first few lectures works exactly the same over $\mathbb{Z}$, and thus over an arbitrary noetherian ring A by basechange). We write $S = k[x_0, \ldots, x_n]$ for the homogeneous coordinate ring of $\mathbb{P}^n$.

The Hilbert scheme of a projective variety X, denoted $\operatorname{Hilb} X$, which we'll define and construct shortly, will be a moduli space parametrizing all closed subschemes of X. That is, we want the closed points (i.e., the $\operatorname{Spec} k$-valued points) to represent closed subschemes of X. This tells us what the closed points of $\operatorname{Hilb} X$ should be, but this doesn't specify a topological structure on them, nor the structure of a ringed space (i.e., the scheme-theoretic data). To get a handle on this data, we want to view $\operatorname{Hilb} X$ as its *functor of points*; that is, we want to define a functor Φ_X such that, for any scheme T, giving a map $T \to \operatorname{Hilb} X$ is the same thing as giving an element of $\Phi_X(T)$. By Yoneda's lemma, this will uniquely specify $\operatorname{Hilb} X$ up to unique isomorphism.

More formally, define the functor
$$\Phi_X : \text{schemes} \to \text{sets}$$
by
$$\Phi_X(T) = \{Z \subset X \times T\colon\ Z \text{ flat over } T\}.$$
The Hilbert scheme $\operatorname{Hilb} X$ will represent this functor, so that giving a map $T \to \operatorname{Hilb} X$ will be the same as giving an element of $\Phi_X(T)$, i.e., a closed subscheme $Z \subset X \times T$ flat over T (functorially).

Now, the task before us is to construct a scheme representing this functor; by uniqueness, whatever scheme we construct representing this functor will be the Hilbert scheme.

The example above, and our intuition, suggests that certain subschemes naturally "fit together" in families, and that a "nice" family of subschemes can't vary too much (e.g., we saw two points could collide to a length-2 point, but we don't expect the entire curve to degenerate to a few points). Above, we see that length (i.e., the number of points, suitably interpreted) is one natural numerical invariant of a family of subschemes. For the case of Hilbert schemes of points, length is indeed the only invariant we'll need to consider; however, taking a more general viewpoint will help clarify how to make our construction:

Definition. Let $Z \subset \mathbb{P}^n$ be a closed subscheme. Then we define a function $H_Z : \mathbb{Z} \to \mathbb{Z}$ by

$$H_Z(m) := S/I_Z,$$

where I_Z is the saturated homogeneous ideal defining I. We say H_Z is the Hilbert function of Z.

It can be easily shown by induction that $H_Z(m)$ is in fact equal to a polynomial $P_Z(m)$ in m for $m \gg 0$. In fact, we see that

$$P_Z(m) = \chi(\mathcal{O}_Z(m)) := \sum (-1)^i h^i(Z, \mathcal{O}_Z(m)).$$

(The proof in fact shows that $H_Z(m)$ is the $\mathbb{Z}$-linear sum of binomial coefficients in m, i.e., elements of the form $\binom{m+a}{a}$, so we just need to take m large enough that these are polynomial.) We say P_Z is the Hilbert polynomial of Z.

Remark. The polynomial P_Z encodes several important numerical invariants of the variety:

(1) $\dim Z = \deg P_Z$; call this d.

(2) $\deg Z =$ (leading coefficient of $P_Z)/d!$.

(3) $P_Z(0) - 1 = p_a(Z)$, the arithmetic genus of Z.

Exercise (easy)**.** Show when $\dim Z = 0$ that $P_Z(0) = \operatorname{length} Z$.

Proposition. *Let T be a noetherian scheme and $Z \subset \mathbb{P}^n \times T$. If Z is flat over T then $P_{Z_{\kappa(t)}}$ (the Hilbert polynomial of $Z_{\kappa(t)} \subset \mathbb{P}^n_{\kappa(t)}$) is locally constant. Moreover, if T is reduced, then the converse is true.*

The converse is quite impressive; intuitively, it says that all that is need for a closed subscheme of $\mathbb{P}^n \times S$ to be flat over S is for each fiber to have the same "numerical data" encoded in the Hilbert polynomial.

Now, we can replace $\mathbb{P}^n$ by any projective scheme X (with a fixed embedding $X \subset \mathbb{P}^n$), and define the Hilbert polynomial in the exact same way (a closed subscheme $Z \subset X$ will be closed in $\mathbb{P}^n$, after all). What the proposition above showed is that if T is connected (so that locally constant implies actually constant) then

$$\Phi_X(T) = \bigcup_P \{Z \subset X \times T \colon Z \text{ flat over } T \text{ and } Z_{\kappa(t)} \subset \mathbb{P}^n_{\kappa(t)} \text{ has Hilbert polynomial } P \text{ for all } t \in T\}$$

Defining

$$\Phi^P_X(T) = \{Z \subset X \times T \colon Z \text{ flat over } T \text{ and } Z_{\kappa(t)} \subset X_{\kappa(t)} \text{ has Hilbert polynomial } P \text{ for all } t \in T\},$$

we see that to represent the Hilbert scheme we just need to represent $\Phi^P_X(-)$ for each P, and that once we do so the Hilbert scheme will decompose as the disjoint union of the representing schemes for each P.

Polynomials P of the correct form to be a Hilbert polynomial are called numerical polynomials.

Exercise. What coefficients occur in Hilbert polynomials? (Hint: calculate Hilbert polynomials for $\mathcal{O}_{\mathbb{P}^n}(a)$ and use the fact that any $\mathcal{O}_Z$ has a finite resolution by direct sums of $\mathcal{O}_{\mathbb{P}^n}(a)$).

Remark. Note that the fixed embedding $X \to \mathbb{P}^n$ matters because the Hilbert polynomial is only defined after fixing a hyperplane section $\mathcal{O}_X(1)$. Thus, the total Hilbert scheme, which represents the functor

$$\Phi^X(T) = \{Z \subset X \times T \colon Z \text{ flat over } T\},$$

is independent of the embedding, but the labeling of its connected components will depend on the embedding.

Example. One might ask why we consider the Hilbert polynomial instead of the Hilbert function. As a basic example, let $Z_1 \subset \mathbb{P}^2$ be the subscheme consisting of three non-collinear points, which we can take to be $\{[1:0:0],[0:1:0],[0:0:1]\}$, and let $Z_2 \subset \mathbb{P}^2$ be the subscheme consisting of three points on a line, say $\{[1:0:0],[0:1:0],[1:1:0]\}$. By our comments on the numerical data encoded by the Hilbert polynomial, it's clear these have the same Hilbert polynomial, 3 (since they're each degree-3 subschemes). However, we claim that they do not have the same Hilbert function.

It's straightforward to calculate that these correspond to homogeneous graded ideals $I = (xy, xz, yz)$ and $J = (z, xy(x-y))$. Then just by counting polynomials in each ideal of a given degree we have

m	$(S/I)_m$	$(S/J)_m$
0	1	1
1	3	2
2	3	3
3	3	3
$\vdots$	$\vdots$	$\vdots$

Thus we see that they do not have the same Hilbert function. But there's an obvious family of subschemes of three points not on a line degenerating to three points on a line, so we want our moduli space to encode this family as well, so we need to use the Hilbert polynomial instead of the Hilbert function.

This suggests, however, that one can the consider the stratification of the Hilbert scheme given by subschemes with a fixed Hilbert function. Indeed, in this case we obtain the following kinds of subschemes:

(1) three distinct noncollinear points.

(2) three distinct collinear points.

(3) A length-2 point, i.e., a point and a tangent vector, and another point not collinear with this tangent vector.

(4) A length-2 point and another point collinear with the tangent vector.

(5) A length-3 subscheme of the form (y, x^3), obtained by three points colliding along a smooth curve (thought of as a point, a tangent vector, and a second-order tangent vector).

(6) A length-3 subscheme of the form (x^2, xy, y^2) (obtained by three points colliding along distinct directions).

These all have Hilbert function either $\{1,2,3,3,\dots\}$ or $\{1,3,3,\dots\}$; (1), (3), and (5) have the latter Hilbert functions and (2), (4), and (6) have the former. (Note that this reflects a difference in regularity between these schemes!)

Remark. This example in fact gives the first glimpse of the important concept of Castelnuovo–Mumford regularity, which we'll see shortly in the existence of the Hilbert scheme: in this guise, the regularity of I and J record how quickly the Hilbert function agrees with the Hilbert polynomial, and thus the regularity of I is lower than that of J. This is reflected in the defining homogeneous ideals, in that I is generated in degree 2 and J in degree ≤ 3. This notion of the degrees of defining equations is very important below.

So, the fundamental existence result we claim is:

Theorem. *Let X be a projective scheme and $\mathcal{O}_X(1)$ a fixed very ample line bundle on X, and let P be a numerical polynomial. The functor Φ_P^X defined above is representable by a projective scheme* $\operatorname{Hilb}_P X$.

(Note that projective implies finite-type!)

Example. Linear subspaces of $\mathbb{P}^n$ of a fixed dimension k are characterized precisely by their Hilbert polynomials, and thus for such a Hilbert polynomial the corresponding Hilbert scheme is just the grassmannian of k-planes in $\mathbb{P}^n$.

Example (Hilbert schemes of points on a curve)**.** Before we go any further, we'll give an example showing how the functor of points viewpoint can clarify our constructions. Let C be a curve (i.e., a 1-dimensional integral projective k-scheme). Intuitively, C doesn't have very many interesting subschemes: note that any 1-dimensional subscheme must C itself (since C is reduced it factors through any closed subscheme with the same support), and thus the only interesting Hilbert polynomial of degree 1 is just $P(m) = m + (p_a(C) + 1)$, and this has $\operatorname{Hilb}^P X = \{\text{pt}\}$.

Exercise. Check this claimed equality using the functor of points.

Now, we take C to be smooth. Consider the degree-0 Hilbert polynomials, which are just positive integers corresponding to the length of a subscheme. Fix one such integer, say n. We claim that $\operatorname{Hilb}^n C = C^{(n)}$, the n-th symmetric power of C; this will in turn show that $\operatorname{Hilb}^n C$ is smooth and irreducible of dimension n. To see the claimed equality, it is easiest to show that $C^{(n)}$ and $\operatorname{Hilb}^n C$ represent the same functor (i.e., their functor of points are naturally isomorphic).

It's not too hard to show that giving a morphism $T \to C^{(n)}$ is the same as giving an effective Cartier divisor D on $T \times C$, flat over T, such that $D|_{\{t\}\times C}$ has degree n.

Exercise. Show this. (Or refer to [ACGH85, p. 165].)

Thus it suffices to show that these two functors agree, which can clearly be checked on fibers.[1] Thus, what we really need to show is that specifying a length-n subscheme of C is the same as specifying a degree-n effective divisor on C. Note that given a length-n subscheme Z we obtain a degree-n divisor D obtained by sending

$$Z \mapsto \sum_{P \in Z} (\operatorname{length}_P Z)[P].$$

(This map will reappear slightly later in the course.) We just need to show that this is an isomorphism; equivalently, that a subscheme concentrated at a point is determined completely by its length. (We've seen this is not true already in dimension > 1; subschemes of length 2 concentrated at a point in $\mathbb{A}^2$ had a whole $\mathbb{P}^1$'s worth of distinct subscheme structure, depending on the choice of "tangent vector".)

But this is immediate, since C is a smooth curve: if $P \in C$ is a point with local ring $\mathcal{O}_P$, $\mathcal{O}_P$ is a DVR with uniformizer π, say, and the subschemes concentrated at P are all of the form $\mathcal{O}_P/\pi^m$, which has length m. Thus, a subscheme concentrated at a point is determined precisely by its length, so the two schemes have isomorphic functor of points and thus $\operatorname{Hilb}^n C = C^{(n)}$.

Exercise. Show that $C^{(n)}$ is smooth. (Hint: first, show that $(\mathbb{A}^1)^{(n)} = \mathbb{A}^n$, and is thus smooth, via the isomorphism $k[x_1, \ldots, x_n]^{S_n} \cong k[e_1, \ldots, e_n]$. Then use that smoothness can be checked after completing, and that completing commutes with taking the subring of invariants, so we can use the Cohen structure theorem and the $\mathbb{A}^1$ case to obtain the result for any smooth curve C.) Note that irreducibility is obvious, as $C^{(n)}$ is the surjective image of the irreducible variety C^n.

Thus, the Hilbert scheme of C is just

$$\left(\bigcup_n C^{(n)}\right) \cup \{\text{pt}\}.$$

Construction of the Hilbert scheme

We'll sketch only the main idea of the construction. Our exposition follows [Mum66, Chapter 15]. The key idea is to recognize the Hilbert scheme as a subscheme of the grassmannian. We recall that the grassmannian $G(r, s)$ represents the functor

$$\operatorname{Gr}_{r,s}(T) = \{\mathcal{O}_T^{\oplus r+1} \to Q : Q \text{ locally free of rank } s\}.$$

[1] Or note that one can merely check on k-points and then use Zariski's main theorem to see that we obtain an isomorphism.

We will give a natural transformation of functors

$$\Phi_P^X(-) \to \mathrm{Gr}_{r,s}(-)$$

for suitable r, s. That is, given a closed subscheme Z of $X \times T$, flat over T with Hilbert polynomial P on each fiber over T, we want to give a locally free quotient $\mathcal{O}_T^{\oplus r} \to Q$.

By general results about Hilbert schemes of points, it in fact suffices to work with T affine.

We first recall the notion of Castelnuovo–Mumford regularity.

Definition. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if $H^i(\mathcal{F}(m-i)) = 0$ for $i > 0$. The regularity of $\mathcal{F}$, denoted $\operatorname{reg}\mathcal{F}$, is the smallest m_0 such that $\mathcal{F}$ is m_0-regular.

If $X \subset \mathbb{P}^n$, we define the regularity of $\mathcal{F}$ by $\operatorname{reg} i_*\mathcal{F}$.

Recall that by Serre vanishing $H^i(\mathcal{F}(l)) = 0$ for $l \gg 0$, so this is a quantitative description of how much you need to twist by to guarantee this vanishing. Knowing the regularity of a sheaf tells you much more than the above description might suggest:

Proposition. *Let $\mathcal{F}$ be m-regular.*

(1) $\mathcal{F}(m)$ is generated by global sections.

(2) $H^0(\mathcal{F}(m)) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathcal{F}(m+1))$ is surjective.

(3) $\mathcal{F}$ is m'-regular for $m' \geq m$.

(Remember: low regularity is good, and high regularity is bad.)

The following lemma makes regularity a useful tool for constructing the Hilbert scheme:

Lemma. *For all n, there exists a polynomial $F(t_0, \dots, t_n)$ such that if $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$ is an ideal sheaf on $\mathbb{P}^n$, and we write*

$$\chi(\mathcal{J}(m)) = \sum a_i \binom{m}{i},$$

then $\mathcal{J}$ is $F_n(a_0, \dots, a_n)$-regular.

Since knowing $\chi(\mathcal{J}(m))$ is equivalent to knowing $\chi(\mathcal{O}_Z(m))$, where Z is the subscheme cut out by $\mathcal{J}$, this says that for a numeric polynomial P there is a constant C such that for *any* subscheme Z with Hilbert polynomial P we have $\operatorname{reg}\mathcal{O}_Z \leq C$. By our above description this says that for any subscheme Z with Hilbert polynomial P that we can uniformly bound the degrees of the generators of the homogeneous ideal I_Z, which is by no means obvious!

Let $\mathcal{J}$ be the ideal sheaf cutting out the closed subscheme Z of $X \times T$; for any $t \in T$, then, we have that $\mathcal{J}_{\kappa(t)} = \mathcal{J} \otimes \kappa(t)$ cuts out a closed subscheme of $\mathbb{P}^n_{\kappa(t)}$ with Hilbert polynomial P, and thus by our regularity bound we can choose m_0, *independently of* t, such that $\mathcal{J}_{\kappa(t)}$ is m_0-regular, and thus such that $H^1(\mathcal{J}_{\kappa(t)}(m_0)) = H^2(\mathcal{J}_{\kappa(t)}(m_0)) = 0$ and $\mathcal{J}_{k(t)}(m_0)$ is generated by global sections. By increasing m_0 we can assume moreover that $H^1(\mathcal{O}_X(m_0)) = 0$.

Write $p : X \times T \to T$. By assumption $\mathcal{O}_Z$ is flat over T, and so we can apply the cohomology and basechange theorems to obtain that:

(1) $p_*(\mathcal{O}_Z(m_0))$ is locally free of rank

$$r = H^0(\mathcal{O}_{X_{\kappa(t)}}(m_0)) - H^0(\mathcal{J}_{\kappa(t)}(m_0)) = \chi(\mathcal{O}_{X_{\kappa(t)}}(m_0)) - \chi(\mathcal{O}_{Z_{\kappa(t)}}(m_0)) = P(m_0),$$

where we used the vanishing of the higher cohomology of both $\mathcal{O}_{X_{\kappa(t)}}$ and $\mathcal{J}_{\kappa(t)}(m_0)$ to obtain the equality of the Euler characteristics and the global sections; note this is independent of t!

(2) $R^1 p_* \mathcal{J}(m_0) = 0$.

Thus, taking the exact sequence

$$0 \to \mathcal{J}(m_0) \to \mathcal{O}_{X \times T}(m_0) \to \mathcal{O}_Z(m_0) \to 0$$

and pushing forward along p we get

$$0 \to p_*\mathcal{J}(m_0) \to \underbrace{p_*\mathcal{O}_{X\times T}(m_0)}_{H^0(X,\mathcal{O}_X(m_0))\otimes\mathcal{O}_T} \to p_*\mathcal{O}_Z(m_0) \to \underbrace{R^1p_*\mathcal{J}(m_0)}_{0}$$

Note that we used affineness of T to say that

$$p_*(\mathcal{O}_{X\times T}(m_0)) = (H^0(\mathcal{O}_{X\times T}(m_0)))^{\sim} = (H^0(X,\mathcal{O}_X(m_0)) \times H^0(T,\mathcal{O}_T))^{\sim} = H^0(X,\mathcal{O}_X(m_0)) \otimes \mathcal{O}_T.$$

Now, set $N+1 = \dim H^0(X,\mathcal{O}_X(m_0))$. Thus, what the above short exact sequence gives is a surjection

$$\mathcal{O}_T^N \to p_*\mathcal{O}_Z(m_0) \to 0,$$

with the target locally free of rank r; note N and r are independent of Z and depend only on P! Thus, we've given a T-valued point of the grassmannian functor, and we thus have a natural transformation of functors $\Phi_P^X(-) \to \mathrm{Gr}_{r,N}(-)$.

Now, what is left to show is that this natural transformation in fact is an embedding of functors, so that there's actually a subscheme of the grassmannian that $\Phi_P^X(-)$ is the functor of points of.

The rest of the proof is much more formal, and so we omit its verification. It essentially consists of giving a criterion for a subfunctor to arise as the actual functor of points of a subscheme, and then verifying that criterion in our case. We sketch the main idea in the following exercises:

Exercise. Give a natural transformation of functors from $\mathrm{Gr}_{r,N}(-)$ to the functor Ψ taking T to the set of all closed subschemes of $X \times T$, as follows: Say T is affine, and start with a locally free quotient $\mathcal{O}_T^N \to \mathcal{E} \to 0$, with $\mathcal{E}$ of rank r, and say the kernel is $\mathcal{K}$. View $\mathcal{O}_T^N$ as $H^0(X,\mathcal{O}_X(m_0)) \otimes \mathcal{O}_T = p_*(\mathcal{O}_{X\times T}(m_0))$, Pull back via $p : X \times T \to T$ to obtain

$$\begin{array}{ccccccc} p^*\mathcal{K} & \longrightarrow & p^*\big(p_*(\mathcal{O}_{X\times T}(m_0))\big) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & \mathcal{O}_{X\times T}(m_0) & & & & \end{array}$$

Now, twist by $\mathcal{O}_{X\times T}(-m_0)$ to obtain a map $p^*\mathcal{K}(-m_0) \to \mathcal{O}_{X\times T}$; call the image of this map $\mathcal{J}$. $\mathcal{J}$ thus defines a sheaf of ideals on $\mathcal{O}_{X\times T}$, giving the desired transformation of functors.

Exercise. Check that $\Psi \circ \Phi_P^X$ is the "obvious" inclusion of flat subschemes of $X \times T$ with the desired Hilbert polynomial into the set of all closed subschemes of $X \times T$.

Exercise. Prove the following: say we have a commutative diagram of functors from schemes to sets

$$\begin{array}{ccc} \Phi(-) & & \\ \big\downarrow & \searrow & \\ & & h_G(-) \\ \big\downarrow & \swarrow & \\ \Psi(-) & & \end{array}$$

where h_G is the functor of points of an actual scheme G (i.e., $h_G(T) = \mathrm{Hom}(T,G)$), such that:

- ($\star$) for all T and $\alpha \in \Psi(T)$, there is a subscheme $Y \subset T$ such that the following are equivalent for any $g : T' \to T$:
  - (a) $g^*(\alpha) \in \Psi(T')$ is in $\Phi(T')$.
  - (b) g factors through Y.

Then there is a subscheme $G_0 \subset G$ such that $\Phi(-) = h_{G_0}(-)$ and the map $\Phi(-) \to h_G(-)$ is the natural map $h_{G_0}(-) \to h_G(-)$ induced by the inclusion $G_0 \to G$.

Exercise. Verify the hypotheses in our case, where $\Phi = \Phi_P^X$ and Ψ is as in the preceding exercise. Translated into our context, this consists of showing that

($\star\star$) for all schemes T and closed subschemes Z of $X \times T$, there is a subscheme $Y \subset T$ such that the following are equivalent for any $g : T' \to T$:

(a) $g^{-1}(Z) \subset X \times T'$ is flat over T' with Hilbert polynomial P on fibers.

(b) g factors through Y.

Use the following lemma, with $\mathcal{O}_Z$ in place of $\mathcal{F}$:

Lemma (existence of flattening stratifications)**.** *Let T be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n \times T$. There is a decomposition $T = T_1 \sqcup \cdots \sqcup T_l$ into locally closed subschemes such for any morphism $g : T' \to T$, we have that $(\mathrm{id}_{\mathbb{P}^n} \times g)^* \mathcal{F}$ is flat over T' if and only if $g : T' \to T$ factors as*

$$T' \to \bigsqcup T_i \hookrightarrow T.$$

Moreover, the T_i can be indexed by numerical polynomials P_i such that $\mathcal{F}|_{\mathbb{P}^n \times T_i}$ has Hilbert polynomial P_i on each fiber, and if $i \neq j$ then $P_i \neq P_j$.

Note that you'll need to say something about the Hilbert polynomial of $g^{-1}(\mathcal{O}_Z)$ (why is it equal to the Hilbert polynomial of $\mathcal{O}_Z$?); for this, use asymptotic cohomology and basechange.

If you're interested, the lemma is proved using Grothendieck's generic freeness lemma; see [Mum66, Chapter 8].

Thus, the Hilbert scheme exists, and is quasiprojective. In fact, it is even projective! To see this, note that it's enough to show it's proper, as this ensures it's a closed subscheme of the grassmannian, which is projective. This is easy, using the valuative criterion for properness and the knowledge of the functor of points of the Hilbert scheme:

Proposition (valuative criterion)**.** *A finite-type morphism $X \to Y$ of noetherian schemes is proper if and only if for all diagrams*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \overset{\exists!}{\nearrow} & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

where R is a DVR and K its fraction field, have a unique morphism $\operatorname{Spec} R \to X$ *making the diagram commute.*

Now, we apply this to structural morphism of the Hilbert scheme $\mathrm{Hilb}_P\, X \to \operatorname{Spec} k$. Translating, this says that if R is a k-algebra DVR and K its field of fractions, given a closed subscheme Z° of $X \times \operatorname{Spec} K$ with Hilbert polynomial P (flatness is automatic), there is a unique closed subscheme Z of $X \times \operatorname{Spec} R$, flat over R, with $Z \cap (X \times \operatorname{Spec} K) = Z^\circ$, i.e., a unique closed subscheme Z restricting to Z° on the open subset $X \times \operatorname{Spec} K$ of $X \times \operatorname{Spec} R$. But this is easy: just take $Z = \overline{Z^\circ}$, the closure of Z° in $X \times \operatorname{Spec} R$.

Exercise. Check that Z is flat over R, using the following criterion for a scheme to be flat over a regular 1-dimensional ring: Z is flat over R if and only if no associated point of Z is sent to the closed point of $\operatorname{Spec} R$ (check that all the associated points of the closure appear already in Z°).

Check moreover that Z is unique, by checking that any other choice of Z' must contain Z as a closed subscheme and that if $Z' \neq Z$ then Z' has an associated point mapped to the closed point of $\operatorname{Spec} R$.

Remark. As we'll see, this construction of the Hilbert scheme is actually of essentially no use in actually working with it. Note that part of the barrier here is the asymptotic vanishing results we've used: $N = H^0(\mathcal{O}_X(m))$ grows like $m^{\dim X}$, and we've fixed m enough to ensure m-regularity and kill many cohomology sheaves. Note that since $r = P(m_0)$ we have that r is on the order of $m^{\dim Z}$. Thus we've embedded the Hilbert scheme into the Grassmannian $G(r, N)$ which has dimension $r(N-r)$, which is of order $m^{\dim X + \dim Z}$, and thus the need to take a high m will give us an embedding into a very high-dimensional grassmannian.

Moreover, the proof doesn't give any indications of how to find the equationns cutting out the Hilbert scheme as a subscheme of the grassmannian.

Example. We give a geometrically intuitive example of a very "nice" situation in which we already see singularities arise purely via geometric reasoning. Consider the Hilbert scheme parametrizing subschemes of $\mathbb{P}^3$ with Hilbert polynomial $3m+1$. These will have dimension 1 and degree 3. We have the following two families of subschemes with this Hilbert polynomial:

(1) rational normal cubic curves: these are open and dense in a 12-dimensional component of the Hilbert scheme.[2] (Recall that by Riemann–Roch a rational normal curve in $\mathbb{P}^3$ corresponds to a choice of basis (up to scaling) for the 4-dimensional space $H^0(\mathcal{O}_{\mathbb{P}^1}(3))$, modulo isomorphisms of $\mathbb{P}^1$; this gives $16-1-3=12$ dimensions.)

(2) The disjoint union of a planar cubic and a point; this gives a 16-dimensional component.[3] (There is a 3-dimensional space of choices of the plane, and a 10-dimensional space of cubics on a plane, and a 3-dimensional space of choices of the point; thus, $10+3+3=16$ dimensions.)

However, it's quite straightforward to give a flat degeneration from a rational cubic curve to a plane cubic with an embedded point. This follows from a more general technique for obtaining flat degenerations: projection from a point. If X is a subscheme of $\mathbb{P}^n$ and we project X from $p=[0:\cdots:0:1]\notin X$ to $V(x_n)$, then we in fact obtain a flat degeneration from X to some (nonreduced!) scheme structure on $\varphi(X)$ as follows: for $\lambda \in k^*$, let $\sigma_\lambda \in \mathrm{Aut}(\mathbb{P}^n)$ be the map $[x_0,\ldots,x_n]\mapsto[x_0,\ldots,x_{n-1},\lambda x_n]$. Let X_λ be $\sigma_\lambda(X)$. This gives a flat family over $\mathbb{A}^1-\{0\}$, which thus has a unique extension to $\mathbb{A}^1$; the fiber of this flat family over 0 lies on $V(x_n)$ and has underlying scheme the projection of X.

In our case, one can check that projecting from a suitably chosen point we obtain exactly the nodal cubic with an embedded point at the node. Thus, the nodal cubic with an embedded point lies on the 12-dimensional irreducible component of the Hilbert scheme corresponding to (limits of) rational cubic curves.

However, it's also very easy to see that the nodal cubic with embedded point is also the limit of the disjoint union of planar cubic and a point: simply take the planar cubic to be the nodal cubic and let the disjoint point approach the node point along the normal direction to the plane.

Thus, we have that the point of the Hilbert scheme corresponding to the nodal cubic with embedded point lies on the intersection of a 12-dimensional component and a 16-dimensional component, and is thus necessarily singular.

Connectedness

In general, the Hilbert scheme is quite badly behaved, as we'll see shortly! It's highly singular and rarely reduced or irreducible. One thing that is always true, however, is that it's connected. This is due to Hartshorne; we give a slightly more modern interpretation via the theory of Gröbner degeneration. Our exposition here is mostly drawn from [Mac07], which describes a proof of [Ree95] in characteristic 0, extended by [Par96] to the positive characteristic case; for more background on monomial orders and Gröbner bases see [Eis95, Chapter 15].

We begin by defining an order on the set of monomials of $S=k[x_0,\ldots,x_n]$. We will discuss only one fixed ordering (the graded lexicographic ordering), but much of what we say works for any ordering.

Write x^α for a monomial, where $\alpha=(\alpha_0,\ldots,\alpha_n)\in\mathbb{Z}^{n+1}$, and write $|\alpha|=\sum\alpha_i$. We define $x^\alpha > x^\beta$ if either $|\alpha|>|\beta|$, or if $|\alpha|=|\beta|$ then if

$$\alpha-\beta=(\alpha_0-\beta_0,\ldots,\alpha_n-\beta_n)$$

has its first nonzero entry positive.

[2] moreover, one can check by computing the global sections of the normal sheaf that these are all smooth points of the Hilbert scheme.

[3] one can check these are nonsingular points of the Hilbert scheme as well.

Example. $x_0x_2 > x_1^2$, since we have $(1,0,1)-(0,2,0)=(1,-2,1)$, which indeed has first nonzero entry positive.

It is straightforward to check that this gives a total order on the set of all monomials. Thus, for $f \in S$, if we write f as a k-linear combination of monomials, then we may define $\operatorname{in} f$ to be the maximal monomial with respect to this order.

Example. $\operatorname{in}(x_1^2 - x_0x_2) = x_0x_2$.

Definition. For an ideal $I \subset S$, define $\operatorname{in} I = \{\operatorname{in} g : g \in I\}$.

Example. If

$$I = I_2 \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix} = (xz - y^2, yw - z^2, xw - yz),$$

the homogeneous ideal of the twisted cubic, then one can check that

$$\operatorname{in} I = (xz, yw, xw).$$

Remark. Note that in general if we have $I = (g_1, \dots, g_r)$, then $\operatorname{in} I$ is *not* generated by the $\operatorname{in} g_i$! A generating set for I whose initial terms generate $\operatorname{in} I$ is called a Gröbner basis; we won't say more about Gröbner basis, but their study underlies a lot of the following material on connectedness.

Clearly $\operatorname{in} I$ is always a monomial ideal, and thus susceptible to many special tools; in particular, there are a great many combinatorial techniques to deal with monomial ideals. Note that in passing to $\operatorname{in} I$ from I, we trade nice *geometric* properties for nice *combinatoric/algebraic* properties; for example, in passing from the ideal of the twisted cubic to its initial ideal, we trade a smooth irreducible curve for three lines intersecting in two points; a "worse" geometric object, but one with a much easier ideal to work with!

Initial ideals are of particular utility in the study of the Hilbert scheme. To start with, it is elementary to show:

Proposition. *Let $I \subset S$ be homogeneous. S/I and $S/\operatorname{in} I$ have the same Hilbert function, and thus the same Hilbert polynomial.*

Proof. We claim first that

$$\{m \in S : m \notin \operatorname{in} I\}$$

is a k-basis for both S/I and $S/\operatorname{in} I$. It's clearly a k-basis for $S/\operatorname{in} I$. To see linear independence in S/I, say $\sum \alpha_i m_i = 0$ in S/I; then $\sum \alpha_i m_i = f \in I$. But then we have that $\operatorname{in} f$ is one of the m_i, a contradiction. To see that they span, we appeal to the division algorithm for Gröbner bases; this is not hard to show at all, but would take us a bit out of our way. In any case, this division algorithm implies that for $f \in S$ we can write

$$f = g + h,$$

with $g \in I$ and h the sum of monomials *not* in $\operatorname{in} I$ (the idea being if some monomial term of h equal to $\operatorname{in} g'$, we would divide further by g'). This then implies that the image of f in S/I can be written in the desired form.

Now, this immediately implies the proposition, since monomials will have the same degree whether viewed in S/I or $S/\operatorname{in} I$. □

Proposition (Gröbner degeneration)**.** *Let $I \subset S$ be a homogeneous ideal. There is an ideal $\widetilde{I} \subset S[t]$ such that $S[t]/\widetilde{I}$ is a free (hence flat) $k[t]$-module. Moreover, we have*

$$S[t]/\widetilde{I} \otimes_{k[t]} k[t]/(t-\alpha) \cong \begin{cases} S/I & \alpha \neq 0 \\ S/\operatorname{in} I & \alpha = 0. \end{cases}$$

Geometrically, the idea is that we have a flat family $Z \subset \mathbb{P}^n \times \mathbb{A}^1$ over $\mathbb{A}^1$, with fiber over 0 the scheme $S/\operatorname{in} I$, and all other fibers isomorphic to S/I.

Proof. First, we need to describe $\widetilde{I}$: To begin with, we want a *weight function* approximating $<$, that is, a function $\lambda : \mathbb{Z}^{n+1} \to Z$ (thought of as assigning a "weight" to each monomial) giving an order resembling $<$. We can define $\mathrm{in}_\lambda I$ in the obvious way; for suitably generic λ, the ideal $\mathrm{in}_\lambda I$ will be monomial. Moreover, once we've chosen I, it's straightforward to show there is a weight function λ such that $\mathrm{in}_\lambda I = \mathrm{in}\, I$.

So, fix such a weight function. For $g \in S$, say m is a monomial appearing in g with $b := \lambda(m)$ maximal. Set

$$\widetilde{g} = t^b g\big(t^{-\lambda(x_0)}x_0, \ldots, t^{-\lambda(x_n)}x_n\big)$$

(compare this to the process of homogenizing an ideal). It's easy to check that

$$\widetilde{g} = \mathrm{in}_< g + tg'(t, x_1, \ldots, x_n)$$

for some polynomial g'. Define $\widetilde{I}$ to be the ideal generated by $\widetilde{g}$ for $g \in I$.

Now, by our earlier characterization of flatness over reduced schemes (flat if and only if constant Hilbert polynomial) and our preceding proposition, we need only show the claimed description of $S[t]/\widetilde{I} \otimes_{k[t]} k[t]/(t-\alpha)$, since S/I and $S/\mathrm{in}\, I$ have the same Hilbert polynomial and this gives flatness. It's clear from our claimed expression of $\widetilde{g} = \mathrm{in}_< g + tg'(t, x_1, \ldots, x_n)$ that $S[t]/\widetilde{I} \otimes_{k[t]} k[t]/t$ is $S/\mathrm{in}\, I$. To see the other fibers are isomorphic to S/I, just note that setting $t = \alpha$ for $\alpha \neq 0$ we see that the fiber is isomorphic to the image of S/I under the automorphism of $\mathbb{P}^n$ given by rescaling the coordinates, i.e.,

$$[x_0, \ldots, x_n] \mapsto [\alpha^{\lambda(x_0)}x_0, \cdots, \alpha^{\lambda(x_n)}x_n];$$

this is obviously an isomorphism of projective space and thus $S[t]/\widetilde{I} \otimes_{k[t]} k[t]/(t-\alpha)$ is abstractly isomorphic to S/I, as desired. □

In fact, the proposition is true even if I is not homogeneous, but the proof takes just slightly more work, and this is the only case we'll need.

The import of this statement is that S/I and $S/\mathrm{in}\, I$ define closed subschemes on the same component of the Hilbert scheme, since there is a flat family over $\mathbb{A}^1$ degenerating one to the other, i.e., a map from $\mathbb{A}^1$ to the Hilbert scheme with $1 \mapsto [\mathrm{Proj}\, S/I]$ and $0 \mapsto [\mathrm{Proj}\, S]$. Thus to show that the Hilbert scheme is connected it's enough to show that every monomial ideal lies on the same connected component.

In fact, we may restrict to a smaller class of ideals, the *Borel-fixed* ideals. Recall that there is a Borel subgroup $\mathcal{B} \subset \mathrm{PGL}(n+1, k)$ consisting of upper-triangular matrices; an ideal J is called Borel-fixed if $\mathcal{B} \cdot J = J$. Note that $\mathcal{B}$ contains the torus $(k^*)^{n+1}$, and thus Borel-fixed ideals must be monomial.

We can obtain Borel-fixed ideals via the notion of the *generic initial ideal*. Note that the initial ideal we've defined depends on a choice of basis for S_1, i.e., on our choice of $x_0, \ldots, x_n$. To remedy that, we have the following theorem:

Theorem. *Let $I \subset S$ be an ideal. There exists a Zariski-open dense set $U \subset \mathrm{PGL}(n+1, k)$ and a monomial ideal J such that for all $g \in U$ we have*

$$\mathrm{in}(gI) = J.$$

Moreover, J is Borel-fixed.

(Recall that $\mathrm{PGL}(n+1, k)$ is $\mathrm{GL}(n+1, k)$ modulo scaling, acting in the obvious way.)

The ideal J is called the generic initial ideal of I, denoted $\mathrm{gin}\, I$.

Example. Let $S = k[x, y]$, so the order on the variables is $x > y$. The initial ideal of the ideal (y) is just (y), of course, but the generic initial ideal is (x), since for an open dense subset of linear automorphisms y is sent to $ax + by$, with $a \neq 0$. In contrast $\mathrm{in}(x) = \mathrm{gin}(x) = (x)$. Note that (x) is clearly Borel-fixed:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(since α_1 cannot be 0).

Example. If $I = (xz - y^2, yw - z^2, xw - yz)$, then we've seen in $I = (xz, yw, xw)$. In contrast, $\operatorname{gin} I = (x^2, xy, y^3, xz)$.

It is clear that the subscheme cut out by the generic initial ideal lies on the same irreducible component of the Hilbert scheme as the subscheme cut out by the ideal: given I an ideal and $g \in \mathrm{PGL}(n+1, k)$, by connectedness of $\mathrm{PGL}(n+1, k)$, there is a flat family $I' \subset S[t]$ with fiber over 0 equal to S/I and fiber over 1 equal to S/gI (let $\gamma(t)$ be a path from 0 to g in $\mathrm{PGL}(n+1)$, and let I' be $\gamma(t)I$), so S/I and S/gI lie on the same component, and we've already seen degenerating to $S/\operatorname{in} g$ also preserves the component.

Thus, one is reduced to showing that all Borel-fixed ideals lie on the same connected component of the Hilbert scheme. The key step here is the process of *distraction*: let $I \subset S$ be a monomial ideal. We define a monomial ideal $p(I)$ of the infinite polynomial ring $k[z_{ij} : 0 \le i \le n, 0 \le j]$ as the ideal generated by

$$\left\{ \prod_{i=0}^{n} \prod_{j=1}^{u_i} z_{ij} : x^u \in I - mI \right\}$$

(that is, for x^u a minimal generator of I). Note $p(I)$ is finitely generated even though $k[z_{ij}]$ is not noetherian.

Example. Let $S = k[x, y]$ and $I = (x^2, xy, y^2)$. Then the minimal generators of I are x^2, xy, y^2, and thus $p(I)$ is generated by

$$\{z_{01}z_{02}, z_{01}z_{11}, z_{11}z_{12}\}.$$

Now, for some choice of scalars $\underline{\alpha} = (\alpha_{ij})$ we define $\sigma_\alpha : k[z_{ij}] \to S$ by $\sigma_\alpha(z_{ij}) = x_i - \alpha_{ij}x_n$. Now, we define the distraction of I as $\sigma_\alpha(p(I))$.

Example. With $I = (x^2, xy, y^2)$ as above, we see that

$$\begin{aligned} \sigma_\alpha(p(I)) &= \{(x - \alpha_{01}y)(x - \alpha_{02}y), (x - \alpha_{01}y)(y - \alpha_{11}y), (y - \alpha_{11}y)(y - \alpha_{12}y)\} \\ &= \{x^2 - (\alpha_{01} + \alpha_{02})y + \alpha_{01}\alpha_{02}y^2, y(x - \alpha_{01}y)(1 - \alpha_{11}), y^2(1 - \alpha_{11})(1 - \alpha_{12})\} \end{aligned}.$$

Note that for $\alpha_{01} = 0, \alpha_{02} = 0$ and $\alpha_{11} \neq 1$, $\alpha_{12} \neq 1$ we recover I as $\operatorname{gin}(\sigma_\alpha(p(I)))$.

Now, one can show that for general (α_{ij}) that there is a Gröbner degeneration from S/I to $S/\sigma_\alpha(p(I))$ (essentially the α_{ij} just need to be chosen so that certain matrix minors don't vanish), and thus the subscheme defined by I and $\sigma_\alpha(p(I))$ lie on the same connected component of the Hilbert scheme.

Thus, we see that starting from an ideal I, taking $\operatorname{gin} I$, taking a generic distraction $\sigma_\alpha(p(\operatorname{gin} I))$, and then taking gin of this, and repeating, we will always stay on the same connected component of the Hilbert scheme. The key idea is that starting with any I this process (for suitably generic α_{ij}) will eventually bring us to a certain point of the Hilbert scheme.

But what point to use? That is, given a Hilbert polynomial P, is there a "canonical" subscheme with this Hilbert polynomial? By a theorem of Macaulay, the answer is yes:

Definition. A monomial ideal I is lexicographic if for $x^\beta \in I$ and $x^\alpha >_{\text{lex}} x^\beta$ for $|\alpha| = \beta|$ we have $x^\alpha \in I$ as well.

(That is, a lexicographic ideal is specified by choosing a smallest monomial in each degree and then adjoining all greater monomials in the lex order.)

Theorem (Macaulay)**.** *Let P be a numerical polynomial. If P occurs as the Hilbert polynomial of a subscheme of $\mathbb{P}^n$, then there is exactly one saturated lexicographic ideal I_{lex} with Hilbert polynomial P. Moreover, I_{lex} has the worst regularity of all ideals defining closed subschemes with Hilbert polynomial P.*

This unique ideal defines as a closed subscheme known as the lexicographic point. (In fact, by a theorem of [RS97], the lexicographic point is smooth and thus lies on a unique component of the Hilbert scheme!)

One can show that for suitably general α_{ij} that $\sigma_\alpha(p(I))$ is radical and has an irreducible decomposition into linear subspaces of $\mathbb{P}^n$. Such a linear component of $\sigma_\alpha(p(I))$ is said to be *lexicographic position* if it's an irreducible component of $\sigma_\alpha(p(I_{\text{lex}}))$ (the distraction of the lexicographic point). Moreover, for general (α_{ij}), $\operatorname{gin}(\sigma_\alpha(I))$ is independent of α!

The key to showing this process terminates is the following proposition:

Proposition. *Let I be a saturated Borel-fixed ideal such that all irreducible components of $\sigma_\alpha(p(I))$ of dimension $\geq i+1$ are in lexicographic position. Let $J = \left(\operatorname{gin}(\sigma_\alpha(p(I))) : m^\infty\right)$ and $\beta = \beta_{ij}$ be general; then $\sigma_\beta(p(J))$ has all irreducible components of dimension $\geq i$ in lexicographic position.*

Example. We return to the example of the Hilbert scheme of subschemes of $\mathbb{P}^3$ with Hilbert polynomial $P = 3m+1$; recall we've seen that there are two distinct components, one of dimension 12 (with open dense subset corresponding to twisted cubics) and another of dimension 15 (corresponding to the disjoint union of a planar cubic curves and a point). One can check that the lexicographic ideal with Hilbert polynomial is

$$I_{\text{lex}} = (x, y^3z, y^4).$$

Note that this has primary decomposition

$$I_{\text{lex}} = (x, y^3) \cap (x, y^4, z),$$

and so corresponds geometrically to a "triple line" (the degree-3 subscheme cut out by (x, y^3), with support just the line (x, y)) with an embedded point (x, y^4, z). Now, we take a randomly chosen ideal corresponding to the union of a planar cubic and a point

$$\begin{aligned} J &= (z-w, y+w, x-w) \cap (w, x^3 - 5x^2z - y^2z + z^3) \\ &= (zw - w^2, yw + w^2, xw - w^2, x^3 - 5x^2z - y^2z + z^3 + 4w^3) \end{aligned}$$

(one can check using the description in the following section that the corresponding point of the Hilbert scheme has tangent space 15, and so is a smooth point of the Hilbert scheme). One can calculate that

$$\operatorname{gin} J = (x^2, xy, y^4, xz, y^3z, xw^2).$$

Now, taking the saturation (i.e., the largest homogeneous ideal defining the same closed subscheme as $\operatorname{gin} J$) we find

$$(\operatorname{gin} J : m^\infty) = (x, y^3z, y^4) = I_{\text{lex}};$$

thus, we definitely know that I_{lex} lies on the irreducible component corresponding to the union of planar cubic with a point (and is even a smooth point of the component, by [RS97])!

Now, take

$$I = (z^2 - yw, yz - xw, y^2 - xz),$$

the ideal of a twisted cubic. As we've said,

$$\operatorname{gin} I = (x^2, xy, y^3, xz) = (x, y^3) \cap (x^2, y, z),$$

so corresponds to the union of a triple line with an embedded point[4]. One can check that this point has tangent space dimension 16, and thus is even more singular than the other points we've considered! One can check that for (α_{ij}) general we have that

$$\operatorname{gin}(\sigma_\alpha(p(\operatorname{gin} I))) = (x^2, xy, y^4, xz, y^3z, xw^2) = \operatorname{gin} J;$$

thus they have the same saturation, which is I_{lex}. (One can check moreover that $\sigma_\alpha(p(\operatorname{gin} I))$ is the union three lines, lying on a plane, and a point not on the plane, which corresponds to a point of the Hilbert scheme with tangent space of dimension 15, which is thus singular). Thus, we confirm that I and J lie on the same connected component of the Hilbert scheme, by connecting them both to I_{lex}.

[4]Note that the difference between $\operatorname{gin} I$ and I_{lex} is that I_{lex} defines a scheme contained in the plane $x = 0$, while $\operatorname{gin} I$ is not planar: the nilpotent "points away" from the plane $x = 0$.

Infinitesimal study of the Hilbert scheme

Because the Hilbert scheme is a moduli space, many questions about the geometry of the Hilbert scheme translate to questions about families of closed subschemes of projective space.

Exercise. Show that for any k-scheme X the tangent space to X at p, $T_pX = (m_p/m_p^2)^\vee$, is

$\mathrm{Hom}_p(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X) := \big\{\varphi \in \mathrm{Hom}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X) : \mathrm{Spec}\, k \to \mathrm{Spec}\, k[\epsilon]/\epsilon^2 \xrightarrow{\varphi} X \text{ has image } p\big\}$.

Write $D = \mathrm{Spec}\, k[\epsilon]/\epsilon^2$, and let $i : \mathrm{Spec}\, k \to D$ the inclusion of the closed point; note that we then have a natural inclusion $i_X : X \to X \times_k D$ for any k-scheme X.

So, if Z is a subscheme of X, so $[Z]$ is a k-valued point of $\mathrm{Hilb}\, X$, then we have that

$$T_{[Z]} \mathrm{Hilb}\, X = \mathrm{Hom}_{[Z]}(D, \mathrm{Hilb}\, X);$$

By construction we know that

$$\mathrm{Hom}_{[Z]}(D, \mathrm{Hilb}\, X) = \{\widetilde{Z} \subset X \times D : \ \widetilde{Z} \text{ flat over } D, \, i_X^{-1}\widetilde{Z} = Z\}.$$

Such a subscheme $\widetilde{Z}$ is called a first-order deformation of Z.

Theorem (subschemes)**.** *Let $Z \subset X$ be a closed inclusion of k-schemes. Then the set of first-order deformations of Z in X is naturally in bijection to*

$$H^0(Z, N_{Z/X})$$

where $N_{Z/X} = (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$ is the normal sheaf of Z in X.

Corollary. *If X is projective and $Z \subset X$ a closed subscheme, $T_{[Z]} \mathrm{Hilb}\, X = H^0(Z, N_X)$.*

Remark (intuition for the proof)**.** Say we're given an ideal $I = (f_1, \ldots, f_n)$ defining a closed subscheme $X = \mathrm{Spec}(R/I)$ in $Y = \mathrm{Spec}\, R$. Naively, we'd expect a "small" deformation of I to look like $(f_1 + \epsilon g_1, \ldots, f_n + \epsilon g_n)$; moreover, the g_i should only be defined as elements as R/I, since clearly adding multiples of something in I doesn't affect the ideal. Thus, a deformation corresponds to an element of $\mathrm{Hom}_R(I, R/I)$. This is naturally an R/I-module, though, so it's equal to

$$\mathrm{Hom}_{R/I}(R/I, \mathrm{Hom}_R(I, R/I)) = \mathrm{Hom}_{R/I}(I/I^2, R/I) = (I/I^2)^\vee = N_{X/Y}.$$

Globalizing this, one can check that a first-order deformation of X in Y is exactly the same thing as a global section of $N_{X/Y}$.

Pathologies of Hilbert schemes

Exercise (tricky)**.** Confirm the singularity of the Hilbert scheme $\mathrm{Hilb}^{3t+1} \mathbb{P}^3$ by calculating $H^0(N_{X_0/\mathbb{P}^3})$, where X_0 is the nodal planar cubic with embedded point; you should get 15.

Example. Now, let's see the first example of a singular Hilbert scheme of points. Let $X = \mathbb{P}^3$ and consider the subscheme Z of $\mathbb{P}^3$ defined in the chart $\mathrm{Spec}\, k[x, y, z]$ by $I = (x, y, z)^2$. Z is thus a length-4 point supported at (x, y, z). Note that since $N_{Z/\mathbb{P}^3}$ is supported at the point (x, y, z), we have that $H^0(N_{Z/\mathbb{P}^3}) = \mathrm{Hom}(I/I^2, k[x, y, z]/I)$. We claim that $\mathrm{Hom}(I/I^2, k[x, y, z]/I)$ has dimension 18. Note that the component of $\mathrm{Hilb}^4 \mathbb{P}^3$ with open dense subset consisting of 4 distinct points has dimension 12. A priori, this implies already that $\mathrm{Hilb}^4 \mathbb{P}^3$ is certainly singular somewhere, since $\mathrm{Hilb}^4 \mathbb{P}^3$ is connected[5]. In fact, $\mathrm{Hilb}^4 \mathbb{P}^3$ is singular at $[Z]$, which can be seen immediately by noting that Z can be obtained as the flat limit of four distinct points coming together, so that Z actually lies on the component of dimension 12, yet has tangent space of dimension 18.

[5]That is, if $[Z]$ were not a singular point of the Hilbert scheme it must lie on a different component, and thus $\mathrm{Hilb}^4 \mathbb{P}^3$ has multiple components, which must intersect by connectedness, and this point of intersection must be singular

Exercise. Check this: let $J_t = (x, y, z) \cap (x - t, y, z) \cap (x, y - t, z) \cap (x, y, z - t)$ and check that setting $t = 0$ gives $(x, y, z)^2$.

So, we just need to verify the dimension claim. Let $R = k[x, y, z]/I$. Then R has k-basis $1, x, y, z$. A map $I/I^2 \to R$ is then specified by giving some assignment

$$\left\{ x^2, xy, xz, y^2, yz, z^2 \right\} \to \left\{ \begin{matrix} 1 \\ x, y, z \end{matrix} \right\}$$

(since of course by R-linearity saying where the generators of I go tells us where things like $x^2 y$, etc., go). Now, we claim first that if $\varphi : I/I^2 \to R$ is a morphism then φ must have image contained in $mR = (x, y, z)$. Say for example $\varphi(x^2) = 1 + (ax + by + cz)$ (we can multiply by a scalar so that this holds). Then since $y\varphi(x^2) = \varphi(x^2 y) = x\varphi(xy)$, we must have that

$$x\varphi(xy) = y + y(ax + by + cz) = y$$

in R. This says that $y = x\varphi(xy) + g$ for $g \in I$; note however that y has degree 1 and I has no elements of degree 1, so this is a contradiction. The same argument holds for any other generator of I.

Now, I claim that any k-linear map $\varphi : k\langle x^2, xy, xz, y^2, yz, z^2 \rangle \to R$ with image contained in m is in fact R-linear: to see this, just note all R-linear dependences of the generators of I have coefficients in mI/I^2, and that $m(mR) = m^2 R = 0$ (e.g., we have a relation $y(x^2) - x(xy) = 0$, but $y\varphi(x^2) = 0 = x\varphi(xy)$ if φ has image in mR).

Thus, the set of R-linear maps $I/I^2 \to R$ is just the 18-dimensional set of k-linear maps $k\langle x^2, xy, xz, y^2, yz, z^2 \rangle \to k\langle x, y, z \rangle$. Thus, our claim is shown.

Example. In contrast, it's easy to see that if X is a smooth projective variety, a point of $\operatorname{Hilb}^m X$ corresponding to m distinct points is a smooth point of the Hilbert scheme: if $Z \subset X$ is a length-m subscheme consisting of distinct points, then Z is a locally complete intersection (locally this just says maximal ideals are generated by regular sequences!). Thus $\mathcal{J}_Z/\mathcal{J}_Z^2$ is locally free of rank n (since $\operatorname{codim} Z = n$), and thus $(\mathcal{J}_Z/\mathcal{J}_Z^2)^\vee$ is as well; since Z is 0-dimensional, we must have that $(\mathcal{J}_Z/\mathcal{J}_Z^2)^\vee = \bigoplus_{p \in Z} \mathcal{O}_p^n$, so that $\dim H^0((\mathcal{J}_Z/\mathcal{J}_Z^2)^\vee) = mn$, which is also the dimension of the component of the Hilbert scheme $[Z]$ lies on, so $[Z]$ is a smooth point.

Remark. We must be careful not to draw the wrong lesson from these examples: in the first two we saw that singular points on the Hilbert scheme arose from singular subschemes, and in the latter we saw that a smooth subscheme gave rise to a smooth point on the Hilbert scheme. Neither of these patterns is true in general:

(1) Singular subschemes can correspond to smooth points of the Hilbert scheme (for example, the family of degree-d hypersurfaces in $\mathbb{P}^n$ is a Hilbert scheme, and isomorphic to $\mathbb{P}^{\binom{n+d}{d}-1}$, which is smooth, even though there are singular hypersurfaces in this family; alternatively, we'll show shortly that $\operatorname{Hilb}^n X$ is smooth for $\dim X = 2$, even though the subschemes not consisting of n distinct points are singular).

(2) Smooth subschemes can correspond to singular points of the Hilbert scheme; this is harder to give an example of.

We see already multiple issues appearing here: singularities, multiple components, nonequidimensional points. This is the rule, not the exception, even for Hilbert schemes of points!

Example. Let $X = \mathbb{P}^4$ and consider $X^{[8]}$, the Hilbert scheme parametrizing length-8 subschemes of $\mathbb{P}^4$. There is the obvious component with open dense subset corresponding to subschemes consisting of 8 distinct points. This clearly has dimension 32.

However, we claim that there's another irreducible component of $X^{[8]}$, consisting of length-8 subschemes with support at a single point. We construct such subschemes as follows: let $R = k[x, y, z, w]$ be the coordinate ring of $\mathbb{A}^4 \subset \mathbb{P}^4$, and let $m = (x, y, z, w)$. Let V be a 7-dimensional subspace of the 10-dimensional R/m-vector space m^2/m^3, and let $I = V + m^3$. Let

$B = R/I$. It's clear that length $B = 8$ (it's easy to check length $R/m^3 = 15$, and then we quotient out 7 more dimensions to get 8) and $\operatorname{supp} B = \{m\}$. Thus, B corresponds to a point of the Hilbert scheme.

It's easy to give a heuristic count for the dimension of the family of such schemes: there is a 4-dimensional family of choices of the closed point, and a $7(10-7)$-dimensional family of choices of V (i.e., just $G(7, 10)$), yielding a 25-dimensional space of such families.

Let's take a specific example: set $I = (x^2, xy, y^2, z^2, zw, w^2, xz - yw) + m^3$ and set $B = R/I$. We claim that

$$\dim \operatorname{Hom}(I/I^2, B) = 25;$$

thus $\operatorname{Spec} B$ is a nonsingular point of the component corresponding to such families; in particular, by dimension reasons it does *not* lie on the irreducible component corresponding to eight distinct points! Thus $\operatorname{Spec} B$ is not obtained as the limit of 8 distinct points coinciding, unlike the previous Hilbert schemes we've seen.

To see why $\dim \operatorname{Hom}(I/I^2, B) = 25$, we work as follows: let $\varphi : I/I^2 \to B$. First we claim that $\varphi(I/I^2) \subset mB$. Note that B has k-basis $\{1, x, y, z, w, yz, xw, xz = yw\}$, and mB is the 7-dimensional subspace spanned by everything except 1. It suffices to show the image of each generator of I in I/I^2 is mapped into mB. Say, for example, that $\varphi(x^2) = 1$. We have that

$$0 = \varphi(x^2 y) - \varphi(x^2 y) = x\varphi(xy) - y \underbrace{\varphi(x^2)}_{1};$$

thus we have

$$x\varphi(xy) = y$$

in B. But one can easily check this is impossible (for example, B is graded, so $\varphi(xy)$ must be degree 0, i.e., a scalar α, but then we obtain a linear form $\alpha x - y$ in I, which isn't the case). Likewise, we'd run into the same problem if $\varphi(xy) = 1$. By symmetry this handles the image of every generator of I/I^2 except $xz - yw$. If $\varphi(xz - yw) = 1$, then we have

$$0 = \varphi(xw(xz - yw)) = xw \cdot \varphi(xz - yw) = xw,$$

but $xw \neq 0$ in B.

Now, note that $\{\psi \in \operatorname{Hom}(I/I^2, B) : \operatorname{im} \psi \subset m^2 B\}$ is an R/m-vector space of dimension 21: all the relation satisfied by the generators of I/I^2 have coefficients in m, but elements of $m^2 B$ are all killed by any element of m, so the relations are automatically satisfied and this is an R/m-vector space; the dimension is obvious since it's clear that $\dim m^2 B = 3$ (spanned by $yz, xw, xz = yw$) so we have a 3-dimensional space of choices for the image of each of the 7 generators of I.

Finally, we note that modulo elements ψ with image in $m^2 B$, the image of φ is determined entirely by $\varphi(xz - yw)$. To see this, say $\varphi(xz - yw) = ax + by + cz + dw$ (since we're looking modulo the elements ψ, we may assume the image is of this form). Call this element θ.

Note that since x, y kill the terms ax and by of θ we have

$$cxz + dxw = x\theta = x\varphi(xz - yw) = \varphi(x^2 z - xyw) = z\varphi(x^2) - w\varphi(xy),$$

and likewise

$$cyz + dyw = y\theta = y\varphi(xz - yw) = \varphi(xyz - y^2 w) = z\varphi(xy) - w\varphi(y^2)$$

(where we've used the equality $xz = yw$ in B to obtain the expression on the left side of each).

Say we have

- $\varphi(x^2) = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 w$.
- $\varphi(xy) = \beta_1 x + \beta_2 y + \beta_3 z + \beta_4 w$.
- $\varphi(y^2) = \gamma_1 x + \gamma_2 y + \gamma_3 z + \gamma_4 w$.

Then

$$z\varphi(x^2) - w\varphi(xy) = \alpha_1 yw + \alpha_2 yz - \beta_1 xw - \beta_2 yw = -yw(\alpha_1 + \beta_2) + \alpha_2 yz - \beta_1 xw$$

and

$$z\varphi(xy) - w\varphi(y^2) = \beta_1 xz + \beta_2 yz - \gamma_1 xw - \gamma_2 yw = -yw(\beta_1 + \gamma_2) + \beta_2 yz - \gamma_1 xw.$$

Thus we have equalities

$$cyz + dyw = -yw(\alpha_1 + \beta_2) + \alpha_2 yz - \beta_1 xw$$

and

$$cyz + dyw = -yw(\beta_1 + \gamma_2) + \beta_2 yz - \gamma_1 xw.$$

Since each term is a k-basis element of B, we can compare coefficients to obtain

(1) $\beta_1 = d$.

(2) $\alpha_1 + \beta_2 = c$.

(3) $\alpha_2 = 0$.

(4) $\beta_1 + \gamma_2 = -d$.

(5) $\beta_2 = c$.

(6) $\gamma_1 = 0$.

This then uniquely specifies the $\alpha_i, \beta_i, \gamma_i$. Doing the symmetric calculation for $\varphi(z^2), \varphi(zw), \varphi(w^2)$ we see that $\varphi(xz - yw)$ indeed uniquely specifies φ modulo $\{\psi : \operatorname{im}\psi \subset m^2 B\}$. There is a 4-dimensional space of choices for $\varphi(xz - yw)$ (corresponding to the 4-dimensional vector space mB/m^2B), and thus we have $\dim \operatorname{Hom}(I/I^2, B) = 4 + 21 = 25$, so $\operatorname{Spec} B$ indeed corresponds to a nonsingular point of the Hilbert scheme.

In the preceding example, the "unexpected" component had dimension less than that of the "obvious" component obtained as the closure of the points parametrizing subschemes of distinct points. This is not always true, as the following example suggests:

Example. Let $X = \mathbb{P}^3$ and consider $X^{[96]}$, the Hilbert scheme parametrizing length-96 subschemes. The "obvious" component has dimension $3 \times 96 = 288$. But consider subschemes of the following form: let $k[x, y, z]$ correspond to $\mathbb{A}^3 \subset \mathbb{P}^3$, and let $m = (x, y, z)$. We have that m^7/m^8 is 36-dimensional; let V be a 24-dimensional sub-vector-space. Let $I = V + m^8$. It's straightforward to check that R/I has length 96 and support $\{m\}$. This gives a family of dimension $3 + 24(36 - 24) = 291$ parametrizing such schemes (3 dimensions for the choice of the point and then a Grassmannian of 24-dimensional subspaces of a 36-dimensional vector space). Thus, in this case we obtain another component, this time of *higher* dimension than the "obvious component".

Remark. Note, also, that it was "easier" to get an extra component of Hilbert scheme in $\mathbb{P}^4$ rather than $\mathbb{P}^3$; in $\mathbb{P}^4$ we needed only consider 8 points, but in $\mathbb{P}^3$ we had to look at 96! This suggests that in smaller dimensions extra componenets take 'longer" to show up as the number of points increases. It is known that:

(1) $(\mathbb{P}^d)^{[n]}$ is irreducible for $n \leq 7$ and any d.

(2) $(\mathbb{P}^d)^{[8]}$ is irreducible for $d = 3$ and reducible for $d \geq 4$.

I'm not sure it's known what the smallest number of points in $\mathbb{P}^3$ is where an extra component appears; it's recently been shown that 11 points in $\mathbb{P}^3$ yield an irreducible Hilbert scheme [CITE].

Example. In fact, even more can go wrong in general. Mumford gave an example of a Hilbert scheme parametrizing very nice geometric subschemes with a component that's generically nonreduced, i.e., with every local ring nonreduced. His example is for $P = 14t - 23$ and $\mathrm{Hilb}_P \mathbb{P}^3$; this Hilbert scheme parametrizes degree-14, genus-24 curves in $\mathbb{P}^3$. There is a component parametrizing such curves contained in smooth cubic surfaces linearly equivalent to $4H + 2L$, L a line on the cubic surface and H a hyperplane section, and this component is nowhere-reduced.

Remark. I don't believe it's known if an example of this occurring for a Hilbert scheme of points exists!

Hilbert schemes of points on surfaces

Theorem (Fogarty 1968)**.** *If X is a smooth (projective) surface then $X^{[n]}$ is smooth and irreducible.*

Note that this immediately implies the case where X is quasiprojective with smooth compactification; in particular, it implies the result for any quasiprojective X in characteristic 0, by resolution of singularities.

Proof. Assume for now that we know $X^{[n]}$ is irreducible. Since $X^{[n]}$ contains the locus of n distinct points as an open subset, which by irreducibility is dense, we know that $\dim X^{[n]} = 2n$. It then suffices to show that if Z is a closed 0-dimensional subscheme of X then $\dim T_{[Z]} X^{[n]} = 2n$. Recall by basic deformation theory on the Hilbert scheme that

$$T_{[Z]} X^{[n]} = H^0(Z, N_{Z/X});$$

thus, our goal is calculate the right side.

Recall that $H^0(Z, N_{Z/X}) = \mathrm{Hom}_{\mathcal{O}_X}(I_Z, O_Z)$, where I_Z is the ideal sheaf defining Z. Take the short exact sequence

$$0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

and apply $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_Z)$ to

$$\begin{aligned}0 \to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Hom}(I_Z, \mathcal{O}_Z) \to \\ \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) \to \\ \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Ext}^2(I_Z, \mathcal{O}_Z) \to 0\end{aligned}$$

Note the first map is tautologically an isomorphism since

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, i_*\mathcal{O}_Z) = \mathrm{Hom}_{\mathcal{O}_Z}(i^*\mathcal{O}_X, \mathcal{O}_Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{O}_Z).$$

Note moreover that $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_Z) = H^i(\mathcal{O}_Z) = 0$ for $i > 0$, since Z is 0-dimensional. Thus, we obtain the following:

(1) $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \cong \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z)$ (note, moreover, that this has dimension n).

(2) $\mathrm{Hom}(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$.

(3) $\mathrm{Ext}^1(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$.

We claim now that

$$\dim \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z) - \dim \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) + \dim \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = 0.$$

This is straightforward: since X is smooth projective and $\mathcal{O}_Z$ 0-dimensional, $\mathcal{O}_Z$ has a resolution

$$0 \to \mathcal{O}_X^{r_2} \to \mathcal{O}_X^{r_1} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Note, moreover, that $\mathcal{O}_Z$ is torsion, so considering the ranks at the generic point we have $r_1 = r_2 + 1$.

Now, define $\chi_Z(\mathcal{F}) = \sum (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{O}_Z)$. We use the following fact:

Lemma. *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is a short exact sequence of coherent sheaves, then*

$$\chi_Z(\mathcal{F}) = \chi_Z(\mathcal{F}') + \chi_Z(\mathcal{F}'').$$

Proof. Easy exercise using the Ext long exact sequence. □

Applying this to the resolution of $\mathcal{O}_Z$ above, we get

$$\chi_Z(\mathcal{O}_Z) = \underbrace{(1 - r_1 + r_2)}_{0} \chi_Z(\mathcal{O}_X) = 0.$$

Thus we must have that

$$\underbrace{\dim \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_Z)}_{n} - \dim \operatorname{Hom}(I_Z, \mathcal{O}_Z) + \dim \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = 0.$$

It now suffices to show that

$$\dim \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \leq n :$$

then $\dim T_{[Z]} X^{[n]} \leq 2n$, and thus must be equal to $2n$ (since $X^{[n]}$ is assumed irreducible).

But note that the surjection $\mathcal{O}_X \to \mathcal{O}_Z$ gives a surjection

$$\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X) \to \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

(by dimensional vanishing of higher Exts), and we may thus show that

$$\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_X) \leq n.$$

Now note that $\mathcal{O}_Z = \bigoplus \mathcal{O}_Z|_{p_i}$, where $\operatorname{Supp} Z = \{p_1, \ldots, p_s\}$, and thus it suffices to show that if $\mathcal{O}_Z|_{p_i}$ has length m then

$$\operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_Z|_{p_i}, \mathcal{O}_X) \leq m.$$

Note that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z|_{p_i}, -) = \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_Z|_{p_i}, (-)_U)$$

for any affine neighborhood $U = \operatorname{Spec} A$ of p_i, and thus we're reduced to the following local claim:

Lemma. *Let* (A, m) *be Gorenstein local ring of dimension* 2 *and* I *an* m*-primary ideal. Then*

$$\operatorname{length} \operatorname{Ext}^2_A(A/I, A) = \operatorname{length} A/I.$$

Proof. Let $(-)^\vee = \operatorname{Hom}_A(-, E_R(R/m))$ be the Matlis dual; recall Matlis duality preserves length, so it suffices to compute $\operatorname{length} \operatorname{Ext}^2_A(A/I, A)^\vee$. But local duality for Gorenstein rings says that

$$\operatorname{Ext}_2(A/I, A) = H^0_m(A/I);$$

by assumption A/I is m-torsion, and thus $H^0_m(A/I) = A/I$ and the result is shown. □

Finally, we just need to see that $X^{[n]}$ is actually irreducible. Recall that we know at least that it's connected. Examining the above argument, note that we've really proved, *in the absence of the irreducibility hypothesis*, is that any point has tangent space of dimension $\leq 2n$. We know that there is one component Y of dimension $2n$ (and thus with tangent space at each point of dimension $2n$). But if there were other components, by connectedness there would have to be one meeting Y, necessarily in a singular point of $X^{[n]}$ lying on Y, which must then have tangent space of dimension $> 2n$, a contradiction. □

In fact, we can say slightly more. Recall from our example calculation of $\operatorname{Hilb}^n C$ for C a smooth curve that we had a morphism $\operatorname{Hilb}^n C \to C^{(n)}$. The same is true more generally:

Proposition. *There is a morphism*

$$\varphi : X^{[n]}_{\mathrm{red}} \to X^{(n)}, \quad [Z] \mapsto \sum_{p \in \operatorname{Supp} Z} \operatorname{length}_p(Z)[p],$$

called the Hilbert–Chow morphism. When X is a surface, so that $X^{[n]}$ is smooth and thus reduced, the morphism

$$\varphi : X^{[n]} \to X^{(n)}$$

is a resolution of singularities.

Proof. We'll assume the existence of the morphism (it's constructed functorially; for a nice account see [FG05]). To see that it's a resolution of singularities, just note what we've already shown: $X^{[n]}$ is projective, smooth, and irreducible, and $\varphi : X^{[n]} \to X^{(n)}$ is clearly an isomorphism over the locus corresponding to n distinct points. □

Remark. In fact, more can be said: not only is this a resolution of singularities, but $X^{(n)}$ is Gorenstein and the morphism φ is *crepant*, i.e., $\varphi^*(\omega_{X^{(n)}}) = \omega_{X^{[n]}}$. For a reference see [BK05, Chapter 7].

Birational geometry of $X^{[n]}$

To conclude, we'll give a brief description of the birational geometry of $X^{[n]}$ for X a smooth surface of irregularity 0; this depends crucially on the existence of the Hilbert–Chow morphism. We follow the exposition of [BC13] for many of the results of this section.

We begin with a very brief overview of some definitions; for more information see [Har77] or [Laz04]. Fix a smooth projective variety X [6] and recall the following:

Definition. A prime divisor D on X is an irreducible codimension-1 closed subvariety of X. A divisor is an element of the free abelian group generated by prime divisors; such a divisor D is said to be effective, denoted $D \geq 0$, if all coefficients of D are nonnegative. Divisors D, D' are said to be linearly equivalent if there's a rational function $\varphi \in k(X)$ with $\operatorname{div} \varphi = D - D'$; the group of divisors modulo linear equivalence is called the class group, and is isomorphic to $\operatorname{Pic} X$, the group of line bundles on X; an explicit bijection between linear equivalence classes of divisors and isomorphism classes of line bundles is given by $D \mapsto \mathcal{O}(D) := \{\text{local regular functions } \varphi \text{ with } \operatorname{div}(\varphi) + D \geq 0\}$.

Definition. A line bundle L (or divisor D) is *ample* if one of the following equivalent conditions is satisfied:

(1) Some power of L is very ample, i.e., gives an embedding $g : X \hookrightarrow \mathbb{P}(H^0(X, L^n))$ with $g^*(\mathcal{O}(1)) = L$.

(2) For any coherent sheaf F on X, $H^i(X, F \otimes L^m) = 0$ for $i > 0$ and $m \gg 0$.

(3) For any coherent sheaf F on X, $F \otimes L^m$ is globally generated for $m \gg 0$.

Recall that we have an intersection theory on X, which given divisors $D_1, \ldots, D_n$ and a subvariety $V \subset X$ of dimension k gives an intersection number $(D_1 \cdots D_k; V) \in \mathbb{Z}$; this gives the following equivalent criterion:

(4) For any subvariety V, of dimension k, $(L^k; V) > 0$.

Now, we can define a notion of numerical equivalence in the obvious way: D is numerically equivalent to D' if $(D \cdot D_1 \cdots D_k; V) = (D' \cdot D_1 \cdots D_k; V)$ for all $V, D_1, \ldots, D_k$ (i.e., D and D' are indistinguishable through their behavior in the intersection product). We write $\operatorname{Num} X$ for the set of numerical equivalence classes of divisors, and $\operatorname{NS}(X)$ for $\operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{Q}$.

[6] Smoothness may be omitted if we are careful to distinguish Weil divisors, Cartier divisors, and line bundles; projectivity is not essential for the definitions although it's our only situation of interest.

By criterion (4) above, it makes sense to talk about a numerical equivalence class being ample, and thus we can define the ample cone in $\mathrm{NS}(X)$ to be the cone generated by ample numerical classes.

Ample divisors are of interest in birational geometry because they correspond to ways of embedding a variety in projective space. More generally, effective divisor classes correspond to rational maps from our variety; by resolving these maps, we obtain different compactifications, which help illustrate or classify the birational geometry of our model. It ends up being advantageous to consider a slightly weakened condition, that of nef divisors.

Definition. A divisor D is nef if $(D \cdot \mathcal{O}_C; C) \geq 0$ for all irreducible curves $C \subset X$.

Clearly we can similarly define what it means for a numerical equivalence class to be nef, and thus we can similarly define the nef cone of divisors in $\mathrm{NS}(X)$. In fact, one can show that the nef cone is the closure of the ample cone; that is, nef divisors can be thought of as limits of ample divisors. Our exposition will follow that of [BC13] quite closely.

First, note that given a line bundle L on X we obtain a S_n-invariant line bundle $p_1^* L \otimes \cdots \otimes p_n^* L$ on X^n, and thus a line bundle $L^{(n)}$ on $X^{(n)}$. The Hilbert–Chow morphism $\varphi : X^{[n]} \to X^{(n)}$ allows us to pull back this line bundle to $L^{[n]} := \varphi^* L^{(n)} \in \operatorname{Pic} X^{(n)}$. Geometrically, if L has a section s cutting out a subscheme $V(s) \subset X$, then $L^{(n)}$ has a section cutting out the subscheme $\{[Z] \in X^{[n]} : Z \cap V(s) \neq \emptyset\}$. (We could also have essentially defined $L^{(n)}$ this way for a line bundle with sections, and then extended by linearity; the key point is that meeting a finite set of points imposes a codimension-1 condition on the Hilbert scheme, giving a Weil divisor, or equivalently a Cartier divisor by smoothness.)

Note that we have another natural divisor (and thus line bundle) on $X^{[n]}$, given by the locus of nonreduced schemes; we write B for the corresponding line bundle. In fact, one can show that B is divisible by 2 in $\operatorname{Pic} X$.

Exercise. Check that the locus of nonreduced schemes actually has codimension 1.

We need to make the technical assumption $q(X) := H^1(X, \mathcal{O}_X) = 0$ (this is satisfied by rational surfaces, for example); the import is that this ensures that numerically trivial line bundles have no infinitesimal deformations. In this case, Fogarty showed the following theorem:

Theorem ([Fog73])**.** $\operatorname{Pic} X^{[n]} = \operatorname{Pic} X \oplus \mathbb{Z}\langle B/2 \rangle$.

Corollary. $\mathrm{NS}(X^{[n]}) = \mathrm{NS}(X) \oplus \mathbb{Q}$.

We can say slightly more: note that if L is ample then $p_1^* L \otimes \cdots \otimes p_n^* L$ is ample (note that if L is very ample and thus defines an embedding $X \hookrightarrow \mathbb{P}^m$ then $p_1^* L \otimes \cdots \otimes p_n^* L$ determines the embedding $X \hookrightarrow \mathbb{P}^m \times \cdots \times \mathbb{P}^m$). Moreover, we have that if $\pi : X^n \to X^{(n)}$ is the quotient morphism, and thus a finite map, we have that $\pi^* L^{(n)} = p_1^* L \otimes \ldots p_n^* L$; that is, $\pi^* L^{(n)}$ is ample, and thus by finiteness of π we must have that $L^{(n)}$ is itself ample (recall that the pullback of a line bundle under a finite map is ample if and only if the line bundle is ample). Finally, we recall that the pullback of an ample divisor under a generically finite proper morphism is nef and big, and thus $L^{[n]}$ is itself nef. It is not ample, however: over a point of $X^{(n)}$ of the form $2p_1 + \cdots + p_{n-1}$ (which form an open dense subset of $X^{(n)}_{\mathrm{sing}}$) the fiber of φ is just a copy of $\mathbb{P}^1$ (consisting of the possible length-2 structures at p_1, which we've seen look like the closed point p_1 with a tangent vector). By choosing a section $s \in H^0(L)$ to avoid $p_1, \ldots, p_{n-1}$, we see that there's an effective divisor in $X^{[n]}$ representing $L^{[n]}$ that doesn't intersect the fiber; thus, by our above criterion for ampleness $L^{[n]}$ cannot be ample. $L^{[n]}$ is thus nef but not ample, and thus is an extremal nef divisor (i.e., it lies on the boundary of the nef cone). Thus, we at least understand the "slice" $B = 0$ of the ample/nef cone $\mathrm{NS}(X)$.

To understand more thoroughly the nef cone, we need to make a construction: say we're given a line bundle L on X with $H^0(X, L) = N > n$. Let $[Z] \in X^{[n]}$ be cut out by the ideal sheaf I_Z; then we have a short exact sequence

$$0 \to L \otimes I_Z \to L \to L \otimes \mathcal{O}_Z \to 0,$$

giving an inclusion

$$H^0(L \otimes I_Z) \to H^0(L);$$

for general $[Z]$, this will be of codimension exactly n, and thus will give an $N-n$-plane in the N-dimensional vector space $H^0(L)$. We thus get a rational map $\varphi_L : X^{[n]} \dashrightarrow \mathrm{Gr}(N-n, N)$. Let $D_L(n) = \varphi_L^*(\mathcal{O}_{\mathrm{Gr}(N-n,N)}(1))$ (note that φ_L is defined in codimension-1, merely since $X^{[n]}$ is smooth and the target projective, so to pull back a line bundle we may restrict to some $U \subset X^{[n]}$ where φ_L is defined, pull back along the *morphism* $\varphi_L|_U$, and then take the closure; by the codimension of $X - U$ there is a unique such line bundle on $X^{[n]}$ restricting appropriately to U). Via Grothendieck–Riemann–Roch we can calculate that $D_L(n) = L^{[n]} - B/2$ in Pic $X^{[n]}$.

Note that $\mathcal{O}_{\mathrm{Gr}(N-n,N)}(1)$ is very ample, hence basepointfree, and thus $\varphi_L^*(\mathcal{O}_{\mathrm{Gr}(N-n,N)}(1))$ is basepointfree on U, and thus the base locus of $\mathcal{O}_{\mathrm{Gr}(N-n,N)}(1)$ is contained in the locus of indeterminacy of φ_L; thus, if φ_L is actually a morphism, then $\varphi_L^* \mathcal{O}_{\mathrm{Gr}(N-n,N)}(1)$ is basepointfree on $X^{[n]}$ and thus nef.

Thus, we can obtain some nef divisors by analyzing when φ_L is a morphism rather than simply a rational map. For φ_L to be a morphism, we need to obtain an $(N-n)$-plane $H^0(L \otimes I_Z) \subset H^0(L)$ for *any* $[Z] \in X^{[n]}$. This leads naturally to the notion of k-very ampleness:

Definition. A line bundle L on a scheme Y is k-very ample if for all subschemes Z of length-$(k+1)$ we have $H^0(L) \to H^0(L \otimes \mathcal{O}_Z)$ is surjective.

Thus 0-very ampleness is just basepointfree, and 1-very ampleness is just very ample (recalling the characterization of a very ample divisor as one separating points and tangent vectors).

Exercise. Check using cohomology and basechange and the universal property of the grassmannian that if L is $(n-1)$-very ample and $H^1(L) = H^2(L) = 0$ then φ_L is a morphism.

Thus, we may study the nef cone of $X^{[n]}$ by studying the $(n-1)$-very ample line bundles L on X with no higher cohomology.

The other tool we need to study the nef cone is a method of constructing suitable curves in $X^{[n]}$: recall that a line bundle is nef if and only if it intersects every line bundle nonnegatively, and thus for each curve Y in the Hilbert scheme we obtain a vector-half-space $\{L \in \mathrm{NS}(X) : (L \cdot \mathcal{O}_Y : Y) \geq 0\}$. The intersection of these half-spaces for all curves Y is precisely the nef cone.

If we can produce enough "test" curves then, and enough $(n-1)$-very ample line bundles with vanishing higher cohomology, we should be able to determine the nef cone (as the largest cone contained in all half-spaces and the smallest cone containing all the nef line bundles).

The construction we'll use is as follows: let $C \subset X$ be a curve admitting a g_n^1, i.e., a degree-n map $C \to \mathbb{P}^1$. Each fiber over $\mathbb{P}^1$ is thus a subscheme of C of length n, and thus we obtain a family of length-n subschemes of X parametrized by $\mathbb{P}^1$, i.e., a map $\mathbb{P}^1 \to X^{[n]}$; we denote the corresponding curve by $C(n)$.

Through this process, one can show the following:

Theorem. *(1) The nef cone of $(\mathbb{P}^2)^{[n]}$ is the closed cone bounded by $(n-1)H^{[n]} - B/2$ and $H^{[n]}$.*

(2) The nef cone of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ is the cone of divisor classes

$$\alpha H_1[n] + \beta H_2[n] + \gamma B/2$$

with $\gamma \leq 0$, $\alpha + (n-1)\gamma \geq 0$, and $\beta + (n-1)\gamma \geq 0$.

One has similar results for del Pezzo and Hirzebruch surfaces; for their description and proof see [BC13], which our exposition is based on.

Proof. We prove only the first one; the second follows similarly once one uses results of [BS90] to see that $\mathcal{O}(a,b)$ is $(n-1)$-ample if $a, b \geq n-1$.

Note first that if X is any surface and F a fiber over a general point of the singular locus of $X^{(n)}$ then $F \cdot (B/2) = -1$ (check this!); thus any nef divisor $aL^{[n]} + bB/2$ must have $b < 0$ (since we've seen already that $L^{[n]} \cdot F = 0$).

Let $Z \subset \mathbb{P}^2$ be a 0-dimensional subscheme. We know by smoothness of $\mathbb{P}^2$ that I_Z has a resolution of the form

$$0 \to \bigoplus \mathcal{O}_{\mathbb{P}^2}(-a_i) \to \bigoplus \mathcal{O}_{\mathbb{P}^2}(-b_i) \to \mathcal{I}_Z \to 0.$$

By basic results on regularity we have that $0 < a_i \leq n+1$; clearly $a_i > b_i$ by minimality as well. Let $d \geq n-1$ and tensor by $\mathcal{O}_{\mathbb{P}^2}(d)$ to obtain

$$0 \to \bigoplus \mathcal{O}_{\mathbb{P}^2}(d-a_i) \to \bigoplus \mathcal{O}_{\mathbb{P}^2}(d-b_i) \to \mathcal{I}_Z(d) \to 0.$$

Taking the long exact sequence in cohomology, we see that $H^1(\mathcal{I}_Z(d)) = H^2(\mathcal{O}_{\mathbb{P}^2}(d-a_i) = 0$, and thus taking the cohomology of

$$0 \to \mathcal{I}_Z(d) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_Z(d) \to 0$$

we obtain a surjection $H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(\mathcal{O}_Z(d))$. Thus $\mathcal{O}_{\mathbb{P}^2}(d)$ is $(n-1)$-very ample if $d \geq n-1$. (And thus the nef cone must contain $(n-1)H^{[n]} - B/2$; we know already it contains $H^{[n]}$).

Note that if C is a line in $\mathbb{P}^2$ then one can check that

$$C(n) \cdot H^{[n]} = 1, \quad C(n) \cdot (B/2) = n-1;$$

thus a nef divisor $aH^{[n]} + b(B/2)$ must $a + (n-1)b \geq 0$; we already know that $b \leq 0$, which implies that the nef cone must be exactly that bounded y $H^{[n]}$ and $(n-1)H^{[n]} - B/2$. □

References

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga. The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability. *Adv. Math.*, 235:580–626, 2013.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*. Springer, New York, 1985.

[BC13] Aaron Bertram and Izzet Coskun. The birational geometry of the Hilbert scheme of points on surfaces. In *Birational geometry, rational curves, and arithmetic*, Simons Symp., pages 15–55. Springer, Cham, 2013.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[BHL+16] Barbara Bolognese, Jack Huizenga, Yinbang Lin, Eric Riedl, Benjamin Schmidt, Matthew Woolf, and Xiaolei Zhao. Nef cones of Hilbert schemes of points on surfaces. *Algebra Number Theory*, 10(4):907–930, 2016.

[BK05] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser, Boston, MA, 2005.

[BS90] Mauro Beltrametti and Andrew J. Sommese. On k-spannedness for projective surfaces. In *Algebraic geometry (L'Aquila, 1988)*, volume 1417 of *Lecture Notes in Math.*, pages 24–51. Springer, Berlin, 1990.

[DJNT17] Theodosios Douvropoulos, Joachim Jelisiejew, Bernt Ivar Utstøl Nødland, and Zach Teitler. The Hilbert scheme of 11 points in $\mathbb{A}^3$ is irreducible. In *Combinatorial algebraic geometry*, volume 80, pages 321–352. Fields Institute, Toronto, ON, 2017.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer, New York, 1995.

[Eis05] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[FG05] Barbara Fantechi and Lothar Göttsche. Local properties and Hilbert schemes of points. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 139–178. Amer. Math. Soc., Providence, RI, 2005.

[Fog68] John Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math*, 90:511–521, 1968.

[Fog73] J. Fogarty. Algebraic families on an algebraic surface. II. The Picard scheme of the punctual Hilbert scheme. *Amer. J. Math.*, 95:660–687, 1973.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[Hui16] Jack Huizenga. Effective divisors on the Hilbert scheme of points in the plane and interpolation for stable bundles. *J. Algebraic Geom.*, 25(1):19–75, 2016.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Mac07] Diane Maclagan. Notes on Hilbert schemes of points, 2007. `http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/HilbertSchemesNotes.pdf`.

[Mum66] David Mumford. *Lectures on curves on an algebraic surface*. Number 59 in Annals of Mathematics Studies. Princeton University Press, 1966.

[Par96] Keith Pardue. Deformation classes of graded modules and maximal Betti numbers. *Illinois J. Math.*, 40(4):564–585, 1996.

[Ree95] Alyson Reeves. The radius of the Hilbert scheme. *J. Algebraic Geom.*, 4(4):639–657, 1995.

[RS97] Alyson Reeves and Mike Stillman. Smoothness of the lexicographic point. *J. Algebraic Geom.*, 6(2):235–246, 1997.

[Vak06] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.