# Singularities

University of Michigan

Fall 2018

Any errors or inaccuracies are likely introduced by myself, Devlin Mallory.

## Resolutions of singularities (September 20, Matt Stevenson)

The setting today is varieties over a field $k$ of characteristic 0.

**Definition.** A resolution of a variety $X$ over $k$ is a proper birational morphism $f : X' \to X$, with $X'$ smooth (equivalently nonsingular, or regular, etc.) over $k$. We often ask for more: we'd like to control the open set $U \subset X$ over which $f$ is an isomorphism, and we'd like for $f^{-1}(X - U)$ to have "nice" geometry. Moreover, we'd like to have a resolution of $X$ and some extra data, which we'll discuss later.

**Example.** If $X$ is dimension 1, then the normalization $X' \to X$ is a resolution of singularities: $X'$ is normal, hence regular in codimension 1, so smooth, and $X' \to X$ is proper.

This, of course, has no hope of working in higher dimension.

**Remark** (The common approach)**.** Many approaches share the same general strategy:

(1) embed $X \subset M$, where $M$ is something we understand (for example, $M$ smooth).

(2) blow up the "worst" singularities of $X$ inside $M$.

(3) Look at the strict transform $X'$ of $X$; if this is smooth we're done, and if not repeat.

This is often called an "embedded" resolution of singularities.

**Example** (cuspidal cubic)**.** Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$ (we know of course that the normalization resolves the singularity, but this remains a good example). Step (1) is complete, so now we blow up $\mathbb{A}^2$ at the origin, obtaining $\operatorname{Proj} k[x, y][u, v]/(xv - yu)$. This is covered by two charts: in one, we have that $f^{-1}(y^2 - x^3) = x^2(v^2 - x)$; thus the strict transform is just the parabola $v^2 - x$, so this gives a resolution of singularities.

**Example** ($D_4$-singularity)**.** Let $X = V(x^2 + y^3 + z^3) \subset \mathbb{A}^3$; this is singular at the origin, so we blow up at the origin, obtaining the map

$$\frac{\operatorname{Proj}(k[x,y,z][u,v,w]}{I_2\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}} \to X$$

In the chart $v \neq 0$ the equation $x^2 + y^3 + z^3$ pulls back to $(uy)^2 + y^3 + (wy)^3 = y^2(u^2 + y(1 + w^3))$. Then the strict transform $X'$ is locally in this chart given by $u^2 + y(1 + w^3)$. But this is still singular when $u = y = 1 + w^3 = 0$, giving three singular points. Now, restrict further to $D(v) \cap D(1 - w + w^2)$ and blow up at $u = v = 0$ and $w = -1$, giving

$$\frac{\operatorname{Proj}(k[u,y,w][r,s,t]}{I_2\begin{pmatrix} u & y & w+1 \\ r & s & t \end{pmatrix}};$$

checking the chart $t \neq 0$ we see the pullback $g^{-1}(\{u^2 + y(1 + w^3) = 0\})$ is defined by $r^2(w+1)^2 + s(w+1)(1+w^3) = (w+1)^2(r^2 + s(1 - w + w^2))$, so the new strict transform is $r^2 + s(1 - w + w^2)$, which is smooth. The other two singular points blow up similarly, so we've found our resolution.

**Example** (Whitney umbrella)**.** Let $X = V(x^2 - y^2 z) \subset \mathbb{A}^3$; this is singular along the line $V(x, y)$, and is even more singular at the origin. If we try to blow up the very bad point at the origin, we get that the equation pulls back to (locally on the chart $w \neq 0$) $(uz)^2 - (vz)^2 z = z^2(u^2 - v^2 z)$. That is, we've reproduced the exact same singularity! If instead we blow up the line, this does resolve the singularity.

This hints that the general problem of constructing resolutions is quite difficult. Thankfully, we have the following general theorem, which gives us a "strong resolution":

**Theorem** (Hironaka 1964)**.** *There exists a projective birational isomorphism $f : X' \to X$, $X'$ smooth, such that:*

*(1) $f$ is an isomorphism over the smooth locus of $X$.*

*(2) $f^{-1}(X_{\text{sing}})$ is a very mildly singular (more precisely, snc) codimension-1 subset.*

*(3) $f$ is a composition of blowups along smooth centers.*

By snc, we mean the following:

**Definition.** If $X$ is smooth over $k$, then an effective divisor $D \subset X$ is snc (or has simple normal crossings) if

(1) each irreducible component $D_i$ of $D$ is smooth, and

(2) the intersections of the $D_i$ are transverse, i.e., at all $p \in D$ we can choose $D = V(x_1 \cdots x_r)$ where $x_1, \cdots x_d \in \mathcal{O}_{X,p}$ are part of a regular system of parameters.

(1) If $X = \mathbb{A}^n$ then $D = V(x_1 \cdots x_r)$ is snc for $r \leq n$.

(2) If $X$ is a smooth toric variety and $D$ is the toric boundary then $D$ is snc.

**Definition.** $D$ has normal crossings (or is nc) if $D$ is étale-locally snc.

**Example.** The nodal cubic $V(y^2 - x^2(x+1))$ is nc but not snc, as can be seen by taking the étale cover $\operatorname{Spec} k[x, y, z]/(z^2 - (x+1)) \to \mathbb{A}^2$, since when we pullback $D$ to this cover it decomposes as $(y - xz)(y + xz)$.

We can, in fact, ask more from resolutions of singularities: we can resolve not only $X$, but $X$ plus some "extra" data. There are two choices of this data, essentially equivalent: we can fix $D$ a Weil divisor on $X$ or $a \subset \mathcal{O}_X$ an ideal sheaf.

**Definition.** A log resolution of $(X, D)$ or $(X, a)$ is a proper birational morphism $f : X' \to X$, with $X'$ smooth, such that

(1) $\operatorname{Exc}(f)$ is a divisor (i.e., pure codimension 1).

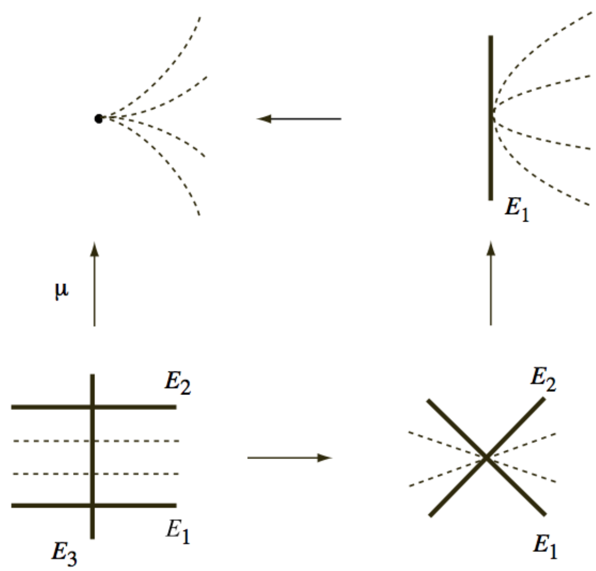
(2a) For $(X, D)$ we demand that $f^*D + \operatorname{Exc}(f)$ is snc[1].

(2b) For $(X, a)$ we demand that $a\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for $F$ an effective divisor with $F + \operatorname{Exc}(f)$ snc.

**Theorem** (Hironaka)**.** *Log resolutions exist.*

[1]When $D$ is not Cartier, we ask instead that the support $f^{-1}D \cup \operatorname{Exc}(f)$ is snc.

**Example.** Consider $D = V(y^2 - x^3) \subset X = \mathbb{A}^2$ (equivalently, this gives a log resolution of $(X, a = (y^2, x^3)))$. We begin by blowing up at the origin; in one chart (where $y = xv$) we get $f_1^{-1}(D) = x^2(v^2 - x) = 0$. Each component is smooth, but they intersect in a length-2 subscheme, i.e., this certainly isn't snc. So, we blow up at the intersection of $V(x)$ and $V(v^2 - x)$. In the interesting chart we have $(f_1 \circ f_2)^{-1}(D) = V(v^3 r^2(v - r))$. This is three lines meeting at the origin, which is not snc, so we blow up one more time, finally obtaining an snc divisor. Thus we've finally computed a log resolution of $(X, D)$. The following illustration of this case is taken from Rob Lazarsfeld's "Positivity in Algebraic Geometry II", where the dotted lines depict general $k$-linear combinations of $x^3$ and $y^2$.




**Remark.** Note that $\operatorname{Exc}(f)$ need not be a divisor for a birational morphism $X' \to X$! For example, take $X = V(xy - uv) \subset \mathbb{A}^4$, and blow up the plane $V(u, v)$ and consider the strict transform $X'$ of $X$. Then we have $X'$ is smooth, but the exceptional locus of the blowup is a $\mathbb{P}^1$, and thus of codimension 2.

We do at least have the following useful fact:

**Theorem.** *If $X$ is normal and $\mathbb{Q}$-factorial (i.e., every Weil divisor has a Cartier multiple) and $X'$ is quasiprojective and smooth, then the exceptional locus of any birational map $f : X' \to X$ is a divisor.*

So, we have resolutions; so, how do we use them to analyze singularities? Classically, given a variety $X$ over $k$, for $x \in X$ we can attach invariants to valuations (or valuation rings) in $k(X)/k$ centered at $x$. Hironaka's theorem thus gives a geometric reformulation of the study of these valuations.

**Theorem** (Zariski)**.** *For all $k$ and all $X$ over $k$, and $v$ a valuation on $k(X)/k$ centered on $X$, the following are equivalent:*

*(1) $v$ is discrete (and Abhyankar).*

*(2) There exists $Y \to X$ proper birational with $Y$ normal and $E \subset Y$ a prime divisor such that $v = c \cdot \operatorname{ord}_E$ for some $c > 0$.*

Of course, in characteristic 0 by resolution of singularities we may take $Y$ smooth by resolving the normal variety $Y$ and considering the strict transform of $E$.

**Example.** If $X = \mathbb{A}^2_k$, then we can define a valuation

$$v\left(\sum_{i,j} a_{ij} x^i y^j\right) = \min_{a_{ij} \neq 0} i + j;$$

this is the same as taking the order of vanishing along the exceptional divisor of the blowup at the origin.

# Invariants of singularities in characteristic 0 (September 27, Devlin Mallory)

The following talk, particularly the discussion of the canonical divisor, is heavily indebted to Miles Reid's *A Young Person's Guide to Canonical Singularities*, Karl Schwede's *Generalized Divisors and Reflexive Sheaves*.

**Question** (understanding singular varieties through their resolutions)**.** Last week Matt showed us that any singular variety $X$ has a resolution of singularities, that is, a smooth variety $Y$ and a proper birational map $Y \to X$. We want to use some data of this resolution to discuss the singularities of $X$ itself. A first guess might be to examine the exceptional divisors occurring on the resolution $Y \to X$. But of course we might have exceptional divisors that we didn't "have" to create; for example, if we take further blowups of $Y$, we introduce exceptional divisors that don't have anything to do with the original singularity or its resolution.

**Remark** (minimal resolution for surfaces)**.** Given a normal surface $X$ we actually have a *minimal* resolution of $X$, i.e., a resolution $f : Y \to X$ such that any other resolution $f' : Y' \to X$ factors through $f$:

$$\begin{array}{ccc} Y' & \xrightarrow{\exists} & Y \\ & \underset{f'}{\searrow} & \downarrow f \\ & & X \end{array}$$

In fact, one can take any resolution $Y_0 \to X$; one can look for $(-1)$-curves on $Y_0$ (i.e., divisors $E \subset Y_0$ with $E^2 = \deg \mathcal{O}_X(E)|_E = -1$, and thus with $E \cong \mathbb{P}^1$); by Castelnuovo's contractibility criterion there is then a morphism $Y_0 \to Y_1$ contracting $E$ such that $Y_1$ is smooth. If in addition $E$ is contracted to a point by the map $Y_0 \to X$, we obtain a morphism $Y_1 \to X$. One can check this process eventually terminates, and the resulting morphism $Y \to X$ can be checked to be minimal.

**Example** (non-minimal resolution of cone over quadric)**.** In higher dimensions, we don't have minimal resolutions of singularity. For a first example, consider

$$X = V(xy - zw) \subset \mathbb{A}^4,$$

the threefold obtained by taking the cone over a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. $X$ has two families of hyperplanes, given by the cones over the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$; choosing $H_1, H_2$ hyperplanes in each family, with $H^1 \cap H^2 = \{p\}$, the vertex of the cone, we have

$$\begin{array}{ccccc} & & \operatorname{Bl}_p X & & \\ & \swarrow & & \searrow & \\ \operatorname{Bl}_{H_1} X & & & & \operatorname{Bl}_{H_2} X \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

Both $\operatorname{Bl}_{H_1} X$ and $\operatorname{Bl}_{H_2} X$ are smooth, and if one knows a little about the cone of curves of a variety one can check that $\operatorname{Bl}_{H_1} X$ cannot factor through any other resolution $Y \to X$, nor can $\operatorname{Bl}_{H_2} X$.

What this suggests is that in general we need to consider data coming from *any* resolution $Y \to X$. Because singular varieties are harder to work on, we'd like to examine data pulled back to $Y$, rather than on $X$. A first candidate may be to consider the relation between $\mathcal{O}_X$ and $\mathcal{O}_Y$, but we have of course that $f^*\mathcal{O}_X = \mathcal{O}_Y$, so there's not much here of interest. A natural idea would be to note that any normal variety $X$ carries a canonical divisor $K_X$; under some suitable assumptions on $X$ (allowing us to pull back $K_X$) we can compare $f^*K_X$ and $K_Y$. Moreover, our comparison will be a more refined version of our "naive" idea to look at the exceptional divisors of the resolution $Y \to X$; we'll consider $K_Y - f^*K_X$, which will be supported on the exceptional divisors $E_i$ of $Y \to X$, but which has additional information contained in the multiplicities of $E_i$ in $K_Y - f^*K_X$.

**Remark** (Weil and Cartier divisors)**.** We recall briefly the notion of Weil and Cartier divisors from both a geometric and algebraic point of view. Fix $X$ a normal variety. A Weil divisor is simply a formal $\mathbb{Z}$-linear combination $\sum a_i D_i$ of codimension-1 irreducible subvarieties $D_i \subset X$. Such a divisor is *effective* if all $a_i \geq 0$; we write $D \geq 0$ if $D$ is effective.

An effective Cartier divisor is simply a locally principal Weil divisor, i.e., a codimension-1 subscheme defined locally by a single equation. A Cartier divisor (in this setting) is the same as a line bundle on $X$, or can be viewed as the data $(U_i, f_{ij})$, with $\{U_i\}$ an open cover of $X$ and $f_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ satisfying the usual cocycle condition (equivalently, a Cartier divisor is a global section of the sheaf $k(X)^*/\mathcal{O}_X^*$).

Recall that (in our setting) every Cartier divisor gives a Weil divisor, and that on a smooth variety Weil and Cartier divisors are the same thing.

It also is useful to consider the corresponding algebraic point of view: locally on some chart $U = \operatorname{Spec} R$, a Cartier divisor is simply a projective $R$-module of rank 1 (and an effective Cartier divisor is simply a locally principal ideal $I \subset R$). A Weil divisor is slightly more subtle: a Weil divisor $D$ corresponds uniquely to a reflexive rank-1 subsheaf $\mathcal{O}_X(D) = \{f \in \operatorname{Frac} R : \operatorname{div} f + D \geq 0\}$ of $\operatorname{Frac} R$ (in fact, the last condition is automatic, so one may simply consider reflexive rank-1 modules).

We have a group operation on Cartier divisors, given by tensor product of line bundles, and the obvious group operation on Weil divisors when viewed as formal combinations of codimension-1 subschemes. One may show that if $D_1, D_2$ are Weil divisors that

$$\mathcal{O}_X(D_1 + D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**}.$$

We will frequently consider also $\mathbb{Q}$-divisors, where we allow coefficients in $\mathbb{Q}$ rather than $\mathbb{Z}$.

**Example** (quadric cone)**.** Consider $V(x^2 - yz) \subset \mathbb{A}^3$, the cone over a smooth quadric in $\mathbb{P}^2$. Let $L = V(x, y) \subset X$ be a line through the vertex of the cone; one can check via the tangent space at the origin that $L$ cannot be cut out by a single equation, but $2L$ is cut out by $y$, so $2L$ is Cartier.

**Definition.** First, let $X$ be a smooth variety of dimension $n$. Recall that in this case $\Omega_{X/k}$ is locally free of rank $n$, so that $\omega_X := \bigwedge^n \Omega_{X/k}$ is a line bundle, the *canonical* bundle. We obtain a canonical *divisor* via the standard line-bundle-to-divisor correspondence, i.e., by choosing a rational section $s \in H^0(X, \omega_X \otimes k(X))$ (i.e., choose some open set $U$ and an isomorphism $\omega_X|_U \cong \mathcal{O}_U$, and take a local section $s \in \mathcal{O}_U$ which we may view as an element of $k(X)$) of $\omega_X$ and taking $K_X = \operatorname{div} s$. This is of course only well-defined up to linear equivalence, but fix one such representative.

Let $X$ be a normal variety. Recall that in this case by Serre's criteria that the singular locus $X_{\text{sing}}$ has codimension $\geq 2$. Let $U = X - X_{\text{sing}}$ be the smooth locus, and fix $K_{X_{\text{sm}}}$ as above. We have an isomorphism $\operatorname{Cl}(X) \cong \operatorname{Cl}(U)$ of class groups since $X_{\text{sing}}$ has codimension $> 1$, and there is thus a unique Weil divisor $K_X$ on $X$ such that $K_X|_U \cong K_U$, called the canonical divisor of $X$. We write $\omega_X$ for the coherent sheaf $\mathcal{O}_X(K_X)$.

Alternatively:

(1) One can define $\omega_X = j_*(\omega_U)$, where $U \hookrightarrow X$ is the inclusion of the smooth locus, and then define $K_X$ by taking div of a rational section of $\omega_X$.

(2) One can define $\omega_X$ by specifying its sections via

$$\Gamma(W, \omega_X) = \{s \in \Omega_{k(X)} : s \text{ is regular on } W \cap X_{\text{sm}}\}.$$

(3) One may take $\omega_X = \left(\bigwedge^n \Omega_{X/k}\right)^{**}$.

**Example** (what's wrong with Kahler differentials?)**.** We can see even in dimension 1 why we don't simply work with a sheaf obtained from the cotangent sheaf $\Omega_{X/k}$. Let $C = V(y^2 - x^3)$ be the cuspidal cubic (again!), and consider $\Omega_{C/k}$. By the conormal sequence for $k[x, y] \to k[x, y]/(y^2 - x^3) =: R$,

$$\Omega_{C/k} = \frac{R\langle dx, dy\rangle}{2y\, dy - 3x^2\, dx}$$

Note then that we have

$$y(2x\,dy - 3y\,dx) = 2xy\,dy - 3y^2\,dx = 2xy\,dy - 3x^3\,dx = x(2y\,dy - 3x^2\,dx) = 0;$$

thus, $\Omega_{C/k}$ is torsion, and thus sections of $\Omega_{C/k}$ can't be thought of as sections of any reasonable geometric bundle.

**Remark.** This example and our third definition above hints at how $\omega_X$ is an improvement over $\bigwedge^n \Omega_{X/k}$: for any normal domain $R$ and finitely generated $R$-module $M$, one may check that $M^{**}$ is reflexive, torsionfree, and S2. The last, in particular, means that if $m \in M_P$ for all $P$ of height 1 in $R$ then $m \in M$. Thus, the passage from $\bigwedge^n \Omega_{X/k}$ to $(\bigwedge^n \Omega_{X/k})^{**}$ kills any torsion and then adds all sections already regular at all codimension-1 local rings.

**Remark** (pulling back divisors)**.** In order to compare the behavior of divisors under birational transformation, we need to be able to pull them back. It is immediate how to pull back Cartier divisors: you can either pull them back as projective modules/line bundles, or you can pull back effective Cartier divisors just by pulling back a local defining equation, and then extend via linearity. There is no obvious way to pull back Weil divisors though.

Thus, we'll need to make certain assumptions going forward on the singularities of the varieties we consider. There is an alternative to this formalism, using the Mather log discrepancy and Nash blowups, but that's a story for a different day.

**Definition** ($\mathbb{Q}$-Cartier and $\mathbb{Q}$-Gorenstein)**.** A Weil divisor $D$ is $\mathbb{Q}$-Cartier if $mD$ is Cartier for some integer $m$. A variety $X$ is $\mathbb{Q}$-Gorenstein if the Weil divisor $mK_X$ is $\mathbb{Q}$-Cartier.

Note that this creates an unfortunate distinction between a Gorenstein variety (each local ring is Gorenstein) and a 1-Gorenstein variety, which is weaker. (Gorenstein is 1-Gorenstein and Cohen–Macaulay.)

**Remark.** Recall that algebraically a Weil divisor $D$ corresponds locally to a rank-1 reflexive module $M$; to be $\mathbb{Q}$-Cartier, we must have that $(M)^{\otimes m})^{**}$ is projective for some $m$. Locally, if $P$ is a height-1 prime, the condition that $[V(P)]$ is $\mathbb{Q}$-Cartier is that some symbolic power $P^{(m)}$ is principal.

**Definition.** We may pull back $\mathbb{Q}$-Cartier divisors in the obvious way: if $mD$ is Cartier on $X$ and $\sigma : Y \to X$, we define $\sigma^*(D) = (1/m)\sigma^*(mD)$.

**Definition.** If $X$ is $\mathbb{Q}$-Gorenstein and $f : Y \to X$ is any birational morphism from a normal variety $Y$, the relative canonical divisor of $Y \to X$ is $K_{Y/X} := K_Y - f^*K_X$.

**Remark** (choice of $K_X$)**.** In the above construction, $K_X$ is only well-defined up to linear equivalence, but the particular choice is of no importance. One may check that if $f : Y \to X$ is a proper birational morphism of normal varieties then $f_*(K_Y)$ is a canonical divisor on $X$ (normality allows us to reduce immediately to when $X, Y$ are smooth, and we then can note that $f^*K_X + E = K_Y$ for $E$ effective and supported on $\operatorname{Exc}(f)$, so $f_*(K_Y) = K_X$ by the projection formula). ) Thus, whenever we speak of a variety $X$ we fix a choice of $K_X$, and whenever we speak of a proper birational morphism $f : Y \to X$ we choose $K_Y$ such that $f_*K_Y = K_X$. One may then check that for such a morphism the relative canonical divisor $K_{Y/X} := K_Y - f^*K_X$ is independent of the choice of $K_X$.

**Example.** If $f : Y \to X$ is a birational morphism of smooth varieties, given locally by some regular functions $f_i$, one has the cotangent sequence is left-exact, i.e.,

$$0 \to \Omega_{X/k} \xrightarrow{(\partial f_i/\partial x_j)} \Omega_{Y/k} \to \Omega_{Y/X} \to 0,$$

and thus taking determinants that $K_{Y/X}$ is just defined locally by the determinant of the Jacobian of the morphism.

To make this concrete, if $\sigma : \mathrm{Bl}_p\,\mathbb{A}^2 \to \mathbb{A}^2$ is the blowup of $\mathbb{A}^2$ at a point, then in one chart this map is given by $k[x, y] \to k[u, v]$, $x \mapsto uv$, $y \mapsto v$, so the Jacobian matrix is

$$\begin{pmatrix} v & 0 \\ u & 1 \end{pmatrix},$$

so the relative canonical divisor is cut out by $v$, i.e., the relative canonical divisor is just the exceptional divisor $E$ of the blowup.

One can check this directly: a section of $K_{\mathbb{A}^2}$ is given by $dx \wedge dy$, which pulls back to $d(uv) \wedge d(v) = v\, du \wedge dv$, which is an element of $K_{\mathrm{Bl}_p \mathbb{A}^2} - E$, and thus we have

$$K_{\mathrm{Bl}_p \mathbb{A}^2} = \sigma^*(K_{\mathbb{A}^2}) + E,$$

so we again see that the relative canonical divisor is $E$.

**Remark.** The exact same proof shows that if we blowup $\mathbb{A}^n$ at a smooth subspace $Z$ of codimension $r$ that we get $K_{\mathrm{Bl}_Z \mathbb{A}^n/\mathbb{A}^n} = (r-1)E$;if $Z = \{p\}$ is a point we get $K_{\mathrm{Bl}_p \mathbb{A}^n/\mathbb{A}^n} = (n-1)E$.

The following is a useful tool for calculating $K_{Y/X}$:

**Remark** (adjunction)**.** If $X$ is normal and $H$ is a normal irreducible effective Cartier divisor, then if $mK_X$ Cartier we have $mK_H = (mK_X + mH)|_H$, and in particular is Cartier

**Definition** (log discrepancies of divisors)**.** Let $X$ be a normal $\mathbb{Q}$-Gorenstein variety and let $Y \to X$ be a birational morphism with $Y$ smooth. Given some divisor $E$ on such a resolution of $X$, we may define a rational number $a_E(X)$, the *discrepancy*, as

$$\mathrm{ord}_E(K_{Y/X}),$$

i.e., the coefficient of $E$ in $K_{Y/X} = K_Y - f^*(K_X)$.

One may check that $a_E(X)$ in fact depends only on the valuation ring $\mathcal{O}_E \subset k(Y) = k(X)$, and thus this definition really is independent of the choice of smooth birational model $Y$ (e.g., if we take $E \subset \mathbb{A}^2 = X$ to be the $x$-axis, we have that $a_E(X) = 0$, which is the same as if we blow up $\mathbb{A}^2$ at the origin and consider $E'$ the strict transform of the $x$-axis).

We say that $E$ is *exceptional* over $X$ if the restriction of $f : Y \to X$ to $E$ is not an isomorphism at the generic point of $E$, or equivalently (since $X$ is normal) if the $f(E)$ is of codimension $> 1$.

We define the discrepancy, denoted discrep$(X)$, as

$$\mathrm{discrep}(X) := \min_{E \text{ exceptional}} a_E(X).$$

This invariant is a measure of the singularities of $X$; the larger discrep$(X)$ is, the milder the singularities of $X$.

We then say that $X$ is:

(1) terminal if discrep$(X) > 0$.

(2) canonical if discrep$(X) \geq 0$.

(3) log terminal if discrep$(X) > -1$.

(4) log canonical if discrep$(X) \geq -1$.

**Remark.** Note that some define instead $a_E(X) = \mathrm{ord}_E(K_{Y/X}) + 1$, and thus shift the entire hierarchy of singularities up by 1; these are often refered to as *log* discrepancies, as opposed to the usual discrepancies..

**Example** (why stop at log canonical)**.** What if discrep$(X) < -1$? One can show that in this case by repeatedly blowing up we can obtain a resolution of singularities $Y_i \to X$ and a divisor $E_i$ on $Y_i$ with $a_{E_i}(X) = -i$, so that discrep$(X) = -\infty$; thus this invariant is only meaningful for discrep$(X) \geq -1$.

**Remark** (what about curves?/why care about pairs)**.** At this point, this may seem like a bad definition; one of the most tractable and studied classes of singularities is curve singularities, but since a normal curve is smooth, the restriction above excludes curve singularities. We'll see later that this can be resolved by considering a curve $C$ singularity embedded in a higher-dimensional smooth space $X$, and considering the singularities of the pair $(X, C)$; for example, we study the singularities of the cusp and node by considering them as divisors in $\mathbb{A}^2$ and studying the singularities of these pairs.

A priori, this definition sounds impossible to calculate: how do we analyze all possible exceptional divisors appearing on smooth birational models of $X$? However, the following allows us to actually calculate these invariants:

**Theorem.** *Let $Y \to X$ be a log resolution of $X$. If $X$ is log canonical, then*

$$\operatorname{discrep}(X) = \min\{1, a_E(X) : E \subset Y, E \text{ exceptional over } X\}.$$

In particular, then, we have the discrepancy is always rational (and computed by a divisor on some smooth model) and that a smooth variety has discrepancy 1.

## Characteristic 0 examples (October 4, Devlin Mallory)

**Example.** Consider $X = V(x^2 + y^2 + z^2)$, the cone in $\mathbb{A}^3$ over a smooth conic in $\mathbb{P}^2$. One can check that a single blowup at the cone point, say $\sigma_X : \widetilde{X} \to X$ (which is the restriction of the blowup $\sigma : \mathrm{Bl}_p \mathbb{A}^3 \to \mathbb{A}^3$ to the strict transform of $X$ under $\sigma$), resolves the singularity, and that the exceptional divisor, say $E$, is a $\mathbb{P}^1$; note $E$ is the intersection of the exceptional divisor of $\sigma$, say $E_0$, with $\widetilde{X}$. Summarized, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{X} = \mathrm{Bl}_p X & \hookrightarrow & \mathrm{Bl}_p \mathbb{A}^3 \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma} \\ X & \hookrightarrow & \mathbb{A}^3 \end{array}$$

To compute $K_{\widetilde{X}/X}$, we use adjunction for $X \subset \mathbb{A}^3$ and $\widetilde{X} \subset \mathrm{Bl}_p \mathbb{A}^3$, and the blowup formula for the blowup $\sigma$ of $\mathbb{A}^3$ at a point. We have equations

$$\begin{aligned} K_X &= (K_{\mathbb{A}^3} + X)|_X, \\ K_{\widetilde{X}} &= (K_{\mathrm{Bl}_p \mathbb{A}^3} + \widetilde{X})|_{\widetilde{X}}, \\ K_{\mathrm{Bl}_p \mathbb{A}^3} &= \sigma^* K_{\mathbb{A}^3} + 2E_0. \end{aligned}$$

The goal, recall, is to compare $K_{\widetilde{X}}$ and $\sigma_X^* K_X$. So, from the second and third equations and the equality $(E_0)|_{\widetilde{X}} = E$, we have

$$\begin{aligned} K_{\widetilde{X}} &= (K_{\mathrm{Bl}_p \mathbb{A}^3})|_{\widetilde{X}} + \widetilde{X}|_{\widetilde{X}} \\ &= (\sigma^* K_{\mathbb{A}^3} + 2E_0)|_{\widetilde{X}} + \widetilde{X}|_{\widetilde{X}} \\ &= (\sigma^* K_{\mathbb{A}^3})|_{\widetilde{X}} + 2E + \widetilde{X}|_{\widetilde{X}} \\ &= \sigma_X^*(K_{\mathbb{A}^3}|_X) + 2E + \widetilde{X}|_{\widetilde{X}}, \end{aligned}$$

where the third equality used that restriction commutes with addition of divisors and the last equality used commutative of the above blowup diagram. We can rewrite the first adjunction above as saying that

$$K_{\mathbb{A}^3}|_X = K_X - X|_X,$$

and thus plugging into our expression above we get

$$\begin{aligned} \sigma_X^*(K_{\mathbb{A}^3}|_X) + \widetilde{X}|_{\widetilde{X}} + 2E &= \sigma_X^*(K_X - X|_X) + \widetilde{X}|_{\widetilde{X}} + 2E \\ &= \sigma_X^*(K_X) - \sigma_X^*(X|_X) + \widetilde{X}|_{\widetilde{X}} + 2E. \\ &= \sigma_X^*(K_X) - \sigma^*(X)|_{\widetilde{X}} + \widetilde{X}|_{\widetilde{X}} + 2E, \\ &= \sigma_X^*(K_X) - (\sigma^*(X) - \widetilde{X})|_{\widetilde{X}} + 2E, \end{aligned}$$

where the second-to-last equality again used commutativity of the diagram.

Now, by standard results on pulling back hypersurfaces under blowing up at points, one has that $\sigma^*(X) - \widetilde{X} = \mathrm{mult}_p(X)E$, which is thus $2E_0$, so $(\sigma^*(X) - \widetilde{X})|_{\widetilde{X}} = 2E$, and thus we obtain that

$$K_{\widetilde{X}} = \sigma_X^*(K_X) - 2E + 2E = \sigma_X^*(K_X)$$

(such resolutions are called crepant). Thus we have that $\mathrm{discrep}(X) = 0$, so $X$ is canonical.

**Example.** We may repeat the exact same calculation for $V(x^3 + y^3 + z^3) \subset \mathbb{A}^3$, the cone over the Fermat elliptic curve. Again, a single blowup resolves the singularity; now the exceptional divisor is a copy of the elliptic curve. Repeating the same calculation, we get that

$$K_{\widetilde{X}} - \sigma^* K_X = -E,$$

so that $X$ is log canonical.

**Example** (why care about pairs II)**.** The cone $\operatorname{Spec} k[x,y,z]/(x^4+y^4+z^4)$ will have

$$K_{\widetilde{X}} - \sigma^* K_X = -2E,$$

so discrep$(X) < -1$ and is thus $-\infty$, so this falls outside our schema above. This may seem like a fairly tractable singularity to exclude from our hierarchy, but we'll see later how we may still get information about the singularities of $X$ by viewing it as a divisor $D$ in $\mathbb{A}^3$, and considering the singularities of the pair $(\mathbb{A}^3, qD)$ for $q \in \mathbb{Q}$.

**Example** (cone over a hypersurface)**.** In general, let $f$ be a homogeneous polynomial of degree $d$ defining a smooth hypersurface in $\mathbb{P}^{n-1}$. The cone $X = V(f) \subset \mathbb{A}^n$ then has an isolated singularity at the origin, and one can check that a single blowup at this cone point, say $\widetilde{X}$, resolves the singularity, with exceptional divisor $E$ a copy of the original hypersurface. Using adjunction and the fact that

$$K_{\mathrm{Bl}_p \mathbb{A}^n} = \sigma^* K_{\mathbb{A}^n} + (n-1)E,$$

we obtain that $K_{\widetilde{X}/X} = (n-1-d)E$. Thus, $X$ is terminal for $d \le n-2$, canonical for $d = n-1$, log canonical for $d = n$,and outside our hierarchy for $d > n$.

**Example** (cones more generally)**.** Let's extend our discussion of cones; let $X_0 \subset \mathbb{P}^{n-1}$ be a smooth variety of positive dimension that is projectively normal (i.e., the homogeneous coordinate ring $k[x_0,\dots,x_n]/I_X$ is normal, and in particular normal at the cone point), and let $X \subset \mathbb{A}^n$ be the cone over $X_0$. $X$ thus has an isolated normal singularity at the origin. We can ask then when $X$ is $\mathbb{Q}$-Gorenstein, and give an answer in terms only of the embedding $X_0 \subset \mathbb{P}^{n-1}$:

We claim first that $\bigoplus_m H^0(X_0, \omega_{X_0}(k))$ is the canonical module for $X$. Since $X$ is affine, we have $\omega_X$ is the sheafification of $H^0(X, \omega_X)$, so it suffices to find the global sections of $\omega_X$. Let $i : U = X - \{(0,\dots,0)\} \hookrightarrow X$ be the inclusion of the smooth locus. Since $X - U$ has codimension $> 2$ and $\omega_X$ is S2, we have that $H^0(X,\omega_X) = H^0(U, \omega_X|_U) = H^0(U, \omega_U)$. But now note that $U$ is an $\mathbb{A}^1 - \{0\}$-bundle over $X_0$, say $\pi : U \to X_0$; one can then check that $\pi^*(\omega_{X_0}) = \omega_U$ (to see this, note that since $\pi$ is a smooth map the determinant of the cotangent sequence gives us that $\omega_U = \pi^*(\omega_Y) \otimes \Omega_{U/Y}$; one can then check that $\Omega_{U/Y}$ is trival, for example by viewing $\mathbb{A}^n - \{(0,\dots,0)\} \to \mathbb{P}^{n-1}$ as an $(\mathbb{A}^1) - \{0\}$-bundle, noting that since $\mathbb{A}^n - \{(0,\dots,0)\}$ has no nontrivial line bundles the relative canonical sheaf of this map is trivial, and using that $\Omega_{U/Y}$ is the restriction of this to $U$ by compatibility of modules of differentials with basechange).

In any case, since $\pi^*(\omega_{X_0}) = \omega_U$, the projection formula says that

$$\pi_*(\pi^*(\omega_{X_0})) = \omega_{X_0} \otimes \pi_* \mathcal{O}_U = \omega_{X_0} \otimes \left( \bigoplus \mathcal{O}_{X_0}(k) \right).$$

We then have that

$$\omega_X = H^0(X, \omega_X) = H^0(U, \omega_U) = H^0(U, \pi^* \omega_X) = H^0(X_0, \pi_* \pi^* \omega_X) = \bigoplus H^0(X_0, \omega_{X_0}(k)).$$

Since the coordinate ring $\bigoplus_m H^0(X, \mathcal{O}_X(m))$ of $X$ is graded[2], a graded module is locally free if and only if it's free, and thus the only way that $\omega_X$ can be locally free is if $\omega_{X_0} = \mathcal{O}_{X_0}(l) = \mathcal{O}_{\mathbb{P}^{n-1}}(l)|_{X_0}$ for some $l \in \mathbb{Z}$. Analogously, we have that

$$\omega_X^{\otimes m} = \bigoplus H^0(X_0, \omega_{X_0}^m(k)),$$

and $\omega_X^{\otimes m}$ is locally free if and only if $\omega_{X_0}^{\otimes m} = \mathcal{O}_{X_0}(mK_{X_0}) = \mathcal{O}_{X_0}(l) = \mathcal{O}_{\mathbb{P}^{n-1}}(l)|_{X_0}$ for some $l$.

To sum up, then, the cone over $X_0 \subset \mathbb{P}^{n-1}$ is $\mathbb{Q}$-Gorenstein exactly when some multiple of the canonical divisor on $X_0$ is a multiple of the hyperplane section from $\mathbb{P}^{n-1}$.

**Remark** (resolving cones)**.** One can check that if $X_0$ is a smooth variety in $\mathbb{P}^{n-1}$, $X$ the cone over $X_0$ in $\mathbb{A}^n$, that blowing up $X$ at the cone point gives a resolution $Y \to X$ with exceptional divisor $E \cong X_0$; in fact, it's not hard to check that this blowup is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{O}_{X_0}(1))$, with $\mathcal{O}_E(E)$ corresponding to $\mathcal{O}_{X_0}(-1)$ under our identification $E \cong X_0$.

[2]Note that this expression is the coordinate ring because $X_0$ is projectively normal

This gives a second view of our above calculations: for example, for the cone over the Fermat cubic, we had that $E^2 = \deg \mathcal{O}_E(E) = \deg \mathcal{O}_{X_0}(-1) = -3$; the adjunction formula

$$2g_E - 2 = K.(K + E)$$

then tells us that since $g_E = 1$ we have that $K^2 = -K.E = 3$, and writing $K_{\tilde{X}} = \pi^* K_X + aE$ for some $a$ and taking the intersection product with $E$ we get that $-3 = 0 + aE^2 = -3a$, so $a = -1$.

**Example.** Let $X_0$ be the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. We know that $\mathcal{O}_{\mathbb{P}^5}(1)|_{X_0} = \mathcal{O}_{\mathbb{P}^2}(2)$, and $\omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. It's thus clear that if $X$ is the cone over this Veronese then $K_X$ is not Cartier but $2K_X$ is. Let $\sigma : Y \to X$ be the blowup at the cone point, giving us a resolution with exceptional divisor $E \cong \mathbb{P}^2$. Write $2K_Y = \sigma^*(2K_X) + aE$. Restricting this to $E$ and use the adjunction $(K_Y + E)|_E = K_E$ (and that $\sigma^*(2K_X)|_E$ is trivial), we get

$$2(K_E - E|_E) = aE|_E,$$

so

$$2K_E = (a+1)E|_E.$$

Since $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^2}(-2)$ and $\mathcal{O}_E(K_E) = \omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$, we get that

$$\mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}_{\mathbb{P}^2}(-2(a+1))$$

so $a = 1/2$.

Thus the cone $X$ over the Veronese is 2-Gorenstein but not Gorenstein, and has discrepancy 1/2, so that $X$ is terminal but not smooth.

**Example.** Consider $X = V(x^2 + y^3 + z^3) \subset \mathbb{A}^3$, the $D_4$-type singularity mentioned last time by Matt. Recall that we constructed a (log) resolution $Y \to X$ last time, with four exceptional divisors $E_1, E_2, E_3, E_4$, all of which are $(-2)$-curves, i.e., copies of $\mathbb{P}^1$ with $\deg \mathcal{O}_{E_i}(E_i) = -2$. By repeated applications of adjunction and our formula for blowups of affine space at points, one can check that in fact $K_{Y/X} = 0$, i.e., as for $V(x^2 + y^2 + z^2)$ we have a crepant resolution, so that $X$ has canonical singularities.

In fact, it's the case that terminal surface singularities are in fact smooth and that canonical surface singularities are precisely the Du Val singularities

$$\begin{aligned} &A_n : x^2 + y^2 + z^{n+1} \\ &D_n : x^2 + y^2 z + z^{n-1} \\ &E_6 : x^2 + y^3 + z^4 \\ &E_7 : x^2 + y^3 + yz^3 \\ &E_8 : x^2 + y^3 + z^5 \end{aligned}$$

(equivalently, the isolated double points whose resolutions are given by blowing up of isolated double points, or the quotient singularities obtained as quotients of $\mathbb{A}^2$ by finite subgroups of $\mathrm{SL}(2, \mathbb{C})$).

**Definition** (singularities of pairs)**.** With the above formalism, it's not much harder to define these notions of singularities for *log pairs* $(X, D)$. A log pair is a normal variety $X$ (not necessarily $\mathbb{Q}$-Gorenstein!) and a $\mathbb{Q}$-Weil divisor $D$ on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. (Thus $(X, 0)$ is a log pair exactly when $X$ is $\mathbb{Q}$-Gorenstein.) Given a birational morphism $f : Y \to X$ with $Y$ smooth, we can define a divisor via $K_{Y/X,D} := K_Y - f^*(K_X + D)$ (note we can pull back $K_X + D$ exactly because it's $\mathbb{Q}$-Cartier). (This notation is completely nonstandard.) For a codimension-1 subvariety $E$ of $Y$ we can then define the log discrepancy $a_E(X, D) = \mathrm{ord}_E(K_{Y/X,D})$. Taking $D = 0$ we of course recover our old definition. We define

$$\mathrm{discrep}(X, D) = \min_{E \text{ exceptional}} a_E(X, D),$$

and say that $(X, D)$ is

(1) terminal if $\operatorname{discrep}(X, D) > 0$.

(2) canonical if $\operatorname{discrep}(X, D) \geq 0$.

(3) Kawamata log terminal if $\operatorname{discrep}(X, D) > -1$ and $\lfloor D \rfloor = 0$.

(4) log canonical if $\operatorname{discrep}(X, D) \geq -1$.

(When $D \neq 0$ it's mostly the last two which are of interest.) The discrepancy can be thought of as simultaneously measuring the singularities of $X$ and $D$ (and their interaction!).

If we demand only that $\operatorname{discrep}(X, D) > 0$, we call $(X, D)$ *purely* log terminal.

Similarly to the case $D = 0$, when finite, $\operatorname{discrep}(X, D)$ can be computed on a single log resolution of $(X, D)$, and is thus a rational number:

**Theorem.** *Let $(X, D = \sum d_i D_i)$ be a pair and $f : Y \to X$ a log resolution of singularities, with $E_j \subset Y$ exceptional divisors.*

*(1) $(X, D)$ is log canonical if and only if* $\min\{-d_i, a_{E_j}(X, D) : i, j\} \geq -1$.

*(2) If $X$ is log canonical and the strict transform of* $\operatorname{Supp} D$ *is smooth (i.e., if the components don't meet!), then* $\operatorname{discrep}(X, D) = \min\{a_{E_j}(X, D), 1 - d_i\}$.

**Example.** Let $D = aV(x) + bV(y)$ be a divisor on $X = \mathbb{A}^2$. This doesn't need a log resolution to apply (1) above: to be log canonical, the above says that we must have $a, b \leq 1$. So, assume this now. We can't use (2) because $\operatorname{Supp} D$ is not smooth, so we blowup to obtain a further resolution; because the strict transforms of the axes do not meet, we can then use this log resolution to compute the discrepancy. The only exceptional divisor is $E$, and we know that

$$K_Y - f^*(K_X + D) = \underbrace{K_Y - f^*K_X}_{E} - f^*D = E - (a+b)E - a\widetilde{V(x)} - b\widetilde{V(y)} = (1-a-b)E - a\widetilde{V(x)} - b\widetilde{V(y)}.$$

Thus we have that $a_E(X, D) = 1 - a - b$, and thus

$$\operatorname{discrep}(X, D) = \min(1 - a, 1 - b, 1 - a - b).$$

**Example.** Let $D = a[C]$, with $C$ the cuspidal cubic, a divisor in $\mathbb{A}^2$. We saw last time how to construct a log resolution of $(X, D)$ via three successive blowups, with exceptional divisors $E_1, E_2, E_3$. Say $\sigma : \widetilde{X} \to X$ is the composition of the three blowups. One can check easily by the formula for the blowup of at a point that

$$K_{\widetilde{X}} - \sigma^* K_X = E_1 + 2E_2 + 4E_3.$$

By examining the multiplicity of $C$ at $p$ and its strict transforms at each successive blowups, we can show that

$$\sigma^*(C) = \widetilde{C} + 2E_1 + 3E_2 + 6E_3,$$

where $\widetilde{C}$ is the strict transform of $C$, and thus $\sigma^*(aD) = a(\widetilde{C} + 2E_1 + 3E_2 + 6E_3)$, Thus we have that

$$\begin{aligned} K_{\widetilde{X}} - \sigma^*(K_X + D) &= (K_{\widetilde{X}} - \sigma^*(K_X)) - \sigma^*(D) \\ &= E_1 + 2E_2 + 4E_3 - a(\widetilde{C} + 2E_1 + 3E_2 + 6E_3) \\ &= -a\widetilde{C} + (1 - 2a)E_1 + (2 - 3a)E_2 + (4 - 6a)E_3 \end{aligned}$$

Then $\operatorname{discrep}(X, D) = \min(1 - 2a, 2 - 3a, 4 - 6a)$. In particular, if $a = 1$, $\operatorname{discrep}(X, D) \leq -2$ and is thus $-\infty$. Note that if $a = 5/6$, though, we have

$$\operatorname{discrep}(X, D) = \min(1 - 10/6, 2 - 15/6, 4 - 5) = \min(-4/6, -3/6, -1) = -1.$$

so $(X, 5/6 \cdot [D])$ is log canonical.

**Remark** (log canonical threshold)**.** Let $(X, D)$ be a log pair. It may well be the case $\operatorname{discrep}(X, D) < -1$, in which case it's $-\infty$ and thus tells us very little. Nonetheless, we can use the formalism above to describe the singularities of $(X, D)$: one may consider $\sup\{q > 0 : (X, qD) \text{ is log canonical}\}$. The higher $q$ is, the less singular $(X, D)$ is.

**Example.** Let $D_1 = V(x) + V(y)$ and $D_2 = V(y^2 - x^3)$. Our calculation for the $D_1$ above shows that $\operatorname{discrep}(X, D_1) = -1$ and that $\operatorname{discrep}(X, qD_1) = -\infty$ for $q > 1$, so $\operatorname{lct}(X, D_1) = 1$, while by our previous calculation $\operatorname{discrep}(X, (5/6)D_2) = -1$ and $\operatorname{discrep}(X, qD_2) = -\infty$ for $q > 5/6$, so $\operatorname{lct}(X, D_2) = 5/6$. Thus, we have that the log canonical threshold of the pair $(X, D)$ differentiates between the curve and the cusp, even though both are curve singularities of multiplicity 2.

**Example.** Now, recall we saw above that if $D$ is a cone over a smooth degree-$d$ hypersurface in $\mathbb{P}^{n-1}$, we have

$$\operatorname{discrep}(D) = n - 1 - d,$$

if this is $\geq -1$ (otherwise it's $-\infty$).

Consider instead the situation where we take $X$ as part of the pair $(\mathbb{A}^n, q \cdot D)$. As we've seen, a log resolution is given by blowing up $\mathbb{A}^n$ at the origin, since the strict transform $\widetilde{D}$ of $D$ meets the exceptional divisor $E$ transversely in a copy of the smooth degree-$d$ hypersurface. We know, of course, that

$$K_{\operatorname{Bl}_p \mathbb{A}^n} = \pi^* K_{\mathbb{A}^n} + (n-1)E.$$

Moreover, we have that $\pi^*(D) = \widetilde{D} + dE$ (since $f$ has multiplicity $d$ at the cone point). Thus we have that

$$K_{\operatorname{Bl}_p \mathbb{A}^n} = \pi^*(K_{\mathbb{A}^n} + q \cdot D) + (n - 1 - qD)E,$$

and thus

$$\operatorname{discrep}(\mathbb{A}^n, qD) = n - 1 - qd$$

if this is $\geq -1$, or $-\infty$ else.

Note that for $q = 1$ this agrees with the calculation of $\operatorname{discrep} D$ above; moreover, we can always choose $q$ small enough that $n - 1 - qd \geq -1$. In fact, we see immediatley that the log canonical threshold of $(\mathbb{A}^n, D)$ is $n/d$.

# An introduction to singularities in characteristic $p$: Frobenius is your friend (October 11, Eloísa Grifo)

Let $R$ be a ring of characteristic $p$, so that the Frobenius map $F : R \to R$, $r \mapsto r^p$ is a ring homomorphism. $R^p := F(R)$ is then the subring of $R$ of all $p$-th powers.

If $R$ is a domain, we can also talk about the inclusion $R \subset R^{1/p}$, where $R^{1/p}$ is the set of $p$-th roots of elements of $R$ taken in some fixed algebraic closure of $\operatorname{Frac} R$. We then have the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{F} & R \\ \big\downarrow{=} & & \big\downarrow{r \mapsto r^{1/p}} \\ R & \hookrightarrow & R^{1/p} \end{array}$$

We can think of $R$ as an $R^p$-module, and thus when $R$ is a domain this is the same as thinking of $R^{1/p}$ as an $R$-module. We write $I^{[p]}$ for $(f^p : f \in I)$, which is an ideal.

**Definition.** $R$ is $F$-finite if $R$ is a finite $R^p$-module (i.e., if $R$ is a domain, $R$ is a finite $R^{1/p}$-module).

**Example.** If $k$ is a perfect field (so $k = k^p$) then $k$ is $F$-finite.

$F$-finiteness is preserved by:

(1) Taking quotients.

(2) Taking finitely generated algebras over an $F$-finite ring.

(3) Localization.

(4) Completion at a maximal ideal.

Thus, for example, a finitely generated algebra (or even an essentially finitely generated algebra) over a perfect field is $F$-finite.

How does the Frobenius $F$ measure singularities?

(1) $F$ measures if $R$ is reduced: $R$ is reduced if and only if $F$ is injective. (From now on, $R$ will be reduced.)

(2) $F$ measures regularity: by a theorem of Kunz from 1969, a local ring $(R, m)$ is regular if and only if the Frobenius map is flat. If $R$ is an $F$-finite domain, this says that $R$ is regular if and only if $R$ is a free $R^p$-module.

**Example.** $R = \mathbb{F}_p[x_1, \ldots, x_d]$ should be free over $R^p = \mathbb{F}_p[x_1^p, \ldots, x_d^p]$, and indeed we see that it is, with basis $x_1^{a_1} \cdots x_d^{a_d}$ for $0 \le a_i < p$ for each $i$.

(3) If $R^p$ is a direct summand of $R$ (via the Frobenius), then $R$ is called "$F$-split", i.e., $R$ is $F$-split if we have an $R$-module map $R \to R$ splitting $F : R \to R$. Equivalently, $R$ is $F$-split if there exists $\varphi \in \operatorname{Hom}_R(R^{1/p}, R)$ with $\varphi(1) = 1$.

**Remark.** We see that regular rings are $F$-split, and that clearly any direct summand of an $F$-split ring is $F$-split: if $R \hookrightarrow S$ splits, we have

$$\begin{array}{ccc} R & \underset{\longleftarrow}{\hookrightarrow} & S \\ \big\uparrow & & \big\uparrow\big\downarrow \\ R^p & \underset{\longleftarrow}{\hookrightarrow} & S^p \end{array}$$

thus, following $R \to S \to S^p \to R^p$, we obtain our splitting.

**Example.** Veronese rings are $F$-split: the inclusion

$$k[\text{monomials of degree } d \text{ in } v \text{ variables}] \hookrightarrow k[v \text{ variables}]$$

splits. More generally, any Veronese subring of an $F$-split ring is $F$-split.

**Exercise.** Show that Veronese subrings are direct summands.

**Example.** The Veronese subring $k[s^2, st, t^2] \hookrightarrow k[s,t]$ can be viewed as the ring $k[x,y,z]/(xz - y^2)$, so this ring is $F$-split[3].

**Definition** (Hochster–Roberts 1974)**.** $R$ is $F$-pure if the Frobenius is pure, i.e., if for all $R$-modules $M$ the map

$$R \otimes M \xrightarrow{F \otimes 1} R \otimes M$$

is injective.

**Exercise.** If $R$ is $F$-finite, then $F$-purity and $F$-splitness are equivalent.

**Example.** If $I$ is a squarefree monomial ideal in a polynomial ring $R$, then $R/I$ is $F$-pure.

(Note that $F$-split implies reducedness, since it demands in particular that the Frobenius is injective, so the squarefree assumption is clearly necessary.)

One way to see this is via the following:

**Theorem** (Fedder's criterion (1983))**.** *If $(R, m)$ is a regular local ring and $I$ is an ideal, then $R/I$ is $F$-pure if and only if $(I^{[p]} : I) \not\subset m^{[p]}$.*

**Example.** If $I = (f)$, then the demand to be $F$-pure is that $(f^p : f) = f^{p-1} \not\subset m^{[p]}$.

Fedder's criterion can taken, equivalently, to be that $(I^{[p^e]} : I) \not\subset m^{[p^e]}$ for some $e$, for all $e$, or for all $e$ large enough.

**Example.** Consider $k[x,y]/(xy(x-y))$. Writing $f = xy(x-y)$, to check $F$-purity we just need to see if $f^{p-1} \in (x^p, y^p)$. But it's easy to see directly that $f^{p-1} = x^{p-1}y^{p-1}(x-y)^{p-1} \in (x^p, y^p)$, and thus for all $p$ this is *not* $F$-pure.

So far, we've said that $R$ is regular if and only if $R^{1/p}$ is free over $R$, and that $R$ is $F$-split if and only if $R$ is a direct summand of $R^{1/p}$. A notion lying in between these is the following:

**Definition** (Hochster–Huneke)**.** An $F$-finite ring $R$ is strongly $F$-regular if for all $f \notin \bigcup \operatorname{Min} R$ the map

$$R \hookrightarrow R^{1/q} \xrightarrow{\cdot f^{1/q}} R^{1/q}$$

splits for some $q = p^e$ (equivalently, for all $e$, or all $e$ large enough). We can rephrase this as asking that there exist $\varphi \in \operatorname{Hom}_R(R^{1/q}, R)$ with $\varphi(f^{1/q}) = 1$.

**Remark.** Direct summands of strongly $F$-regular rings are strongly $F$-regular, by a similar argument as in the $F$-pure case. Moreover, regular rings are strongly $F$-regular, and strongly $F$-regular rings are $F$-pure.

**Theorem** (Hochster–Huneke)**.** *Strongly $F$-regular rings are normal and Cohen–Macaulay.*

Note that $F$-purity does not imply Cohen–Macaulay! Nor does being $F$-pure and Cohen–Macaulay imply strong $F$-regularity.

**Example.** By our previous discussion, Veronese subrings (of polynomial rings) are strongly $F$-regular, as they're subrings of strongly $F$-regular rings.

We have a similar criterion to Fedder's criterion for strong $F$-regularity:

**Theorem** (Glassbrenner's criterion (1994))**.** *If $(R, m)$ is an $F$-finite regular local ring and $I$ is an ideal, then $R/I$ is strongly $F$-regular if and only if for all $c \notin \min I$ we have $c(I^{[q]} : I) \not\subset m^{[q]}$ for some (equivalently, large enough) $q = p^e$.*

[3] Recall that we've seen this ring already (up to a change of variables) as the cone over a smooth projective conic, and shown that it has canonical singularities.

**Example.** (1) If $I$ is a squarefree monomial ideal, then $R/I$ is strongly $F$-regular if and only if $I$ is generated by variables. To show this, one shows that strongly $F$-regular rings are products of domains, so $R/I$ is a domain and thus $I$ must be generated by variables.

(2) If $p \neq 2$, $R = k[x, y, z]/(x^2 + y^2 + z^2)$ is strongly $F$-regular.

(3) $R = k[x, y, z]/(x^2 + y^2 + z^2)$ is *not* strongly $F$-regular for $p \equiv 1 \bmod 3$, though it is $F$-pure.

(4) Regular rings are strongly $F$-regular (the proof is a neat application of Nakayama's lemma!).

**Theorem** (Hochster–Roberts)**.** *Rings of invariants of linearly reductive groups are Cohen–Macaulay.*

Linearly reductive groups are algebraic groups such that every representation is completely reducible; in positive characteristic this includes tori, finite groups of order not divisible by $p$, and extensions of these examples.

This is proved in part by showing that such rings are in fact strongly $F$-regular, hence (as we've said) normal and Cohen–Macaulay, and then using our above observation about direct summands. Hochster and Roberts then use reduction to characteristic $p$ to show that rings of invariants are Cohen–Macaulay even in characteristic 0!

## Why characteristic $p$ is better (October 18, Monica Lewis)

**Question** (direct summand conjecture)**.** If $(R, m)$ is a regular ring and $S$ is a module-finite extension of $R$, does the map $R \to S$ split?

In answering this question in characteristic $p$, we'll see how the notion of tight closure naturally arises.

By a straightforward reduction, we can reduce to the case where $R$ and $S$ are complete local domains.

**Proposition** (splitting criterion in Gorenstein rings)**.** *If $R$ is complete and Gorenstein and $M$ is any $R$-module, then a map $R \to M$ splits if and only if:*

*(1) For some (equivalently, for all) systems of parameters $x_1, \ldots, x_d$ of $R$, if $I_t = (x_1^t, \ldots, x_n^t)R$, then the map $R/I_t \to M/I_t M$ is injective for all $t$. If $M = S$ is an $R$-algebra, this is equivalent to the equality $I_t S \cap R = I_t$.*

*(2) If $u_t$ represents a generator of $\operatorname{ann}_M(R/I_t)$, then $u_t \notin I_t S \cap R$.*

The first criterion is shown by taking the limit over all $t$, identifying it as the tensor with the Matlis dual, and using that the original map is split if and only if it's injective after tensoring with the Matlis dual.

From now on, $R, S$ are complete local domains.

**Remark.** Here's one potential approach: take a system of parameters $(x_1, \ldots, x_d)$, let $I_t = (x_1^t, \ldots, x_d^t)$, so $u_t = x_1^{t-1} \cdots x_d^{t-1}$. We know $R \hookrightarrow S$ splits if and only if $u_t \notin I_t S \cap R$, so let's try assuming that it doesn't split and thus that $u_t \in I_t S \cap R$ for some $t$. This is equivalent to saying that

$$x_1^{t-1} \cdots x_n^{t-1} = \sum x_1^t s_1 + \cdots + x_d^t s_d$$

for $s_1, \ldots, s_d \in S$. *If* $x_1, \ldots, x_d$ were $S$-regular, this is already a contradiction, and we'd be done! So, maybe since $x_1, \ldots, x_d$ are $R$-regular, they remain so in $S$? In fact, this rarely happens.

This is the same as asking if $S$ is a Cohen–Macaulay $R$-algebra, which is the condition that a system of parameters in $R$ is $S$-regular. Unfortunateley, this is generally false. But it's actually enough to embed $S$ into a Cohen–Macaulay algebra for $R$; can we do so? At least in characteristic $p$, this is true, but how?

The idea is to forget about the particular module-finite extension $S$ entirely.

**Definition** (intuition)**.** The absolute integral closure of $R$ is the ring $R^+$ given by the union of all module-finite extension domains, so some kind of "universal integral extension". Equivalently, it's the union over all elements integral over $R$.

**Definition** (formal definition)**.** The absolute integral closure of $R$ is the integral closure of $R$ in an algebraic closure of $\operatorname{Frac} R$.

**Theorem** (Hochster–Huneke 1991)**.** *If $R$ is a complete local domain of characteristic $p$, then $R^+$ is a big Cohen–Macaulay algebra.*

This then solves our problem (since $R^+$ contains $S$ and is a Cohen–Macaulay algebra), and yields a proof of the direct summand conjecture in characteristic $p$. This theorem is actually false in characteristic 0, although by using Artin approximation it allows us to construct a big Cohen–Macaulay algebra in equal characteristic 0.

So, why is this a big Cohen–Macaulay algebra? If $x_1, \ldots, x_t$ is part of a system of parameters on $R$, we want this to be a regular sequence on $R^+$. Note that $mR^+ \not\subset R^+$, since $R^+$ is an integral extension of $R$ and thus lying-over (which requires no noetherianity on the extension) gives a prime ideal of $R^+$ lying over $m$. We'll now use induction on $t$. If $t = 1$, then $x_1$ is a nonzerodivisor on $R^+$ since $R^+$ is a domain (it was defined inside a field!). How do we do the inductive step? Suppose

$x_1, \ldots, x_{t-1}$ are a regular sequence on $S$, and $x_t$ is a counterexample, i.e., $x_t$ is a zerodivisor in $R^+/(x_1, \ldots, x_{t-1})R^+$. Equivalently, there exists $u \in R^+$ such that

$$ux_t \in \underbrace{(x_1, \ldots, x_{t-1})}_{J}$$

but

$$u \notin J.$$

Equivalently, this says that $(J + Ru)/J \neq 0$; let $P$ be a minimal prime in its support as an $R$-module. Localizing at $P$, we get that $(J_P + R_P u)/J_P \neq 0$, and $J_P = (x_1, \ldots, x_{t-1})(R^+)_P$; one can check that $(R^+)_P = R_P^+$ (in fact, formation of $R^+$ commutes with any localization). Thus we have that

$$J_P = (x_1, \ldots, x_{t-1})(R_P)^+;$$

thus, given our original counterexample, we've produced a counterexample where the maximal ideal is a minimal prime in the support of $(J + Ru)/J$, so we'll assume that's the case (i.e., $P = m$). Thus $\operatorname{Supp}((J + Ru)/J) = \{m\}$, and $(J + Ru)/J$ is clearly finitely generated, and thus it must actually be $m$-torsion and thus

$$m^a u \in J$$

for some $t$. Equivalently, this says that

$$H_m^0(R^+/J) = \Gamma_m(R^+/J) \neq 0.$$

**Lemma.** $H_m^i(R^+/(x_1, \ldots, x_h)R^+) = 0$ *whenever* $i + h < \dim R$.

The $i = 0$ case then implies our desired result by giving a contradiction above.

*Proof.* The proof is by induction on $h$. Assume we have the statement for $h$. Let $S = R^+/(x_1, \ldots, x_h)R^+$ and let $x = x_{h+1}$. We have a short exact sequence

$$0 \to S \xrightarrow{x} S \to S/xS \to 0$$

(note that we use a double induction here to ensure that multiplication by $x$ is actually injective).

The resulting long exact sequence gives

$$H_m^i(S) \xrightarrow{x} H_m^i(S) \to H_m^i(S/x) \to H_m^{i+1}(S),$$

where $i + (h + 1) < \dim R$. Certainly $i + h < i + (h + 1) = (i + 1) + h$, and thus $H_m^i(S) = 0$, and $H_m^{i+1}(S) = 0$ as well, and thus $H^i(S/xS) = 0$. Thus the inductive step really isn't too had.

The base case is the case $h = 0$. This is the statement that $H_m^i(R^+) = 0$ for all $i < \dim R$. This is the heart of the argument and very hard; we'll attempt to show why characteristic $p$ becomes useful.

We claim that it's sufficient to show that for any complete local domain $S$ there is a module-finite extension domain $T$ such that the natural map $H_m^i(S) \to H_m^i(T)$ is zero for $i < \dim S$. Since $R^+$ is the directed limit of such extensions and local cohomology commutes with formation of the direct limit, this claim then says that $H_m^i(R^+)$ is zero for $i < \dim S$.

By Matlis duality, this is equivalent to showing that

$$H_m^i(T)^\vee \to H_m^i(S)^\vee$$

is the zero map; applying local duality, and writing $S$ as a module-finite extension of a regular local ring $A$ of dimension $d$, we obtain that this map is just

$$\operatorname{Ext}_A^{d-i}(T, A) \to \operatorname{Ext}_A^{d-1}(S, A).$$

Note that if we extend $T$ to $T'$, and if we let $V_T$ be the image of this map, we get $V_{T'} \subset V_T$, thus we want to take further extensions such that this image shrinks to 0.

Fix $T_0$ extending $S$, and let $P_1, \ldots, P_n \subset S$ be the associated primes of $V_{T_0}$. We want to kill the non-maximal associated primes one at a time, to obtain a finite-length image. We won't say how, but we can pick $T_1$ such that

$$\operatorname{Ass}(V_{T_1}) \subset \operatorname{Ass}(V_{T_0})$$

but $P_1$ is excluded (when not maximal), and so on. Eventually, we get $T$ such that $m_S$ is the only associated prime of $V_T$, and thus $V_T$ has finite-length (since $\operatorname{Ext}_A^{d-1}(T, A)$ is finitely generated), and thus its Matlis dual does as well.

Thus, we know that we can ensure that the image of $H_m^i(S) \to H_m^i(T)$ is finite-length (just doing it one $i$ at a time and taking further and further extensions).

So far, we've made no reference to characteristic $p$. This is where characteristic becomes important.

Let $G$ be a monic polynomial in the Frobenius with coefficients in $S$; we claim that $G$ can then act on local cohomology.

**Lemma.** *The action of the Frobenius on local cohomology is compatible with natural maps induced by ring maps.*

Thus, the image of $H_m^i(S) \to H_m^i(T)$ is stable under the action of the Frobenius. The following propositions are relatively easy to show directly:

**Proposition.** *If $u$ is an element of $H_m^i(S)$ such that $G(u) = 0$ for some monic $G$, then there exists a module-finite extension domain $T$ such that $H_m^i(S) \to H_m^i(T)$ kills $u$.*

The proof is elementary and arises just by making explicit module-finite extensions arising from the data of the Cech complex.

**Proposition.** *If $M$ is a finitely generated $F$-stable submodule of $H_m^i(S)$, there is a module-finite extension domain $T$ of $S$ such that $H_m^i(S) \to H_m^i(T)$ is zero.*

Since the image of $H_m^i(S) \to H_m^i(T)$ was already finite length and $F$-stable, there's some further extension $S \to T \to T'$ such that $H_m^i(S) \to H_m^i(T')$ is the zero map. □

Thus, we see that $R^+$ is a big Cohen–Macaulay algebra, and the key was the ability to keep going up along "natural" module-finite extensions and killing relations.

We've already shown that $H_m^i(R^+) = 0$ for $i < d = \dim R$; since $R^+$ is a big Cohen–Macaulay algebra, it follows then that $H_m^d(R^+) \neq 0$ (this can be seen by embedding $R^+/I_1 R^+ \hookrightarrow R^+/I_t R^+ \hookrightarrow \cdots \hookrightarrow H_m^d(R^+)$. For a complete local domain $R$, the statement that $H_m^d(S) \neq 0$ for an $R$-algebra $S$ is equivalent to saying that there exists an $R$-linear map $\theta : S \to R$ such that $\theta(1) \neq 0$; this is a property referred to as "solidity", and thus we see that big CM algebras are solid.

Thus, we have a map $R^+ \to R$ with $\theta(1) \neq 0$, but this map is very hard to describe and unintuitive.

Now, we return to the condition that $u \in IR^+ \cap R$. This is equivalent to saying that $u$ is in the contraction of some expansion to a module-finite extension; if $I = (f_1, \ldots, f_n)$ we can write $u = f_1 s_1 + \cdots + f_n s_n$ for $s_i \in R^+$. This is the condition that $u$ is in the *plus-closure* of $I$, which is very hard to describe since $R^+$ is so hard to work with.

Instead, note that the statement above says that

$$u^q = f_1^q s_1^q + \cdots + f_n^q s_n^q$$

for all $q = p^e$; applying $\theta$, we obtain that $\theta(1) = f_1^q \theta(s_1^q) + \cdots + f_n^q \theta(s_n^q)$. Thus, this is the same as saying that there is some $c \neq 0$ such that $cu^q \in I^{[q]}$ for all $q \geq 0$, and this condition is the tight closure.

# Characteristic-$p$ continued (October 25, Monica Lewis)

**Remark.** Recall for a positive-characteristic domain $R$ that $R^+$ is the integral closure of $R$ in an algebraic closure of $\operatorname{Frac} R$. We write $R^\infty \subset R^+$ for the union $\bigcup_{q=p^e} R^{1/q}$, the "perfection" of $R$. If furthermore $R$ is a complete local domain, by the Cohen structure theorem we can write $R$ as a module-finite local extension of a regular complete local ring $A$. For $a \in A$, we can define $\operatorname{ord}(a) = \max\{k : a \in m_A^k\}$. This extends to $R^\infty$ in a fairly obvious way: $\operatorname{ord}(a^{1/a}) = \operatorname{ord}(a)/q$. This gives a $\mathbb{Z}[1/p]$-valued valuation on $A^\infty$, and can actually be extended to a $\mathbb{Q}$-valued valuation on $A^+ = R^+$. Since $\operatorname{ord}(0) = \infty$ and $\operatorname{ord}(1) = 0$, we want to think of elements with very "small" orders as being "close" to units.

Now, with this valuation defined, we can recall our remarks from last time: for $R$ a complete local domain of dimension $d$ and characteristic $p > 0$, we noted that $H_m^i(R^+) = 0$ for $i < d$ and that $H_m^d(R^+) \neq 0$. From the former vanishing, we saw that $R^+$ was a big Cohen–Macaulay algebra, and thus that if $x_1, \dots, x_{k+1}$ is part of a system of parameters, and if

$$r x_{k+1} = r_1 x_1 + \cdots + r_k x_k,$$

then there exists a module-finite extension domain $R \hookrightarrow S$ such that $r \in (x_1, \dots, x_k)S \cap R$.

**Definition.** The plus closure of an ideal $I \subset R$ is the ideal

$$I^+ := IR^+ \cap R.$$

Thus, the statement preceding the definition is then equivalent to saying that $r \in (x_1, \dots, x_k)^+$. Moreover, the hypothesis that $r x_{k+1} = r_1 x_1 + \cdots + r_k x_k$ is the same as saying that

$$r \in (x_1, \dots, x_k) : x_{k+1}.$$

Thus, what we've seen is that

$$((x_1, \dots, x_k) : x_{k+1}) \subset (x_1, \dots, x_k)^+;$$

this is referred to by saying "plus closure captures colons". o

Now, from the latter vanishing statement we deduce a "solidity" property: there exists an $R$-linear map $\theta : R^+ \to R$ such that $\theta(1) \neq 0$. Thus, if $r \in I^+$, with $I = (a_1, \dots, a_n)$, then $r = a_1 s_1 + \cdots + a_n s_n$ for $s_i \in R^+$; applying $\theta$ and using $R$-linearity, we obtain

$$r\theta(1) = a_1\theta(s_1) + \cdots + a_n\theta(s_n).$$

Writing $\theta(1) = c$, we get that $cr \in (a_1, \dots, a_n)$. But what if $c$ is "close to 0", i.e., what if the valuation of $c$ is very large. However, if we take $q$-th powers, obtaining

$$r^q = a_1^q s_1^q + \cdots + a_n^q s_n^q,$$

and apply $\theta$, we get

$$r^q c = a_1^q \theta(s_1^q) + \cdots + a_n^q \theta(s_n^q),$$

and thus

$$r c^{1/q} = a_1 \theta(s_1^q)^{1/q} + \cdots + a_n \theta(s_n^q)^{1/q};$$

thus, we have that $c^{1/q} r \in IR^{1/q} \subset IR^\infty$. Moreover, we have $\operatorname{ord}(c^{1/q}) = (1/q)\operatorname{ord}(c) \to 0$ as $q \to \infty$.

**Remark.** Recalling the direct summand conjecture: if $R \hookrightarrow S$ is a module-finite extension of a regular local ring $R$, then there's a splitting $\theta : S \to R$ with $\theta(1) = 1$ splitting $R \hookrightarrow S$. Then if $r \in I^+$ for $I \subset R$ a regular ring, so

$$r = a_1 s_1 + \cdots + a_n s_n$$

for $a_i \in I$, then $r = r\theta(1) = a_1\theta(s_1) + \cdots + a_n\theta(s_n) \in I$, and thus $I^+ = I$. Thus, every ideal in a regular local ring is plus-closed.

Now, we'll define tight-closure, which will have analogous ideals to the colon-capturing and regularity properties of plus closure.

Let $R$ be a complete local domain for now.

**Definition.** We say that an element $u \in R$ is in the *tight closure* of an ideal $I$, written $u \in I^*$, if $u$ is "almost" in $IR^\infty$, i.e., if one of the following equivalent statements holds:

(1) There is a sequence of elements $\delta_n \in R^\infty - \{0\}$ such that $\operatorname{ord}(\delta_n) \to 0$ as $n \to \infty$ such that $\delta_n u \in IR^\infty$ for all $n$.

(2) There is one element $c \in R - \{0\}$ such that $c^{1/q}u \in IR^{1/q} \subset IR^\infty$ for all $q \gg 0$.

(3) There's $c \in R - \{0\}$ (depending on $u$) such that $cu^q \in I^{[q]}$ for all $q \gg 0$.

**Remark.** We've seen already that $I^+ \subset I^*$; we have the following comparison:

**Theorem.** *Almost membership in $IR^\infty$ is equivalent to almost membership in $IR^+$, i.e., the following are equivalent:*

*(1) There is a sequence of elements $\delta_n \in R^\infty - \{0\}$ such that $\operatorname{ord}(\delta_n) \to 0$ as $n \to \infty$ such that $\delta_n u \in IR^\infty$ for all $n$.*

*(2) There is a sequence of elements $\delta_n \in R^+ - \{0\}$ such that $\operatorname{ord}(\delta_n) \to 0$ as $n \to \infty$ such that $\delta_n u \in IR^+$ for all $n$.*

We claim that tight closure shares a lot of great properties with plus closure:

(1) (colon-capturing) If $x_1, \dots, x_{k+1}$ is part of a system of parameters, we know

$$((x_1, \dots, x_k) : x_{k+1}) \subset (x_1, \dots, x_k)^+ \subset (x_1, \dots, x_k)^*,$$

and thus tight closure captures colons.

(2) In a regular local ring $R$ all ideals are tightly closed: recall that by Kunz the Frobenius is flat in this case, and that if $R \to S$ is flat then expansion commutes with the formation of colon ideals, i.e., $(IS :_S JS) = (I :_R J)S$. Then combining these we have

$$(I^{[q]} : J^{[q]}) = (I : J)^{[q]};$$

if $u \in I^* - I$ we have $c \neq 0$ such that $cu^q \in I^q$ for $q \gg 0$. We can restate this as

$$c \in (I^{[q]} : u^q) = (I : u)^{[q]}$$

for all $q$, and thus

$$c \in \bigcap_q (I : u)^{[q]}.$$

Since $c \notin I$, we have that $(I : u)$ is proper, thus contained in $m$, and thus the intersection is contained in $\bigcap m^{[q]} \subset \bigcap m = 0$, thus giving the result.

**Definition.** A complete local domain $R$ is called $F$-rational if some (equivalently every) ideal generated by a system of parameters is tightly closed.

It is thus immediately seen that $F$-rational implies Cohen–Macaulay.

**Theorem** (Smith 1994)**.** *If $I$ is generated by a part of a system of parameters, then $I^+ = I^*$.*

Part of the original reason for interest in this question is that plus closure can be easily seen to commute with localization, while it was not known at the time if tight closure did; this says that at least it commutes with localization for systems of parameters. (Later, in 2007, Monsky showed that in fact tight closure does not commute with localization.)

Now, we just want to restate the definition of tight closure in more generality:

**Definition.** If $N \subset M$, define

$$N^{[q]} := \operatorname{im}\bigl(R^{1/q} \otimes_R N \to R^{1/q} \otimes M\bigr);$$

$u^q$ will be the image of $1 \otimes u$.

**Definition.** If $R$ is a noetherian ring of characteristic $p$, and $N \subset M$ are $R$-modules, then we say $u \in M$ belongs to $N^*_M$, the tight closure of $N$ inside $M$, if there exists $c \in R - \bigcup_{P \in \operatorname{Min} R} P$ such that $cu^q \in N^{[q]}$ for $q \gg 0$.

**Definition.** A ring $R$ is weakly $F$-regular if all ideals are tightly closed.

Note that the following are equivalent:

(1) $R$ is weakly $F$-regular.

(2) $R_m$ is weakly $F$-regular for all maximal ideals in $R$.

**Definition.** If moreover $R_P$ is weakly $F$-regular for all prime ideals $P$ in $R$, then we say $R$ is $F$-regular.

**Proposition.** *Regular rings are $F$-regular.*

(In fact, they're also stronger $F$-regular.) Thus, tight closure measures the failure of a ring to be regular.

**Definition.** A ring $R$ is strongly $F$-regular if for all inclusions

$$N \subset M,$$

not necessarily finitely generated, $N = N^*_M$.

**Remark.** If $R$ is $F$-finite, this is equivalent to the earlier definition in terms of Frobenius splittings.

**Remark.** The following are equivalent:

(1) $R$ is strongly $F$-regular.

(2) $R_m$ is strongly $F$-regular for all maximal ideals $m$.

(3) $R_P$ is strongly $F$-regular for all prime ideals $P$.

(4) 0 is tightly closed in $E_R(R/P)$ for all prime ideals $P$.

(5) 0 is tightly closed in $E_R(R/m)$ for all maximal ideals $m$.

**Theorem.** *If $R$ is Gorenstein, then the following properties are equivalent:*

*(1) strong $F$-regularity.*

*(2) $F$-regularity.*

*(3) weak $F$-regularity.*

*(4) $F$-rational.*

In general, it is not known if weak $F$-regularity implies strong $F$-regularity.
We conclude by making some definitions that will recur in future weeks:

**Definition.** An element $c \in R - \bigcup_{P \in \operatorname{Min} R} R$ is called a test element if for all ideals $I$ and all $u \in I^*$, $cu^{[q]} \in I^{[q]}$ for all $q \geq 0$.

**Theorem.** *If $R$ is essentially of finite type over an excellent semilocal ring, then $R$ has a (completely stable big) test element.*

Thus complete local rings, for example, have test elements.

The following is an implication of the Jacobian–Lipman–Sathaye theorem:

**Theorem.** *If $R$ is a finitely generated geometrically reduced equidimensional $k$-algebra, then every element of $J_{R/k}$ not lying in any minimal prime is a (completely stable big) test element.*

**Definition.** The test ideal (in a ring with test elements) is the ideal generated by test elements, and is denoted $\tau_R$.

**Remark.** If $R$ is weakly $F$-regular then $\tau_R = R$. One can show that in an $F$-pure ring the test ideal is radical.

**Example.** If $p \equiv 1 \bmod 3$ and $R = \mathbb{F}_p[[x, y, z]]/(x^3 + y^3 + z^3)$ is $F$-split/$F$-pure, the test ideal will be radical while $J_{R/\mathbb{F}_p} = (x^2, y^2, z^2)R$ is not radical, so this isn't the full set of test elements; the test ideal will be $(x, y, z)$.

# TEST IDEALS FOR PAIRS VIA GENERALIZED TIGHT CLOSURE

TAKUMI MURAYAMA

The goal of this talk is to develop a theory of test ideals for pairs $(R, \mathfrak{a}^t)$, and to give some applications that do not mention test ideals. See [ST12] and [TW18, §5] for overviews of the theory. The theory is originally due to Hara–Yoshida [HY03] and Hara–Takagi [HT04], and generalizes the tight closure theory developed by Hochster–Huneke [HH90].

We start by describing our applications. First, recall that the *integral closure* $\overline{\mathfrak{a}}$ of an ideal $\mathfrak{a} \subseteq R$ is the set of all elements $r \in R$ such that $r$ satisfies a polynomial $f(x) = \sum_{i=0}^{d} c_{d-i} x^i \in R[x]$ where $c_0 = 1$ and $c_i \in \mathfrak{a}^i$. A version of the Briançon–Skoda theorem says that if $R$ is regular, then

$$\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^n,$$

where $\ell$ is the number of generators of $\mathfrak{a}$. This inclusion was shown by Skoda–Briançon for smooth $\mathbf{C}$-algebras [SB74, Thm. 3], by Lipman–Sathaye for all regular rings [LS81, Thm. 1″], and by Hochster–Huneke for weakly $F$-regular rings [HH90, Thm. 5.4]. We will prove a version of this result in §2. This result has some interesting corollaries, which we will not be able to prove:

(1) If $f \in \mathbf{C}\{z_1, z_2, \ldots, z_n\}$ is a convergent power series in $n$ variables that defines a hypersurface with an isolated singularity at the origin, then $f^n \in (\partial f/\partial z_1, \partial f/\partial z_2, \ldots, \partial f/\partial z_n)$ [SB74, Cor.]. This answers a question of Mather.

(2) If $f_1, f_2, \ldots, f_{n+1} \in R$, where $R$ is regular of dimension $n$, then

$$f_1^n f_2^n \cdots f_n^n \in (f_1^{n+1}, f_2^{n+1}, \ldots, f_{n+1}^{n+1}).$$

See [Hoc14, Thm. on p. 29].

Second, recall that the *$n$th symbolic power* of an ideal $\mathfrak{a} \subseteq R$ is

$$\mathfrak{a}^{(n)} \coloneqq R \cap \bigcap_{\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})} \mathfrak{a}^n R_{\mathfrak{p}}.$$

We will prove the following uniform comparison theorem for symbolic powers on regular rings:

$$\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$$

where $h$ is the maximal height of the associated primes of $\mathfrak{a}$. This uniform comparison theorem is originally due to Ein–Lazarsfeld–Smith for smooth $\mathbf{C}$-algebras [ELS01, Thm. A], to Hochster–Huneke for regular rings containing a field [HH02, Thm. 1.1(a)], and to Ma–Schwede for mixed characteristic regular rings [MS18, Thm. 7.4]. We will prove a version of this result in §3.

**Notation.** All rings will be commutative with identity, noetherian, and of characteristic $p > 0$. If $R$ is a ring, then $R^\circ$ denotes the complement of the union of the minimal primes of $R$. We denote the Frobenius morphism by $F \colon R \to F_* R$. For every integer $e \geq 0$, the $e$th iterate of the Frobenius morphism is denoted by $F^e \colon R \to F^e_* R$.

*Date*: November 1, 2018 at the Singularities reading group at the University of Michigan, Fall 2018.

## 1. Definition and preliminaries

We start by defining with the analogue of tight closure and test ideals for pairs $(R,\mathfrak{a}^t)$, following Hara–Yoshida [HY03] and Hara–Takagi [HT04].

**Definition 1.1** [HY03, Def. 6.1; HT04, Def. 1.4]**.** Let $R$ be a ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Let $\iota\colon N \hookrightarrow M$ be an inclusion of $R$-modules. For every $t \in \mathbf{R}_{\geq 0}$, the $\mathfrak{a}^t$-*tight closure* is

$$N_M^{*\mathfrak{a}^t} \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ c\mathfrak{a}^{\lceil p^e t\rceil} \otimes x \subseteq \operatorname{im}\bigl(\operatorname{id}\otimes\iota\colon F^e_*R\otimes_R N \to F^e_*R \otimes_R M\bigr) \end{array} \right\}.$$

Now let $E \coloneqq \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum of the injective hulls of the residue fields $R/\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subseteq R$. The (*non-finitistic* or *big*) *test ideal* is

$$\tau(\mathfrak{a}^t) \coloneqq \operatorname{Ann}_R\bigl(0_E^{*\mathfrak{a}^t}\bigr).$$

Now let $\mathfrak{b} \subseteq R$ be another ideal such that $\mathfrak{b} \cap R^\circ \neq \emptyset$. For every $s \in \mathbf{R}_{\geq 0}$, we similarly define

$$N_M^{*\mathfrak{a}^t\mathfrak{b}^s} \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes x \subseteq \operatorname{im}\bigl(\operatorname{id}\otimes\iota\colon F^e_*R\otimes_R N \to F^e_*R \otimes_R M\bigr) \end{array} \right\}$$

in which case the test ideal is

$$\tau(\mathfrak{a}^t\mathfrak{b}^s) \coloneqq \operatorname{Ann}_R\bigl(0_E^{*\mathfrak{a}^t\mathfrak{b}^s}\bigr).$$

Setting $\mathfrak{a} = R$ and $t = 1$, one obtains the usual notion of tight closure for modules due to Hochster–Huneke [HH90, Def. 8.2], and the (non-finitistic or big) test ideal $\tau(R)$ defined by Lyubeznik–Smith [LS01, §7].

*Remark* 1.2. The definition in [HY03] is the analogue of the (finitistic) test ideal defined in [HH90, Def. 8.22] for pairs $(R,\mathfrak{a}^t)$. There is also a version of tight closure for pairs $(R,\Delta)$, where $\Delta$ is an effective $\mathbf{R}$-Weil divisor [Tak04, Def. 2.1], or triples $(R,\Delta,\mathfrak{a}^t)$ [Sch10, Def. 2.14]. The test ideal $\tau(\mathfrak{a}^t)$ can also be described in terms of an appropriate version of test elements for $\mathfrak{a}^t$-tight closure if $R$ is $F$-finite [Sch10, Thm. 2.22].

We now state some basic properties of test ideals that we will use often.

**Proposition 1.3** (cf. [HY03, Props. 1.3 and 1.11; LS01, Prop. 2.9])**.** *Let $R$ be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $R$ intersecting $R^\circ$.*

(*i*) $\tau(\mathfrak{a}^t \cdot \mathfrak{b}^s)\cdot\mathfrak{b} \subseteq \tau(\mathfrak{a}^t\cdot\mathfrak{b}^{s+1})$.
(*ii*) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^t) \subseteq \tau(\mathfrak{b}^t)$. Equality holds if $\mathfrak{b} \subseteq \overline{\mathfrak{a}}$.*
(*iii*) *If $s < t$, then $\tau(\mathfrak{a}^s) \supseteq \tau(\mathfrak{a}^t)$.*
(*iv*) *For every $m \in \mathbf{N}$, we have $\tau((\mathfrak{a}^n)^t) = \tau(\mathfrak{a}^{nt})$.*
(*v*) *$R$ is strongly $F$-regular if and only if $\tau(R) = R$.*

*Proof.* It suffices to show the corresponding dual statements for $N_M^{*\mathfrak{a}^t}$ by setting $N = 0$ and $M = E$ and taking annihilators, where for (*i*), we use the fact that $\operatorname{Ann}_R(N :_M \mathfrak{a}) \supseteq \operatorname{Ann}_R(N)\cdot\mathfrak{a}$.

For (*i*), we have

$$N_M^{*\mathfrak{a}^t\mathfrak{b}^{s+1}} \subseteq \bigl(N_M^{*\mathfrak{a}^t\mathfrak{b}^s} :_M \mathfrak{b}\bigr) \tag{1}$$

since if $x \in N_M^{*\mathfrak{a}^t\mathfrak{b}^{s+1}}$ with multiplier $c \in R^\circ$, then

$$\begin{aligned} c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes \mathfrak{b}\cdot x &= c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil}\mathfrak{b}^{[p^e]} \otimes x \\ &\subseteq c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil + p^e} \otimes x \\ &= c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e (s+1)\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\operatorname{id}\otimes\iota\colon F^e_*R\otimes_R N \to F^e_*R \otimes_R M\bigr) \end{aligned}$$

for all $e \gg 0$.

The first part of (*ii*) and (*iii*) follow from the fact that $N_M^{*\mathfrak{a}^t} \supseteq N_M^{*\mathfrak{b}^t}$ and $N_M^{*\mathfrak{a}^s} \subseteq N_M^{*\mathfrak{a}^t}$. For the second part of (*ii*), it suffices to show that $N_M^{*\mathfrak{a}^t} \subseteq N_M^{*\mathfrak{b}^t}$. Recall from [HS06, Cor. 1.2.5] that $\mathfrak{b} \subseteq \overline{\mathfrak{a}}$ if and only if there exists an integer $r > 0$ such that

$$\mathfrak{b}^{r+1} = \mathfrak{a}\mathfrak{b}^r,$$

i.e., if and only if $\mathfrak{a}$ is a *reduction* of $\mathfrak{b}$. By [HS06, Rem. 1.2.3], this implies $\mathfrak{b}^{r+s} \subseteq \mathfrak{a}^s$ for every integer $s > 0$. Now consider $x \in N_M^{*\mathfrak{a}^t}$. Setting $s = \lceil p^e t \rceil$, we see that for any choice of $d \in \mathfrak{b}^r \cap R^\circ$ and all $x \in M$, we have

$$\begin{aligned} cd\mathfrak{b}^{\lceil p^e t\rceil} \otimes x &\subseteq c\mathfrak{b}^{\lceil p^e t\rceil + r} \otimes x \\ &\subseteq c\mathfrak{a}^{\lceil p^e t\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F^e_* R \otimes_R N \to F^e_* R \otimes_R M\bigr) \end{aligned}$$

hence $x \in N_M^{*\mathfrak{a}^t}$ implies $x \in N_M^{*\mathfrak{b}^t}$.

For (*iv*), we note that $\lceil p^e t\rceil - 1 \le p^e t \le \lceil p^e t\rceil$, hence multiplying by $n$ throughout and applying ceilings again, we have

$$n\lceil p^e t\rceil - n \le \lceil p^e nt\rceil \le n\lceil p^e t\rceil.$$

We therefore have the inclusions

$$\mathfrak{a}^{n\lceil p^e t\rceil - n} \supseteq \mathfrak{a}^{\lceil p^e nt\rceil} \supseteq \mathfrak{a}^{n\lceil p^e t\rceil}.$$

The right inclusion already implies $N_M^{*\mathfrak{a}^{nt}} \subseteq N_M^{*(\mathfrak{a}^n)^t}$. On the other hand, if $x \in N_M^{*(\mathfrak{a}^n)^t}$ with multiplier $c \in R^\circ$, then for any choice of $d \in \mathfrak{a}^n \cap R^\circ$, we have

$$\begin{aligned} cd\mathfrak{a}^{\lceil p^e nt\rceil} \otimes x &\subseteq cd\mathfrak{a}^{n\lceil p^e t\rceil - n} \otimes x \\ &\subseteq c\mathfrak{a}^{n\lceil p^e t\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F^e_* R \otimes_R N \to F^e_* R \otimes_R M\bigr) \end{aligned}$$

for all $e \gg 0$, hence $N_M^{*(\mathfrak{a}^n)^t} \subseteq N_M^{*\mathfrak{a}^{nt}}$.

Finally, (*v*) follows since $R$ is strongly $F$-regular if and only if $0_E^* = 0$ [LS01, Prop. 2.9]. □

*Remark* 1.4. Motivated by Proposition 1.3(*v*), we say that $(R, \mathfrak{a}^t)$ is *strongly $F$-regular* if $\tau(\mathfrak{a}^t) = R$.

The following results are more subtle. The corresponding results for $\tau(R)$ are due to Lyubeznik–Smith [LS01, Thms. 7.1(2) and 7.1(3)], and do not require the full strength of $F$-finiteness.

**Proposition 1.5** [HT04, Props. 3.1 and 3.2]**.** *Let $R$ be a reduced $F$-*finite *ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$.*

(*i*) *For every multiplicative subset $W \subseteq R$, we have $\tau(\mathfrak{a}^t)W^{-1}R = \tau((\mathfrak{a}W^{-1}R)^t)$.*
(*ii*) *If $(R, \mathfrak{m})$ is a local ring, then $\tau(\mathfrak{a}^t)\widehat{R} = \tau((\mathfrak{a}\widehat{R})^t)$.*

*Proof Sketch.* Both of these properties follow from the fact (see [HT04, Lem. 2.1]) that

$$\tau(\mathfrak{a}^t) = \sum_{e \ge 0} \sum_{\phi^{(e)}} \phi^{(e)}\bigl(F^e_*(c\mathfrak{a}^{\lceil p^e t\rceil})\bigr),$$

where $\phi^{(e)}$ ranges over all elements of $\operatorname{Hom}_R(F^e_* R, R)$, and $c \in R^\circ$ is an appropriate version of a completely stable big test element for $\mathfrak{a}^t$-tight closure. □

Before we can move on to applications, we need the following version of Matlis duality:

**Proposition 1.6** (Matlis duality, see [Hoc07, Prop. on p. 242])**.** *Let $(R, \mathfrak{m})$ be a local ring, and let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. If $\mathfrak{a} \subseteq R$ is an ideal, then* $\operatorname{Ann}_R(0 :_E \mathfrak{a}) = \mathfrak{a}$*. If $R$ is complete and $N \subseteq E$ is a submodule, then $(0 :_E \operatorname{Ann}_R(N)) = N$.*

Proposition 1.6 has the following consequence:

**Lemma 1.7.** *Let $R$ be a complete local ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. For every ideal $\mathfrak{b} \subseteq R$ and every $t \in \mathbf{R}_{\geq 0}$, we have*

$$\bigl(0_E^{*\mathfrak{a}^t} :_E \mathfrak{b}\bigr) \supseteq \bigl(0 :_E \tau(\mathfrak{a}^t) \cdot \mathfrak{b}\bigr).$$

*Proof.* If $x \in (0 :_E \tau(\mathfrak{a}^t) \cdot \mathfrak{b})$, then $\mathfrak{b} \cdot x \subseteq (0 :_E \tau(\mathfrak{a}^t)) = 0_E^{*\mathfrak{a}^t}$ by Matlis duality (Proposition 1.6), hence $x \in (0_E^{*\mathfrak{a}^t} :_E \mathfrak{b})$. □

## 2. Briançon–Skoda

Our next goal is to prove an application of the machinery developed so far. We first prove a version of the Briançon–Skoda theorem involving test ideals.

**Theorem 2.1** (Skoda's theorem, cf. [HT04, Thms. 4.1 and 4.2])**.** *Let $R$ be a reduced $F$-finite ring or a complete local ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $R$ intersecting $R^\circ$. Let $\ell$ be the number of generators of an ideal $\mathfrak{c}$ such that $\overline{\mathfrak{a}} = \overline{\mathfrak{c}}$. Then, for every $t \in \mathbf{R}_{\geq 0}$, we have*

$$\tau(\mathfrak{a}^\ell \mathfrak{b}^t) = \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t) \cdot \mathfrak{a}.$$

*Proof.* The inclusion $\supseteq$ follows from Proposition 1.3(*i*), hence it suffices to show the reverse inclusion $\subseteq$. If $R$ is not complete local, we can reduce to the complete local case using Proposition 1.5.

By Matlis duality (Proposition 1.6), it suffices to show the chain of inclusions

$$0_E^{*\mathfrak{a}^\ell \mathfrak{b}^t} \supseteq \bigl(0_E^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} :_E \mathfrak{a}\bigr) \supseteq \bigl(0 :_E \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t) \cdot \mathfrak{a}\bigr).$$

The right inclusion holds by Lemma 1.7, hence it suffices to show the left inclusion.

By (a slight generalization of) the equality statement in Proposition 1.3(*ii*), we may assume that $\mathfrak{a}$ is generated by $\ell$ elements. Let $x \in (0_E^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} :_E \mathfrak{a})$, in which case there exists $c \in R^\circ$ such that

$$c\mathfrak{a}^{p^e(\ell-1)}\mathfrak{b}^{\lceil p^e t\rceil} \otimes \mathfrak{a}x = c\mathfrak{a}^{p^e(\ell-1)}\mathfrak{a}^{[p^e]}\mathfrak{b}^{\lceil p^e t\rceil} \otimes x = 0$$

in $F_*^e R \otimes_R E$ for all $e \gg 0$. Since $\mathfrak{a}$ is generated by $\ell$ elements, the pigeon-hole principle implies $\mathfrak{a}^{p^e\ell} = \mathfrak{a}^{p^e(\ell-1)}\mathfrak{a}^{[p^e]}$. Thus, we have $c\mathfrak{a}^{p^e\ell}\mathfrak{b}^{\lceil p^e t\rceil} \otimes x = 0$, and therefore $x \in 0_E^{*\mathfrak{a}^\ell\mathfrak{b}^t}$. □

We can now prove a version of the promised application.

**Corollary 2.2** (Modified Briançon–Skoda, cf. [HY03, Thm. 2.1 and Rem. 2.2])**.** *Let $R$, $\mathfrak{a}$, and $\ell$ be as in Theorem 2.1. Then, we have*

$$\tau(\mathfrak{a}^{n+\ell-1}) \subseteq \mathfrak{a}^n \tag{2}$$

*for all $n \geq 0$.*

*In particular, with $R$, $\mathfrak{a}$, and $\ell$ as above, if $R$ is strongly $F$-regular, then*

$$\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^n. \tag{3}$$

*Proof.* For the first inclusion, we have

$$\tau(\mathfrak{a}^{n+\ell-1}) = \tau(\mathfrak{a}^{n+\ell-2}) \cdot \mathfrak{a} = \cdots = \tau(\mathfrak{a}^{n+\ell-(n+1)}) \cdot \mathfrak{a}^n \subseteq \mathfrak{a}^n$$

by applying Theorem 2.1 $n$-times. The second inclusion follows from Propositions 1.3(*v*), 1.3(*i*), and 1.3(*ii*) and (2), since

$$\overline{\mathfrak{a}^{n+\ell-1}} = \tau(R) \cdot \overline{\mathfrak{a}^{n+\ell-1}} \subseteq \tau\bigl(\overline{\mathfrak{a}^{n+\ell-1}}\bigr) = \tau(\mathfrak{a}^{n+\ell-1}) \subseteq \mathfrak{a}^n.$$ □

*Remark* 2.3. The last inclusion (3) for smooth $\mathbf{C}$-algebras was first proved by Skoda and Briançon using analytic methods [SB74, Thm. 3], and was shown for all regular rings by Lipman and Sathaye [LS81, Thm. 1″]. The last inclusion (3) can also be obtained using the usual notion of tight closure, and in fact holds for *weakly* $F$-regular rings; see [HH90, Thm. 5.4].

## 3. Subadditivity and Symbolic Powers

For our second application, we first need some more technical results about test ideals.

**Proposition 3.1** (cf. [Tak06, Props. 2.1 and 2.2])**.** *Let $R$ be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $R$ intersecting $R^\circ$. Then, for every $t, s \in \mathbf{R}_{\geq 0}$, we have*

$$\big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}\big) \supseteq \big(0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)\big).$$

*Proof.* Let $x \in (0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t))$, in which case there exists $c \in R^\circ$ such that

$$c\mathfrak{b}^{\lceil p^e s\rceil} \otimes \tau(\mathfrak{a}^t)x = c\mathfrak{b}^{\lceil p^e s\rceil}\tau(\mathfrak{a}^t)^{[p^e]} \otimes x = 0$$

in $F^e_* R \otimes_R E$ for all $e \gg 0$. To show that $x \in (0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t})$, we want to show that for every element $r \in \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}$, we have $rx \in 0_E^{*\mathfrak{a}^t\mathfrak{b}^s}$. By definition of $\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}$, there exists $d \in R^\circ$ such that $d\mathfrak{a}^{\lceil p^e t\rceil}r^{p^e} \subseteq \tau(\mathfrak{a}^t)^{[p^e]}$ for all $e \gg 0$. Thus, we have

$$cd\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes rx = cd\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil}r^{p^e} \otimes x \subseteq c\mathfrak{b}^{\lceil p^e s\rceil}\tau(\mathfrak{a}^t)^{[p^e]} \otimes x = 0,$$

hence $x \in (0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t})$. □

**Theorem 3.2** (Subadditivity, cf. [HY03, Thm. 6.10(2); Tak06, Thm. 2.4])**.** *Let $R$ be a regular ring that is $F$-finite or complete local, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $R$ intersecting $R^\circ$. Then, for every $t, s \in \mathbf{R}_{\geq 0}$, we have*

$$\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s).$$

*Proof.* We note that if $R$ is not complete local, then we can reduce to the complete local case using Proposition 1.5.

We first claim that it suffices to show that

$$\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t} = R. \tag{4}$$

By Proposition 3.1 and Lemma 1.7, we have

$$0_E^{*\mathfrak{a}^t\mathfrak{b}^s} = \big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E R\big) = \big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}\big) \supseteq \big(0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)\big) \supseteq \big(0 :_E \tau(\mathfrak{a}^t)\cdot\tau(\mathfrak{b}^s)\big).$$

Taking annihilators, Matlis duality (Proposition 1.6) implies $\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t)\cdot\tau(\mathfrak{b}^s)$.

To show (4), it suffices to show that

$$\mathfrak{a}^{\lceil p^e t\rceil} \subseteq \tau(\mathfrak{a}^t)^{[p^e]}$$

for $e \gg 0$ by the definition of $\mathfrak{a}^t$-tight closure. By [HY03, Thms. 1.7(2) and 6.4], since $R$ is regular, 1 is a test element for $\mathfrak{a}^t$-tight closure. Thus, we have

$$0 = \mathfrak{a}^{\lceil p^e t\rceil} \otimes 0_E^{*\mathfrak{a}^t}$$

in $F^e_* R \otimes_R E$. By Matlis duality (Proposition 1.6), we then have

$$0 = \mathfrak{a}^{\lceil p^e t\rceil} \otimes \big(0 :_E \tau(\mathfrak{a}^t)\big) = \mathfrak{a}^{\lceil p^e t\rceil} \cdot \big(0 :_E \tau(\mathfrak{a}^t)^{[p^e]}\big). \tag{5}$$

Here, the second equality follows from the isomorphism

$$F^e_* R \otimes_R E \simeq E$$

coming from the fact that $R \to F^e_* R$ is flat [Kun69, Thm. 2.1] and the description of $E$ as $H^d_{\mathfrak{m}}(R)$. Under this identification, we have $F^e_* R \otimes (0 :_E \tau(\mathfrak{a}^t)) \simeq (0 :_E \tau(\mathfrak{a}^t)^{[p^e]})$. Finally, (5) implies

$$\mathfrak{a}^{\lceil p^e t\rceil} \subseteq \operatorname{Ann}_R\big(0 :_E \tau(\mathfrak{a}^t)^{[p^e]}\big) = \tau(\mathfrak{a}^t)^{[p^e]}$$

by Matlis duality (Proposition 1.6). □

We now give the second application of the theory.

**Theorem 3.3** (cf. [HH02, Thm. 1.1($a$)])**.** *Let $R$ be a regular ring with reduced formal fibers, let $\mathfrak{a} \subseteq R$ be an ideal intersecting $R^\circ$, and let $h$ be the maximal height of the associated primes of $\mathfrak{a}$. Then, $\mathfrak{a}^{(hn+kn)} \subseteq (\mathfrak{a}^{(k+1)})^n$ for all integers $n \geq 1$ and $k \geq 0$. In particular, $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

*Proof.* By replacing $R$ with $R[X]$ for an indeterminate $X$, we may assume that the residue fields of $R$ are infinite [HH02, Discussion 2.3]. The proof below is written for $F$-finite regular rings $R$, although if $R$ is not $F$-finite, then one may reduce to the case where $R$ is complete local using the assumption on formal fibers and the strategy in [MS18, Thm. 7.4].

Since $R$ is regular, it is also strongly $F$-regular, hence Proposition 1.3($v$) and subadditivity (Theorem 3.2) imply

$$\mathfrak{a}^{(hn+kn)} \subseteq \tau(\mathfrak{a}^{(hn+kn)}) \subseteq \Big(\tau\big((\mathfrak{a}^{(hn+kn)})^{1/n}\big)\Big)^n$$

for all $n \geq 0$. It therefore suffices to show $\tau((\mathfrak{a}^{(hn+kn)}R_\mathfrak{p})^{1/n}) \subseteq \mathfrak{a}^{k+1}R_\mathfrak{p}$. We have

$$\tau((\mathfrak{a}^{(hn+kn)}R_\mathfrak{p})^{1/n}) = \tau\big((\mathfrak{a}^{hn+kn}R_\mathfrak{p})^{1/n}\big) = \tau(\mathfrak{a}^{h+k}R_\mathfrak{p}) \subseteq \mathfrak{a}^{k+1}R_\mathfrak{p}$$

by the definition of symbolic powers, Proposition 1.3($iv$), and (2) in the modified Briançon–Skoda theorem (Corollary 2.2), where we use the infinite residue field to use [HS06, Prop. 8.3.7]. □

*Remark* 3.4. The proof here follows [Har05, Thm. 2.21]. This uniform containment theorem was originally proved by Ein–Lazarsfeld–Smith for smooth **C**-algebras using multiplier ideals [ELS01, Thm. A]. Hochster–Huneke generalized their result to regular rings containing a field using tight closure and reduction modulo $p$ [HH02, Thm. 1.1($a$)]. The mixed characteristic case (with the addititional assumption on formal fibers) was not proved until recently by Ma–Schwede using perfectoid techniques [MS18, Thm. 7.4].

## References

[ELS01] L. Ein, R. Lazarsfeld, and K. E. Smith. "Uniform bounds and symbolic powers on smooth varieties." *Invent. Math.* 144.2 (2001), pp. 241–252. DOI: 10.1007/s002220100121. MR: 1826369. 1, 6

[Har05] N. Hara. "A characteristic $p$ analog of multiplier ideals and applications." *Comm. Algebra* 33.10 (2005), pp. 3375–3388. DOI: 10.1080/AGB-200060022. MR: 2175438. 6

[HH90] M. Hochster and C. Huneke. "Tight closure, invariant theory, and the Briançon-Skoda theorem." *J. Amer. Math. Soc.* 3.1 (1990), pp. 31–116. DOI: 10.2307/1990984. MR: 1017784. 1, 2, 4

[HH02] M. Hochster and C. Huneke. "Comparison of symbolic and ordinary powers of ideals." *Invent. Math.* 147.2 (2002), pp. 349–369. DOI: 10.1007/s002220100176. MR: 1881923. 1, 6

[Hoc07] M. Hochster. *Foundations of tight closure theory.* Lecture notes from a course taught at the University of Michigan, Fall 2007. URL: http://www.math.lsa.umich.edu/~hochster/711F07/fndtc.pdf. 3

[Hoc14] M. Hochster. *Lectures on integral closure, the Briançon–Skoda theorem and related topics in commutative algebra.* Lecture notes from a course taught at the University of Michigan, Winter 2014. URL: http://www.math.lsa.umich.edu/~hochster/615W14/615.pdf 1

[HS06] C. Huneke and I. Swanson. *Integral closure of ideals, rings, and modules.* London Math. Soc. Lecture Note Ser., Vol. 336. Cambridge: Cambridge Univ. Press, 2006. URL: http://people.reed.edu/~iswanson/book/SwansonHuneke.pdf. MR: 2266432. 3, 6

[HT04] N. Hara and S. Takagi. "On a generalization of test ideals." *Nagoya Math. J.* 175 (2004), pp. 59–74. DOI: 10.1017/S0027763000008904. MR: 2085311. 1, 2, 3, 4

[HY03] N. Hara and K. Yoshida. "A generalization of tight closure and multiplier ideals." *Trans. Amer. Math. Soc.* 355.8 (2003), pp. 3143–3174. DOI: 10.1090/S0002-9947-03-03285-9. MR: 1974679. 1, 2, 4, 5

[Kun69] E. Kunz. "Characterizations of regular local rings for characteristic $p$." *Amer. J. Math.* 91 (1969), pp. 772–784. DOI: 10.2307/2373351. MR: 0252389. 5

[LS81] J. Lipman and A. Sathaye. "Jacobian ideals and a theorem of Briançon–Skoda." *Michigan Math. J.* 28.2 (1981), pp. 199–222. DOI: 10.1307/mmj/1029002510. MR: 0616270. 1, 4

[LS01] G. Lyubeznik and K. E. Smith. "On the commutation of the test ideal with localization and completion." *Trans. Amer. Math. Soc.* 353.8 (2001), pp. 3149–3180. DOI: 10.1090/S0002-9947-01-02643-5. MR: 1828602. 2, 3

[MS18] L. Ma and K. Schwede. "Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers." *Invent. Math.* 214.2 (2018), pp. 913–955. DOI: 10.1007/s00222-018-0813-1. MR: 3867632. 1, 6

[SB74] H. Skoda and J. Briançon. "Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbf{C}^n$." *C. R. Acad. Sci. Paris Sér. A* 278 (1974), pp. 949–951. URL: https://gallica.bnf.fr/ark:/12148/bpt6k6236817d/f315.item. MR: 0340642. 1, 4

[Sch10] K. Schwede. "Centers of $F$-purity." *Math. Z.* 265.3 (2010), pp. 687–714. DOI: 10.1007/s00209-009-0536-5. MR: 2644316. 2

[ST12] K. Schwede and K. Tucker. "A survey of test ideals." *Progress in commutative algebra 2*. Berlin: Walter de Gruyter, 2012, pp. 39–99. DOI: 10.1515/9783110278606.39. MR: 2932591. 1

[Tak04] S. Takagi. "An interpretation of multiplier ideals via tight closure." *J. Algebraic Geom.* 13.2 (2004), pp. 393–415. DOI: 10.1090/S1056-3911-03-00366-7. MR: 2275023. 2

[Tak06] S. Takagi. "Formulas for multiplier ideals on singular varieties." *Amer. J. Math.* 128.6 (2006), pp. 1345–1362. DOI: 10.1353/ajm.2006.0049. MR: 2047704. 5

[TW18] S. Takagi and K.-i. Watanabe. "$F$-singularities: applications of characteristic $p$ methods to singularity theory." Translated from the Japanese by the authors. *Sugaku Expositions* 31.1 (2018), pp. 1–42. DOI: 10.1090/suga/427. MR: 3784697. 1

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# Multiplier ideals (September 27, Devlin Mallory)

Everything today is over $\mathbb{C}$. If $D$ is a $\mathbb{Q}$-divisor, we write $[D]$ for the round-down of $D$. We'll discuss only the case of an ambient smooth variety; the definition can be modified to work in the context of a pair $(X, D)$ with $X$ normal and $K_X + D$ $\mathbb{Q}$-Cartier, and many of the nice formal properties carry over, but some important ones do not.

## Definitions and examples

We'll actually begin by giving a "sheafy" definition, before restricting to the affine case for some examples.

**Definition.** Let $X$ be a smooth variety and $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor. The multiplier ideal sheaf $\mathcal{J}(D)$ is defined as follows: if $f : Y \to X$ is a log resolution of $D$, then

$$\mathcal{J}(D) := f_*\mathcal{O}_Y(K_{Y/X} - [D]).$$

Similarly, if $\mathfrak{a} \subset \mathcal{O}_X$ is an ideal sheaf and $t \in \mathbb{R}_{\geq 0}$, we define the multiplier ideal sheaf $\mathcal{J}(\mathfrak{a}^t)$ as follows: if $f : Y \to X$ is a log resolution of $\mathfrak{a}$, with $A$ the divisor such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$, then

$$\mathcal{J}(\mathfrak{a}^t) := f_*\mathcal{O}_Y(K_{Y/X} - [tA])$$

**Remark.** Thankfully, it's easy to verify that if $t \in \mathbb{N}$, then $\mathcal{J}(\mathfrak{a}^t)$ is the same as $\mathcal{J}((\mathfrak{a}^t)^1)$.

**Remark.** This definition leaves several things in need of verification: first, it's not at all obvious that these are even well-defined, since they depended on our choice of a log resolution; perhaps even more troublingly, why are these multiplier *ideals*? This at least we can answer immediately: recall that since for any resolution $Y \to X$ we have that $f_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$. Since $f_*$ is left-exact, we have immediately that $f_*\mathcal{O}_Y(K_{Y/X} - [tA]) \subset \mathcal{O}_X$, so it's at least a genuine ideal sheaf.

**Remark.** Note that if $A$ is any integral divisor, then $\mathcal{J}(X, A) = \mathcal{O}_X(-A)$: in this case given any log resolution $f$ of $A$ we have $[f^*A] = f^*A$, and thus applying the projection formula to the definition and using that $f_*\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$ we get simply $\mathcal{O}_X(-A)$. Thus, multiplier ideals are interesting because of the interaction of the rounding operation with the pullback of divisors.

**Remark.** Similarly, it's not hard to see that at least $\mathfrak{a} \subset \mathcal{J}(\mathfrak{a})$.

**Remark.** How do we actually calculate these ideal sheaves? Let's say we want to calculate $\mathcal{J}(X, tD)$. Since these are sheaves they can be computed locally, so assume $X = \operatorname{Spec} R$ is affine and $f : Y \to X$ is a log resolution of $(X, D)$ (note that $Y$ will almost never be affine in this scenario!). We can write

$$K_{Y/X} = \sum k_i E_i, \quad f^*D = \sum a_i E_i, \quad a_i \geq 0$$

for some divisors $E_i$ on $Y$. (These are not all exceptional, since they will include the strict transforms of components of $D$!) Then obviously

$$K_{Y/X} - [tD] = \sum (k_i - [ta_i]) E_i.$$

By definition,

$$f_*\mathcal{O}_Y(K_{Y/X} - [tf^*D]) = \mathcal{O}_Y(K_{Y/X} - [tf^*D]) \subset \mathcal{O}_Y(K_{Y/X})$$

consists of the rational functions $\varphi$ on $X$ (i.e., elements of $\operatorname{Spec} R$) such that

$$\operatorname{div}_Y(\varphi) + K_{Y/X} - [tf^*D] \geq 0,$$

i.e., such that for all divisors $E$ on $Y$ we have

$$\operatorname{ord}_E(\varphi) + \operatorname{ord}_E(K_{Y/X} - [tf^*D]) = \operatorname{ord}_E(\varphi) + \operatorname{ord}_E\left(\sum \big(k_i - [ta_i]\big) E_i\right) \geq 0.$$

In words, we simply take a rational function on $X$, reconsider it as a rational function on $Y$ via pullback, examine the order of vanishing along divisors on $Y$, and compare it to the "known" divisor $K_{Y/X} - [tf^*D]$.

**Example.** Let $D = L_1 + L_2 \subset \mathbb{A}^2$, where $L_1, L_2$ are distinct lines through the origin, say $V(x), V(y)$. A log resolution of $D$ is given by blowing up the origin of $\mathbb{A}^2$; call this blowup $f : Y = \mathrm{Bl}_p \mathbb{A}^2 \to X = \mathbb{A}^2$. We know $K_{Y/X} = E$, and clearly $f^*(tD) = t(\widetilde{L}_1 + \widetilde{L}_2 + 2E)$, since $D$ has multiplicity 2 at the origin. Thus we have that

$$[f^*(tD)] = [t]\widetilde{L}_1 + [t]\widetilde{L}_2 + [2t]E$$

and thus

$$K_{Y/X} - [f^*(tD)] = (1 - [2t])E - [t]\widetilde{L}_1 - [t]\widetilde{L}_2.$$

Now, let $f \in k[x, y]$ be a function on $\mathbb{A}^2$. We can write $f = x^a y^b f_0$, with $x, y \nmid f_0$. It's easily seen that $\mathrm{div}_Y(f) = a\widetilde{L}_1 + b\widetilde{L}_2 + (\mathrm{mult}_0 f)E$. Thus we have $f \in \mathcal{J}(X, tD)$ if and only if

$$\begin{aligned} 0 &\leq a\widetilde{L}_1 + b\widetilde{L}_2 + (\mathrm{mult}_0 f)E + (1 - [2t])E - [t]\widetilde{L}_1 - [t]\widetilde{L}_2 \\ &= (1 - [2t] + \mathrm{mult}_0 f)E + (a - [t])\widetilde{L}_1 + (b - [t])\widetilde{L}_2 \end{aligned}$$

That is, exactly when $a, b \geq [t]$ and $\mathrm{mult}_0 f + 1 \geq [2t]$; since $\mathrm{mult}_0 f = a + b + \mathrm{mult}_0 f_0$, this latter condition follows from the first two. Thus, we have that $\mathcal{J}(\mathbb{A}^2, tD) = (xy)^{[t]}$, i.e., $\mathcal{J}(\mathbb{A}^2, tD) = k[x, y]$ for $t < 1$, $\mathcal{J}(X, tD) = (xy)$ for $1 \leq t < 2$, etc.

**Example.** One can use the numerical data of the log resolution of $(\mathbb{A}^2, D = V(x^2 - y^3))$ given on page 13, where we say that $K_{Y/X} = E_1 + 2E_2 + 4E_3$ and $\sigma^* D = 2E_1 + 3E_2 + 6E_3$ to see that

$$\mathcal{J}(\mathbb{A}^2, tD) = \begin{cases} (1) & 0 \leq t < 5/6, \\ (x, y) & 5/6 \leq t < 1, \\ (x^2 - y^3) & 1 \leq t < 11/6, \\ (x, y) \cdot (x^2 - y^3) & 11/6 \leq t < 2. \end{cases}.$$

Thus, we see that the multiplier ideals can differentiate between the singularities of the cusp and the node in a way that multiplicity can't: the multiplier ideal of the cusp becomes nontrivial at $t = 5/6$, while that of the node becomes nontrivial at $t = 1$. We'll give a formal definition of this shortly.

**Example.** There is some connection of multiplier ideals with multiplicity, though: if $D$ is an effective $\mathbb{Q}$-divisor on $X$ (assumed smooth, as always), and $\mathrm{mult}_x D \geq e + p - 1$ for $x \in X$ (where the multiplicity of a $\mathbb{Q}$-divisor is defined by $\mathbb{Q}$-linearity from the definition for a Cartier divisor), then it's easy to check that $\mathcal{J}(D)_x \subset m_x^e \mathcal{O}_{X,x}$ (to see this, construct a log resolution of $(X, D)$ by first blowing up $x \in X$ and then resolving further). That is, high multiplicity at a point of $D$ forces nontriviality of the multiplier ideal of $(X, D)$.

**Remark.** As we've seen above, for small $t$, $\mathcal{J}(X, tD)$ is trivial, and it becomes nontrivial as we increase $t$. Moreover, it appears to be constant on intervals of the form $[a, b)$. Both of these observations follow in general very easily: As above, let $f : Y \to X$ be a log resolution of $D$, with

$$K_{Y/X} = \sum k_i E_i, \quad f^* D = \sum a_i D_i.$$

Note since $X$ is smooth (hence Gorenstein and terminal) we have the $k_i$ are positive integers. Since $\mathcal{J}(X, tD) = f_* \mathcal{O}_Y(K_{Y/X} - [tf^*D]) = f_* \mathcal{O}_Y\left(\sum(k_i - [ta_i])E_i\right)$, we have

$$1 \in \mathcal{J}(X, tD)$$

exactly when

$$\underbrace{\mathrm{div}(1)}_{0} + \sum(k_i - [ta_i])E_i \geq 0.$$

That is, $\mathcal{J}(X, tD)$ *fails* to be trivial exactly when some

$$k_i - [ta_i] < 0,$$

i.e., when

$$k_i < [ta_i],$$

and thus $\mathcal{J}(X, tD)$ becomes nontrivial exactly when

$$\frac{k_i + 1}{a_i} \le t$$

for some $i$. Thus, the smallest $t$ such that $\mathcal{J}(X, tD)$ is nontrivial is just

$$\min\left(\frac{k_i + 1}{a_i}\right)$$

for $k_i, a_i$ arising from some resolution of singularities.

Note, moreover, that

$$a_{E_i}(X, tD) := \mathrm{ord}_{E_i}(K_Y - f^*(K_X + tD)) = \mathrm{ord}_{E_i}(K_{Y/X} - tf^*(D));$$

thus, if $t = \min((k_i + 1)/a_i)$, then $\min(a_{E_i}(X, tD)) \ge -1$ for all $i$, so $(X, tD)$ is log canonical, while if $t > \min((k_i + 1)/a_i)$ then the discrepancy is $-\infty$, so $(X, tD)$ is not log canonical.

**Definition.** We define the log canonical threshold of a pair $(X, D)$ as

$$\max(t : (X, tD) \text{ is log canonical});$$

note that this is a rational number since it can be calculated on a single log resolution.

What we've just seen is that

$$\mathrm{lct}(X, D) = \min(t : \mathcal{J}(X, tD) \ne \mathcal{O}_X).$$

Using this as inspiration, we can similarly a local version of the log canonical threshold via

$$\mathrm{lct}_x(X, D) = \min(t : \mathcal{J}(X, tD)_x \ne \mathcal{O}_{X,x})$$

(this corresponds to considering only divisors centered over $x$ in our calculation of the minimal log discrepancies).

Moreover, the multiplier ideals allow us to give a scheme structure to the points where $(X, D)$ is not log canonical, and shows that by definition these form a closed set.

**Remark.** In fact, our above analysis showed more: not only was $\mathrm{lct}(X, D)$ of the form $(k_i + 1)/a_i$ for $k_i, a_i$ arising from numerical data of a single resolution of singularities, but since $\mathcal{J}(X, tD) = f_*\mathcal{O}_Y((k_i - [ta_i])E_i)$, we have that $\mathcal{J}(X, tD)$ is constant if $t$ increases very slightly; moreoveor, $\mathcal{J}(X, tD)$ can only change at rational numbers of the form $(k_i + m)/a_i$. The numbers where $\mathcal{J}(X, tD)$ *does* change are called the jumping numbers of $(X, D)$, and can be thought of as a whole family of invariants generalizing the log canonical threshold. Moreover, these relate to other perspectives on singularities; for example, when $X = \mathbb{A}^n$, the (negatives of) jumping numbers in $(0, 1]$ will be roots of the Bernstein–Sato polynomial of $(\mathbb{A}^n, D)$, a subtle invariant of the singularities of a hypersurface arising from the study of $D$-modules.

**Remark.** One reason we need only consider the jumping numbers in the interval $(0, 1]$ above is that all jumping numbers are determined by those lying in $[0, 1]$; to see this, we note that if $D$ is effective then

$$\mathcal{J}(X, (1 + t)D) = \mathcal{J}(X, tD)(-D).$$

**Remark.** More generally, it's easily seen via the projection formula that if $A$ is an integral divisor and $D$ a $\mathbb{Q}$-divisor then

$$\mathcal{J}(X, D + A) = \mathcal{J}(X, D)(-A).$$

**Example.** This may seem to be a somewhat unmotivated definition (an important similarity with test ideals). Here we sketch a brief example of how they might naturally arise: First, we recall a vanishing theorem:

**Theorem** (Kawamata–Viehwig vanishing)**.** *Let $X$ be a smooth projective complex variety, $L$ an integer divisor and $D$ an snc $\mathbb{Q}$-divisor with $L - D$ big and nef (positivity properties giving a "birational" analogue of ampleness). Then*

$$H^i(X, \mathcal{O}_X(K_X + L - [D])) = 0, \quad i > 0.$$

Now, say $X$ is a smooth projective variety, $L$ is an integer divisor, $D$ an effective $\mathbb{Q}$-divisor with $L - D$ big and nef, but such that $D$ isn't snc. We want to obtain a Kawamata–Viehwig type statement

$$H^i(X, \mathcal{O}_X(K_X + L - [D])) = 0, \quad i > 0.$$

We want to apply the above vanishing theorem, so let $f : Y \to X$ be a log resolution of $D$; $f^*(L-D)$ will be big and nef still. We can then apply Kawamata–Viehwig vanishing to obtain

$$H^i(Y, \mathcal{O}_Y(K_Y + f^*L - [f^*D])) = 0, \quad i > 0.$$

The Leray–Serre spectral sequence is

$$H^i\big(X, R^j f_*\mathcal{O}_Y(K_Y + f^*L - [f^*D])\big) \implies H^{i+j}\big(Y, \mathcal{O}_Y(K_Y + f^*L - [f^*D])\big),$$

and we know the right side is 0 for $i + j > 0$.

Note that we can rewrite

$$K_Y + f^*L - [f^*D] = f^*(K_X + L) + K_{Y/X} - [f^*D],$$

so that (using the projection formula)

$$\begin{aligned} R^j f_*\mathcal{O}_Y(K_Y + f^*L - [f^*D]) &= R^j f_*\mathcal{O}_Y(f^*(K_X + L) + K_{Y/X} - [f^*D]) \\ &= \mathcal{O}_X(K_X + L) \otimes R^j f_*(\mathcal{O}_Y(K_{Y/X} - [f^*D])). \end{aligned}$$

For $j = 0$ this is exactly

$$\mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D),$$

while one can show that $R^j f_*(\mathcal{O}_Y(K_{Y/X} - [f^*D])) = 0$ for $j > 0$ (see the later statement of local vanishing). Altogether, this gives a vanishing statement, not quite for the higher cohomology of $\mathcal{O}_X(K_X + L - [D])$, but for that of $\mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)$. Thus, multiplier ideals arise naturally as "correction terms" when one wants to apply vanishing theorems that work for snc divisors to non-snc divisors.

**Example** (monomial ideals)**.** Let $\mathfrak{a} \subset k[x_1, \ldots, x_n]$ be a monomial ideal. There is a nice combinatorial description of $\mathcal{J}(\mathfrak{a}^t)$ in this case, due to Howald: we define the newton polyhedron $P = P(\mathfrak{a})$ to be the convex hull of the exponent vectors in $\mathbb{R}^n$ representing monomials in $\mathfrak{a}$. We write $t \cdot P$ for the scaling of $P$ by $t$, and write $\int(tP)$ for the topological interior of $tP$ as a subset of Euclidean space. Write $\mathbf{1}$ for the vector $(1, \ldots, 1)$.

**Theorem.** *$\mathcal{J}(a^t)$ is the monomial ideal generated by monomials $x^v$ with $v + 1 \in \operatorname{int}(P(\mathfrak{a}^t))$.*

**Example.** Let $\mathfrak{a} = (x^2, y^3)$. Examining the Newton polyhedron it's clear that $\mathcal{J}(\mathfrak{a}^t) = (1)$ for $t < 5/6$ and that $\mathcal{J}(\mathfrak{a}^t) = (x, y)$ for $5/6 \le t < 7/6$. This seems to resemble the multiplier ideal of $x^2 + y^3$, at least for small $t$.

**Remark.** In fact, this resemblance can be generalized to suitably general hypersurfaces in $\mathbb{C}^n$. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$, write $D$ for the hypersurface defined by $f$, and let $\mathfrak{a} = \mathfrak{a}_f$ be the term ideal of $f$, i.e., the ideal generated by monomials appearing with nonzero coefficient in $f$. Let $P = P(\mathfrak{a}_f)$, which we refer to as the Newton polyhedron of $f$. We'll always have that $\mathcal{J}(\mathfrak{a}^t_f) \subset \mathcal{J}(tD_f)$ for $0 < t < 1$; equality will hold when the coefficients of $f$ are "suitably" general. To be more precise: Given any face $\sigma$ of $P(\mathfrak{a}_f)$, we write $f_\sigma$ for the sum of terms of $f$ whose exponent vector occur in $\sigma$. We say $f$ is *nondegenerate with respect to its Newton polyhedron* if for every face $\sigma$ of $P(\mathfrak{a}_f)$ the 1-form $d(f_\sigma)$ vanishes only along the coordinate hyperplanes, i.e., $d(f_\sigma)$ is nonvanishing on the torus $(\mathbb{C}^\times)^n$.

**Theorem.** *If $f$ is nondegenerate with respect to its Newton polyhedron and $0 < t < 1$, then*

$$\mathcal{J}(\mathfrak{a}_f^t) = \mathcal{J}(tD_f).$$

**Example.** Consider $f = x^2 + y^3$. The Newton polyhedron has three faces, with corresponding $f_\sigma$'s $x^2 + y^3$, $x^2$, $y^3$. The 1-forms are then $2x\,dx + 3y^2\,dy$, $2x\,dx$, $3y^2\,dy$, and each of these can vanish only along the $x$- and $y$-axes.

Similarly, any "diagonal" polynomial $x_1^{a_1} + \cdots + x_n^{a_n}$ will be nondegenerate with respect to its Newton polyhedron, and thus to calculate the multiplier ideals and log canonical thresholds of these hypersurfaces we can work with the monomial ideal

$$(x_1^{a_1}, \ldots, x_n^{a_n}),$$

which are easy to calculate.

## Briançon–Skoda

Similarly to the story with test ideals, we want to use multiplier ideals to give a proof of Briançon–Skoda in the complex case. Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf of a smooth variety $X$ of dimension $n$, or equivalently think of $\mathfrak{a} \subset R$ an ideal of a regular ring of characteristic 0.

**Theorem** (Briançon–Skoda)**.** *If $m \geq n$, then*

$$\overline{\mathfrak{a}^m} \subset \mathfrak{a}^{m+1-n}.$$

We will deduce this from the following:

**Theorem** (Skoda's theorem)**.** *If $m \geq n$ is an integer, then $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1})$, and thus $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}^{m-n+1}\mathcal{J}(\mathfrak{a}^{n-1})$.*

The interesting inclusion is $\mathcal{J}(\mathfrak{a}^m) \subset \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1})$, sine $\mathfrak{a}\mathcal{J}(\mathfrak{a}^{m-1}) \subset \mathcal{J}(\mathfrak{a}^m)$ is immediate from definition.

**Remark.** In fact, Skoda's theorem, and the proof we give, can be easily extended to show the inclusion $\mathcal{J}(\mathfrak{a}^m\mathfrak{b}^c) = \mathfrak{a}^{m-n+1}\mathcal{J}(\mathfrak{a}^{n-1}\mathfrak{b}^c)$, for $\mathfrak{b}$ another ideal sheaf and $c$ an arbitrary rational number.

Our proof of Skoda's theorem requires a brief discussion of integral closures from the perspective of birational geometry. For now, let $X$ be a normal algebraic variety. The following can be thought of as a birational version of the valuative criterion for membership in the integral closure:

**Definition.** Let $v : X^+ \to X$ be the normalized blowup of $X$ along $\mathfrak{a}$, with exceptional divisor $E$, so $a \cdot \mathcal{O}_{X^+} = \mathcal{O}_{X^+}(-E)$. The integral closure of $\overline{a}$ is $v_*(\mathcal{O}_{X^+}(-E))$.

The following result is an immediate consequence of the universal property of blowing up:

**Lemma.** *If $\mathfrak{a} \subset \mathcal{O}_X$, and if $f : Y \to X$ is a proper birational morphism of normal varieties such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$ with $A$ a Cartier divisor, then $f_*\mathcal{O}_Y(-A) = \overline{a}$.*

**Lemma.** *If $f : Y \to X$ is a proper birational morphism of normal varieties, and $D$ is an effective Cartier divisor on $Y$, then $\mathfrak{a} := f_*\mathcal{O}_Y(-D) \subset \mathcal{O}_X$ is integrally closed.*

*Proof.* Replacing $Y$ be a further birational modification $g : Y' \to Y$ such that $\mathfrak{a} \cdot \mathcal{O}'_Y = \mathcal{O}_{Y'}(-D-E)$ (i.e., taking a log resolution of $\mathfrak{a} \cdot \mathcal{O}_Y$), we may assume that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D-E)$. Then by the preceding lemma we have $\bar{a} = f_*(\mathfrak{a} \cdot \mathcal{O}_Y)$ □

We then have:

**Corollary.** *$\mathcal{J}(X, \mathfrak{a}^t)$ is integrally closed, and thus $\bar{\mathfrak{a}} \subset \mathcal{J}(X, \mathfrak{a})$*

*Briançon–Skoda from Skoda's theorem.* Say $m \geq n$. We have that $\overline{\mathfrak{a}^m} \subset \mathcal{J}(\mathfrak{a}^m)$ by the preceding corollary; Skoda's theorem then says that $\mathcal{J}(\mathfrak{a}^m) = a^{m-n+1}\mathcal{J}(\mathfrak{a}^{n-1}) \subset a^{m-n+1}$, and thus $\overline{a^m} \subset a^{m+n-1}$. □

The goal, then, is to prove Skoda's theorem, which is done by constructing the Skoda complexes and showing their exactness using cohomological properties of multiplier ideals. The key cohomological input is the following:

**Theorem** (local vanishing)**.** *Let $\mathfrak{a} \subset \mathcal{O}_X$, and choose a log resolution $Y \to X$ of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$. Then*

$$R^j f_* \mathcal{O}_Y(K_{Y/X} - [tA]) = 0$$

*for $j > 0$.*

The analogous statement holds for the case of a divisor $D$ in place of $\mathfrak{a}$.

When $j = 0$, of course, this is simply the definition of multiplier ideals. We omit the proof; the case when $Y, X$ are projective uses the Kawamata–Viehwig vanishing theorem, while the general case proceeds from this case using a suitably chosen projective compactification and the Leray–Serre spectral sequence for a further resolution of the compactification.

*Proof of Skoda's theorem.* We will actually prove something slightly stronger: we will show that for any reduction $\mathfrak{r}$ of $\mathfrak{a}$ that we have $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{r}\mathcal{J}(\mathfrak{a}^{m-1})$, which will imply our result (since $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{r}\mathcal{J}(\mathfrak{a}^{m-1}) \subset \mathfrak{a}\mathcal{J}(\mathfrak{a}^{m-1}) \subset \mathcal{J}(\mathfrak{a}^m)$, so everything is an inclusion). Recall that $\mathfrak{r} \subset \mathfrak{a}$ is a reduction if $\overline{\mathfrak{r}} = \overline{\mathfrak{a}}$.

So, say $\mathfrak{r}$ is a reduction of $\mathfrak{a}$, and that $s_1, \ldots, s_r$ generate $\mathfrak{r}$. Let $V$ the vector space with basis the $s_i$. Let $f : Y \to X$ be a log resolution of $\mathfrak{a}$, with $\mathcal{O}_Y(-A) = \mathfrak{a} \cdot \mathcal{O}_Y$. It's clear by definition that the $S_i := f^* s_i$ generate $\mathcal{O}_Y(-A)$. Take the Koszul complex on the $S_i$, yielding a complex

$$\Lambda^r V \otimes \mathcal{O}_Y(rA) \to \ldots \Lambda^2 V \otimes \mathcal{O}_Y(2A) \to V \otimes \mathcal{O}_Y(A) \to \mathcal{O}_Y \to 0$$

Twisting by $\mathcal{O}_Y(K_{Y/X} - mA)$ (which is a line bundle), we get

$$\begin{aligned} \Lambda^r V \otimes \mathcal{O}_Y(K_{Y/X} - (m-r)A) \to \cdots \to \Lambda^2 V \otimes \mathcal{O}_Y(K_{Y/X} - (m-2)A) \\ \to V \otimes \mathcal{O}_Y(K_{Y/X} - (m-1)A) \to \mathcal{O}_Y(K_{Y/X} - mA) \to 0. \end{aligned}$$

Finally, if we apply the pushforward $f_*$, we get the $m$-th Skoda complex

$$\begin{aligned} \Lambda^r V \otimes \underbrace{f_* \mathcal{O}_Y(K_{Y/X} - (m-r)A)}_{\mathcal{J}(\mathfrak{a}^{m-r})} \to \cdots \to \Lambda^2 V \otimes \underbrace{f_* \mathcal{O}_Y(K_{Y/X} - (m-2)A)}_{\mathcal{J}(\mathfrak{a}^{m-2})} \\ \to V \otimes \underbrace{f_* \mathcal{O}_Y(K_{Y/X} - (m-1)A)}_{\mathcal{J}(\mathfrak{a}^{m-1})} \to \underbrace{f_* \mathcal{O}_Y(K_{Y/X} - mA)}_{\mathcal{J}(\mathfrak{a}^{m})} \to 0. \end{aligned}$$

That is, we obtain a complex

$$(\mathrm{Skod}_m^{\mathfrak{r}}) : \quad \Lambda^r V \otimes \mathcal{J}(\mathfrak{a}^{m-r}) \to \Lambda^2 V \otimes \mathcal{J}(\mathfrak{a}^{m-2}) \to \Lambda^1 V \otimes \mathcal{J}(\mathfrak{a}^{m-1}) \to \mathcal{J}(\mathfrak{a}^m) \to 0;$$

the maps are given by multiplication by the $s_i$, so in particular the image of the rightmost map is

$$\tau \mathcal{J}(\mathfrak{a}^{m-1}) \subset \mathcal{J}(\mathfrak{a}^m).$$

We want equality here, which is to say we want exactness at the rightmost term. In fact, we have:

*Claim*: If $s \leq m$, then $\mathrm{Skod}_m^{\mathfrak{r}}$ is exact.

This isn't hard at all, though: there's a spectral sequence computing the cohomology of $\mathrm{Skod}_m^{\mathfrak{r}}$, and the vanishing of the higher pushforwards

$$R^i f_* \mathcal{O}_Y(K_{Y/X} - dA) = 0$$

for $d \geq 0$ implies that the cohomology of the complex is trivial (as long as $m - s \geq 0$).

This then implies Skoda's theorem: by standard results on reductions, locally there's a reduction of $\mathfrak{a}$ generated by $\leq n = \dim X$ elements, and thus we have $s \leq n \leq m$, so the result follows. □

# The multiplier ideal is a universal test ideal (November 15, Eric Canton)

Our goal is to prove the following informal claim: "If $(R, m)$ is local, $\mathbb{Q}$-Gorenstein, normal, and essentially of finite type over a field $k$ of characteristic 0, then $\mathcal{J}(R)$ mod $p = \tau(R \text{ mod } p)$ for all $p \gg 0$." To formalize this, we need some preliminary discussion about the following:

(1) Reduction to positive characteristic.

(2) Canonical covers.

(3) Discuss concepts of canonical modules in commutative algebra and algebraic geometry.

## Reduction to positive characteristic

We start with a ring $R$ essentially of finite type over a field $k$ of characteristic 0. We want to construct a (essentially) finite type $\mathbb{Z}$-algebra $A \subset k$, and a subring $R_A \subset R$ essentially of finite type over $A$, such that $R_A \otimes_A k \cong R$. Once we've constructed these objects, given $\mu$ a maximal ideal of $A$, $\mathbb{Z} \to A/\mu$ must factor through $\mathbb{Z}/p\mathbb{Z}$ for some $p$, where $(p) = \mu \cap \mathbb{Z}$. In particular, $A/\mu \otimes_A R_A$ is a characteristic-$p$ "model" for $R$, denoted $R_p$ or $R_\mu$.

We'll discuss only the case of localizations of affine algebras:

**Example.** (1) If $R = \mathbb{Q}[x_1, \ldots, x_n]$, just take $A = \mathbb{Z}$ and $R_A = \mathbb{Z}[x_1, \ldots, x_n]$, so $R_p = \mathbb{F}_p[x_1, \ldots, x_n]$.

(2) If $R = \mathbb{C}[x, y]/(\pi x - \sqrt{2} y^5)$, we take $A = \mathbb{Z}[\varpi, t]/(t^2 - 2)$, with $\varpi$ viewed as a formal variable. Clearly $A \to \mathbb{C}$ via $t \mapsto \sqrt{2}$ and $\pi \mapsto \varpi$. We then take $R_A = A[x, y]/(\varpi x - t y^5)$.

(3) If $\{g_1, \ldots, g_s\} \subset R = S^{-1}\big(k[x_1, \ldots, x_n]/(f_1, \ldots, f_l)\big)$, then we can choose lifts $\widetilde{g}_1, \ldots, \widetilde{g}_s \in k[x_1, \ldots, x_n]$; writing $\mathcal{C}$ for the set of coefficients of $f_i$, $\widetilde{g}_i$, we can take $A = \mathbb{Z}[\mathcal{C}] \subset k$. Then $f_1, \ldots, f_l \in A[x_1, \ldots, x_n]$, so we can take $R_A = A[x_1, \ldots, x_n]/(f_1, \ldots, f_l)$ (or really, some localization) and furthermore $\{g_1, \ldots, g_s\} \subset R_A$.

(4) We can also consider reductions of projective morphisms (in particular resolutions of singularities): say $X = \operatorname{Spec} R$ and we have a projective morphism $\pi : Y \to X$. We can then write

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}^N_k \times X \\ & \overset{\pi}{\searrow} & \downarrow \\ & & X \end{array}$$

so that

$$Y = \operatorname{Proj}\left(\frac{R[T_0, \ldots, T_n]}{(G_1, \ldots, G_l)}\right).$$

The coefficients of the $G_i$ give some finite set inside $R$, and we can thus produce the desired subring $R_A \subset R$ containing all these coefficients, so that we can write $\pi_A : Y_A = \operatorname{Proj}\big(R_A[T_0, \ldots, T_n]/(G_1, \ldots, G_l)\big) \to \operatorname{Spec} R_A$, which becomes $Y \to X$ after tensoring with $k$.

Moreover, if $Y \to \operatorname{Spec} k$ is a smooth morphism (e.g., if it's a resolution of singularities) then $Y_A \times_A \operatorname{Spec}(\operatorname{Frac} A) \to \operatorname{Spec}(\operatorname{Frac} A)$ is smooth, and then there exists an open dense subset $U \subset \operatorname{Spec} A$ such that $Y_P$ is smooth for all $P \in U$. If $\mu \in \operatorname{Spec} A$ and $\mu \cap \mathbb{Z} \neq (p)$, then $Y_p$ is smooth for all $p \gg 0$.

**Remark.** We were somewhat imprecise with localization: we can write $S^{-1}B = \varinjlim_{f \in S} B[f^{-1}]$, and we can rewrite this as

$$\varinjlim B[u]/(uf - 1),$$

and then it's clear how to proceed, at least if we require only that $A$ is flat instead of essentially finite type.

The takeaway from this all is that given $R$ essentially of finite type over a field $k$ of characteristic 0, we get a finite-type (or flat) $\mathbb{Z}$-algebra $A$ and a subring $R_A \subset R$, which we can reduce modulo $p$, and we can likewise reduce a resolution of $\operatorname{Spec} R$ to characteristic $p$.

## Canonical modules in algebraic geometry and commutative algebra

Let $(R, m)$ be normal, local, and essentially finite type over $k$. Write $R = S/I$ for $(S, n)$ a regular local ring essentially of finite type over $k$. Say $d = \dim R$ and $n = \dim S$, and $n = d = c$ is the codimension of $\operatorname{Spec} R$ in $\operatorname{Spec} S$. Write $\omega_S = \bigwedge^n \Omega_{S/k}$, which is a rank-1 free $S$-module, hence isomorphic to $S$ as an $S$-module. We want to show that $\operatorname{Ext}^c_S(R, \omega_S)$ has the following two properties:

*First property: If $E_R$ is the injective hull of $R/m$, then*

$$H^d_m(R) \cong \operatorname{Hom}_R(\operatorname{Ext}^c_S(R, \omega_S), E_R),$$

*and thus $\operatorname{Ext}^c_S(R, \omega_S)$ is a canonical module for $R$ (in the sense of commutative algebra).*

We'll need the following:

**Theorem** (local duality)**.** *If $M$ is a finitely generated $S$-module then*

$$H^i_n(M) \cong \operatorname{Hom}_S(\operatorname{Ext}^i_S(M, \omega_S), E_S).$$

We also need that if $R = S/I$, then $\operatorname{Hom}_S(R, E_S) \cong E_R$.

Putting these together we have that

$$H^d_n(R) \cong \operatorname{Hom}_S(\operatorname{Ext}^{n-d}_S(R, \omega_S), E_S);$$

since $\operatorname{Ext}^{n-d}_S(R, \omega_S)$ is an $R$-module, $\operatorname{Ext}^{n-d}_S(R, \omega_S) = \operatorname{Ext}^{n-d}_S(R, \omega_S) \otimes_S R$, and then tensor-hom duality says that

$$\operatorname{Hom}_S(\operatorname{Ext}^{n-d}_S(R, \omega_S), E_S) = \operatorname{Hom}_R(\operatorname{Ext}^c_S(R, \omega_S), \operatorname{Hom}_S(R, E_S)) = \operatorname{Hom}_R(\operatorname{Ext}^c_S(R, \omega_S), E_R),$$

showing the first property.

*Second property: Say $R$ is normal, so $\omega_R := \Gamma(X, \mathcal{O}_X(K_X))$ can be defined as previously (i.e., $\mathcal{O}_X(K_X)$ is the pushforward of $\bigwedge^d \Omega_{X_{\mathrm{reg}}/k}$ from the regular locus). We claim $\omega_R = \operatorname{Ext}^{n-d}_S(R, \omega_S)$.*

**Remark** (adjunction formula)**.** If $R = S/I$ is regular, then there's a short exact sequence

$$0 \to I/I^2 \to \Omega_S \otimes_S R \to \Omega_R \to 0,$$

and then taking the top exterior power we get

$$\omega_S \otimes_S R = \Big(\bigwedge^{\mathrm{top}} I/I^2\Big) \otimes \omega_R;$$

since $\bigwedge^{\mathrm{top}} I/I^2$ is locally free of rank 1 on $R$, we can move it to the other side to get

$$\omega_R \cong (\omega_S \otimes R) \otimes \Big(\bigwedge^{\mathrm{top}} I/I^2\Big).$$

We write $\omega_S \otimes R = i^* \omega_S$ and $I/I^2 = \mathcal{N}$.

If $R$ is $S/f$, and $\Delta = \operatorname{div} f$, this can be written as $K_R = (K_S + \Delta)|_R$, and this requires only that $R$ is normal.

**Proposition.** *$\operatorname{Ext}^c_S(R, \omega_S)$ is a reflexive $R$-module.*

To see this, one can check that at the regular locus of $R$ this is the same as $\omega_R$, and that via the Koszul complex for $R$ the depth is at least 2.

**Proposition.** *If $R$ is regular local, then* $\mathrm{Ext}^c_S(R,\omega_S) \cong i^*\omega_S \otimes \bigwedge^c \mathcal{N} \cong \omega_R$.

This statement is called the fundamental local isomorphism, and is given in Proposition III.7.2 of "Residues and Duality".

*Proof.* Since $R = S/I$ is regular, we can write $I = (x_1, \dots, x_c)$ for $x_1, \dots, x_c$ a regular sequence on $S$; we can then compute $\mathrm{Ext}^c_S(R,\omega_S)$ using the Koszul complex on the $x_i$. The end of the complex looks like

$$0 \to S \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_c \end{pmatrix}} \bigoplus S \to \cdots,$$

with the last term generated by $\varepsilon_1 \wedge \cdots \wedge \varepsilon_c$; applying $\mathrm{Hom}_S(-,\omega_S)$, we get

$$\cdots \to \bigoplus \mathrm{Hom}_S(S,\omega_S) \xrightarrow{(x_1 \ \cdots \ x_c)} \mathrm{Hom}(S,\omega_S) \to 0.$$

Thinking of the last term of the Koszul complex as $S(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r)$ on the $\varepsilon_i$, when we take cohomology we just get

$$S/(x_1,\dots,x_c) \otimes_S \omega_S \otimes S(\varepsilon_1 \wedge \cdots \wedge \varepsilon_c)^* \cong \left(\omega_S \otimes_S R\right) \otimes_R \bigwedge^c \underbrace{(\bigoplus_{i=1}^c R\varepsilon_i)^*}_{I/I^2}$$

But this is how to calculate $\mathrm{Ext}^c_S(R,\omega_S)$, so we get the desired equality. □

Since everything commutes with localization, we have a reflexive module $\mathrm{Ext}^c_S(R,\omega_S)$ agreeing with $\omega_R$ at all regular points of $R$; by normality of $R$ this occurs at all codimension-1 points, and thus by reflexivity we must have that the two modules agree.

# The multiplier ideal is a universal test ideal (November 29, Eric Canton)

Some necessary results for today are Mehta–Srinivas 1997 and Karen Smith 1997, "$F$-rational rings have rational singularities".

For today, $(R, m)$ will be local noetherian normal and essentially of finite type over a field $k$, and we'll write $Y = \mathrm{Spec}\, R$.

Recall that last time we showed that if $\mathbb{Q} \subset k$ and $\{f_1, \dots, f_r\} \in R.$, then there's $A \subset k$, finite type over $\mathbb{Z}$, and $R_A \subset R$ with $R_A \otimes_A k \cong R$ and $\{f_1,\dots,f_r\} \in R_A$. Moreover, given $\pi : X \to Y$ a resolution of singularities, there exists $\pi_A : X_A \to Y_A$ with $X_A \to \mathrm{Spec}\, A$ smooth.

Therefore, given $M$ a finitely generated $R$-module, choosing a free presentation given by a matrix $(f_{ij})$ of $M$, we can choose $\{f_{ij}\} \subset R_A$, and thus we obtain $M_A = \mathrm{coker}(f_{ij})$. Thus we can transport resolution of singularities and finitely generated modules to objects defined over $A$.

The other thing we did last time was sort out the different meanings of "canonical module". If $R$ is normal, and we write $R = S/I$ for $(S, n)$ a regular local ring essentially of finite type over $k$, and $d = \dim R$, $n = \dim S$, and $c = n - d$, then if we define

$$\omega_R := \mathrm{Ext}^c_S(R, \bigwedge^{\mathrm{top}} \Omega_{S/k}),$$

we showed that:

(1) $\omega_R \cong \Gamma(Y, \mathcal{O}_Y(K_Y))$;

(2) $\mathrm{Hom}_R(\omega_R, E_R) \cong H^d_m(R)$;

It is thus a canonical module in both the sense of algebraic geometry and commutative algebra.

Today, we'll discuss canonical covers and multiplier and test ideals (to reduce to the 1-Gorenstein case), and then discuss the main theorem.

## Canonical covers

Assume that $R$ is $m$-Gorenstein, i.e., we have that

$$\Gamma(Y, \mathcal{O}_Y(mK_Y)) = (\omega_R^{\otimes m})^{**} =: \omega_R^{(m)} \cong R.$$

Fix a specific isomorphism of $\omega_R^{(m)}$ with $R$. Then the canonical cover of $R$ is the (spectrum of the) ring

$$B = R \oplus \omega_R \oplus \omega_R^{(2)} \oplus \cdots \oplus \omega_R^{(m-1)}.$$

This is an $R$-algebra via the maps $\omega_R^{(i)} \otimes \omega_R^{(j)} \to \omega_R^{(i+j \bmod m)}$ given by the identification of $\omega_R^{(m)}$ with $R$. We need the following facts about $B$:

(1) $B$ is normal (check R1 and S2 directly, using that $\omega_R$ is a line bundle at all height-1 primes).

(2) $B$ is local, with maximal ideal $m \oplus \omega_R \oplus \cdots \oplus \omega_R^{(m-1)}$.

(3) $B$ is a finitely generated $R$-module, and thus $\operatorname{Hom}_R(B, \omega_R) = \omega_B$. Moreover, we have

$$\operatorname{Hom}_R(\omega_R^{(i)}, \omega_R) \cong \omega_R^{(i-1 \bmod m)};$$

thus, it's immediate that $\omega_B \cong B$, so $B$ is 1-Gorenstein (or quasi-Gorenstein).

Now, continue to assume $R$ is $m$-Gorenstein, and assume moreover that $k \supset \mathbb{Q}$. Let $Y' = \operatorname{Spec} B$, so we have a diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \operatorname{Spec} R = Y & \xleftarrow[f]{} & Y' \end{array}$$

where $X'$ is obtained from $X \times_Y Y'$ by taking an irreducible component dominating $Y$ and $X$, and taking a resolution of singularities of this component. Thus, we have that $X' \to Y'$ is also a resolution of singularities.

**Lemma.** $\mathcal{J}(Y) = \pi_* \mathcal{O}_X(\lceil K_X - \pi^* K_Y \rceil) = f_* \pi'_* \mathcal{O}_{X'}\big(\lceil K_{X'} - (\pi')^* K_{Y'} \rceil\big) = f_* \mathcal{J}(Y')$.

Karen writes this as $\mathcal{J}(R) = \mathcal{J}(B) \cap R$. Note that this uses that $Y$ is normal, so that $\mathcal{J}(Y) \subset \pi_* \mathcal{O}_X = R$.

If instead $R$ is $m$-Gorenstein and $k \supset \mathbb{F}_p$, with $p \nmid m$, we have the following analogue:

**Lemma.** $\tau(R) = \tau(B) \cap R$.

We omit the proofs of both of these lemmas, although they are interesting!

Now, we return to the case $\mathbb{Q} \subset k$. Build $A \subset k$ starting with $\mathbb{Z}[1/m]$. For $B$, we want to have $\mathcal{J}(B) = \pi'_*(\omega X') = \pi'_* \bigwedge^{\text{top}} \Omega_{X'/k}$. We want $A$ such that if $\mathcal{J}(B) = (g_1, \ldots, g_t)B$ then $(g_1, \ldots, g_t)B_A = (\pi'_A)_* \omega_{X_A}$; this adjoins only finitely many coefficients, so it's okay.

By our lemmas on behavior under canonical covers, we may assume that $R = B$. Now, mod out by $\mu \in \operatorname{Max} A$. Resetting notation, we may assume that $(R, m)$ is 1-Gorenstein, normal, and essentially finite type over $k$, with $\operatorname{char} k = p$. Note then that $Y = \operatorname{Spec} R$ has a resolution of singularities $\pi : X \to Y = \operatorname{Spec} R$, since we brought this with us from characteristic 0. Write $Z = \pi^{-1}(m)$ for the closed fiber. The 1-Gorenstein assumption says that

$$\mathcal{J}(R) = \pi_* \omega_X,$$

and thus we get a short exact sequence

$$0 \to \underbrace{\pi_* \omega_X}_{\mathcal{J}(R)} \to \underbrace{\omega_R}_{R} \to R/\mathcal{J}(R) \to 0.$$

Applying Matlis duality, we get

$$0 \leftarrow \mathrm{Hom}_R(\pi_*\omega_X, E) \leftarrow \mathrm{Hom}_R(\omega_R, E) \leftarrow \mathrm{Hom}_R(R/\mathcal{J}(R), E) \leftarrow 0$$

Now, we have that $\mathrm{Hom}_R(\omega_R, E) = H^d_m(R)$, while $\mathrm{Hom}_R(\pi_*\omega_X, E) = H^d_Z(\mathcal{O}_X)$. Finally, we have $\mathrm{Hom}_R(R/\mathcal{J}(R), E) = (0 :_E J)$. But since $R$ is 1-Gorenstein, we also have that $\mathrm{Hom}(\omega_R, E) \cong E$, so $E \cong H^d_m(R)$.

Now, we use the positive characteristic. Let $F : R \to R$ be the Frobenius and likewise for $F : \mathcal{O}_X \to \mathcal{O}_X$. We get (compatible) maps

$$F : H^d_m(R) \to H^d_m(R), \quad F : H^d_Z(\mathcal{O}_X) \to H^d_Z(\mathcal{O}_X).$$

For $\eta \in H^d_m(R)$ we write $F^e(\eta) = \eta^{p^e}$; then we have an isomorphism

$$(c \otimes \eta \mapsto c\eta^{p^e}) : F^e_* R \otimes H^d_m(R) \to H^d_m(R),$$

which is what people use to study tight closure inside local cohomology modules.

Finally, let $\delta$ be the above map $E = H^d_m(R) \to H^d_Z(\mathcal{O}_X)$.

**Theorem.** $\ker \delta = 0^*_E$, *hence* $\mathcal{J}(R) = \tau(R)$.

*Proof.* By a previous paper of Karen Smith, $\delta$ is $F$-linear, i.e., $F\delta = \delta F$. Moreover, she showed that $0^*_E$ is the largest $F$-stable submodule of $E$.

Now, if $\eta \in \ker\delta$ then clearly $\delta(F(\eta)) = F\delta(\eta) = 0$, so the kernel is $F$-stable and thus $\ker\delta \subset 0^*_E$.

On the other hand, since $F(0^*_E) \subset 0^*_E$, we have $\delta(0^*_E) \subset H^d_Z(\mathcal{O}_X)$ is $F$-stable. However, Mehta–Srinivas showed that $H^d_Z(\mathcal{O}_X)$ has no proper nonzero $F$-stable submodules, and thus if we can show that $\delta(0^*_E) \neq H^d_Z(\mathcal{O}_X)$ then it must be zero and thus the other desired inclusion holds.

So, let $c \in \tau(R) - (0)$; then by definition $c \cdot 0^*_E = 0$. Since $\delta$ is $R$-linear, $c \cdot \delta(0^*_E) = \delta(c0^*_E) = 0$. Thus, $\mathrm{ann}_R(\delta(0^*_E)) \supset \tau(R)$. However, $\mathrm{ann}_R(H^d_Z(\mathcal{O}_X)) = (0)$, and thus $\delta(0^*_E)$ cannot be all of $H^d_Z(\mathcal{O}_X)$, and the result follows. □

Thus, we have that $0^*_E = \ker(\delta) = (0 :_E \mathcal{J}(R))$, and thus $\mathcal{J}(R) \subset \tau$. Applying $\mathrm{Hom}_R(-, E)$ one more time, we have that $(0 :_R 0^*_E) = \tau\widehat{R} = \mathcal{J}(R)\widehat{R}$. Thus we have that $\tau/\mathcal{J}(R) \otimes_R \widehat{R} = 0$, and since $\widehat{R}$ is faithfully flat we must have that $\tau/\mathcal{J}(R) = 0$, so $\tau = \mathcal{J}(R)$.

## Alterations (Haoyang, December 6)

References for this talk include Abramovich and Oort's notes on alterations and de Jong's original paper on smoothness, semistability, and alterations.

Throughout let $k$ be a field. Alterations are a "substitute" for resolution of singularities. Recall first the notion of resolution of singularities:

**Theorem** (Hironaka 1964)**.** *Let* $\mathrm{char}\, k = 0$ *and let* $X$ *be an integral, finite-type scheme over* $k$*; let* $Z \subset X$ *be a closed subvariety. Then there exists a finite sequence of blowups* $X_n \to X_{n-1} \to \cdots \to X_0 = X$ *such that* $X_n$ *is regular and the strict transform of* $Z$ *is an snc divisor.*

Recall that blowups are projective birational morphisms. It is still unknown whether such a theorem holds true in characteristic $p$. Alterations are obtained by weakening the birationality requirement: we ask only that each $X_i \to X_{i-1}$ is generically finite (i.e., it induces a finite extension on fraction fields). In this case, we *do* have such a result, even in positive characteristic:

**Theorem** (de Jong 1995)**.** *Let* $X$ *be an integral, finite-type scheme over* $k$*, and let* $Z \subset X$ *be a closed subscheme. There then exists a morphism* $X' \to X$ *that is separable, proper, surjective, and generically finite, such that* $X'$ *is regular and quasiprojective, and the strict transform of* $Z$ *is snc in* $X'$*.*

**Definition.** A morphism $f : X' \to X$ is called an *alteration* if it's proper, surjective, and generically finite.

The proof involves transforming $X$ into a family of nodal curves, since nodal curves can be resolved by blowing up in arbitrary characteristic.

**Example.** Say $\operatorname{char} k \neq 2$. Let $\mathbb{A}^1 = \operatorname{Spec} k[t]$, and consider the family $X \to \mathbb{A}^1$ defined by $\operatorname{Spec} k[t][x,y]/(xy - t^2)$. This gives a family of curves, with the only singularity occurring at $(0,0,0)$, i.e., the origin of the fiber over $t = 0$. Moreover, each curve is nonsingular except the fiber over 0, which is nodal; thus we say the family is nodal. It's then clear how to resolve $X$: blowing up at $x = (0,0,0)$, we obtain a smooth family.

Inspired by this, the goal is to find an alteration $X' \to X$, with $X' \to P$ a nodal family of curves (with $P$ possibly singular). Then by induction we can take $P$ regular, and thus we can resolve $X'$ by blowing up.

## Sketch of proof

### Step 1: reduction

We can replace $(X, Z)$ by $(X', Z')$ such that $X'$ is projective and normal and $Z'$ is a divisor on $X'$: To do so, we begin by blowing up $Z$ inside of $X$; then use Chow's lemma to obtain a quasiprojective model. These are both clearly alterations. If we now can obtain a regular alteration of the closure of $X$ inside projective space, we can restrict it to the open subset over $X$ itself, and thus may assume that $X$ is projective itself. Finally, we can just take the normalization of $X$, which is finite (hence projective and an alteration).

### Step 2: find a projection

We know that there's a closed immersion $X \hookrightarrow \mathbb{P}^m$.

**Lemma.** *(1) If $\dim X < m - 1$, then there's an open subset of points in $\mathbb{P}^m$ such that projection from these points maps $X$ birationally onto its image.*

*(2) If $\dim X = m - 1$ then there's an open subset of points in $\mathbb{P}^m$ such that projection to a hyperplane maps $X$ generically étale over its image $\mathbb{P}^{\dim X}$.*

We thus have a generically étale map $\pi : X \to \mathbb{P}^d$, with $\dim X = d$. There's a closed locus $B \subset \mathbb{P}^d$ such that $\pi$ is not étale on $B$, and an open subset inside $U \subset \mathbb{P}^d$ such that $\pi(Z)$ maps generically étale onto its image under projection from points in $U$.

Now, choose $p \in U - B$, and consider $\operatorname{Bl}_{\pi^{-1}(p)} X$; this is the incidence correspondence

$$\{(x, l) \in X \times \mathbb{P}^{d-1} : \pi(x) \in l\}$$

(where $\mathbb{P}^{d-1}$ is seen as the space of lines through $\mathbb{P}^d$ passing through $p$).

Thus, we have $f : \operatorname{Bl}_{\pi^{-1}(p)}(X) \hookrightarrow \mathbb{P}^{d-1}$, and one can check this is a family of curves.

### Step 3: improvement of the fibration

There's $U \subset \mathbb{P}^{d-1}$ such that $f^{-1}(U)$ is smooth of dimension 1 over $U$, and thus certainly a "family of nodal curves".

Now, consider the functor from schemes to sets that takes a scheme $T$ to the isomorphism class of families of nodal curves over $T$. We want this to be representable, because this would allow us to enlarge the family $U$. However, to do so, we need to "rigidify" the functor to make it representable. That is, we change the functor to return the family of stable $n$-pointed nodal curves.

**Proposition** (stable extension theorem)**.** *Given an open inclusion $U \subset P$ and a family of nodal curves $\mathbb{C} \to U$, there's an alteration $P' \to P$ with $\mathcal{C}' \to P'$ a family of stable $n$-pointed nodal curves such that the pullback of $\mathcal{C} \to U$ agrees with the restriction.*

We then will obtain in our case a diagram

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & P' \times_P X \\ & \searrow \quad \swarrow & \\ & P' & \end{array}$$

which we claim can be extended to an alteration.

Thus, we can replace $X$ by $\mathcal{C}$; note that $\dim P' = d - 1$ and $X$ is a family of nodal curves over $\mathcal{C}$.

### Induction

By induction, we can then find an alteration $P' \to P$ with $P'$ regular (and such that the pullback of the pull of $Z$ to the fiber product is an snc divisor). Finally, one can resolve singularities by hand via blowing up, obtaining a smooth variety.